Course notes for Math 6170: Algebraic geometry

Cliff Blakestad, Matt Grimes, Jonathan Lamar,
Paul Lessard, Keli Parker, Mason Pelfrey, Hao Song, Jonathan Wise

April 28, 2013

Contents

1 Introduction

These notes are intended as a companion to lecture and the other textual references. They are not complete, but I hope that they will become so with your contributions over the duration of the semester. One of the ways you can contribute is by discovering—or correcting!—the errors that will inevitably appear in this text.

2 Category theory

2.1 References

See [Vak, Chapter 2], especially §2.2 and §2.3. Grothendieck's Tôhoku paper [Gro1, Chapitre I] is a very thorough introduction to abelian categories. Vakil's §2.3 contains a number of excellent exercises about universal properties; you may want to work some of those exercises and contribute solutions to these notes!

2.2 Introduction

The principle underlying category theory is that what are truly important are the relationships between mathematical objects, as opposed to the objects themselves.

2.3 The definition of a category

Scholium 1. *There are some set theoretic technicalities associated with category theory that will be ignored in these notes. The technical problem is that the collection of objects of a category, such as the category* Sets, *is too large to form* *a set itself. One therefore requires a language in which to speak about such collections. This can be frustrating as the additional language adds a layer of technicality to the discussion yet seems to add nothing to it.*

*We will avoid them by assuming the existence of an uncountable universe. A set is called **small** if it is a member of this universe and large otherwise. All sets will be assumed small unless otherwise specified, so that all rings, groups, etc. without qualification are likewise assumed small. A set that is not assumed to be small is called a **class**.*

Definition 2.3.1. A **category** C consists of the following data:

CAT1 a class of **objects** $\mathbf{Ob}(C)$,

CAT2 for each pair $X, Y \in \mathbf{Ob}(C)$, a set $\mathrm{Hom}_C(X, Y)$ of **morphisms** from X to Y,

CAT3 for each $X \in \mathbf{Ob}(C)$, a distinguished morphism $\mathrm{id}_X \in \mathrm{Hom}_C(X, X)$,

CAT4 for each $X, Y, Z \in \mathbf{Ob}(C)$, a **composition law** $\mathrm{Hom}_C(X, Y) \times \mathrm{Hom}_C(Y, Z) \to \mathrm{Hom}_C(X, Z) : (f, g) \mapsto g \circ f$,

subject to the following conditions:

CAT5 composition is associative: $h \circ (g \circ f) = (h \circ g) \circ f$ when the expression makes sense, and

CAT6 $f \circ \mathrm{id}_X = \mathrm{id}_X \circ f = f$ whenever the expression makes sense.

We denote by $\mathbf{Mor}(\mathscr{C})$ the class $\coprod_{X,Y \in \mathbf{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(X, Y)$ of all morphisms of $\mathscr{C}$.

A **subcategory** of a category $\mathscr{C}$ is a category $\mathscr{D}$ such that $\mathbf{Ob}(\mathscr{D}) \subset \mathbf{Ob}(\mathscr{C})$ and $\mathbf{Mor}(\mathscr{C}) \subset \mathbf{Mor}(\mathscr{D})$ and the composition law of $\mathscr{D}$ is the same as that of $\mathscr{C}$. We say that $\mathscr{D}$ is a **full subcategory** if $\mathrm{Hom}_{\mathscr{D}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$ for every $X, Y \in \mathbf{Ob}(\mathscr{D})$.

ex:category-examples

Exercise 2.3.2. The following are categories:

(i) The objects of Sets are sets and the morphisms are all functions.

(ii) The objects of Grp are groups and the morphisms are group homomorphisms.

(iii) The objects of Ab are abelian groups and the morphisms are group homomorphisms. This is a full subcategory of Grp.

(iv) The objects of Rng are rings and the morphisms are ring homomorphisms.

(v) The objects of ComRng are commutative rings and the morphisms are homomorphisms of rings. This is a full subcategory of Rng.

(vi) For a fixed group G, let $\mathbf{Ob}(G\text{-}\mathsf{Sets})$ be the class of all pairs (S, α) where S is a set and $\alpha : S \times G \to S$ is a right action of G on S. Define $\mathrm{Hom}_{G\text{-}\mathsf{Sets}}((S, \alpha), (T, \beta))$ to be the set of functions $f : S \to T$ such that $\beta(f(s), g) = f(\alpha(s, g))$ for all $s \in S$ and $g \in G$.

(vii) For a fixed ring R, define $\mathbf{Ob}(R\text{-}\mathsf{Mod})$ to be the class of left R-modules and $\mathbf{Mor}(R\text{-}\mathsf{Mod})$ to be R-module homomorphisms.

(viii) Let $\mathbf{Ob}(C)$ be the collection of all pairs (R, M) where R is a ring and M is a left R-module. Define $\mathrm{Hom}_C((R, M), (R', M'))$ to be the set of all pairs (φ, ψ) where $\varphi : R \to R'$ is a ring homomorphism and $\psi : M \to M'$ is a function such that $\psi(\lambda.x) = \varphi(\lambda).\psi(x)$ for all $\lambda \in R$ and $x \in M$.

(ix) The objects of Top are topological spaces and the morphisms are continuoks functions.

(x) Let C be a category and X an object of C. Define $\mathbf{Ob}(C/X)$ to be the class of pairs (Y, φ) where Y is an object of C and $\varphi : Y \to X$ is a morphism of C. Define $\mathrm{Hom}_{C/X}((Y, \varphi), (Z, \psi))$ to be the set of $\alpha : Y \to Z$ in $\mathrm{Hom}_C(Y, Z)$ such that $\psi\alpha = \varphi$.

A morphism of a category is called a **isomorphism** if it has an left and right inverse. It is called an **automorphism** if its source and target are the same.

ex:cat-aut-gp

Exercise 2.3.3. Let $\mathscr{C}$ be a category and X an object of $\mathscr{C}$. Define $\mathrm{Aut}_{\mathscr{C}}(X)$ to be the set of automorphisms of X. Show that the composition law of $\mathscr{C}$ makes $\mathrm{Aut}_{\mathscr{C}}(X)$ into a group.

Solution. To show that $\mathrm{Aut}_{\mathscr{C}}(X)$ is a group under the composoition law $\circ$ of $\mathscr{C}$ we must show that $\mathrm{Aut}_{\mathscr{C}}(X)$ is (i) closed under $\circ$, (ii) contains an identity element, (iii) contains inverses. We must also show that (iv) $\circ$ is associative.

(i) By **CAT4** $\circ$ is a map from $\mathrm{Hom}_{\mathscr{C}}(X) \times \mathrm{Hom}_{\mathscr{C}}(X) \to \mathrm{Hom}_{\mathscr{C}}(X)$. Now the composition of two automorphisms is an automorphism by the following argument.

We need to show that if $f, g \in \mathrm{Aut}_{\mathscr{C}} X$ then $f \circ g \in \mathrm{Aut}_{\mathscr{C}} X$. Now, since $f, g \in \mathrm{Aut}_{\mathscr{C}} X$ there exists isomorphisms $f^{-1}, g^{-1} \in \mathrm{Aut}_{\mathscr{C}} X$ such that $f \circ f^{-1} = f^{-1} \circ f = \mathrm{id}$ and $g \circ g^{-1} = g^{-1} \circ g = \mathrm{id}$.

Consider the homomorphism $g^{-1} \circ f^{-1}$. Composing this with $f \circ g$ we get $(f \circ g) \circ (g^{-1} \circ f^{-1})$. By **CAT5**,

$$\begin{aligned}
(f \circ g) \circ (g^{-1} \circ f^{-1}) &= f \circ (g \circ (g^{-1} \circ f^{-1})) \\
&= f \circ ((g \circ g^{-1}) \circ f^{-1}) \\
&= f \circ (\mathrm{id} \circ f^{-1}) \\
&= f \circ f^{-1} \\
&= \mathrm{id}
\end{aligned}$$

Similarly, $(g^{-1} \circ f^{-1}) \circ (f \circ g) = \mathrm{id}$. Therefore, since $f \circ g$ has a left and right inverse, $f \circ g \in \mathrm{Aut}_{\mathscr{C}}(X)$.

Therefore, $(f \circ g)$ is an isomorphism from X to itself and we have that $\circ$ is actually a map from $\mathrm{Aut}_{\mathscr{C}}(X) \times \mathrm{Aut}_{\mathscr{C}}(X) \to \mathrm{Aut}_{\mathscr{C}}(X)$. Thus, $\mathrm{Aut}_{\mathscr{C}}(X)$ is closed under $\circ$.

(ii) By **CAT4**, $\mathrm{Aut}_{\mathscr{C}}(X)$ contains an identity element.

(iii) Let $\varphi \in \mathrm{Aut}_{\mathscr{C}}(X)$. By the defintion of an isomorphism, there exists and isomorphism $\tau \in \mathrm{Hom}_{\mathscr{C}}(X)$ such that $\varphi \circ \tau = \tau \circ \varphi = \mathrm{id}_X$. Since τ is also a map from X to X, i.e. an endomorphism, it must be an automorphism. Thus, $\mathrm{Aut}_{\mathscr{C}}(X)$ contains inverses.

(iv) We have by **CAT5** that $\circ$ is associative.

Therefore, the composition law of $\mathscr{C}$ makes $\mathrm{Aut}_{\mathscr{C}}(X)$ into a group. □

`ex:BG` **Exercise 2.3.4.** Let G be any group. Construct a category $\mathscr{C}$ with one object X such that $\mathrm{Hom}(X, X) = G$ and composition in $\mathscr{C}$ corresponds to multiplication in G. This category is sometimes known as BG.

The opposite category

If $\mathscr{C}$ is any category, we can build an **opposite category** $\mathscr{C}^\circ$ by the following construction. Let $\mathbf{Ob}(\mathscr{C}^\circ) = \mathbf{Ob}(\mathscr{C})$ and let $\mathrm{Hom}_{\mathscr{C}^\circ}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X)$. If $u \in \mathrm{Hom}_{\mathscr{C}^\circ}(X, Y)$ and $v \in \mathrm{Hom}_{\mathscr{C}^\circ}(Y, Z)$ define the composition $v \circ u$ in $\mathscr{C}^\circ$ to be the element of $\mathrm{Hom}_{\mathscr{C}^\circ}(X, Z) = \mathrm{Hom}_{\mathscr{C}}(Z, X)$ obtained by composing $u \circ v$ in $\mathscr{C}$.

`ex:opp-cat` **Exercise 2.3.5.** Verify that $\mathscr{C}^\circ$ is indeed a category.

`ex:BG-op` **Exercise 2.3.6.** What is $(BG)^\circ$?

2.4 Functors

Definition 2.4.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **covariant functor** $F : \mathscr{C} \to \mathscr{D}$ consists of the following data:

F1 for each object X of $\mathscr{C}$, an object $F(X)$ of $\mathscr{D}$, and

F2 for each pair of objects X and Y of $\mathscr{C}$, a function $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$,

subject to the condition that

F3 whenever u and v are composable morphisms of $\mathscr{C}$, we have $F(u \circ v) = F(u) \circ F(v)$.

A **contravariant functor** from $\mathscr{C}$ to $\mathscr{D}$ is a covariant functor from $\mathscr{C}^\circ$ to $\mathscr{D}$.

A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be **faithful** if, for any objects X and Y, the function

$$F : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(FX, FY)$$

is *injective*. If it is *bijective*, we say that F is **fully faithful**.

`ex:delta-n` **Exercise 2.4.2.** For each non-negative integer n, let Δ^n be the category with objects $\mathbf{Ob}(\Delta^n) = \{0, 1, \dots, n\}$. We declare that $\mathrm{Hom}_{\Delta^n}(i, j)$ is empty if $i > j$ and has a unique element if $i \leq j$.

(i) Show that Δ^n is a category for each n.

(ii) Show that, for any category $\mathscr{C}$, to give a functor $\Delta^n \to \mathscr{C}$ is the same as to give a sequence of morphisms $f_1, \dots, f_n$ of $\mathscr{C}$ such that the target of f_i is the source of f_{i+1}.

`ex:poset-cat` **Exercise 2.4.3.** This exercise generalizes Exercise 2.4.2. Let P be any partially ordered set.[1] Construct a category $\mathscr{C}$ with $\mathbf{Ob}(\mathscr{C})$ being the set of elements of P and

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) = \begin{cases} \{*\} & X \leq Y \\ \varnothing & \text{otherwise.} \end{cases}$$

Here $\{*\}$ is a one-point set. Prove that $\mathscr{C}$ is a category.

Exercise 2.4.4. If $\mathscr{D}$ is a subcategory of $\mathscr{C}$ then the inclusion functor $\mathscr{C} \to \mathscr{D}$ is faithful. It is fully faithful if and only if $\mathscr{D}$ is a full subcategory of $\mathscr{D}$.

Exercise 2.4.5 (Forgetful functors)**.** (i) Let $F : \mathsf{Grp} \to \mathsf{Sets}$ be defined to take a group G to its underlying set $F(G)$ and a homomorphism φ to its underlying function $F(\varphi)$. Show that F is a functor and that it is *faithful*. We say that F "forgets the group structure".

(ii) Let R be a ring and $F : R\text{-}\mathsf{Mod} \to \mathsf{Ab}$ the functor that sends an R-module M to its underlying abelian group $F(M)$. We say that this functor "forgets the R-module structure".

Exercise 2.4.6. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Define a new category $\mathscr{C} \times \mathscr{D}$ with

$$\mathbf{Ob}(\mathscr{C} \times \mathscr{D}) = \mathbf{Ob}(\mathscr{C}) \times \mathbf{Ob}(\mathscr{D})$$
$$\mathbf{Mor}(\mathscr{C} \times \mathscr{D}) = \mathbf{Mor}(\mathscr{C}) \times \mathbf{Mor}(\mathscr{D}).$$

The composition law is $(f, g) \circ (f', g') = (f \circ f', g \circ g')$, provided the arrows are composable. Verify that this is a category.

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. We write $\mathrm{Hom}(\mathscr{C}, \mathscr{D})$ for the class of functors from $\mathscr{C}$ to $\mathscr{D}$. This turns out to be the set of objects of a category that we will, somewhat abusively, also call $\mathrm{Hom}(\mathscr{C}, \mathscr{D})$. Let $\mathbf{Mor}(\mathrm{Hom}(\mathscr{C}, \mathscr{D})) = \mathrm{Hom}(\mathscr{C} \times \Delta^1, \mathscr{D})$. A morphism φ from a functor F to a functor G is thus a choice, for each object X of $\mathscr{C}$, of an element $\varphi_X \in \mathrm{Hom}_{\mathscr{D}}(F(X), G(X))$, such that for every $f : X \to Y$ in $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, the diagram

$$\begin{array}{ccc} FX & \xrightarrow{\varphi_X} & GX \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FY & \xrightarrow{\varphi_Y} & GY \end{array}$$

[1] In fact, this exercise will work if P is a pre-ordered set.

is commutative. The composition of $\varphi : F \to G$ and $\psi : G \to H$ is defined by taking $(\psi \circ \varphi)_X = \psi_X \circ \varphi_X$ for each $X \in \mathbf{Ob}(\mathscr{C})$. The morphisms between functors are known as **natural transformation**.

Exercise 2.4.7. Verify that, with the definitions above, $\mathrm{Hom}(\mathscr{C}, \mathscr{D})$ is a category.

Exercise 2.4.8. Let $\mathscr{C}$ be a category. For each object X of $\mathscr{C}$, let $h_X : \mathscr{C}^\circ \to \mathsf{Sets}$ be the functor $h_X(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X)$.

(a) Verify that h_X is a functor.

(b) Suppose $f : X \to Y$ be a morphism of $\mathscr{C}$. For each $Z \in \mathbf{Ob}(\mathscr{C})$, define $h_f(Z) : h_X(Z) \to h_Y(Z)$ to be the map sending $g : Z \to X$ to $f \circ g : Z \to Y$. Verify that h_f is a natural transformation from h_X to h_Y.

(c) Define $F : \mathscr{C} \to \mathrm{Hom}(\mathscr{C}^\circ, \mathsf{Sets})$ by $F(X) = h_X$. Show that F is a functor.

Exercise 2.4.9. Let $F, G : \mathscr{C}^\circ \to \mathsf{Sets}$ be two functors. Construct a new functor $F \times G$ by defining $(F \times G)(X) = F(X) \times G(X)$.

