Quotienting a Monad by Splitting an Idempotent

Ralph Sarkis
June 18, 2021

Abstract

When studying the monad $\mathcal{C}^{\downarrow}$ which appears in a paper of mine, I worked out a very general construction that yields a quotient of a monad starting with a natural family of idempotent homomorphisms. I do not think the result is very surprising, but I think it may have some applications I am not seeing.

Remark 1. You can skip the parts where we discuss $\mathcal{C}^{\downarrow}$ and $\mathcal{C}(- + \mathbf{1})$, this example was simply our starting point.

The following results were motivated by observing that the free $\mathcal{C}^{\downarrow}$–algebra on X can be nicely embedded into the free $\mathcal{C}(\cdot + \mathbf{1})$–algebra on X and thus, there might be a way to view $\mathcal{C}^{\downarrow}$ as a *submonad* of $\mathcal{C}(\cdot + \mathbf{1})$ or relate their categories of algebras and their presentations. It turns out we will not obtain $\mathcal{C}^{\downarrow}$ as an actual submonad of $\mathcal{C}(\cdot + \mathbf{1})$, but there is still an interesting relation between the two.

What we mean by nicely embedded is that there is an idempotent homomorphism $\mathcal{C}(X + \mathbf{1}) \to \mathcal{C}(X + \mathbf{1})$ of $\perp$–closure which splits through $\mathcal{C}^{\downarrow}X$. In the following, we try to use general abstract assumptions similar to this to attain our goal of relating the presentations of the two monads. In particular, what we call K below is supposed to be an operation akin to $\perp$–closure.

The first restrictions we put on K were not general enough as we later found that $\perp$–closure did not satisfy them. Still, we start by discussing them because they give another way to see our construction.

1 General Construction

Let (M, η, μ) be any monad on a category $\mathbf{C}$, the full subcategory of $\mathrm{EM}(M)$ consisting of only free M–algebras can be identified with the Kleisli category $\mathbf{C}_M$ whose objects are objects of $\mathbf{C}$ and morphisms in $\mathrm{Hom}_{\mathbf{C}_M}(X, Y)$ are morphisms in $\mathrm{Hom}_{\mathbf{C}}(X, MY)$ with composition given by $f \circ_M g = \mu_Y \circ Mf \circ g$.

In this context, a natural family of idempotents like the K_X above is just an idempotent natural transformation $K : \mathrm{id}_{\mathbf{C}_M} \Rightarrow \mathrm{id}_{\mathbf{C}_M}$.

Remark 2. We will soon see that this K induces an idempotent natural transformation $\overline{K} : M \Rightarrow M$ and this may be a better starting point because it is enough for our purposes. However, since $\overline{K}$ does not induce the K described above, we still start from K and at some point we only use $\overline{K}$.

Let us show some properties of K. First, idempotence says that $K_X \circ_M K_X = K_X$, or equivalently,

$$\mu_X \circ M(K_X) \circ K_X = K_X. \tag{1}$$

Next, we can apply naturality of K to different morphisms in $\mathbf{C}_M$ to obtain different identities. Let us use squiggly arrows to denote Kleisli morphisms in diagrams.

$$\begin{tikzcd} MX \arrow[r, squiggly, "\mathrm{id}_{MX}"] \arrow[d, squiggly, "K_{MX}"'] & X \arrow[d, squiggly, "K_X"] \\ MX \arrow[r, squiggly, "\mathrm{id}_{MX}"'] & X \end{tikzcd} \tag{2}$$

$$\begin{tikzcd} MMX \arrow[r, squiggly, "\mu_X"] \arrow[d, squiggly, "K_{MMX}"'] & X \arrow[d, squiggly, "K_X"] \\ MMX \arrow[r, squiggly, "\mu_X"'] & X \end{tikzcd} \tag{3}$$

$$\begin{tikzcd} X \arrow[r, squiggly, "\eta_X"] \arrow[d, squiggly, "K_X"'] & X \arrow[d, squiggly, "K_X"] \\ X \arrow[r, squiggly, "\eta_X"'] & X \end{tikzcd} \tag{4}$$

$$\begin{tikzcd} MX \arrow[r, squiggly, "Mf"] \arrow[d, squiggly, "K_{MX}"'] & Y \arrow[d, squiggly, "K_Y"] \\ MX \arrow[r, squiggly, "Mf"'] & Y \end{tikzcd} \tag{5}$$