(i) Show that the map $F(X) \times G(X) \to F(X)$ gives a natural transformation $F \times G \to F$. Obtain similarly a natural transformation $F \times G \to G$.

(ii) Show that $F \times G$ has the universal property of a product.

Theorem 2.4.10 (The Yoneda Lemma)**.** *Let $\mathscr{C}$ be a category. For any object X of $\mathscr{C}$ and any functor $F : \mathscr{C}^\circ \to \mathsf{Sets}$ there is a unique bijection*

$$\varphi_{X,F} : F(X) \to \mathrm{Hom}_{\mathrm{Hom}(\mathscr{C}^\circ, \mathsf{Sets})}(h_X, F)$$

such that

(i) $\varphi_{X,F}$ is natural with respect to X and F, and

(ii) $\varphi_{X,h_X}(\mathrm{id}_X) = \mathrm{id}_{h_X}$.

ex:yoneda **Exercise 2.4.11.** Prove the Yoneda lemma. This main difficulty in the proof is notational, so we begin by simplifying the notation. If $\varphi_{X,F} : F(X) \to \mathrm{Hom}_{\mathrm{Hom}(\mathscr{C}^\circ, \mathsf{Sets})}(h_X, F)$ is a function then for each $\xi \in F(X)$, we get a *natural transformation* $\varphi_{X,F}(\xi) : h_X \to F$. That is, for each $Y \in \mathscr{C}$, we get a *function* $\varphi_{X,F}(\xi)_Y : h_X(Y) \to F(Y)$. In order to unclutter the notation, we will notate both the natural transformation $h_X \to F$ and the individual functions $\xi^\varphi : h_X(Y) \to F(Y)$. We will have to depend on context to know exactly which symbol is meant.

If $f : X \to Y$ is a map of $\mathscr{C}$ then we get a map $F(f) : F(Y) \to F(X)$. Our second notational simplification will be to write f^* rather than $F(f)$.

(i) Prove that there is a unique function $\varphi_{X,F} : F(X) \to \mathrm{Hom}_{\mathrm{Hom}(\mathscr{C}^\circ, \mathsf{Sets})}(h_X, F)$ that is natural in both X and F. (Hint: show that, whenever $\xi \in F(X)$ and $f : Y \to X$ is an element of $h_X(Y)$, one must have $\xi^\varphi(f) = f^*(\xi)$ and then verify that this formula actually does define a natural transformation.)

(ii) Prove that there is a unique function $\eta_{X,F} : \mathrm{Hom}_{\mathrm{Hom}(\mathscr{C}^\circ,\mathsf{Sets})}(h_X, F) \to F(X)$ that is natural in both F and X and satisfies $\eta_{X,h_X}(\mathrm{id}_{h_X}) = \mathrm{id}_X$.

(iii) Prove that $\varphi_{X,F} \circ \eta_{X,F} : \mathrm{Hom}(h_X, F) \to \mathrm{Hom}(h_X, F)$ is the identity.

(iv) Prove that $\eta_{X,F} \circ \varphi_{X,F} : F(X) \to F(X)$ is the identity.

Solution.

(i) Suppose that $\varphi_{X,F}$ is an isomorphism natural in both X and F, let $\xi \in F(X)$, and $f : Y \to X$ be in $h_X(Y)$. Also, assume that $\varphi_{X,h_X}(\mathrm{id}_X) = \mathrm{id}_{h_X}$.

First, observe that $\xi^\varphi : h_X \to F$ is a natural transformation of functors, and so by hypothesis[2] we have the commutative diagram: ←2

$$\begin{array}{ccc} h_X(X) & \xrightarrow{\varphi_{X,h_x}} & \mathrm{Hom}(h_X, h_X) \\ \big\downarrow{\scriptstyle \xi^\varphi} & & \big\downarrow{\scriptstyle \mathrm{Hom}(h_X,\xi^\varphi)} \\ F(X) & \xrightarrow[\varphi_{X,F}]{} & \mathrm{Hom}(h_X, F). \end{array}$$

By assumption, the element id_X of $h_X(X)$ is mapped to $\mathrm{id}_{h_X} \in \mathrm{Hom}(h_X, h_X)$. Further, by commutativity, we have

$$\varphi_{X,F}(\xi^\varphi(\mathrm{id}_X)) = \xi^\varphi.$$

But $\varphi_{X,F}$ is an isomorphism and so we have $\xi^\varphi(\mathrm{id}_X) = \xi$.

Now, by naturality in X, we have the commutative diagram:

$$\begin{array}{ccc} h_X(X) & \xrightarrow{\xi^\varphi} & F(X) \\ \big\downarrow{\scriptstyle -\circ f} & & \big\downarrow{\scriptstyle f^*} \\ h_X(Y) & \xrightarrow{\xi^\varphi} & F(Y). \end{array}$$

By commutativity we have

$$f^* \left(\xi^\varphi(\mathrm{id}_X)\right) = \xi^\varphi \left(\mathrm{id}_X \circ f\right).$$

By the previous argument, this implies that

$$f^*\xi = \xi^\varphi(f).$$

Let $f : Y \to X$ and $\xi \in F(X)$ be given. To check that the putative $\varphi_{X,F}$ is natural in X amounts to checking the commutativity of the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\varphi_{X,F}} & \mathrm{Hom}(h_X, F) \\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle \mathrm{Hom}(h_f,F)} \\ F(Y) & \xrightarrow{\varphi_{Y,F}} & \mathrm{Hom}(h_Y, F), \end{array}$$

[2]which hypothesis? You are using the naturality of $\varphi_{X,F}$ in F

where the far right arrow sends a natural transformation τ to $\tau \circ h_f = \tau \circ (f \circ -)$. Thus, going right and then down we have

$$\xi \mapsto \xi^{\varphi} \mapsto \xi^{\varphi} \circ h_f \in \mathrm{Hom}(h_Y, F).$$

On the other hand, going down and then right we have

$$\xi \mapsto f^* \xi \mapsto (f^* \xi)^{\varphi} \in \mathrm{Hom}(h_Y, F).$$

To check that these two elements agree, fix a $g : Z \to Y$. Then we have

$$\begin{aligned}
(f^* \xi)^{\varphi}(g) &= g^* f^* \xi, \\
&= (f \circ g)^* \xi, \\
&= \xi^{\varphi}(f \circ g), \\
&= \xi^{\varphi} \circ h_f(g).
\end{aligned}$$

Thus, we conclude that φ is natural in X.

Let $\psi : F \to G$ be a natural transformation of functors. Fix $X \in \mathscr{C}$. We must check the commutativity of the diagram

$$\begin{array}{ccc}
F(X) & \xrightarrow{\varphi_{X,F}} & \mathrm{Hom}(h_X, F) \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{Hom}(h_X, \psi)} \\
G(X) & \xrightarrow{\varphi_{X,G}} & \mathrm{Hom}(h_X, G).
\end{array}$$

Let $\xi \in F(X)$ be given. Fix a map $f : Y \to X$. Then we have

$$\begin{aligned}
\psi(\xi^{\varphi}(f)) &= (\psi \circ F(f))\xi, & \\
&= (G(f) \circ \psi)\xi, & \text{naturality of } \psi \\
&= (\psi \xi)^{\varphi}(f). &
\end{aligned}$$

Thus, φ is natural in F.

□

Solution. The proof proceeds in two parts. In Part I we construct the promised bijection and in Part II we prove this bijection to be natural in both objects and functors.

Part I: Let $\mathscr{C}$ be a category, X an object of $\mathscr{C}$, and F a functor $\mathscr{C}^{op} \longrightarrow \mathsf{Sets}$. We define the map $\eta_{X,F}$:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathrm{Hom}(\mathscr{C}^{\circ}, \mathsf{Sets})}(h_X, F) & \longrightarrow & \mathrm{Hom}_{\mathsf{Sets}}(h_X(X), F(X)) & \longrightarrow & F(X) \\
\phi & \longmapsto & \phi_X & \longmapsto & \phi_X(\mathrm{id}_X).
\end{array}$$

Conversely, we define the promised map,

$$\varphi_{X,F} : F(X) \longrightarrow \mathrm{Hom}_{\mathrm{Hom}(\mathscr{C}^{\circ}, \mathsf{Sets})}(h_X, F),$$

by sending a point $\xi \in F(X)$ to the putative natural transformation ξ^{φ} which we define by its components. For any object Z of $\mathscr{C}$ we set

$$\begin{aligned} \xi^{\varphi} : h_X(Z) &\longrightarrow F(Z) \\ (g : Z \to X) &\longmapsto F(g)(\xi)\,. \end{aligned}$$

The collection of morphisms ξ^{φ} is a natural transformation: Let $f : Z \longrightarrow Y$ be a morphism of $\mathscr{C}$ and consider the diagram below,

$$\begin{array}{ccc} h_X(Z) & \xrightarrow{\ \xi^{\varphi}\ } & F(Z) \\ {\scriptstyle h_X(f)}\uparrow & & \uparrow{\scriptstyle F(f)} \\ h_X(Y) & \xrightarrow[\ \xi^{\varphi}\ ]{} & F(Y)\,. \end{array}$$

Now, let $g \in h_X(Y)$ and see that,

$$\begin{array}{ccc} & & (F(f) \circ \xi^{\varphi})(g) \\ & & \uparrow{\scriptstyle F(f)} \\ g & \xmapsto{\ \xi^{\varphi}\ } & \xi^{\varphi}(g) \end{array} \qquad \begin{array}{ccc} h_X(f) & \xmapsto{\ \xi^{\varphi}\ } & \xi^{\varphi}(g \circ f)\ . \\ {\scriptstyle h_X(f)}\uparrow & & \\ g & & \end{array}$$

But from our definition and since F is a contravariant functor,

$$\xi^{\varphi} \circ h_X(f)(g) = \xi^{\varphi}(g \circ f) = F(g \circ f)(\xi) = F(f) \circ F(g)(\xi)\,.$$

So as,

$$(F(f) \circ \xi^{\varphi})(g) = F(f) \circ F(g)(\xi)\,,$$

the square commutes and ξ^{φ} is a natural transformation h_X to F.

Moreover, this naturality implies the uniqueness assertion. Replacing $f : Z \longrightarrow Y$ with any morphism $f : Y \longrightarrow X$ we may follow the identity around the diagram, which by the above commutes. We find that $\xi^{\varphi}(f) = F(f)(\xi)$ as $\xi^{\varphi}(\mathrm{id}_X)$ is ξ by construction and $f \circ \mathrm{id}_X = f$.

Now we prove $\varphi_{X,F}$ a bijection. It is by definition that for any $g \in h_X(Y)$ the component of the natural transformation $\varphi_{X,F}(\eta_{X,F}(\vartheta))$ at Y takes g to the element $F(g)(\vartheta_X(\mathrm{id}_X))$. Since ϑ is a natural transformation however, the diagram

$$\begin{array}{ccc} h_X(X) & \xrightarrow{\ \vartheta_X\ } & F(X) \\ {\scriptstyle h_X(g)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ h_X(Y) & \xrightarrow[\ \vartheta_Y\ ]{} & F(Y)\,, \end{array}$$

commutes so,

$$F(g)(\vartheta_X(\mathrm{id}_X)) = (F(g) \circ \vartheta_X)(\mathrm{id}_X) = (\vartheta_Y \circ h_X(g))(\mathrm{id}_X)\,.$$

Then, since $h_X(g)(\mathrm{id}_X) = g$, we get $\vartheta_Y(g) = \varphi_{X,F}(\eta_{X,F}(\vartheta))_Y(g)$ for all Y object of $\mathscr{C}$ and $g \in h_X(Y)$; $\varphi_{X,F} \circ \eta_{X,F}$ is the identity on $\mathrm{Hom}_{\mathrm{Hom}(\mathscr{C}^\circ, \mathsf{Sets})}(h_X, F)$.

Conversely, if $\xi \in F(X)$, then

$$\eta_{X,F}(\varphi_{X,F}(\xi)) = \xi^{\varphi}(\mathrm{id}_X) = F(\mathrm{id}_X)(\xi) = \xi,$$

so $\eta_{X,F} \circ \varphi_{X,F} = \mathrm{id}_{F(X)}$.

Part II: We show that $\eta_{X,F}$ is natural in X, which forces $\varphi_{X,F}$ to be natural in the same. To that end suppose $f : Y \longrightarrow X$ a morphism of $\mathscr{C}$, and consider the diagram,

$$\begin{array}{ccc}
\mathrm{Hom}(h_X, F) & \xrightarrow{\mathrm{Hom}(h_f, F)} & \mathrm{Hom}(h_Y, F) \\
\Big\downarrow{\scriptstyle \eta_{X,F}} & & \Big\downarrow{\scriptstyle \eta_{Y,F}} \\
F(X) & \xrightarrow[F(f)]{} & F(Y).
\end{array}$$

For any $\vartheta \in \mathrm{Hom}(h_X, F)$ we find,

$$\begin{array}{ccc}
\vartheta & & \\
\Big\downarrow{\scriptstyle \eta_{X,F}} & & \\
\vartheta_X(\mathrm{id}_X) & \overset{}{\underset{F(f)}{\longmapsto}} & F(f)(\vartheta_X(\mathrm{id}_X))
\end{array}
\qquad
\begin{array}{ccc}
\vartheta & \overset{\mathrm{Hom}(h_f, F)}{\longmapsto} & \vartheta \circ h_f \\
 & & \Big\downarrow{\scriptstyle \eta_{Y,F}} \\
 & & (\vartheta \circ h_f)_Y(\mathrm{id}_Y).
\end{array}$$

The component of a composition of natural transformations is the composition of the components: $(\vartheta \circ h_f)_Y = \vartheta_Y \circ (h_f)_Y$. Moreover,

$$(h_f)_Y(\mathrm{id}_Y) = f \circ \mathrm{id}_Y = f = \mathrm{id}_X \circ f \in h_X(Y),$$

so $(\vartheta \circ h_f)_Y(\mathrm{id}_Y) = \vartheta_Y(\mathrm{id}_X \circ f)$. But $\mathrm{id}_X \circ f = h_X(f)(\mathrm{id}_X)$, so

$$(\vartheta \circ h_f)_Y(\mathrm{id}_Y) = \vartheta_Y(h_X(f)(\mathrm{id}_X)).$$

Remember though that ϑ is a natural transformation; the diagram,

$$\begin{array}{ccc}
h_X(X) & \xrightarrow{\vartheta_X} & F(X) \\
\Big\downarrow{\scriptstyle h_X(f)} & & \Big\downarrow{\scriptstyle F(f)} \\
h_X(Y) & \xrightarrow[\vartheta_Y]{} & F(Y),
\end{array}$$

commutes. Thus $\vartheta_Y \circ h_X(f) = F(f) \circ \vartheta_X$, and in particular

$$\vartheta_Y(h_X(f)(\mathrm{id}_X)) = F(f)(\vartheta_X(\mathrm{id}_X));$$

thus $\eta_{X,F}$ is natural is objects.

Lastly, we attend to naturality in F. Letting ψ be a natural transformation F to G, functors $\mathscr{C}^\circ \longrightarrow \mathsf{Sets}$, and ϑ to be one h_X to F, we show the diagram,

$$\begin{array}{ccc} \mathrm{Hom}\,(h_X, F) & \xrightarrow{\eta_{X,F}} & F\,(X) \\ {\scriptstyle \mathrm{Hom}(h_X,\psi)} \downarrow & & \downarrow {\scriptstyle \psi_X} \\ \mathrm{Hom}\,(h_X, G) & \xrightarrow[\eta_{X,G}]{} & G\,(X)\,, \end{array}$$

commutes. Following ϑ,

$$\begin{array}{ccc} \vartheta & \longmapsto & \vartheta_X\,(\mathrm{id}_X) \\ \downarrow & & \downarrow \\ \phi \circ \vartheta & \longmapsto & \psi_X\,(\vartheta_X\,(\mathrm{id}_X))\,, \end{array}$$

and $\eta_{X,F}$ is natural in F. □

Corollary 2.4.11.1. *The functor* $\mathscr{C} \to \mathrm{Hom}(\mathscr{C}^\circ, \mathsf{Sets})$ *sending* X *to* h_X *is fully faithful. That is, the function* $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{Hom}(\mathscr{C}^\circ, \mathsf{Sets})}(h_X, h_Y)$ *is a bijection.*

Exercise 2.4.12. Prove the corollary from Yoneda's lemma.