Now, recall that these diagrams live in $\mathbf{C}_M$, thus, the following equations are derived from each diagram.

$$\mu_X \circ M(K_X) \circ \mathrm{id}_{MX} = \mu_X \circ M(\mathrm{id}_{MX}) \circ K_{MX} \tag{2}$$
$$\mu_X \circ M(K_X) \circ \mu_X = \mu_X \circ M(\mu_X) \circ K_{MMX} \tag{3}$$
$$\mu_X \circ M(K_X) \circ \eta_X = \mu_X \circ M(\eta_X) \circ K_X \tag{4}$$
$$\mu_Y \circ M(K_Y) \circ Mf = \mu_Y \circ MMf \circ K_{MX} \tag{5}$$

From this, we will construct a monad M^K whose free algebra on X is the image of K applied to the free M–algebra on X. We will also construct a monad map $M \Rightarrow M^K$ expressing M^K as a quotient of M.

Proposition 3. *Defining $\overline{K}_X : MX \to MX = \mu_X \circ M(K_X)$, we can show that $\overline{K}_X$ is idempotent.*

Proof. We show that $\mu_X \circ M(K_X) \circ \mu_X \circ M(K_X) = \mu_X \circ M(K_X)$ by paving the following diagram.

$$\begin{tikzcd} MX \arrow[rr, "M(K_X)"] \arrow[d, "M(K_X)"'] \arrow[dr, phantom, "\text{(a)}"] & & MMX \arrow[dd, "\mu_X"] \\ MMX \arrow[r, "MM(K_X)"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\text{(b)}"] & MMMX \arrow[ur, "M(\mu_X)"] \arrow[d, "\mu_{MX}"] \arrow[r, phantom, "\text{(c)}"] & {} \\ MX \arrow[r, "M(K_X)"'] & MMX \arrow[r, "\mu_X"'] & MX \end{tikzcd} \tag{6}$$

(a) Apply M to (1).

(b) Naturality of μ.

(c) Associativity of μ.

□

Furthermore, since this $\overline{K}_X$ is the image of K_X under the embedding $\mathbf{C}_M \to \mathrm{EM}(M)$, we obtain that $\overline{K}_X$ is an M–algebra homomorphism. This is restated and proven for completeness below.

Lemma 4. *For any X, we have $\overline{K}_X \circ \mu_X = \mu_X \circ M(\overline{K}_X)$.*

Proof. We pave the following diagram.

$$
\begin{tikzcd}
MMX \arrow[rr, "M(\overline{K}_X)"] \arrow[dr, "MM(K_X)"'] \arrow[ddd, "\mu_X"'] & \text{(a)} & MMX \arrow[ddd, "\mu_X"] \\
\text{(b)} & MMMX \arrow[ur, "M(\mu_X)"'] \arrow[d, "\mu_{MX}"'] & \text{(c)} \\
& MMX \arrow[dr, "\mu_X"] & \\
MX \arrow[ur, "M(K_X)"] \arrow[rr, "\overline{K}_X"'] & \text{(d)} & MX
\end{tikzcd}
\tag{7}
$$

(a) Def of $\overline{K}_X$ and functoriality of M.

(b) Naturality of μ.

(c) Associativity of μ.

(d) Def of $\overline{K}_X$.

$\square$

Now, if $\mathbf{C}$ has equalizers,[1] we can define a subfunctor M^K of M as follows. For $X \in \mathbf{C}_0$, let $M^K X$ be the equalizer of $\overline{K}_X, \mathrm{id}_{MX} : MX \to MX$. Namely, there is (a monic) $\iota_X : M^K X \to MX$ satisfying $\overline{K}_X \circ \iota_X = \iota_X$ such that for any $e : Y \to MX$ satisfying $\overline{K}_X \circ e = e$, there is a unique morphism $! : Y \to M^K X$ making (8) commute.

$$
\begin{tikzcd}
Y \arrow[d, dashed, "!"'] \arrow[dr, "e"] & & \\
M^K X \arrow[r, "\iota_X"'] & MX \arrow[r, shift left, "\overline{K}_X"] \arrow[r, shift right, "\mathrm{id}_{MX}"'] & MX
\end{tikzcd}
\tag{8}
$$

In order to give the action of M^K on morphisms, we need the following lemma.

Lemma 5. *$\overline{K}$ is a natural transformation $M \Rightarrow M$.*

Proof. We need to show that for any $f : X \to Y$, $\overline{K}_Y \circ Mf = Mf \circ \overline{K}_X$. We have the following derivation.

$$
\begin{aligned}
\overline{K}_Y \circ Mf &= \mu_Y \circ M(K_Y) \circ Mf && \text{def. } \overline{K} \\
&= \mu_Y \circ MMf \circ K_{MX} && \text{by (5)} \\
&= Mf \circ \mu_X \circ K_{MX} && \text{naturality of } \mu \\
&= Mf \circ \mu_X \circ M(K_X) && \text{by (2)} \\
&= Mf \circ \overline{K}_X && \text{def. } \overline{K}
\end{aligned}
$$

$\square$

Remark 6. From this point, we do not have to use any hypothesis about K. Thus, starting with an idempotent natural transformation $\overline{K} : M \Rightarrow M$ such that $\overline{K}_X : MX \to MX$ is an M–algebra homomorphism (with the free algebra structure on MX), we can develop the rest of the section. Another very close starting point will be used in the application section.

[1] I am pretty sure it is enough if idempotents split in $\mathbf{C}$.

Now, for any $f : X \to Y$, we know that both squares on the R.H.S. of diagram (9) commute (id is trivially a natural transformation).

$$\begin{tikzcd} M^K X \arrow[r, "\iota_X"] \arrow[d, dashed, "M^K f"'] & MX \arrow[r, shift left, "\overline{K}_X"] \arrow[r, shift right, "\mathrm{id}_{MX}"'] \arrow[d, "Mf"'] & MX \arrow[d, "Mf"] \\ M^K Y \arrow[r, "\iota_Y"'] & MY \arrow[r, shift left, "\overline{K}_Y"] \arrow[r, shift right, "\mathrm{id}_{MY}"'] & MY \end{tikzcd} \tag{9}$$

From this, we can infer that $Mf \circ \iota_X$ equalizes $\overline{K}_Y$ and id_{MY}. Indeed, we have

$$\begin{aligned} \overline{K}_Y \circ Mf \circ \iota_X &= Mf \circ \overline{K}_X \circ \iota_X \\ &= Mf \circ \mathrm{id}_{MX} \circ \iota_X \\ &= \mathrm{id}_{MY} \circ Mf \circ \iota_X. \end{aligned}$$

Then, from the universality of $M^K Y$, there is a unique morphism $M^K f : M^K X \to M^K Y$ making (9) commute. The uniqueness of $M^K f$ in (9) also shows that $M^K(f \circ g) = M^K f \circ M^K g$, thus M^K is a functor $\mathbf{C} \to \mathbf{C}$.

Proposition 7. *The family $\{\iota_X \mid X \in \mathbf{C}_0\}$ is a natural transformation $\iota : M^K \Rightarrow M$ with monic components, so M^K is a subfunctor of M.*

Proof. The naturality follows trivially from the commutativity of left square in (9). The monicity comes from the standard result that equalizers are monic. $\square$

Observe that by idempotence, $\overline{K}_X$ also equalizes $\overline{K}_X$ and id_{MX}, so we get the following diagram.

$$\begin{tikzcd} MX \arrow[d, dashed, "\widehat{K}_X"'] \arrow[dr, "\overline{K}_X"] & & \\ M^K X \arrow[r, "\iota_X"'] & MX \arrow[r, shift left, "\overline{K}_X"] \arrow[r, shift right, "\mathrm{id}_{MX}"'] & MX \end{tikzcd} \tag{10}$$

In the sequel, we will denote by $\widehat{K}_X$ the unique morphism satisfying $\iota_X \circ \widehat{K}_X = \overline{K}_X$.

Lemma 8. *For any $X \in \mathbf{C}_0$, $\widehat{K}_X \circ \iota_X = \mathrm{id}_{M^K X}$.*

Proof. Using the definitions of $\widehat{K}_X$ and ι_X, we have

$$\iota_X \circ \widehat{K}_X \circ \iota_X = \overline{K}_X \circ \iota_X = \mathrm{id}_{MX} \circ \iota_X = \iota_X \circ \mathrm{id}_{M^K X}.$$

The lemma follows by monicity of ι_X. $\square$

Remark 9. One way to summarize this is to say that $\iota_X \circ \widehat{K}_X$ is the splitting of the idempotent $\overline{K}_X$.