Exercise 2.4.13. State and prove a version of Yoneda's lemma for covariant functors.

2.5 Representable functors

def:representable

Definition 2.5.1. Let $F : \mathscr{C}^\circ \to \mathsf{Sets}$ be a contravariant functor from a category $\mathscr{C}$ to Sets. If F is isomorphic to h_X for some $X \in \mathscr{C}$ then we say that X is **representable in** $\mathscr{C}$, or that it is **represented by** X. If ξ is an element of $F(X)$ such that $\xi^\varphi : h_X \to F$ is an isomorphism,[3] then we say (X, ξ) **represents** F. We apply the same terminology to covariant functors.

We can make a useful, though not universally accepted, generalization of this definition when we are given a functor $S : \mathscr{D} \to \mathrm{Hom}(\mathscr{C}^\circ, \mathsf{Sets})$. We say in this case F is representable by an object of $\mathscr{D}$ if F is isomorphic to $S(X)$ for some $X \in \mathscr{D}$.

ex:rep-funcs

Exercise 2.5.2. Find a commutative ring representing each of the functors $\mathsf{ComRng} \to \mathsf{Sets}$ below:

(i) $\mathbf{A}^1(A) = A$.

(ii) $\mathbf{G}_m(A) = A^*$, where A^* is the set of invertible elements of A.

(iii) $\mathbf{A}^n(A) = A^n$, the product of the underlying set of A with itself n times (n need not be finite here).

[3] See Exercise 2.4.11 for notation

(iv) $X(A) = \{(x,y) \in A^2 \mid y^2 = x^3 + x\}$.

(v) $\mu_n(A) = \{x \in A \mid x^n = 1\}$.

(vi) $\alpha_n(A) = \{x \in A \mid nx = 0\}$.

(vii) $F(A) = \{*\}$ is a set with one element for each A.

ex:univ-derivation **Exercise 2.5.3.** Let A be a commutative ring and B a commutative A-algebra. If M is a B-module, an A-derivation from B into M is a function $\delta : B \to M$ such that

der:1 **DER1** $\delta(a) = 0$ for all $a \in A$, and

def:2 **DER2** $\delta(fg) = f\delta(g) + g\delta(f)$ for each $f, g \in B$.

Let $\mathrm{Der}_A(B, M)$ denote the set of A-derivations from B into M. Let $F : B\text{-}\mathsf{Mod} \to \mathsf{Sets}$ be the functor $F(M) = \mathrm{Der}_A(B, M)$. Construct a B-module representing F.

We write $d : B \to \Omega_{B/A}$ for the univeral derivation.

ex:not-rep-ex **Exercise 2.5.4.** This exercise may be difficult until we have more tools.

Let n be a positive integer and define

$$F(A) = \big\{(x_1, \ldots, x_n) \in A^n \bigm| (x_1, \ldots, x_n)A = A\big\}.$$

The notation is meant to indicate that the ideal generated by the elements $x_1, \ldots, x_n$ is the unit ideal. Show that F is representable if and only if $n = 0, 1$.

ex:rep-iso **Exercise 2.5.5** (Very important to understand)**.** Show, using the Yoneda lemma, that if $F : \mathscr{C}^\circ \to \mathsf{Sets}$ is a functor that is represented by (X, ξ) and (Y, η) then there is a unique isomorphism $\alpha : X \to Y$ such that $\alpha^*(\eta) = \xi$.[4]

2.6 Initial and final objects

Definition 2.6.1. Let $\mathscr{C}$ be a category. An object X of $\mathscr{C}$ is called **initial** if, for every object Y of $\mathscr{C}$, the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ consists of *exactly one* element. We say that X is **final** if for every Y in $\mathscr{C}$, the set $\mathrm{Hom}_{\mathscr{C}}(Y, X)$ has exactly one element.

Exercise 2.6.2. Determine which of the following categories have initial and final objects, and what those objects are.

(i) Sets

(ii) Grp

(iii) Rng

(iv) Top

[4] See Exercise 2.4.11 for notation.

2.7 Universal properties

A very powerful technique for constructing an object in a category is by characterizing it with a universal property. Intuitively, a universal object with some property is the "smallest" or "largest" object with that property. Universal properties can be incredibly useful and incredibly confusing. One reason for the confusion is that the terminology is misleading: First of all, a universal property is not a property at all, but rather a collection of data! Second, there are two types of universal properties—initial and final—and when speaking of universal objects, we frequently fail to distinguish between them.

Example 2.7.1 (Products)**.** Let X and Y be sets. The universal (final) example of a set S with functions $p : S \to X$ and $q : S \to Y$ is the product $X \times Y$ with the projections $P(x, y) = x$ and $Q(x, y) = y$.

If S is any set with the *additional data* $p : S \to X$ and $q : S \to Y$, we can build a *unique* map $(p, q) : S \to X \times Y$ with $(p, q)(s) = (p(s), q(s))$ such that $P \circ (p, q) = P$ and $Q \circ (p, q) = Q$. Thus $(X \times Y, P, Q)$ is the *final example* of a set with a map to X and a map to Y.

Notice that in the example, it would make no sense to say that $X \times Y$ is universal without making mention of the maps P and Q.

One way of treating universal properties is to define a new category that keeps track of the extra data involved. A universal object is then an initial or final object of this category.

Example 2.7.2 (Products, again)**.** Let X and Y be sets. Define $\mathscr{C}$ to be the category whose objects are triples (S, p, q) where S is a set and $p : S \to X$ and $q : S \to Y$ are functions. A morphism $(S, p, q) \to (S', p', q')$ is a function $f : S \to S'$ such that $p'f = p$ and $q'f = q$.

Then $\mathscr{C}$ has the initial object $(\varnothing, p_\varnothing, q_\varnothing)$ where $p_\varnothing : \varnothing \to X$ and $q_\varnothing : \varnothing \to Y$ are the unique maps with empty source. There is also a final object $(X \times Y, P, Q)$ where $P(x, y) = x$ and $Q(x, y) = y$.

A second way to treat universal properties is by trying to represent a functor that encodes the extra data.

Example 2.7.3 (Products, yet again)**.** Let X and Y be sets. Define a functor $F : \mathsf{Sets}^\circ \to \mathsf{Sets}$ by

$$F(Z) = \big\{(p, q) \,\big|\, p \in \mathrm{Hom}_{\mathsf{Sets}}(Z, X) \text{ and } q \in \mathrm{Hom}_{\mathsf{Sets}}(Z, Y)\big\}.$$

This functor is represented by the pair $(X \times Y, (P, Q))$.

Exercise 2.7.4. This problem shows how to translate between the two different definitions of universal properties given above.

Let $\mathscr{C}$ be a category and let $F : \mathscr{C}^\circ \to \mathsf{Sets}$ be a contravariant functor from $\mathscr{C}$ to Sets. Construct a category $\mathscr{D}$ whose objects are pairs (X, ξ) with $X \in \mathbf{Ob}(\mathscr{C})$ and $\xi \in F(X)$. Define

$$\mathrm{Hom}_{\mathscr{D}}((X, \xi), (Y, \eta)) = \big\{f \in \mathrm{Hom}_{\mathscr{C}}(X, Y) \,\big|\, f^*\eta = \xi\big\}.$$

Prove that (X, ξ) is a final object of $\mathscr{D}$ if and only if (X, ξ) represents F.

ex:bilin **Exercise 2.7.5.** Let R be a commutative ring. Let A, B, and C be R-modules. A function (not an R-module homomorphism!) $\mu : A \times B \to C$ is called **bilinear** if

BILIN1 $\mu(a + a', b) = \mu(a, b) + \mu(a', b)$ for all $a, a' \in A$ and $b \in B$,

BILIN2 $\mu(a, b + b') = \mu(a, b) + \mu(a, b')$ for all $a \in A$ and $b, b' \in B$, and

BILIN3 $\mu(\lambda a, b) = \mu(a, \lambda b) = \lambda \mu(a, b)$ for all $\lambda \in R$ and $a \in A$ and $b \in B$.

Fix R-modules A and B and define a functor $F : R\text{-}\mathsf{Mod} \to \mathsf{Sets}$

$$F(C) = \big\{\text{bilinear maps } \mu : A \times B \to C\big\}.$$

(i) Verify that this actually is a functor. (If $C \to C'$ is an R-module homomorphism, what is the map $F(C) \to F(C')$?)

(ii) Construct an R-module X and a bilinear map $A \times B \to X$ representing F.

Solution. (i) Let $\varphi : C \to C'$ be an R–linear map, and define $F(\varphi) : F(C) \to F(C')$ as follows: For each bilinear map $f : A \times B \to C$, let $F(\varphi)(f) = \varphi \circ f$. We claim that $\varphi \circ f$ is bilinear. Let $a, a' \in A$, $b \in B$, and let $r \in R$. Then,

$$\begin{aligned}(\varphi \circ f)(a + ra', b) &= \varphi\left(f(a + ra', b)\right)\\ &= \varphi\left(f(a, b) + rf(a', b)\right)\\ &= (\varphi \circ f)(a, b) + r(\varphi \circ f)(a', b),\end{aligned}$$

so $\varphi \circ f$ is linear in the first coordinate. By an identical argument, $\varphi \circ f$ is linear in the second coordinate, so it is bilinear. Now, for any R–linear maps $\varphi : C \to C'$ and $\psi : C' \to C''$ and any bilinear map $f : A \times B \to C$, we have

$$F(\psi\varphi)(f) = (\psi\varphi)(f) = \psi(\varphi(f)) = F(\psi)F(\varphi)(f),$$

so $F(\psi\varphi) = F(\psi)F(\varphi)$, whence F respects composition of morphisms and is therefore a covariant functor $R\text{-}\mathsf{Mod} \to \mathsf{Sets}$.

(ii) Let $X = A \otimes B$ be the tensor product of A and B over R, i.e., X is the quotient F/S, where F is the free abelian group with basis $A \times B$ and S is the subgroup of F generated by elements of the form

$$\begin{aligned}&(a + a', b) - (a, b) - (a', b),\\ &(a, b + b') - (a, b) - (a, b'), \text{ and}\\ &(r \cdot a, b) - (a, r \cdot b),\end{aligned}$$

for all $a \in A$, $b \in B$, and $r \in R$. For each $(a, b) \in A \times B$, let $a \otimes b$ denote the coset $(a, b) + S$ (and note that every element of X can be expressed as a finite sum of the form $\sum_i a_i \otimes b_i$). Give X an R–module structure by linearly extending the action $r \cdot (a \otimes b) = ra \otimes b = a \otimes rb$. Finally, define the map $\otimes : A \times B \to X$ by $(a, b) \mapsto a \otimes b$. Then we claim that X and $\otimes$ represent F, that is to say, there exist natural transformations $\tau : F \to h^X$ and $\sigma : h^X \to F$ such that $\tau\sigma$ is the identity transformation $F \to F$ and $\sigma\tau$ is the identity transformation $h^X \to h^X$.

For each $C \in R\text{-}\mathsf{Mod}$, define $\tau_C : \mathrm{Hom}(X, C) \to F(C)$ by $\tau_C(f)(a, b) = f(a \otimes b)$, and define $\sigma_C : F(C) \to \mathrm{Hom}(X, C)$ by linearly extending the map $\tau(g)(a \otimes b) = g(a, b)$. Then we check that for any $f \in F(C)$ and $g \in \mathrm{Hom}(X, C)$, $\tau_C(f)$ is bilinear and $\sigma_C(g)$ is well–defined. First, for any $a \in A$ and $b \in B$,

$$\begin{aligned}
\tau_C(f)(a + ra', b) &= f((a + ra') \otimes b) = f(a \otimes b + ra' \otimes b) \\
&= f(a \otimes b) + f(ra' \otimes b) \\
&= f(a \otimes b) + rf(a' \otimes b) \\
&= \tau_C(f)(a, b) + r\tau_C(f)(a', b),
\end{aligned}$$

so $\tau_C(f)$ is linear in the first coordinate. An identical argument shows that $\tau_C(f)$ is linear in the second coordinate as well, so $\tau_C(f)$ is a bilinear map $A \times B \to C$. Next, if $g \in F(C)$ and $\sum_i a_i \otimes b_i = \sum_j a'_j \otimes b'_j$, then by definition, $\sum_i (a_i, b_i)$ and $\sum_j (a'_j, b'_j)$ differ by a finite sum of elements of S. But this implies that $\sum_i g(a_i, b_i) = \sum_j g(a'_j, b'_j)$, since the bilinearity of g ensures that $g(S) = \{0\}$ Thus,

$$\sigma_C(g) \left(\sum_i a_i \otimes b_i \right) = \sigma_C(g) \left(\sum_j a'_j \otimes b'_j \right),$$

so $\sigma_C(g)$ is well–defined.

Now, let $\varphi : C \to C'$ be a morphism in $R\text{-}\mathsf{Mod}$ and consider the following diagram:

$$\begin{array}{ccccccc}
F(C) & \xrightarrow{\sigma_C} & \mathrm{Hom}(X, C) & \xrightarrow{\tau_C} & F(C) & \xrightarrow{\sigma_C} & \mathrm{Hom}(X, C) \\
\Big\downarrow{\scriptstyle F(\varphi)} & & \Big\downarrow{\scriptstyle \varphi_*} & & \Big\downarrow{\scriptstyle F(\varphi)} & & \Big\downarrow{\scriptstyle \varphi_*} \\
F(C') & \xrightarrow[\sigma_{C'}]{} & \mathrm{Hom}(X, C') & \xrightarrow[\tau_{C'}]{} & F(C') & \xrightarrow[\sigma_{C'}]{} & \mathrm{Hom}(X, C')
\end{array}.$$

If $f \in F(C)$ and $\sum_i a_i \otimes b_i \in X$, then

$$\begin{aligned}(\varphi_* \sigma_C)(f)\left(\sum_i a_i \otimes b_i\right) &= \varphi \circ \left(\sum_i f(a_i, b_i)\right) \\ &= \sum_i (\varphi \circ f)(a_i, b_i) \\ &= \sum_i F(\varphi)(f)(a_i, b_i) \\ &= (\sigma_{C'} F(\varphi))(f)\left(\sum_i a_i \otimes b_i\right).\end{aligned}$$

Thus, the left and right squares commute. Similarly if $f \in \mathrm{Hom}(X, C)$ and $(a, b) \in A \times B$, then

$$\begin{aligned}(F(\varphi)\tau_C)(f)(a, b) &= F(\varphi)(f(a \otimes b)) \\ &= (\varphi \circ f)(a \otimes b) \\ &= \varphi_*(f)(a \otimes b) \\ &= (\tau_{C'} \varphi_*)(f)(a, b),\end{aligned}$$

so the middle square commutes, whence τ and σ are natural transformations. Finally, for all $f \in \mathrm{Hom}(X, C)$, $g \in F(C)$, $\sum_i a_i \otimes b_i \in X$, and $(a, b) \in A \times B$, we have

$$\begin{aligned}(\sigma_C \tau_C)(f)\left(\sum_i a_i \otimes b_i\right) &= \sum_i \tau_C(f)(a_i, b_i) \\ &= \sum_i f(a_i \otimes b_i) \\ &= f\left(\sum_i a_i \otimes b_i\right),\end{aligned}$$

so $\sigma_C \tau_C = \mathrm{id}_{\mathrm{Hom}(X,C)}$. Similarly,

$$\begin{aligned}(\tau_C \sigma_C)(g)(a, b) &= \sigma_C(g)(a \otimes b) \\ &= g(a, b),\end{aligned}$$

so $\sigma_C \tau_C = \mathrm{id}_{F(C)}$. Therefore, F is represented by X.

□

ex:QmodZtensorQmodZ

Exercise 2.7.6. Use the previous exercise to prove the following assertions:

(i) If I and J are ideals of a commutative ring R then $R/I \otimes_R R/J \cong R/(I + J)$.

(ii) $\mathbf{Q}/\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z} \cong 0$.

Solution. (i) Define the map $\varphi : R/I \times R/J \to R/(I+J)$ by $\varphi(r+I, s+J) = rs + (I+J)$. Then by the universal property for tensor products, there is a unique linear map $R/I \otimes R/J \to R/(I+J)$ for which $(r+I)\otimes(s+J) \mapsto rs + (I+J)$.