Proposition 10. *The family $\{\widehat{K}_X \mid X \in \mathbf{C}_0\}$ is a natural transformation with epic components $\widehat{K} : M \Rightarrow M^K$.*

Proof. First, we claim that for any $f : X \to Y$, $M^K f \circ \widehat{K}_X = \widehat{K}_Y \circ Mf$. We have the following derivation.

$$\begin{aligned} \iota_Y \circ M^K f \circ \widehat{K}_X &= Mf \circ \iota_X \circ \widehat{K}_X && \text{naturality of } \iota \\ &= Mf \circ \overline{K}_X && \text{def of } \widehat{K}_X \end{aligned}$$

$$\begin{aligned} &= \overline{K}_Y \circ Mf && \text{naturality of } \overline{K} \\ &= \iota_Y \circ \widehat{K}_Y \circ Mf && \text{def of } \widehat{K}_Y \end{aligned}$$

The claim follows since ι_Y is a monomorphism. The components $\widehat{K}_X$ are epimorphisms because they have ι_X as a right inverse by Lemma 8. □

1.1 Monadicity of M^K

Next, we want to show that M^K is a monad with unit $\eta^K := \widehat{K} \cdot \eta$ and multiplication $\mu^K := \widehat{K} \cdot \mu \cdot (\iota \diamond \iota)$.[2] We divide the proof in multiple lemmas.

Lemma 11. *For any $X \in \mathbf{C}_0$, $\mu_X \circ M(\iota_X) = \overline{K}_X \circ \mu_X \circ M(\iota_X)$.*

Proof. We pave the following diagram.

$$\begin{tikzcd}
 & MMX \arrow[rr, "\mu_X"] \arrow[d, "M(\overline{K}_X)"'] \arrow[dr, "MM(K_X)"] & \arrow[d, phantom, "\text{(c)}"] & MX \arrow[d, "M(K_X)"] \\
MM^KX \arrow[ur, "M(\iota_X)"] \arrow[dr, "M(\iota_X)"'] \arrow[r, phantom, "\text{(a)}"] & \arrow[r, phantom, "\text{(b)}"] & MMMX \arrow[r, "\mu_{MX}"'] \arrow[dl, "M(\mu_X)"] \arrow[d, phantom, "\text{(d)}"] & MMX \arrow[d, "\mu_X"] \\
 & MMX \arrow[rr, "\mu_X"'] & & MX
\end{tikzcd} \tag{11}$$

(a) By defintion of M^K.

(b) Functoriality of M and def of $\overline{K}_X$.

(c) By naturality of μ.

(d) Associativity of μ.

□

Lemma 12. *Anagolously to* (2)*, we also have $\mu_X \circ M(\overline{K}_X) = \mu_X \circ \overline{K}_{MX}$.*

Proof. We pave the following diagram.

$$\begin{tikzcd}
MMX \arrow[r, "MM(K_X)"] \arrow[rr, bend left, "M(\overline{K}_X)"] \arrow[dr, "M(K_{MX})"] \arrow[dd, "\overline{K}_{MX}"'] & MMMX \arrow[d, phantom, "\text{(a)}"] & MMX \arrow[dd, "\mu_X"] \\
\text{(b)} & MMMX \arrow[ur, "M(\mu_X)"'] \arrow[dl, "\mu_{MX}"] & \text{(c)} \\
MMX \arrow[rr, "\mu_X"'] & & MX
\end{tikzcd} \tag{12}$$

(a) Apply M to (2).

(b) Def of $\overline{K}$.

(c) Associativity of μ.

□

[2] We write $\diamond$ for the horizontal composition of natural tranformations.

Now we can prove one side of the unit diagram for the monad M^K commutes.

Lemma 13. *For any $X \in \mathbf{C}_0$, $\mu^K_X \circ M^K(\eta^K_X) = \mathrm{id}_{M^K X}$.*

Proof. We will show that $\iota_X \circ \mu^K_X \circ M^K(\eta^K_X) = \iota_X$ from which the result follows by monicity of ι_X. We pave the following diagram.