We claim that φ is injective. Suppose $\varphi(\sum_i (r_i + I \otimes s_i + J)) = 0$. Then it follows that $\sum_i r_i s_i \in I + J$, so $\sum_i r_i s_i = t + u$ for some $t \in I$ and $u \in J$. Thus,

$$\begin{aligned}\sum_i (r_i + I \otimes s_i + J) &= \left(\sum_i r_i s_i + I\right) \otimes 1 + J \\ &= (t+u) + I \otimes 1 + J \\ &= u + I \otimes 1 + J \\ &= 1 + I \otimes u + J \\ &= 0,\end{aligned}$$

so φ is injective

Finally, φ is surjective, since $\varphi((r+I)\otimes(1+J)) = r + I + J$ for all $r \in R$. Therefore, φ is an isomorphism between $R/I \otimes R/J$ and $R/I + J$.

(ii) It suffices to show that any simple tensor is zero. Let $p/q + \mathbf{Z} \otimes r/s + \mathbf{Z} \in \mathbf{Q}/\mathbf{Z} \otimes \mathbf{Q}/\mathbf{Z}$. Then,

$$\begin{aligned}\frac{p}{q} + \mathbf{Z} \otimes \frac{r}{s} + \mathbf{Z} &= \frac{p}{q} + \mathbf{Z} \otimes \frac{rq}{sq} + \mathbf{Z} \\ &= \frac{p}{q} + \mathbf{Z} \otimes q\left(\frac{r}{sq} + \mathbf{Z}\right) \\ &= q\left(\frac{p}{q} + \mathbf{Z}\right) \otimes \frac{r}{sq} + \mathbf{Z} \\ &= \frac{p}{1} + \mathbf{Z} \otimes \frac{r}{sq} + \mathbf{Z} \\ &= 0.\end{aligned}$$

Therefore, each simple tensor is zero, so $\mathbf{Q}/\mathbf{Z} \otimes \mathbf{Q}/\mathbf{Z} = 0$.

□

Fiber products

Perhaps the single most important scheme theoretic construction is that of a *fiber product*.[5] It is therefore very important to understand the universal property that defines the fiber product.

[5] Fiber products are also known as pullbacks, pullback squares, and cartesian squares.

Let $\mathscr{C}$ be a category and suppose given a diagram

$$\begin{array}{ccc} & & Y \\ & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z. \end{array}$$

For $W \in \mathscr{C}$, let $F(W)$ be the set of completions of this diagram to a *commutative* diagram

$$\begin{array}{ccc} W & \xrightarrow{q} & Y \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z. \end{array}$$

In other words,

$$F(W) = \big\{(p,q) \,\big|\, p \in \mathrm{Hom}_{\mathscr{C}}(W,X),\ q \in \mathrm{Hom}_{\mathscr{C}}(W,Y),\ fp = gq\big\}.$$

Should F be representable by an object $(W,(p,q))$ of $\mathscr{C}$ then W is called the **fiber product** of X with Y over Z.[6] We frequently denote the fiber product $(X \times_Z Y, (p_1, p_2))$.

Exercise 2.7.7. Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in a category $\mathscr{C}$. Construct $X \times_Z Y$ when $\mathscr{C}$ is each of the following categories:

(i) Sets

(ii) Top (Remember, you have to say what the topology is, not just the underlying set!)

(iii) Grp

(iv) Rng

(v) Rng°

(vi) Grp°

(vii) R-Mod (where R is a ring)

(viii) R-Mod°

In general, rather than speak of fiber products in $\mathscr{C}^\circ$, we instead refer to cofiber products[7] in $\mathscr{C}$. That is, given a diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow & & \\ Z & & \end{array}$$

[6] Of course, this is an abuse of terminology. It does not technically make sense to speak of the fiber product with mention of the maps $p : W \to X$ and $q : W \to Y$.

[7] or fiber coproducts, or fibered coproducts, or pushouts, or cocartesian diagrams, or amalgamated sums, or ...

in $\mathscr{C}$, the cofiber product of Y and Z along X is the universal completion of the diagram to a commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

The cofiber product is often denoted $Y \amalg_X Z$, with the two maps $Y \to Y \amalg_X Z$ and $Z \to Y \amalg_X Z$ remaining tacit.

Exercise 2.7.8. Suppose that $F \to H$ and $G \to H$ are two natural transformations of functors $\mathscr{C} \to \mathscr{D}$. Assume that $\mathscr{D}$ possesses all fiber products. Construct the fiber product $F \times_H G$.

Exercise 2.7.9. Recall that if a group G acts from the right on a group K, with the action of an element $g \in G$ on an element $k \in K$ denoted k^g (thus $(k^g)^{g'} = k^{gg'}$), we can construct the *semidirect product* $G \ltimes K$: As a set, $G \ltimes K$ is the set of symbols gk with $g \in G$ and $k \in K$; the multiplication law is

$$(gk)(g'k') = gg'k^{g'}k'.$$

(i) If you haven't already done so at some point in your life, verify that the semidirect product is a group.

(ii) Let $G \to H$ be a homomorphism of groups with kernel K. Denote by i the inclusion homomorphism of K in G. Show that $(G \times_H G, (p_1, p_2))$ is isomorphic to $(G \ltimes K, (p, q))$ where $G \ltimes K$ is the semidirect product with G acting on K by $k^g = g^{-1}kg$, and

$$p(gk) = g$$
$$q(gk) = gi(h).$$

Exercise 2.7.10. Let R be a ring and I an R-algebra, possibly without unit (for example, an ideal of R). Construct a new ring $R + I$ whose elements are formal sums $x + y$ with $x \in R$ and $y \in I$. Define the ring structure

$$(x + y) + (x' + y') = (x + x') + (y + y')$$
$$(x + y)(x' + y') = xx' + (xy' + x'y + yy').$$

(i) Verify that this is a ring structure.

(ii) Let $R \to S$ be a ring homomorphism with kernel I, and let $i : I \to R$ be the inclusion. Show that $(R \times_S R, (p_1, p_2))$ is isomorphic to $(R + I, (p, q))$ where

$$p(x + y) = x$$
$$q(x + y) = x + i(y).$$

ex:fiber-prod-rep **Exercise 2.7.11.** Suppose that we have maps $X \to Z$ and $Y \to Z$ in a category $\mathscr{C}$. Show that $X \times_Z Y$, if it exists, represents $h_X \times_{h_Z} h_Y$.

ex:magic-square **Exercise 2.7.12** ([Vak, Exercise 2.3.R])**.** This is a surprisingly important exercise.

(i) Let $Y \to Z$ be a morphism in a category $\mathscr{C}$. Assuming that $Y \times_Z Y$ exists, construct a morphism $\delta : Y \to Y \times_Z Y$ using the universal property. (Hint: There is only one construction that will work in any category.) This morphism is known as the **diagonal**.

(ii) Let $X \to Y \to Z$ be a sequence of morphisms in $\mathscr{C}$ and assume that all of the fiber products in the diagram below exist. Prove that the diagram commutes and is cartesian.

$$\begin{array}{ccc} X \times_Y X & \longrightarrow & X \times_Z X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\delta} & Y \times_Z Y. \end{array}$$

(In other words, you are supposed to show that there is a natural map $X \times_Y X \to Y \times_{Y \times_Z Y} (X \times_Z X)$ and that this map is an isomorphism.)

(Hint: First show the diagram of functors

$$\begin{array}{ccc} h_X \times_{h_Y} h_X & \longrightarrow & h_X \times_{h_Z} h_X \\ \downarrow & & \downarrow \\ h_Y & \longrightarrow & h_Y \times_{h_Z} h_Y \end{array}$$

is cartesian. This is really just a question about sets!)

Solution. (i) For $f \in \mathrm{Hom}(Y, Z)$, we are given that the functor $F : \mathscr{C}^\circ \to \mathsf{Sets}$, defined as

$$F(W) = \{(p, q) | p, q \in \mathrm{Hom}_{\mathscr{C}}(W, Y), fp = fq\},$$

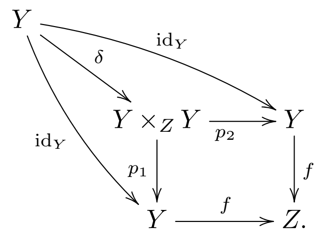
is represented by $Y \times_Z Y, (p_1, p_2)$. Unpacking what this means, we have the element $(p_1, p_2) \in F(Y \times_Z Y)$ giving a natural isomorphism

$$(p_1, p_2)^\varphi : h_{Y \times_Z Y} \to F.$$

In any category, we certainly have $\mathrm{id}_Y \in \mathrm{Hom}_{\mathscr{C}}(Y, Y)$, and thus the commutative square

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{id}_Y} & Y \\ {\scriptstyle \mathrm{id}_Y}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[f]{} & Z. \end{array}$$

Therefore it follows to let $\delta \in h_{Y \times_Z Y}(Y)$ be given as the unique element $((p_1, p_2)^{\varphi}_{Y})^{-1}(\mathrm{id}_Y, \mathrm{id}_Y)$, completing the commutative diagram



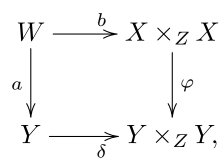
(ii) We are given morphisms

$$X \xrightarrow{g} Y \xrightarrow{f} Z,$$

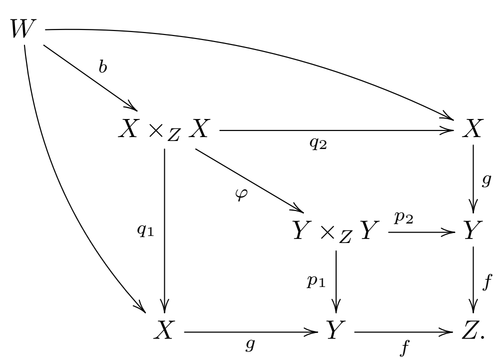
and we wish to construct an isomorphism between the two functors represented by $X \times_Y X$ and $Y \times_{Y \times_Z Y} X \times_Z X$. Let these functors be denoted F and F', respectively. Consider an element

$$(a, b) \in F'(W).$$

This gives a commutative square



and implicitly the following diagram commutes:



This now gives that

$$gq_1b = p_1\varphi b = p_1\delta a = a = p_2\delta a = p_2\varphi b = gq_2b.$$

As a result, $(q_1 b, q_2 b) \in F(W)$. This then suggests a natural isomorphism $\eta : F' \to F$, where

$$\eta_W(a, b) = (q_1 b, q_2 b),$$

and $\eta^{-1} : F \to F'$

$$\eta_W^{-1}(\alpha, \gamma) = (g\alpha, b'),$$

where b' is the map given by the universal property and the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\gamma} & X \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle fg} \\ X & \xrightarrow[fg]{} & Z. \end{array}$$

It remains to check that η is natural. For any morphism $\mu : U \to W$, it follows that

$$F(\mu)\eta_W(a, b) = F(\mu)(q_1 b, q_2 b) = (q_1 b\mu, q_2 b\mu) = \eta_U(a\mu, b\mu) = \eta_U F'(\mu)(a, b),$$

and so the diagram

$$\begin{array}{ccc} F'(W) & \xrightarrow{\eta_W} & F(W) \\ {\scriptstyle F'(\mu)}\downarrow & & \downarrow{\scriptstyle F(\mu)} \\ F'(U) & \xrightarrow[\eta_U]{} & F(U) \end{array}$$

commutes. Similarly,

$$F'(\mu)\eta_W^{-1}(\alpha, \gamma) = F'(\mu)(g\alpha, b') = (g\alpha\mu, b'\mu) = \eta_U^{-1}(\alpha\mu, \gamma\mu) = \eta_U^{-1} F(\mu)(\alpha, \gamma),$$

so there is a commutative diagram for η^{-1}, and thus η is a natural isomorphism between F' and F. We have shown that there is a natural isomorphism between F' and $h_{X \times_Y X}$, so by exercise 2.5.5 there is a unique isomorphism between $X \times_Y X$ and $Y \times_{Y \times_Z Y} X \times_Z X$ as desired. □

2.8 Limits and colimits

Definition 2.8.1. A **diagram shape**[8] $\mathscr{D}$ consists of a set of objects $\mathbf{Ob}(\mathscr{D})$ and a set of morphisms $\mathrm{Hom}_{\mathscr{D}}(X, Y)$ for each pair of objects $X, Y \in \mathbf{Ob}(\mathscr{D})$. A morphism of diagram shapes $F : \mathscr{D} \to \mathscr{D}'$ consists of a function $F : \mathbf{Ob}(\mathscr{D}) \to \mathbf{Ob}(\mathscr{D}')$, together with functions $Hom_{\mathscr{D}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}'}(FX, FY)$.

Noting that a category $\mathscr{C}$ is a diagram shape we define a **diagram** in $\mathscr{C}$ is a pair $(\mathscr{D}, D)$ where $\mathscr{D}$ is a diagram shape and $D : \mathscr{D} \to \mathscr{C}$ is a morphism of diagram shapes. We frequently omit reference to $\mathscr{D}$ and refer simply to a diagram D in $\mathscr{C}$.

[8] This definition is non-standard.

A slightly abusive but very convenient notation is to write X_i, $i \in \mathscr{D}$ or X_i, $i \in D$ to refer to a diagram $(\mathscr{D}, D)$ where $D(i) = X_i$.

We can illustrate diagram shapes with pictures. The following diagram shapes are rather important:

(i)
$$\begin{array}{ccc} & & 1 \\ & & \downarrow \\ 2 & \longrightarrow & 3 \end{array}$$

(ii) $1 \rightrightarrows 2$

(iii) $0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$

When discussing diagrams, we frequently just draw a diagram rather than say explicitly what the diagram shape and the morphism of diagram shapes are. For example, one might say "Consider a diagram in $\mathscr{C}$ of the following form:" and draw a picture like this:

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{q} & Z. \end{array}$$

The diagram shape $\mathscr{D}$ here is

$$\begin{array}{ccc} & & 1 \\ & & \downarrow{\scriptstyle u} \\ 2 & \xrightarrow{v} & 3 \end{array}$$

and the morphism of diagrams is

$$\begin{aligned} D(1) &= X \\ D(2) &= Y \\ D(3) &= Z \\ D(u) &= p \\ D(v) &= q. \end{aligned}$$

We construct a functor associated to a diagram $(\mathscr{D}, D)$. Let $F : \mathscr{C}^\circ \to \mathsf{Sets}$ be the functor with

$$F(X) = \left\{ (x_i)_{i \in \mathbf{Ob}(\mathscr{D})} \,\middle|\, \forall i,\ x_i \in \mathrm{Hom}(X, D(i)); \forall \varphi \in \mathrm{Hom}_{\mathscr{D}}(i,j),\ D(\varphi) \circ x_i = x_j \right\}.$$

If $u : X \to Y$ is a morphism of $\mathscr{C}$, the map $F(u) : F(Y) \to F(X)$ is

$$F(u)((x_i)_{i \in \mathbf{Ob}(\mathscr{D})}) = (u^* x_i)_{i \in \mathbf{Ob}(\mathscr{D})}.$$

If F is representable by $(X, (x_i)_{i \in \mathbf{Ob}(\mathscr{D})})$ then we call X the **limit** of the diagram D.

Exercise 2.8.2. What is a more familiar name for the limit of a diagram with no morphisms?

Exercise 2.8.3. Verify that a limit of a diagram with shape

$$\begin{array}{ccc} & & 1 \\ & & \downarrow \\ 2 & \longrightarrow & 3 \end{array}$$

is the same as a fiber product.

Exercise 2.8.4. Consider a category $\mathscr{C}$ that has a notion of kernel. Show that the kernel of a morphism $u : X \to Y$ is the same as the limit of the diagram

$$X \underset{0}{\overset{u}{\rightrightarrows}} Y$$

where 0 denotes the zero morphism.

Exercise 2.8.5. Show how to construct the limit of any diagram in the category of sets.

We may similarly define

$$G(X) = \left\{ (x_i)_{i \in \mathbf{Ob}(\mathscr{D})} \,\middle|\, \forall i,\ x_i \in \mathrm{Hom}(D(i), X)); \forall \varphi \in \mathrm{Hom}_{\mathscr{D}}(i,j),\ x_j \circ D(\varphi) = x_i \right\}$$

An pair $(X, (x_i)_{i \in \mathbf{Ob}(\mathscr{D})})$ representing G is called a **colimit** of G.

Exercise 2.8.6. Show how to construct the colimit of any diagram in the category of sets.

Exercise 2.8.7. Find a way to express the cokernel of a morphism as the colimit of a diagram.

Exercise 2.8.8. Show that each of the following categories has limits and colimits of all diagrams:

(i) Sets,

(ii) Grp,

(iii) ComRng,

(iv) Ab,

(v) R-Mod.

Exercise 2.8.9. A **pushout** is a colimit of a diagram with the following shape:

$$\begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & & \\ 3 & & \end{array}$$

Show that the category of fields does not have all pushouts. (Hint: Try to form the coproduct of $\mathbf{C}$ with itself over $\mathbf{R}$.)

Exercise 2.8.10. What is the significance, in more familiar terms, of the limit of an empty diagram? The colimit?

Definition 2.8.11. A diagram $(\mathscr{D}, D)$ in a category $\mathscr{C}$ is said to be **filtered** if it satisfies the following two conditions:

FIL1 For any $i, j \in \mathscr{D}$ there is some $k \in \mathscr{D}$ and maps $i \to k$ and $j \to k$ in $\mathscr{D}$.

FIL2 If $u, v : i \to j$ are two maps in $\mathscr{D}$ then there is some map $w : j \to k$ in $\mathscr{D}$ such that $D(w) \circ D(u) = D(w) \circ D(v)$.

A diagram $(\mathscr{D}, D)$ is said to be **cofiltered** if the opposite diagram is filtered.

A category $\mathscr{C}$ is said to be filtered or cofiltered if the diagram $(\mathscr{C}, \mathrm{id}_{\mathscr{C}})$ has the same property.[9]

Colimits of filtered diagrams, as well as limits of cofiltered diagrams, are said to be **directed**.

Exercise 2.8.12. Show that a filtered colimit of sets X_i, $i \in \mathscr{D}$ can be computed by the following construction

$$X = \coprod_{i \in \mathscr{D}} X_i \Big/ \sim$$

where $(i, \xi) \sim (j, \eta)$ if there is some $k \in \mathscr{D}$ and maps $u : i \to k$ and $v : j \to k$ with $X_u(\xi) = X_v(\eta)$.

2.9 Epimorphisms and monomorphisms

Exercise 2.9.1. Let R be an integral domain and F its field of fractions. Show that the map $R \to F$ is an epimorphism in the category of commutative rings but not in the category of sets.

2.10 Abelian categories

Definition 2.10.1. A category A is called **additive** if

AB0 A possesses finite products and finite coproducts and these coincide.

In an additive category we use the symbol $X \oplus Y$ to denote an object that is simultaneously the product and coproduct of X and Y.

This has many non-obvious consequences: the empty product coincides with the empty coproduct, which is therefore an object of A that is simultaneously initial and final. We choose one such object and call it 0.

Every $X \in A$ has a distinguished map to and from 0. Therefore for any X and Y in A we get a map $X \to 0 \to Y$ in $\mathrm{Hom}(X, Y)$. We call this map 0. This means that $\mathrm{Hom}(X, Y)$ is a pointed set for all $X, Y \in A$.

[9] Usually the definitions above are given only for categories, not for diagrams.

If $f, g : X \to Y$ are morphisms of A then we get a map $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} : X \oplus X \to Y \oplus Y$. Composing the sequence

$$X \xrightarrow{\begin{pmatrix} \mathrm{id}_X \\ \mathrm{id}_X \end{pmatrix}} X \oplus X \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} Y \oplus Y \xrightarrow{\begin{pmatrix} \mathrm{id}_Y & \mathrm{id}_Y \end{pmatrix}} Y.$$

We denote this composition $f + g : X \to Y$.

Exercise 2.10.2. Verify that with these definitions, $\mathrm{Hom}_A(X, Y)$ has the structure of an abelian group. This is the origin of the name additive.

Exercise 2.10.3. An alternate definition of an additive category may be found in [Vak, §2.6.1]. Verify that that definition is equivalent to this one. (Show, in other words, that a category is additive in this sense if and only if it is additive in Vakil's sense.)

Definition 2.10.4. An additive category A is said to be **abelian** if it satisfies the axioms below [Gro1]:

AB1 every morphism $f : X \to Y$ in A has a kernel and cokernel, and

AB2 for every $f : X \to Y$, the canonical map from the coimage of f to the image of f is an isomorphism.

3 Commutative algebra

3.1 References

See [AM] for an efficient introduction to commutative algebra well-suited to the prerequisites of this course. For a more thorough introduction, see [Mat].

3.2 Localization

Let f be an element of a commutative ring A. Define

$$A[f^{-1}] = A[g]/(fg - 1).$$

ex:loc-basic

Exercise 3.2.1. (i) Show that every element of $A[f^{-1}]$ can be represented as a/f^n for some $n \in \mathbf{Z}_{\geq 0}$.

(ii) Show that the map $a \mapsto a/f^0$ defines a homomorphism $A \to A[f^{-1}]$ that we will call the **canonical homomorphism**.

ex:loc-epi

Exercise 3.2.2. Show that $A \to A[f^{-1}]$ is an *epimorphism* in the category of rings.

Solution. Let $\varphi : A \to A[f^{-1}]$ be the canonical morphism, and suppose $h, g \in \mathrm{Hom}(A[f^{-1}], B)$ such that

$$h\varphi = g\varphi.$$

We are given that for any $a \in A$,

$$h\left(\frac{a}{f^0}\right) = g\left(\frac{a}{f^0}\right).$$

Note that $h\varphi(f) = g\varphi(f)$ is a unit as

$$h\left(\frac{f}{f^0}\right) h\left(\frac{1}{f}\right) = h(1) = 1 = g(1) = g\left(\frac{f}{f^0}\right) g\left(\frac{1}{f}\right),$$

so it follows that

$$h\left(\frac{1}{f}\right) = g\left(\frac{1}{f}\right).$$

Thus for any $a/f^n \in A[f^{-1}]$, we must have

$$h\left(\frac{a}{f^n}\right) = h\left(\frac{a}{f^0}\right) h\left(\frac{1}{f}\right)^n = g\left(\frac{a}{f^0}\right) g\left(\frac{1}{f}\right)^n = g\left(\frac{a}{f^n}\right),$$

and φ is an epimorphism. □

ex:loc-univ **Exercise 3.2.3.** Show that $A[f^{-1}]$ satisfies the following universal property: For any commutative ring B define $F(B) = \{\varphi \in \mathrm{Hom}_{\mathsf{ComRng}}(A, B) \mid \varphi(f) \in B^*\}$. Show that $A[f^{-1}]$ together with the canonical homomorphism $A \to A[f^{-1}]$ represents F.

More generally, if S is any subset of A, we define

$$S^{-1}A = A[\{f' \mid f \in S\}]/(f'f - 1).$$

ex:loc-basic2 **Exercise 3.2.4.** Formulate and prove analogues of Exercises 3.2.1 and 3.2.2.

ex:loc-univ2 **Exercise 3.2.5.** Let S be a subset of A and let $F : \mathsf{ComRng} \to \mathsf{Sets}$ be the functor $F(B) = \{\varphi \in \mathrm{Hom}_{\mathsf{ComRng}}(A, B) \mid \varphi(S) \subset B^*\}$. Show that $S^{-1}A$ represents F.

Let $\mathfrak{p}$ be a prime ideal of A. We denote by $A_{\mathfrak{p}}$ the ring $S^{-1}A$ where $S = A \smallsetminus \mathfrak{p}$.

ex:loc-local **Exercise 3.2.6.** Check that $A_{\mathfrak{p}}$ has a unique maximal ideal $\mathfrak{m}$ generated by the image of $\mathfrak{p}$ under the canonical homomorphism $A \to A_{\mathfrak{p}}$. Show also that if $f : A \to A_{\mathfrak{p}}$ denotes the canonical homomorphism then $f^{-1}(\mathfrak{m}) = \mathfrak{p}$.

ex:loc-kernel **Exercise 3.2.7.** Show that the kernel of the canonical homomorphism $A \to A[f^{-1}]$ is $\{x \in A \mid \exists n \geq 0,\ f^n x = 0\}$.

Exercise 3.2.8. What is $S^{-1}A$ when $S = A$?

Exercise 3.2.9. Let $\mathfrak{p}$ be a prime ideal of $A_{\mathfrak{p}}$ and $\varphi : A \to A_{\mathfrak{p}}$ a homomorphism of commutative rings.

(i) Suppose that I is an ideal of $A_{\mathfrak{p}}$. Show that $\varphi^{-1}(I)$ is an ideal of A contained in $\mathfrak{p}$.

(ii) Show that φ^{-1} determines a bijection betweeen the set of non-unit ideals of $A_{\mathfrak{p}}$ and the set of ideals of A contained in $\mathfrak{p}$.

Localization of modules

Let M be an A-module and S a subset of A. We denote by $S^{-1}M$ the $S^{-1}A$-module

$$S^{-1}M = S^{-1}A \underset{A}{\otimes} M$$

where the map $A \to S^{-1}A$ is the canonical one.

ex:loc-mod-basic **Exercise 3.2.10.** (i) Show that the elements of $S^{-1}M$ can be represented as $f_1^{-1} \cdots f_k^{-1} x$ with $f_1, \ldots, f_k \in S$ and $x \in M$.

(ii) Show that the map $x \mapsto 1 \otimes x$ from M to $S^{-1}M$ is an A-module homomorphism. We call this map **canonical**.

ex:loc-mod-kernel **Exercise 3.2.11.** Show that the kernel of the canonical map $M \to S^{-1}M$ consists of all $x \in M$ that are annihilated by some finite product of elements of S.

ex:loc-mod-functor **Exercise 3.2.12.** Suppose $u : M \to M'$ is an A-module homomorphism and $S \subset A$. Show that the map $f_1^{-1} \cdots f_k^{-1} x \mapsto f_1^{-1} \cdots f_k^{-1} u(x)$ defines an $S^{-1}A$-module homomorhism $S^{-1}u : S^{-1}M \to S^{-1}M'$. Verify that this maps S^{-1} into a functor from A-Mod to S^{-1}-Mod.

prop:loc-mod-exact **Proposition 3.2.13.** *Let A be a commutative ring and S a subset of A. Suppose that*

$$0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$$

is an exact sequence of A-modules. Then the sequence of $S^{-1}A$-modules

$$0 \to S^{-1}M' \xrightarrow{S^{-1}u} S^{-1}M \xrightarrow{v} S^{-1}M'' \to 0$$

is exact.

Proof (sketch): please add more details. By the right exactness of tensor product, it is sufficient to show that $S^{-1}u : S^{-1}M' \to S^{-1}M$ is injective. Suppose that $S^{-1}u(f_1^{-1} \cdots f_k^{-1} x) = 0$. Then $S^{-1}u(x) = 0$ so x lies in the kernel of the canonical map $M \to S^{-1}M$. Therefore there are $g_1, \ldots, g_\ell \in S$ such that $g_1 \cdots g_\ell u(x) = 0$. But $g_1 \cdots g_\ell u(x) = u(g_1 \cdots g_\ell x)$ and u is injective, so this means $g_1 \cdots g_\ell x = 0$. That is x lies in the kernel of the canonical map $M' \to S^{-1}M'$. Hence x is zero in $S^{-1}M'$ and a fortiori, so is $f_1^{-1} \cdots f_k^{-1} x$. □

ex:annihilator

Exercise 3.2.14. Let M be an A-module. The **annihilator** of M is the set of all $f \in A$ such that $fM = 0$. We denote the annihilator of M by $\operatorname{Ann}_A(M)$.

(i) Show that the annihilator of M is an ideal.

(ii) Show that $\operatorname{Ann}_{S^{-1}A}(S^{-1}M) = S^{-1}\operatorname{Ann}_A(M)$.

(iii) Show that M is the zero module if and only if $\operatorname{Ann}_A(M) = 0$.

prop:loc-mod-zero

Proposition 3.2.15. *An A-module M is the zero module if and only if $M_{\mathfrak{p}} = 0$ for all maximal ideals $\mathfrak{p} \subset A$.*

ex:loc-mod-zero

Exercise 3.2.16. Prove the proposition (you will want to use Exercise 3.2.14:

(i) Let I be the annihilator ideal of M. Show that $M = 0$ if and only if $I = A$.

(ii) Show that $I = A$ if and only if I is not contained in any maximal ideal of A.

(iii) Show that I is contained not contained in a maximal ideal $\mathfrak{p}$ of A if and only if $I_{\mathfrak{p}} = A_{\mathfrak{p}}$.

(iv) Show that $I_{\mathfrak{p}}$ is the annihilator ideal of $M_{\mathfrak{p}}$ for each prime ideal of A. Deduce that $I_{\mathfrak{p}} = A_{\mathfrak{p}}$ if and only if $M_{\mathfrak{p}} = 0$.

prop:loc-mod-morph

Proposition 3.2.17. *An A-module homomorphism $\varphi : M \to N$ is (i) injective, (ii) surjective, (iii) an isomorphism if and only if the $A_{\mathfrak{p}}$-module homomorphism $\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ induced from φ is (i) injective, (ii) surjective, (iii) an isomorphism for all maximal ideals $\mathfrak{m}$ of A.*

ex:loc-mod-morph

Exercise 3.2.18. Prove the proposition. (Hint: Use the first two parts to deduce the third. For the first two, consider the kernel and cokernel of φ and use the last proposition.)

sec:nullstellensatz

3.3 The Nullstellensatz

lem:going-up

Lemma 3.3.1 (The "going up" theorem)**.** *Suppose B is an integral extension of a commutative ring A. Then any prime (resp. maximal) ideal $\mathfrak{p}$ of A is the intersection of some prime (resp. maximal) ideal $\mathfrak{P}$ of B with A.*

Proof. The prime ideal $\mathfrak{p}$ is the kernel of a homomorphism $A \to K$ where K is a field. Let $R = B \otimes_A K$. Then R is an integral extension of K, hence is not the zero ring. Hence R has a maximal ideal $\mathfrak{m}$. We have $\mathfrak{m} \cap K = (0)$ since that is the only proper ideal of K. Let us take $\mathfrak{P}$ to be the pre-image of $\mathfrak{m}$ in B. That is, $\mathfrak{P} = \mathfrak{m} \times_R B$. Then $\mathfrak{P} \cap A$ coincides with the pre-image of $\mathfrak{P} \cap K$ in A, which is $\mathfrak{p}$ by assumption.

If $\mathfrak{p}$ were a maximal ideal of A then $A \to K$ could be chosen to be surjective. Then $B \to R$ will also be surjective so the pre-image of a maximal ideal of R will be a maximal ideal of B. □

lem:alg-ring-ext **Lemma 3.3.2.** *Let A be an integral domain and $B \supset A$ a finitely generated, algebraic A-algebra extension. Then there is some non-zero $a \in A$ such that $B[a^{-1}]$ is an integral $A[a^{-1}]$-algebra.*

Proof. The meaning of *algebraic* in the statement of the lemma is that $B \otimes_A K$ is an algebraic extension of K when K is taken to be the field of fractions of A.

Let $x_1, \ldots, x_n$ be generators of B over A. By assumption, each x_i satisfies some non-zero polynomial with coefficients in A. Let a_i be the leading coefficient of a non-zero polynomial satisfied by x_i. Take $a = \prod a_i$. Then each x_i satisfies a monic polynomial over $A[a^{-1}]$, so $B[a^{-1}]$ is generated by integral elements, hence is integral. □

Theorem 3.3.3. *Let K be a field and L a field that is finitely generated as a K-algebra. Then L is finite dimensional as a K-vector space.*

Proof. Assume to the contrary that $\dim_K L = \infty$. Let $x_1, \ldots, x_n$ be a transcendence basis for L over K. Then L is a finitely generated algebraic extension of $K(x_1, \ldots, x_n)$. The number n must be ≥ 1 since a finitely generated algebraic extension of K is finite dimensional over K.

Note that L is also finitely generated as an extension of $K(x_1, \ldots, x_{n_1})$ and we have $\dim_{K(x_1,\ldots,x_{n-1})} L = \infty$ as well. We can therefore replace K with $K(x_1, \ldots, x_{n-1})$, take $x = x_n$, and assume that L is algebraic over $K[x]$.