$$
\begin{tikzcd}
M^K X \arrow[r, "M^K(\eta_X)"] \arrow[d, "\iota_X"] \arrow[rr, bend left, "M^K(\eta^K_X)"] \arrow[dddd, bend right, "\iota_X"'] \arrow[dr, phantom, "(a)"] & M^K M X \arrow[r, "M^K(\widehat{K}_X)"] \arrow[d, "\iota_{MX}"] \arrow[dr, phantom, "(b)"] & M^K M^K \arrow[d, "\iota_{M^K X}"] \arrow[dddd, bend left, "\mu^K_X"] \\
MX \arrow[r, "M(\eta_X)"] \arrow[dd, "\mathrm{id}_{MX}"] \arrow[d, phantom, "(c)"] & MMX \arrow[r, "M(\widehat{K}_X)"] \arrow[d, "M(\overline{K}_X)"] \arrow[ddl, "\mu_X"] \arrow[r, phantom, "(d)"'] & MM^K X \arrow[dl, "M(\iota_X)"'] \arrow[d, "M(\iota_X)"] \arrow[d, phantom, "(e)"'] \\
\arrow[r, phantom, "(f)"] & MMX \arrow[ddl, "\mu_X"] \arrow[dl, phantom, "(g)"] \arrow[d, phantom, "(h)"] & MMX \arrow[l, "M(\overline{K}_X)"'] \arrow[d, "\mu_X"] \\
MX \arrow[d, "\overline{K}_X"] & & MX \arrow[dll, "\overline{K}_X"] \arrow[d, "\widehat{K}_X"] \arrow[dl, phantom, "(i)"] \\
MX & & M^K X \arrow[ll, "\iota_X"]
\end{tikzcd}
\tag{13}
$$

(a) Naturality of ι.

(b) Naturality of ι.

(c) Monadicity of (M, η, μ).

(d) Apply M to $\overline{K}_X = \iota_X \circ \widehat{K}_X$.

(e) Apply M to $\overline{K}_X \circ \iota_X = \iota_X$.

(f) $\overline{K}_X \circ \iota_X = \iota_X$.

(g) Lemma 4.

(h) Lemma 4.

(i) Def of $\widehat{K}$.

□

Now for the other side of the unit diagram.

Lemma 14. *For any $X \in \mathbf{C}_0$, $\mu^K_X \circ \eta^K_{M^K X} = \mathrm{id}_{M^K X}$.*

Proof. Alternatively, we pave the following diagram.

$$
\begin{tikzcd}
M^K X \arrow[rr, "\eta_{M^K X}"] \arrow[dd, "\mathrm{id}_{M^K X}"'] \arrow[dr, "\iota_X"] & & MM^K X \arrow[rr, "\widehat{K}_{M^K X}"] \arrow[d, "M(\iota_X)"'] \arrow[dr, "M(\iota_X)"] \arrow[dl, phantom, "(a)"] \arrow[dr, phantom, "(b)"'] & & M^K M^K X \arrow[d, "M^K(\iota_X)"'] \arrow[dd, bend left, equal] \\
 & MX \arrow[r, "\eta_{MX}"] \arrow[dr, "\mathrm{id}_{MX}"'] \arrow[dl, phantom, "(c)"] \arrow[r, phantom, "(d)"', shift right=4] & MMX \arrow[d, "\mu_X"] \arrow[r, phantom, "(e)"] & MMX \arrow[r, "\widehat{K}_{MX}"] \arrow[dr, "\overline{K}_{MX}"'] \arrow[r, phantom, "(f)"', shift right=4] & M^K MX \arrow[d, "\iota_{MX}"] \\
M^K X & & MX \arrow[ll, "\widehat{K}_X"] & & MMX \arrow[ll, "\mu_X"]
\end{tikzcd}
\tag{14}
$$

(a) Naturality of η.

(b) Naturality of $\widehat{K}$.

(c) Lemma 8.

(d) Monadicity of (M, μ, η).

(e) Lemma **??**.

(f) Definition of $\widehat{K}_{MX}$.

$\square$

Lastly, we show that μ^K is associative.

Lemma 15. *For any $X \in \mathbf{C}_0$, $\mu^K \circ M^K\mu^K = \mu^K \circ \mu^K M^K$.*

Proof. We pave the following diagram.

$$\begin{tikzcd}
M^K M^K M^K \arrow[r, "M^K\iota M^K"] \arrow[rr, bend left, "M^K(\iota\diamond\iota)"] \arrow[d, "\iota M^K M^K"'] \arrow[dd, bend right=60, "(\iota\diamond\iota)M^K"'] \arrow[dr, phantom, "(a)"] & M^K M M^K \arrow[r, "M^K M\iota"] \arrow[d, "\iota M M^K"'] \arrow[dr, phantom, "(b)"] & M^K M M \arrow[r, "M^K\mu"] \arrow[d, "\iota MM"'] \arrow[dr, phantom, "(c)"] & M^K M \arrow[r, "M^K\widehat{K}"] \arrow[d, "\iota M"'] \arrow[dr, "M^K\overline{K}"'] \arrow[r, phantom, shift right=3ex, "(d)"] & M^K M^K \arrow[d, "M^K\iota"] \arrow[dd, bend left=60, "\iota\diamond\iota"] \\
M M^K M^K \arrow[r, "M\iota M^K"'] \arrow[d, "M\iota M^K"'] & M M M^K \arrow[r, "MM\iota"'] \arrow[dr, "MM\iota"'] \arrow[dddd, "\mu M^K"'] & M M M \arrow[r, "M\mu"'] \arrow[dr, phantom, "(f)"] & M M \arrow[dr, "M\overline{K}"'] \arrow[r, phantom, "(e)"] & M^K M \arrow[d, "\iota M"] \\
M M M^K \arrow[d, "\mu M^K"'] \arrow[r, phantom, "(g)"] & & M M M \arrow[rr, "M\mu"'] \arrow[dd, "\mu M"'] & & M M \arrow[d, "\mu"] \\
M M^K \arrow[d, "\widehat{K} M^K"'] \arrow[dr, "\overline{K} M^K"] \arrow[ddr, phantom, "(j)"] & \arrow[r, phantom, "(h)"] & \arrow[r, phantom, "(i)"] & & M \arrow[d, "\widehat{K}"] \\
M^K M^K \arrow[r, "\iota M^K"'] \arrow[rr, bend right, "\iota\diamond\iota"'] & M M^K \arrow[r, "M\iota"'] & M M \arrow[r, "\mu"'] & M \arrow[r, "\widehat{K}"'] \arrow[ur, equal] & M^K
\end{tikzcd}$$

(a) Naturality of ι.

(b) Naturality of ι.

(c) Naturality of ι.

(d) Def of $\widehat{K}$.

(e) Naturality of ι.

(f) Lemma 11 acted on the left by M.

(g) Lemma 11 acted on the right by M^K.

(h) Naturality of μ.

(i) Associativity of μ.

(j) Def of $\widehat{K}_{M^K X}$.

$\square$

Theorem 16. *The triple (M^K, η^K, μ^K) is a monad.*

Proof. We have to show the following diagrams commute.

$$\begin{tikzcd}
M^K \arrow[r, Rightarrow, "M\eta^K"] \arrow[dr, Rightarrow, "\mathbb{1}_{M^K}"'] & (M^K)^2 \arrow[d, Rightarrow, "\mu^K"] & M^K \arrow[l, Rightarrow, "\eta^K M^K"'] \arrow[dl, Rightarrow, "\mathbb{1}_{M^K}"] \\
& M^K &
\end{tikzcd} \qquad (15)$$

$$\begin{tikzcd}
(M^K)^3 \arrow[r, Rightarrow, "M^K\mu^K"] \arrow[d, Rightarrow, "\mu^K M^K"'] & (M^K)^2 \arrow[d, Rightarrow, "\mu^K"] \\
(M^K)^2 \arrow[r, Rightarrow, "\mu^K"'] & M
\end{tikzcd} \qquad (16)$$

Lemmas 13, 14 and 15 respectively show the commutativity of the L.H.S. of (15), the R.H.S. of (15) and (16). $\square$

1.2 Relating M–algebras and M^K–algebras

We already have natural transformations ι and $\widehat{K}$ between M and M^K in both directions, but it is not enough to relate their algebras. For that, we would need for ι and $\widehat{K}$ to be monad maps. Unfortunatley, while $\widehat{K}$ is a monad map as shown below, we have to proceed differently for the other direction.

Theorem 17. *The natural transformation $\widehat{K} : M \Rightarrow M^K$ is a monad map.*

Proof. We have to show the following diagrams commute.

$$\begin{tikzcd}
\mathrm{id}_{\mathbf{C}} \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta^K"'] & M \arrow[d, Rightarrow, "\widehat{K}"] \\
& M^K
\end{tikzcd} \qquad (17) \qquad\qquad
\begin{tikzcd}
M^2 \arrow[r, Rightarrow, "\widehat{K}\circ\widehat{K}"] \arrow[d, Rightarrow, "\mu"'] & (M^K)^2 \arrow[d, Rightarrow, "\mu^K"] \\
M \arrow[r, Rightarrow, "\widehat{K}"'] & M^K
\end{tikzcd} \qquad (18)$$