Now, by Lemma 3.3.2, there is some $a \in K[x]$ such that L is an *integral* $K[x, a^{-1}]$-algebra (note that $L[a^{-1}] = L$ since L is a field). But $K[x]$ has infinitely many prime ideals.[10] The prime ideals of $K[x, a^{-1}]$ correspond to the prime ideals of $K[x]$ that do not contain a. Since $K[x]$ is a principal ideal domain, and in particular a unique factorization domain, there are only finitely many prime ideals containing a. It follows that $K[x, a^{-1}]$ has infinitely many prime ideals. Therefore, by the going up theorem (Lemma 3.3.1) it follows that L has infinitely many prime ideals. But this contradicts the assumption that L was a field! □

Remark 3.3.4. This proof approximates one originally due to Zariski. The version given here is adapted from [AM, §5, Exercise 18].

cor:nulls-empty **Corollary 3.3.4.1.** *Let K be a field, $X = V(I)$ for an ideal $I \subset K[x_1, \ldots, x_n]$. If $X(\overline{K}) = \varnothing$ for an algebraic closure $\overline{K}$ of K then $X = \varnothing$.*

Proof. If $X \neq \varnothing$ then $I \neq K[x_1, \ldots, x_n]$ so I is contained in a maximal ideal $\mathfrak{m}$. Then if we put $L = K[x_1, \ldots, x_n]/\mathfrak{m}$ we have an element $\xi \in X(L)$. But L is algebraic over K so there is an embedding $L \subset \overline{K}$. Taking the image of ξ in $X(\overline{K})$ shows that $X(\overline{K}) \neq \varnothing$. □

Let I be an ideal in a commutative ring A. Its radical $\sqrt{I}$ is the set of all $f \in A$ such that $f^n \in I$ for some integer $n \geq 0$.

[10] This is obvious if K is an infinite field. If K is finite, note that $K[x]$ contains irreducible polynomials of all degrees.

cor:nulls-radical

Corollary 3.3.4.2. *Let K be an algebraically closed field and $I \subset A$ an ideal in a finitely generated K-algebra A. Let $X = V(I)$ and define $\overline{I}$ to be the set of $f \in A$ such that $f(x) = 0$ for all $x \in X(K)$.*[11] *Then $\overline{I} = \sqrt{I}$.*

Proof. Suppose $f \in \sqrt{I}$. Then $f^n \in I$, so $f(x)^n = 0$ for all $x \in X(K)$. But K is a field, so $f(x)^n = 0$ implies $f(x) = 0$, which means $f \in \overline{I}$.

Suppose $f \in I$. Then let $Y = D(f) \subset X$. We have $Y(K) = \varnothing$. By Corollary 3.3.4.1, this means that $Y = \varnothing$. But $Y = A/I[f^{-1}]$, so this means $A/I[f^{-1}]$ is the zero ring. Thus f must be nilpotent in A/I, which means $f^n \in I$. □

Remark 3.3.5. The proof of Corollary 3.3.4.2 is known as Rabinowitsch's trick.

Exercise 3.3.6. Let K be an algebraically closed field. Show that the maximal ideals of $K[x_1, \ldots, x_n]$ are all of the form $(x_1 - \lambda_1, \ldots, x_n - \lambda_n)$ with $\lambda_1, \ldots, \lambda_n \in K$.

Exercise 3.3.7. Give an example of a homomorphism of commutative rings $\varphi : A \to B$ that is *not an isomorphism* such that for every field K, the map $\mathrm{Hom}(\varphi, K) : \mathrm{Hom}(B, K) \to \mathrm{Hom}(A, K)$ is a bijection.

Deduce that while the set of solutions to a polynomial equation can be determined from the quotient ring, the quotient ring is not determined by its set of points.

sec:flatness

3.4 Flatness

Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories.[12] A functor $F : \mathscr{C} \to \mathscr{D}$ is called **additive** if it preserves direct sums: $F(X \oplus Y) \cong F(X) \oplus F(Y)$.

Exercise 3.4.1. Let A and B be commutative rings and let $F : A\text{-}\mathsf{Mod} \to B\text{-}\mathsf{Mod}$ be the functor $F(M) = M \otimes_A B$. Show that F is an additive functor.

An additive functor $F : \mathscr{C} \to \mathscr{D}$ is said to be **exact** if it preserves exact sequences: Whenever

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \tag{1}$$

eq:src-exact

is exact, so is

$$F(M') \xrightarrow{F(u)} F(M) \xrightarrow{F(v)} F(M''). \tag{2}$$

eq:im-exact

We say that F is **faithfully exact** if the exactness of one of (1) and (2) implies the exactness of the other.

[11] In other words, $\overline{I}$ is the set of all $f \in A$ such that, for all $x \in X(K)$ corresponding to $\varphi : A \to A/I \to K$, we have $\varphi(f) = 0$.

[12] If you do not know what an abelian category is, feel free to imagine $\mathscr{C} = A\text{-}\mathsf{Mod}$ and $\mathscr{D} = B\text{-}\mathsf{Mod}$. More generally, you may think of any familiar category in which there is a notion of an exact sequence.

Definition 3.4.2. Let A be a commutative ring. An A-module N is called **flat** if the functor

$$A\text{-}\mathsf{Mod} \to A\text{-}\mathsf{Mod} : M \mapsto M \underset{A}{\otimes} N$$

is exact. If the functor is **faithfully exact** then we say that N is faithfully flat. We apply the same definition to an A-algebra by regarding it as an A-module.

Exercise 3.4.3 ([Vak, 25.5.A])**.** Show that an A-module N is faithfully flat if and only if the condition $M \otimes_A N = 0$ implies that $M = 0$.

Flatness examples

ex:direct-sum-flat **Exercise 3.4.4.** Suppose that N and N' are (i) flat, or (ii) faithfully flat A-modules. Show that $N \oplus N'$ has the same property.

ex:flat-free **Exercise 3.4.5.** Recall that an A-module F is called **free** if it is isomorphic as an A-module to a direct sum of copies of A. Show that a free A-module is faithfully flat.

ex:field-flat **Exercise 3.4.6.**

(i) Let K be a field and N any K-vector space. Show that N is flat as a K-module and faithfully flat if $N \neq 0$.

(ii) Deduce that a non-zero K-algebra is faithfully flat.

Each of the following three exercises generalizes the previous one. They are very important.

ex:loc-flat **Exercise 3.4.7.** Let f be an element of a commutative ring A.

(i) Show that $A[f^{-1}]$ is a flat A-algebra.

(ii) Show that $A[f^{-1}]$ is not faithfully flat unless $f \in A^*$ by giving an example of a *non-exact* sequence of A-modules E such that $E \otimes_A A[f^{-1}]$ is exact.

ex:loc-flat-2 **Exercise 3.4.8.** Let f_1 and f_2 be elements of A such that $(f_1, f_2)A = A$. Show that $A[f_1^{-1}] \oplus A[f_2^{-1}]$ is a *faithfully flat* A-module.

ex:loc-flat-3 **Exercise 3.4.9.** Let $S \subset A$ be any subset and let

$$M = \bigoplus_{f \in S} A[f^{-1}].$$

Show that M is a flat A-module.

ex:loc-flat-4 **Exercise 3.4.10.** Let S be any subset of A.

(i) Show that $S^{-1}A$ is a flat A-algebra.

(ii) Deduce that $A_{\mathfrak{p}}$ is a flat A-algebra for each prime ideal $\mathfrak{p} \subset A$.

(iii) Show that $\bigoplus_{\mathfrak{p} \subset A \text{ prime}} A_{\mathfrak{p}}$ is a *faithfully* flat A-module.

Preliminaries on descent

Proposition 3.4.11. *Let B be an A-algebra such that there exists an A-algebra homomorphism $B \to A$. Then for any A-module M, the sequence below is exact.*

$$
\begin{array}{ccccccccc}
0 \longrightarrow M & \xrightarrow{d} & M \otimes_A B & \xrightarrow{d} & M \otimes_A B \otimes_A B & \xrightarrow{d} & M \otimes_A B \otimes_A B \otimes_A B \\
m & \longmapsto & m \otimes 1 & & & & \\
 & & m \otimes b & \longmapsto & m \otimes b \otimes 1 & & \\
 & & & & - m \otimes 1 \otimes b & & \\
 & & & & m \otimes b \otimes b' & \longmapsto & m \otimes b \otimes b' \otimes 1 \\
 & & & & & & - m \otimes b \otimes 1 \otimes b' \\
 & & & & & & + m \otimes 1 \otimes b \otimes b'
\end{array}
\tag{3}
$$

eqn:1

Proof. Let $s : B \to A$ be an A-algebra homomorphism. This induces a map $s : M \otimes_A B \to M$ given by $m \otimes b \mapsto s(b)m$. But then $s \circ d : M \to M$ is the identity and therefore d is injective.

Suppose now that $x \in M \otimes_A B$ lies in the kernel of d. Let $s : M \otimes_A B \otimes_A B \to M \otimes_A B$ be the map $s(m \otimes b \otimes b') = s(b)m \otimes b'$. Then $sd(x) = 0$. But we can easily verify that $ds - sd : M \otimes_A B \to M \otimes_A B$ is the identity map, which means that $x = ds(x)$, i.e., x lies in the image of d.

Finally, let us suppose that $x \in M \otimes_A B \otimes_A B$ lies in the kernel of d. Define $s : M \otimes_A B \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A B$ by

$$s(m \otimes b \otimes b' \otimes b'') = s(b)m \otimes b' \otimes b''$$

We can now verify that the map $sd - ds : M \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A B$ coincides with the identity. Thus if $d(x) = 0$ we have $x = ds(x) + sd(x) = ds(x)$ so x lies in the image of d. $\square$

It is convenient to say that a diagram

$$M_0 \xrightarrow{i} M_1 \underset{v_1}{\overset{v_0}{\rightrightarrows}} M_2$$

is **exact** if i is injective and $x \in M_1$ lies in the image of i if and only if $v_0(x) = v_1(x)$. Then the proposition above has the following corollary:

Exercise 3.4.12. In the setting of the proposition, define $F_{-1} = M$ and $F_n = F_{n-1} \otimes_A B$ for all positive $n \in \mathbf{Z}$. Define maps

$$d : F_n \to F_{n+1} : m \otimes b_0 \otimes \cdots \otimes b_n \mapsto \sum_{i=-1}^{n} (-1)^{n-i} m \otimes b_0 \otimes \cdots \otimes b_i \otimes 1 \otimes b_{i+1} \otimes \cdots \otimes b_n.$$

Generalize the proposition above to show that for all n the sequence

$$F_{n-1} \to F_n \to F_{n+1}$$

is exact.[13]

`prop:descent-sequence` **Proposition 3.4.13.** *The sequence* (3) *is exact for any faithfully flat A-algebra B and any A-module M.*

Proof. By faithful flatness, we can replace A by B and B by $B \otimes_A B$ and then apply the proposition (where $s : B \otimes_A B \to B$ sends $b_1 \otimes b_2$ to $b_1 b_2$). ☐

`cor:descent-sequence` **Corollary 3.4.13.1.** *Let M be an A-module and B a faithfully flat A-algebra. Then the sequence*

$$M \xrightarrow{i} B \otimes_A M \underset{v_1}{\overset{v_0}{\rightrightarrows}} B \otimes_A B \otimes_A M,$$

$$i(x) = 1 \otimes x \qquad v_0(b \otimes x) = b \otimes 1 \otimes x \qquad v_1(b \otimes x) = 1 \otimes b \otimes x$$

is exact.

Theorem 3.4.14 (Faithfully flat descent for morphisms)**.** *Let $A \to B_i$, $i \in I$ be a family of faithfully flat ring homomorphisms. Let $B_{ij} = B_i \otimes_A B_j$ and $B_{ijk} = B_i \otimes_A B_j \otimes_A B_k$.*

`descent:morph-eq` *(i) Suppose that M and N are A-modules, and $u, v : M \to N$ are A module homomorphisms. If $u \otimes_A B_i = v \otimes_A B_i$ for all $i \in I$ then $u = v$.*

`descent:morph` *(ii) Suppose that M and N are a A-modules and $u_i : M \otimes_A B \to N \otimes_A B$ is a B-module homomorphism. For each $i, j \in I$, let $u_{ij} = u_i \otimes_{B_i} B_{ij}$. If $u_{ij} = u_{ji}$ for all $i, j \in I$ then there is an A-module homomorphism $u : M \to N$ such that $u_i = u \otimes_A B_i$.*

Proof. Proof of (i): To show that $u = v$ is the same as to show that $u - v = 0$. Let $w = u - v$ and let P be the image of w. Then for each i we have $B_i \otimes_A P = \text{image}(B_i \otimes_A w)$. But $B_i \otimes_A w = 0$ for all i, so $B_i \otimes_A P = 0$ for all i. Therefore $P = 0$ because the B_i are faithfully flat.

Proof of (ii): We have a commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & M & \longrightarrow & \bigoplus_i M \otimes_A B_i & \xrightarrow{d} & \bigoplus_{j,k} M \otimes_A B_{jk} \\
& \downarrow & & \downarrow \scriptstyle{\bigoplus u_i} & & \downarrow \scriptstyle{\bigoplus u_{jk} = \bigoplus u_{kj}} \\
0 \longrightarrow & N & \longrightarrow & \bigoplus_i N \otimes_A B_i & \longrightarrow & \bigoplus_{j,k} N \otimes_A B_{jk}
\end{array}$$

For $b \in B_i$, the map d is

$$d(m \otimes b) = \sum_j m \otimes 1_{B_j} \otimes b - \sum_k m \otimes b \otimes 1_{B_k}.$$

[13] This exercise has important consequences concerning the cohomology of quasi-coherent sheaves in the étale and flat topologies. Among other things, it implies Hilbert's Theorem 90.

The universal property of the kernel provides the map $M \to N$. We necessarily have $u_i(m \otimes 1) = u(m) \otimes 1$, which gives $u_i = u \otimes_A B_i$. □

Exercise 3.4.15. Let L be a Galois extension of a field K with Galois group $G = \mathrm{Gal}(L/K)$. Use the notation $x \mapsto x^g$ for the action of $g \in G$ on $x \in L$. For each $g \in G$, define a homomorphism $\varphi_g : L \otimes_K L \to L$ by $\varphi_g(x \otimes y) = xy^g$. Use this to obtain a homomorphism

$$\varphi : L \underset{K}{\otimes} L \to \prod_{g \in G} L : x \mapsto (\varphi_g(x))_{g \in G}.$$

We will show below that this map is an isomorphism and use this fact to prove half of the fundamental theorem of Galois theory.

(i) Prove that $L \otimes_K L$ is isomorphic to a product of fields. (Hint: Use the fact that $L \otimes_K K[x]/(f(x)) = L[x]/(f(x))$ and that there is a tower of fields $K = K_0 \subset K_1 \subset \cdots \subset K_n = L$ with $K_i = K_{i-1}[x_i]/(f(x_i))$ such that $f(x_i)$ splits in L; then apply the Chinese remainder theorem.)

(ii) Prove that the map

$$\mathrm{Hom}_K(L \underset{K}{\otimes} L, A) \xleftarrow{\varphi^*} \mathrm{Hom}_K(\prod_{g \in G} L, A) \tag{4}$$ eqn:2

is a bijection when A is a field.

(iii) Deduce that the map (4) is a bijection when A is a product of fields.

(iv) Conclude that φ is an isomorphism. (Hint: Apply the Yoneda lemma to the category of commutative algebras that are products of fields.)

(v) Use Corollary 3.4.13.1 to deduce that $L^G = K$. (Here L^G is the set of $x \in L$ such that $x^g = x$ for all $g \in G$.)

(vi) Conclude the apparently stronger statement that if $K \subset K' \subset L$ is any intermediate field then $K' = L^{\mathrm{Gal}(L/K')}$.

Flat descent

Let B be a faithfully flat A-algebra. Let us write $B_i = B \otimes_A \cdots \otimes_A B$ (with $i + 1$ factors). We have maps

$$v_0 : B_0 \to B_1 : b \mapsto b \otimes 1$$
$$v_1 : B_0 \to B_1 : b \mapsto 1 \otimes b$$

By **standard homomorphisms** from B_i to B_j we will mean homomorphisms induced by inclusions of components, such as $b_1 \otimes b_2 \mapsto b_1 \otimes 1 \otimes b_2$. The homomorphisms v_0 and v_1 above are standard homomorphisms.