(17) is trivial because that is the definition of η^K. For (18), we pave the following diagram.

$$\begin{tikzcd}
MMX \arrow[rr, "\widehat{K}_{MX}"] \arrow[rrrr, Rightarrow, bend left, "\widehat{K}\circ\widehat{K}"] \arrow[drr, "\overline{K}_{MX}"'] \arrow[ddr, "\overline{K}_{MX}"] \arrow[dddd, "\mu_X"'] & & M^KMX \arrow[rr, "M^K(\widehat{K}_X)"] \arrow[d, "\iota_{MX}"] \arrow[drr, "M^K(\overline{K}_X)"] \arrow[dl, phantom, "\text{(a)}"] & & \mu^K_X \arrow[d, "M^K(\iota_X)"] \arrow[dddd, bend left, "\mu^K_X"] \arrow[dl, phantom, "\text{(b)}"] \\
& \text{(c)} & MMX \arrow[dl, "\overline{K}_{MX}"'] \arrow[drr, "M(\overline{K}_X)"] & \text{(d)} & M^KMX \arrow[d, "\iota_{MX}"] \\
\text{(e)} & MMX \arrow[d, "\mu_X"] \arrow[drrr, "\mu_X"] & & \text{(f)} & MMX \arrow[d, "\mu_X"] \\
& MX \arrow[drrr, "\widehat{K}_X"] & & \text{(g)} & MMX \arrow[d, "\widehat{K}_X"] \\
MX \arrow[ur, "\overline{K}_X"'] \arrow[rrrr, "\widehat{K}_X"'] & & \text{(h)} & & M^KX
\end{tikzcd} \qquad (19)$$

(a) Def of $\widehat{K}_{MX}$.

(b) Def of $\widehat{K}_X$.

(c) Idempotence of $\widehat{K}_{MX}$.

(d) Naturality of ι.

(e) Lemmas 4 and 12.

(f) Lemma 12.

(g) Paths are equal.

(h) Lemma 8 and $\overline{K}_X = \iota_X \circ \widehat{K}_X$.

□

From a standard result, we obtain a functor $U^K : \mathrm{EM}(M^K) \to \mathrm{EM}(M)$ that sends an algebra (A, α) to $(A, \alpha \circ \widehat{K}_A)$ and acts trivially on morphisms. It is fully faithful because $\widehat{K}$ has epic components.

To go in the other direction, our first attempt was to use the embedding $\iota : M^K \Rightarrow M$ in the following way. Given an M–algebra $\alpha : MA \to A$, we expected that the composition $M^KA \xrightarrow{\iota_A} MA \xrightarrow{\alpha} a$ was the natural M^K–algebra on A corresponding to α.

However, in general $\alpha \circ \iota_A$ is not an M^K–algebra because it might not satisfy the unit law, that is,

$$\alpha \circ \iota_A \circ \eta^K = \mathrm{id}_A.$$

In other words, ι is possibly not a monad map (as we will see in the application in to $\mathcal{C}^{\downarrow}$).

2 Constructing $\mathcal{C}^{\downarrow}$

Let $M = \mathcal{C}(\cdot + \mathbf{1})$ be the monad of non-empty finitely generated convex sets of subdistributions, we will show that the monad $\mathcal{C}^{\downarrow}$ can be constructed with the procedure detailed above. The main idea is that the operation of $\perp$–closure satisfies the properties of $\overline{K}$.

Definition 18. Let X be a set and let $S \in \mathcal{C}(X + \mathbf{1})$. We say that S is $\perp$–closed if for all $\varphi \in S$,

$$\{\psi \in \mathcal{D}(X + \mathbf{1}) \mid \forall x \in X, \psi(x) \leq \varphi(x)\} \subseteq S.$$

For a set X, we define $K_X : X \to \mathcal{C}(X + \mathbf{1}) = x \mapsto cc\left(\{\delta_x, \delta_\star\}\right)$. We will first show that $\overline{K}_X = \mu_X \circ \mathcal{C}(K_X + \mathbf{1})$ is the operation of $\perp$–closure, then that $\overline{K}_X$ satisfies the properties described in the previous sections and finally detail the monad we obtain.

Lemma 19. *Let X be a set, for any $S \in \mathcal{C}(X + \mathbf{1})$, $\overline{K}_X(S)$ is the smallest $\perp$–closed set containing S.*

Proof. See Theorem 35 here. ☐

Lemma 20. *The family $K_X : X \to \mathcal{C}(X + \mathbf{1})$ is natural.*

Proof. For any $f : X \to Y$, we have

$$K_Y(f(x)) = cc\left(\left\{\delta_{f(x)}, \delta_\star\right\}\right) = \mathcal{C}(f + \mathbf{1})(cc\left(\{\delta_x, \delta_\star\}\right)) = \mathcal{C}(f + \mathbf{1})(K_X(x))).$$

☐

Lemma 21. *The family $\overline{K}_X : \mathcal{C}(X + \mathbf{1}) \to \mathcal{C}(X + \mathbf{1})$ satisfies the following properties:*

1. *it is natural,*
2. *each component is idempotent, and*
3. *each component is a homomorphism between the free $\mathcal{C}(\cdot + \mathbf{1})$–algebras.*

Proof. 1. This is a corollary of $K : \mathrm{id}_{\mathbf{Set}} \Rightarrow \mathcal{C}(\cdot + \mathbf{1})$ being natural as shown in the following derivation. We need to show that for any $f : X \to Y$, we have $\overline{K}_Y \circ \mathcal{C}(f + \mathbf{1}) = \mathcal{C}(f + \mathbf{1}) \circ \overline{K}_X$. This follows from the following derivation.

$$\begin{aligned}
\overline{K}_Y \circ \mathcal{C}(f + \mathbf{1}) &= \mu_Y \circ \mathcal{C}(K_Y + \mathbf{1}) \circ \mathcal{C}(f + \mathbf{1}) && \text{def of } \overline{K}_Y \\
&= \mu_Y \circ \mathcal{C}(\mathcal{C}(f + \mathbf{1}) + \mathbf{1}) \circ \mathcal{C}(K_X + \mathbf{1}) && \text{nat of } K \\
&= \mathcal{C}(f + \mathbf{1}) \circ \mu_X \circ \mathcal{C}(K_X + \mathbf{1}) && \text{nat of } \mu \\
&= \mathcal{C}(f + \mathbf{1}) \circ \overline{K}_X && \text{def of } \overline{K}_X
\end{aligned}$$

2. Since $\overline{K}_X(S)$ is $\perp$–closed, it is the smallest $\perp$–closed containing itself, thus $\overline{K}_X(\overline{K}_X(S)) = \overline{K}_X(S)$.

3. This holds because $\overline{K}_X$ is the image of K_X (seen as a Kleisli morphism) under the embedding of the Kleisli category of $\mathcal{C}(\cdot + \mathbf{1})$ into $\mathrm{EM}(\mathcal{C}(\cdot + \mathbf{1}))$.

$\square$

Remark 22. Apart from the second point, the above proof is very general. Namely, it shows that starting from a natural transformation $K : \mathrm{id}_{\mathbf{C}} \Rightarrow M$ such that $\overline{K}$ is idempotent, we can derive all the previous sections.

We find that $\mathcal{C}^{\downarrow}$ is the monad of non-empty finitely generated $\perp$–closed convex sets of subdistributions with the unit being $x \mapsto K_X(x) = \overline{K}_X \circ \eta_X$. For the multiplication, there is a slight surprise; it turns out that the multiplication of $\perp$–closed sets is already $\perp$–closed, so there is no need to apply $\perp$–closure again as in the general case.

In particular, this means the inclusion $\iota : \mathcal{C}^{\downarrow} \Rightarrow \mathcal{C}(\cdot + \mathbf{1})$ is not a monad map *only* because it does not commute with the units of the two monads.

3 Conclusion

Let (M, η, μ) be a monad on a category $\mathbf{C}$ where idempotents split. If you have a natural family of idempotent homomorphisms of free M–algebras $MX \to MX$ given in either of the following ways, then you obtain a monad M^K by splitting these idempotents.

- An idempotent natural transformation $K : \mathrm{id}_{\mathbf{C}_M} \Rightarrow \mathrm{id}_{\mathbf{C}_M}$.
- An natural transformation $\overline{K} : M \Rightarrow M$ such that $\overline{K}_X$ is a homomorphism.[3]
- A natural transformation $K : \mathrm{id}_{\mathbf{C}} \Rightarrow M$ such that $\mu \circ MK$ is idempotent.

[3]Equivalently, a natural transformatin $F^M \Rightarrow F^M$, where F^M is the free algebra functor.