A **descent datum** $\mathscr{M}$ for B relative to A is a collection data of the following form:

descent:1 **DESC1** a B_i-module $\mathscr{M}_i$ for each i,[14]

descent:2 **DESC2** for each standard homomorphism $\varphi : B_i \to B_j$ homomorphism $\mathscr{M}_\varphi : \mathscr{M}_i \to \mathscr{M}_j$ for each of the maps introduced above, such that

descent:3 **DESC3** $\mathscr{M}_\varphi \circ \mathscr{M}_\psi = \mathscr{M}_{\varphi\circ\psi}$ and

descent:4 **DESC4** the maps $B_j \otimes_{B_i} \mathscr{M}_i \to \mathscr{M}_j : b \otimes x \mapsto b\varphi(x)$ are isomorphisms.

Descent data form a category in a natural way; we denote this category by $\mathscr{D}$. A morphism of descent data from $\mathscr{M}$ to $\mathscr{M}'$ consists of maps $u_i : \mathscr{M}_i \to \mathscr{M}'_i$ such that the diagrams

$$\begin{array}{ccc} \mathscr{M}_i & \xrightarrow{u_i} & \mathscr{M}'_i \\ {\scriptstyle \mathscr{M}_\varphi}\downarrow & & \downarrow{\scriptstyle \mathscr{M}'_\varphi} \\ \mathscr{M}_j & \xrightarrow{u_j} & \mathscr{M}'_j \end{array}$$

commute whenever $\varphi : B_i \to B_j$ is a standard homomorphism. We construct functors

$$\Phi : A\text{-}\mathsf{Mod} \to \mathscr{D}$$
$$\Psi : \mathscr{D} \to A\text{-}\mathsf{Mod}$$

by setting $\Phi(M)_i = M \otimes_A B_i$ and $\Phi(M)_\varphi$ to be the map $M \otimes_A B_i \to M \otimes_A B_j : x \otimes b \mapsto x \otimes \varphi(b)$. We define $\Psi(\mathscr{M})$ to be the set of all $x \in \mathscr{M}_0$ such that $v_0(x) = v_1(x)$, where v_0 and v_1 denote the two maps $\mathscr{M}_0 \to \mathscr{M}_1$ associated to $v_0, v_1 : B_0 \to B_1$. In order to indicate the dependence on A and B, we write $\Phi_{B/A}$ and $\Psi_{B/A}$.

These functors make sense if B is any A-algebra.

Exercise 3.4.16. If there is an A-algebra homomorphism $B \to A$ then Φ and Ψ are inverse equivalences of categories.

Theorem 3.4.17. *If B is faithfully flat over A then $\Phi_{B/A}$ and $\Psi_{B/A}$ are inverse equivalences of categories.*

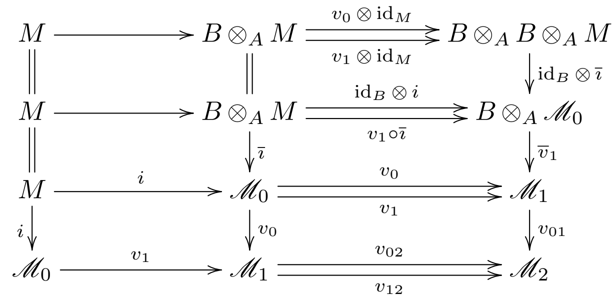
Proof. The fact that $\Psi\Phi(M) \cong M$ is the exactness of the diagram

$$\begin{array}{ccccc} M & \longrightarrow & B \otimes_A M & \overset{v_0}{\underset{v_1}{\rightrightarrows}} & B \otimes_A B \otimes_A M \\ \downarrow & & \| & & \| \\ \Psi\Phi(M) & \longrightarrow & \Phi(M)_0 & \overset{v_0}{\underset{v_1}{\rightrightarrows}} & \Phi(M)_1. \end{array}$$

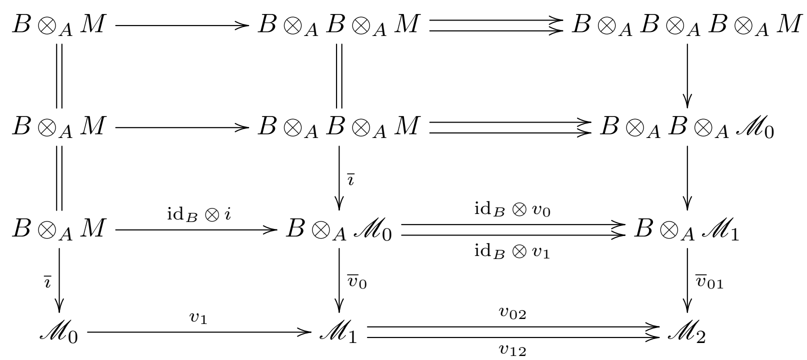
Let $M = \Psi(\mathscr{M})$, let i denote the inclusion of M in $\mathscr{M}_0$, and adopt the convention that, if f is an A-module homomorphism $N \to N'$, where N' is a B-module, then

[14] It suffices to restrict to $i = 0, 1, 2$ here.

$\overline{f}$ denotes the induced map $B \otimes_A N \to N'$; consider the following commutative diagram.



Applying $B \otimes (-)$ to the top three rows we obtain the diagram below.



By assumption, the vertical arrows $\overline{v}_0$, $\overline{v}_{01}$, and $\overline{v}_1$ are all isomorphisms, so $\bar{\imath}$ must also be an isomorphism. Therefore all of the vertical arrows are isomorphisms. But the top row of the diagram above is $\Phi \circ \Psi(\mathscr{M})$ and the bottom row is $\mathscr{M}$. Thus $\Phi \circ \Psi(\mathscr{M}) \cong \mathscr{M}$. $\square$

3.5 Projective modules

ex:projective **Exercise 3.5.1.** Prove that the following are equivalent properties of an A-module M: [15]

proj:1 **PROJ1** The functor $h^M : A\text{-}\mathsf{Mod} \to A\text{-}\mathsf{Mod}$ is exact.

proj:2 **PROJ2** The functor h^M preserves surjections.

proj:3 **PROJ3** Every A-module surjection $N \to M$ has a section.

proj:4 **PROJ4** There is a free A-module F such that $F \simeq M \oplus N$ for some A-module N. [16]

[15] More generally, you may wish to prove that these conditions are equivalent in any abelian category; see the next note, however.

[16] While the previous conditions make sense in an arbitrary abelian category, this last one does not.

def:projective-module

Definition 3.5.2. An A-module is said to be **projective** if it satisfies the equivalent conditions of Exercise 3.5.1.

def:finite-gen-pres

Definition 3.5.3. An A-module is M said to be **finitely generated** if it admits an A-module surjection $A^n \to M$ for some positive integer n. It is said to be **finitely presented** if there is an exact sequence

$$A^m \to A^n \to M \to 0$$

with both m and n finite.

Exercise 3.5.4. Show that an A-module M is finitely presented if and only if, for any A-module surjection $p : A^n \to M$, the A-module $\ker(p)$ is finitely generated.

Exercise 3.5.5. Say that M is **locally finitely generated** or **locally finitely presented** if for every prime ideal $\mathfrak{p}$ of A there is some $f \in A \smallsetminus \mathfrak{p}$ with $A[f^{-1}] \otimes_A M$ respectively finitely generated or finitely presented as an $A[f^{-1}]$-module.

(i) Show that a locally finitely generated A-module is finitely generated. (Hint: Observe that an A-module M is the union of its finitely generated submodules, then find elements $f_1, \ldots, f_n$ of A and finitely generated submodules $M_1, \ldots, M_n$ of M such that the maps $A[f_i^{-1}] \otimes_A M_i \to A[f_i^{-1}] \otimes_A M$ are all surjective. Then show that M is generated as an A-module by $M_1, \ldots, M_n$.)

(ii) Conclude that a locally finitely presented A-module is finitely presented.

Exercise 3.5.6. Show that a locally free A-module is projective:

(i) Observe that a locally free A-module is locally finitely presented and conclude therefore that a locally free A-module is finitely presented.

(ii) Suppose that M is a finitely presented A-module. Show that M is projective if and only if $M_{\mathfrak{p}}$ is a projective $A_{\mathfrak{p}}$-module for all prime ideals $\mathfrak{p}$ of A.

4 Algebra and geometry

4.1 References

[Har, Chapter 1, §1–2] discusses affine and projective space, though we will not treat all the topics discussed there until later in this course. Early familiarity with them won't hurt.

Some of the discussion in Section 4.14 is borrowed from Mumford's "Red Book" [Mum].

4.2 Functorial language

Suppose we wish to study the geometry of the solutions to the equation $x^2+y^2 = -1$. Our first reaction may be that we must first decide where we want to look for solutions. In the real numbers, for example, there are no solutions at all. We may therefore be tempted to say that the algebraic variety over the real numbers defined by $x^2 + y^2 = -1$ is empty, indistinguishable from the empty set.

However, this idea is very difficult to square with our algebraic intuition about the equation defining this variety. The equation $x^2 + y^2 + 1 = 0$ is an irreducible polynomial, and other such polynomials define 1-dimensional subsets of $\mathbf{R}^2$. How are we supposed to distinguish algebraically between those polynomials that define the empty set and those that behave as expected? What is one to do about polynomials like $x^2 + y^2 = 0$, that have just one solution in $\mathbf{R}^2$?

Our answer is that equations like $x^2 + y^2 + 1 = 0$ *do* have a 1-dimensional space of solutions, but that those solutions are invisible when one looks only for solutions in $\mathbf{R}$; they reveal themselves only when we look for solutions in $\mathbf{C}$. We will use the following notation: for any $\mathbf{R}$-algebra A, we define $X(A)$ to be the set of solutions to the equation $x^2 + y^2 + 1 = 0$. Then $X(\mathbf{R}) = \varnothing$, as we saw before, but $X(\mathbf{C})$ is a 1-dimensional subset of $\mathbf{C}^2$. Only by considering the solutions in extensions of $\mathbf{R}$ were we able to see all of this algebraic variety.

Here is another example to illustrate the phenomenon. What is the difference between the equations $x^q - x = 0$ and $0 = 0$ in $\mathbf{F}_q[x]$? If we look for solutions to these equations in $\mathbf{F}_q$, we find that they have exactly the same solution set—namely, all of $\mathbf{F}_q$. However, let's take X to be the solutions to $x^q - x$ and Y to be the solutions to $0 = 0$. Then $X(\mathbf{F}_{q^2}) = \mathbf{F}_q$, while $Y(\mathbf{F}_{q^2}) = \mathbf{F}_{q^2}$. Once again, we could only distinguish between X and Y when looking for solutions in the extension field $\mathbf{F}_{q^2}$.

One may naturally ask how large a field extension one needs in order to be able to distinguish between two equations that are genuinely different. The Nullstellensatz (see Section 3.3) gives a partial answer to this question. It says, in effect, that if f and g are polynomials over a field k that have different solution sets over some field extension K then they will have different solution sets over the algebraic closure $\overline{k}$.

However, even this does not enable one to distinguish entirely between different equations. Consider, for example, the equations $x = 0$ and $x^2 = 0$ in $k[x]$, for any field k. Let X denote solutions to the former equation and Y denote solutions to the latter. For any field extension K of k, both $X(K)$ and $Y(K)$ consist of exactly one element. One can only see the difference when looks at $X(k[\epsilon]/(\epsilon^2))$ and $Y(k[\epsilon]/(\epsilon^2))$. Indeed, the former consists of just one element, while the latter is a 1-dimensional k-vector space.

You may be tempted to dismiss this example at first, since the difference between X and Y above does not feel very geometric. However, we will see later that non-reduced rings like $k[\epsilon]/(\epsilon^2)$ have a very important geometric role to play. It turns out that $X(k[\epsilon]/(\epsilon^2))$ can be interpreted as the *tangent space*[17] of X and in the example above, the difference between X and Y is that the former has a 0-dimensional tangent space while the latter has a 1-dimensional tangent space. That is, X is really a point, while Y only sometimes resembles one. In fact, Y is sometimes called a *fat point*.

Thus we need to be interested in $X(A)$ even for commutative rings that are not fields. Which commutative rings do we need to allow? With some work, we could find some restrictions on the commutative rings that need to be considered, but this is only creating more work for ourselves. We will simply allow all commutative rings.

The price of being so inclusive is that we must find a suitable language in which to talk about the solutions to an equation *in all commutative rings at once*. We will do this in the following way: Assume that we have a system of equations, F, with coefficients in $\mathbf{Z}$. For each commutative ring A, we denote by $X(A)$ the set of solutions to F in A. More generally, if our equations have coefficients in a commutative ring k then for every commutative k-algebra we will have a set $X(A)$.

Now, suppose that $\varphi : A \to B$ is a homomorphism of commutative k-algebras. If we have a solution x to F in A then $\varphi(x)$ will be a solution to F in B. Therefore we get a function $X(\varphi) : X(A) \to X(B)$, and whenever $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ is a sequence of k-algebra homomorphisms, we will have $X(\psi)\circ X(\varphi) = X(\psi\circ\varphi)$.

In short, every system of equations will define a *covariant functor*

$$X : k\text{-}\mathsf{Alg} \to \mathsf{Sets}.$$

We will think of this functor as being the "solution set" of our system of equations. One of our challenges, in this section and in this course, will be in learning to make sense of such functors as geometric objects.

One general method for constructing such a functor is to begin with a commutative k-algebra R. Then we define $X(A) = \mathrm{Hom}_{k\text{-}\mathsf{Alg}}(R, A)$. Such functors (or, more generally functors isomorphic to functors like these) are be called **representable** by commutative rings. When we learn about schemes later in the course, we say that these functors are representable by **affine schemes**.

In this section, we will encounter many such functors and practice working with them individually. We will also see examples of functors like these that are *not* representable by commutative rings. Later we will see that these functors are *representable by schemes*.

4.3 An example

For each commutative ring A, let us define $X(A)$ to be the set of solutions to the equation $x^2 + y^2 = 1$ in A. That is,

$$X(A) = \big\{(x, y) \in A^2 \,\big|\, x^2 + y^2 = 1\big\}.$$

[17] or, more accurately, the *tangent bundle*

We also define

$$\mathbf{A}^1(A) = A.$$

We are of course familiar with $X(\mathbf{R})$, which is a circle. Notice that if we delete a point—say $(0,1)$—of $X(\mathbf{R})$ we get a topological space that is homeomorphic to $\mathbf{A}^1(\mathbf{R}) = \mathbf{R}$. We will see momentarily that this identification is no merely topological: it is algebraic.

Here is a geometric construction of the bijection between $X(\mathbf{R}) \smallsetminus \{(1,0)\}$ and $\mathbf{A}^1(\mathbf{R})$. Let $L(\mathbf{R})$ be the line with equation $y = 0$. For any point $(x,y) \in X(\mathbf{R})$ other than $(x,y) = (0,1)$ we can draw a unique line connecting $(0,1)$ to (x,y). This line intersects $L(\mathbf{R})$ at a single point $(t,0)$. If we define $f(x,y) = t$ then this gives a function from $X(\mathbf{R}) \smallsetminus \{0\}$ to $\mathbf{A}^1(\mathbf{R})$.

Exercise 4.3.1. Let K be a field. Show that three points $P_i = (x_i, y_i)$ in K^2 are colinear if and only if the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}$$

has determinant 0.

Using the exercise, we see that (x,y) and t are related by the equation

$$\det \begin{pmatrix} 0 & t & x \\ 1 & 0 & y \\ 1 & 1 & 1 \end{pmatrix} = 0.$$

The determinant is $x - t + ty$ so we get the equation

$$t = \frac{x}{1-y}.$$

Notice that this is a well-defined function on $X(\mathbf{R}) \smallsetminus \{(0,1)\}$ because the denominator $1-y$ cannot vanish except at $(x,y) = (0,1)$.

We can also obtain formulas for x and y in terms of t as well. Substitute $x = (1-y)t$ into the equation $x^2 + y^2 = 1$ to obtain

$$(1+y)(1-y) = (1-y)^2 t^2.$$

As we have assumed $1 - y \neq 0$, we therefore obtain

$$1 + y = (1-y)t^2,$$

which may be rearranged to give

$$y = \frac{t^2-1}{t^2+1}.$$

Note that t^2+1 never vanishes on $\mathbf{A}^1(\mathbf{R})$, so that the expression on the right is legitimate. We similarly obtain

$$x = \frac{2t}{t^2+1}$$

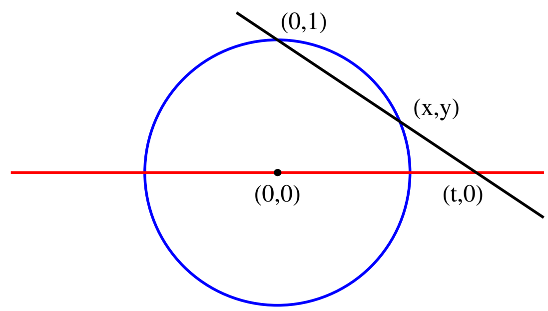


Figure 1: The "bijection" between the unit circle and a line. fig:circle_line

Remarkably, these formulas make sense even if $\mathbf{R}$ is replaced by any field K! We must take care, however, that the equation $t^2 + 1 = 0$ may have solutions in K even though it does not have solutions in $\mathbf{R}$.

Exercise 4.3.2. Verify that if K is a field of characteristic other than 2 then the maps

$$f(x, y) = \frac{x}{1-y}$$
$$g(t) = \Big(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\Big)$$

are inverse bijections between $X(K) \smallsetminus \{(0,1)\}$ and $\mathbf{A}^1(K) \smallsetminus \{\pm\sqrt{-1}\}$.

For any ring A, there is a map $p_A : X(A) \to \mathbf{A}^1(A)$ sending (x, y) to x. For $A = \mathbf{R}$, this is of course the map that projects the unit circle onto the x-axis. We will see how to visualize this map for $A = \mathbf{C}$ below.

Visualizing the bijection over the complex numbers

As mentioned above, $X(\mathbf{R}) = \big\{(x, y) \in \mathbf{R}^2 | x^2 + y^2 = 1\big\}$ is a unit circle and it can be almost identified with a real line by a "bijection except a point" (see Fig. 1).

This "bijection" can generalized to $A = \mathbf{C}$. Then $X(\mathbf{C})$ is almost parametrized by t with $x = \frac{2t}{t^2+1}$, $y = \frac{t^2-1}{t^2+1}$ and conversely $t = \frac{x}{1-y}$. These functions are well-defined except when $t = \pm i$ or $y = 1$. So the map gives an exact pairing between $X(\mathbf{C}) \smallsetminus \{(0,1)\}$ and $\mathbf{A}^1(\mathbf{C}) \smallsetminus \{\pm i\}$. Then, we can visualize the complex unit circle $X(\mathbf{C})$ by the Riemann sphere of the parameter t as shown in Fig. 2 with two points $P(\pm i)$ excluded. The north pole $P(\infty)$ corresponds to $\{(0,1)\} \in X(\mathbf{C})$.

The common boundary (circle) of the front hemisphere and back hemisphere is $X(\mathbf{R})$, which is mapped onto the real segment $-1 \le x \le 1$ under the projection $p : X \to \mathbf{A}^1$ defined by $p(x, y) = x$. Further, take a look at four special points under this projection map: $t = -1$ goes to $x = -1$, $t = 1$ goes to $x = 1$,

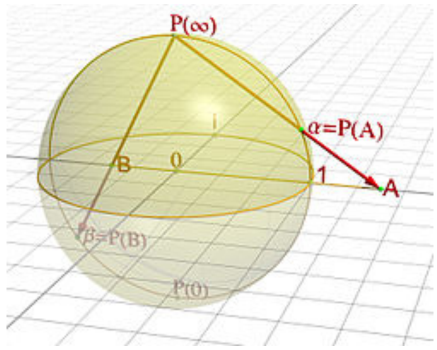


Figure 2: Stereographic projection of $t \in \mathbf{C}$ onto a point α of the Riemann sphere. (The figure is from: wikipedia.org/wiki/Riemann_sphere.) This figure illustrates that one may put the points of $\mathbf{C}$ in bijection with the points of a sphere (excepting only the north pole). The construction in the text gives a bijection between $X(\mathbf{C})$ and $\mathbf{C} \setminus \{\pm 1\}$, which corresponds via stereographic projection to the complement of the points $P(\pm i)$ and $P(\infty)$ in the image above. fig:RiemannS

$t = \infty$ goes to $x = 0$ and $t = 0$ goes to $x = 0$. Therefore, the projection is almost a bijection from each hemisphere to the Riemann sphere of x except on the boundary. Each hemisphere is folded such that the top and bottom semicircles of its boundary overlaps, both mapped onto $-1 \leq x \leq 1$.

It is also interesting to notice that $t = \pm i$ corresponds to $x = \infty$ and $y = \infty$.

4.4 Affine space

Let A be a commutative ring and n a positive integer. Denote by $\mathbf{A}^n(A)$ the set A^n of n-tuples $(x_1, \ldots, x_n)$ with each $x_i \in A$. Notice that if $A \to B$ is a homomorphism of commutative rings then we get a *function* $\mathbf{A}^n(A) \to \mathbf{A}^n(B)$ for each positive integer n.

While A^n can be given a lot of structure (as a commutative group, a ring, etc.), we will ignore all of that structure when we use the notation $\mathbf{A}^n(A)$. That is, we think of $\mathbf{A}^n$ as being simply a *set*.

ex:An **Exercise 4.4.1.** Verify that each element $(x_1, \ldots, x_n)$ of $\mathbf{A}^n(A)$ corresponds to a unique homomorphism

$$\mathbf{Z}[t_1, \ldots, t_n] \to A$$

sending t_i to x_i.

Exercise 4.4.2. If you are already familiar with the language of category theory, verify that $\mathbf{A}^n$ is a *covariant functor* from the category of commutative rings to the category of sets.

4.5 Solutions to polynomial equations

Let f be a polynomial with coefficients in $\mathbf{Z}$. Then for any commutative ring A, define $X(A)$ to be the set of $(x_1, \ldots, x_n) \in \mathbf{A}^n(A)$ such that $f(x_1, \ldots, x_n) = 0$. It is customary to write $X = V(f)$, although without care this can lead to the unpleasant notation $V(f)(A)$.

More generally, if S is any collection of polynomials in $\mathbf{Z}[t_1, \ldots, t_n]$, we let $V(S)$ be the set of points of $\mathbf{A}^n$ that are simultaneously zeroes of all polynomials in S. Writing $X = V(S)$ (to avoid unpleasant notation) this means

$$X(A) = \{x \in \mathbf{A}^n(A) \mid \forall f \in S,\ f(x) = 0\}.$$

Exercise 4.5.1. Suppose that S and S' generate the same ideal in $\mathbf{Z}[t_1, \ldots, t_n]$. Show that $V(S) = V(S')$. That is, if $X = V(S)$ and $X' = V(S')$ you should verify that $X(A) = X'(A)$ as subsets of $\mathbf{A}^n(A)$ for all commutative rings A.

ex:V-hom

Exercise 4.5.2. Let I be an ideal in $\mathbf{Z}[t_1, \ldots, t_n]$ and set $X = V(I)$. Verify that the identification between $\mathbf{A}^n(A)$ and $\mathrm{Hom}(\mathbf{Z}[t_1, \ldots, t_n], A)$ discussed in Exercise 4.4.1 induces an identification between $X(A)$ with $\mathrm{Hom}(\mathbf{Z}[t_1, \ldots, t_n]/I, A)$.

4.6 Working over a base

Of course, plenty of perfectly legitimate geometry can arise from polynomials that don't have integer coefficients. How do we incorporate such polynomials into this theory?

We introduce a coefficient ring k to stand in for the role played by $\mathbf{Z}$ in the sections above. Usually k will be a field, such as $\mathbf{Q}$ or $\mathbf{C}$, but it does not have to be. We define

$$\mathbf{A}^n_k(A) = \mathrm{Hom}(k[t_1, \ldots, t_n], A)$$

to be the functor represented by $k[t_1, \ldots, t_n]$. The homomorphism

$$k \to k[t_1, \ldots, t_n]$$

gives rise to a morphism of functors $\mathbf{A}^n_k \to h^k$. The map

$$\mathbf{Z}[t_1, \ldots, t_n] \to k[t_1, \ldots, t_n]$$

gives a morphism $\mathbf{A}^n \to \mathbf{A}^n_k$.

Exercise 4.6.1. Show that the maps defined above induce an isomorphism $\mathbf{A}^n_k \simeq h^k \times \mathbf{A}^n$.

If S is any subset of $k[t_1, \ldots, t_n]$ then we define $X = V(S)$ as above: for any commutative ring A,

$$X(A) = \{x \in \mathbf{A}^n_k(A) \mid \forall f \in S,\ f(x) = 0\}.$$

Exercise 4.6.2. With notation as above, verify that $X(A) = \mathrm{Hom}_k(k[t_1, \ldots, t_n]/(S), A)$ where (S) denotes the ideal generated by the elements of S. Deduce that X is representable by $k[t_1, \ldots, t_n]/(S)$.

Theorem 4.6.3 (Hilbert basis theorem)**.** *If k is a noetherian ring then $k[T]$ is also noetherian.*

Corollary 4.6.3.1. *If k is a noetherian ring and $S \subset k[t_1, \ldots, t_n]$ is any set of polynomials then $V(S) = V(f_1, \ldots, f_m)$ for some finite collection of polynomials $(f_1, \ldots, f_m)$.*

4.7 Topological fields

Some of our most familiar examples of fields, such as $\mathbf{R}$ and $\mathbf{C}$, come with topologies. Therefore so do the sets $\mathbf{A}^n(\mathbf{R})$ and $\mathbf{A}^n(\mathbf{C})$ and all subsets of these spaces have induced topologies. In particular, suppose I is an ideal in $\mathbf{Z}[t_1, \ldots, t_n]$ and $X = V(I)$. Then we regard $X(\mathbf{R})$ and $X(\mathbf{C})$ as topological spaces with their subspace topologies from $\mathbf{A}^n(\mathbf{R}) = \mathbf{R}^n$ and $\mathbf{A}^n(\mathbf{C}) = \mathbf{C}^n$, respectively.

Proposition 4.7.1. *If $K = \mathbf{C}$ or $K = \mathbf{R}$, then for polynomials $f_1, \ldots, f_m \in K[t_1, \ldots, t_n]$, the set $V(f_1, \ldots, f_m) \subset K^n$ is closed.*

4.8 Affine examples

Exercise 4.8.1. Working over the base $\mathbf{Z}$, consider the polynomial $f = y^2 - x^3 + x \in \mathbf{Z}[x, y]$. Let $X = V(f)$.

(i) Plot $X(\mathbf{R})$.

(ii) Describe $X(\mathbf{C})$ as a topological surface.

Exercise 4.8.2. Consider a polynomial $f = y^2 - p(x)$ where $p(x)$ is a polynomial of degree n with complex coefficients. Regard f as a polynomial in $\mathbf{C}[x, y]$ and let $X = V(f)$. Describe $X(\mathbf{C})$ as a topological surface. Your answer will depend on n.

Exercise 4.8.3. Let R be a commutative ring, S an R-algebra, and T an S-algebra. Suppose that f is a polynomial with coefficients in R. We may also view f as a polynomial with coefficients in S since S is an R-algebra.

Let $X = V(f)$ with f regarded as a polynomial with coefficients in R and let $X' = V(f)$ with f regarded as a polynomial with coefficients in S. Show that $X(T) = X'(T)$ as subsets of $\mathbf{A}^n(T)$.

Conclude that there is no ambiguity in using the notation $V(f)$, even though we could think of the coefficients of f as lying in R or in S.

Exercise 4.8.4. Let I be an ideal in $R[t_1, \ldots, t_n]$ and let $X = V(I)$. Show that $X(A \times B) = X(A) \times X(B)$ for all R-algebras A and B.

Exercise 4.8.5. If $\alpha : A \to B$ is a ring homomorphism, show there is a map $\alpha^* : h^B \to h^A$ induced by α.

Solution. As h^A and h^B are both functors, we must exibit a natural transformation between them. The map α induces a map $\alpha^* : \mathrm{Hom}(B,R) \to \mathrm{Hom}(A,R)$ defined by $\alpha^*(f) = f \circ \alpha$. If $\varphi : R \to S$ is a ring homomorphism, we get a sequence of maps

$$\mathrm{Hom}(B,R) \to \mathrm{Hom}(A,R) \to \mathrm{Hom}(A,S)$$

taking

$$f \mapsto f \circ \alpha \mapsto \varphi \circ (f \circ \alpha)$$

as well as a second sequence of maps

$$\mathrm{Hom}(B,R) \to \mathrm{Hom}(B,S) \to \mathrm{Hom}(A,S)$$

taking

$$f \mapsto \varphi \circ f \mapsto (\varphi \circ f) \circ \alpha.$$

Since composition of functions is associative, we have $\varphi \circ (f \circ \alpha) = (\varphi \circ f) \circ \alpha$, making the diagram

$$\begin{array}{ccc} \mathrm{Hom}(B,R) & \xrightarrow{\alpha^*} & \mathrm{Hom}(A,R) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{Hom}(B,S) & \xrightarrow{\alpha^*} & \mathrm{Hom}(A,S) \end{array}$$

commute. Hence α^* induces a natural transformation $h^B \to h^A$. □

ex:ell-curve

Exercise 4.8.6. Let $f(x,y) = y^2 - p(x)$ where p is a polynomial of degree d and define $X = V(f) \subset \mathbf{A}^2$.

(i) Show that $\varphi(x,y) = x$ is a well-defined (natural) function from X to $\mathbf{A}^1$.

(ii) Let K be a field. Show that for each $x \in \mathbf{A}^1(K)$ the set $\varphi^{-1}(x)$ consists of either 0, 1, or 2 points.

(iii) Suppose $x \in \mathbf{A}^1(K)$ where K is a field of characteristic 2. Show that $\varphi^{-1}(x)$ consists of at most one point, and if K is perfect it consists of exactly one point.

(iv) Suppose that $x \in \mathbf{A}^1(K)$ where K is a field of characteristic other than 2. Show that $\varphi^{-1}(x)$ consists of one point if and only if $p(x) = 0$. For how many points $x \in \mathbf{A}^1(K)$ does this occur?

(v) With the same assumptions as in the last part, show that $\varphi^{-1}(x)$ consists of two points if and only if $p(x)$ is a nonzero square in K.

(vi) Assume that p has no repeated complex roots. The topological space $X(\mathbf{C})$ is the complement of a finite set of points in an orientable surface. Determine the surface. (Your answer will depend on d.)

4.9 Projective space

Recall the topological spaces $\mathbf{R}\mathrm{P}^n$ and $\mathbf{C}\mathrm{P}^n$, whose points, respectively, correspond to the 1-dimensional linear subspaces of the vector spaces $\mathbf{R}^{n+1}$ and $\mathbf{C}^{n+1}$. If we take K to be $\mathbf{R}$ or $\mathbf{C}$ as the case warrants, these spaces are constructed as $(K^{n+1} \smallsetminus \{0\})/K^*$, where the group K^* acts on K^{n+1} diagonally:

$$\lambda.(t_0, \ldots, t_n) = (\lambda t_0, \ldots, \lambda t_n).$$

In fact, the same definition works over any field. For any field K, define

$$\mathbf{P}^n(K) = (K^{n+1} \smallsetminus \{0\})/K^*.$$

The definition of $\mathbf{P}^n(A)$, when A is an arbitrary commutative ring, will appear rather complicated. It will reveal itself to be quite natural when we arrive at Section 15, but we will have to accept it for the moment as a rather opaque formula for the moment.

We can however give a somewhat heuristic explanation for the definition of projective space. One typically views the points of $\mathbf{R}\mathrm{P}^n$ or $\mathbf{C}\mathrm{P}^n$ as corresponding to 1-dimensional linear subspaces of $\mathbf{R}^{n+1}$ or $\mathbf{C}^{n+1}$. However, it is more convenient from an algebro-geometric perspective to instead view these points as corresponding to 1-dimensional *quotient* spaces. Indeed, if L is a 1-dimensional subspace of, say, $\mathbf{C}^{n+1}$ then $L^\vee$ is a 1-dimensional quotient of $(\mathbf{C}^{n+1})^\vee \cong \mathbf{C}^{n+1}$. Thus there is a one-to-one correspondence between 1-dimensional subspaces of $\mathbf{C}^{n+1}$ and 1-dimensional quotients of $\mathbf{C}^{n+1}$.

We want to define $\mathbf{P}^n(A)$ to be the set of 1-dimensional quotients of A^{n+1}, but we must say exactly what we mean by 1-dimensional. It turns out that it is not reasonable to define a 1-dimensional A-module to be a module that is isomorphic to A. We will see exactly why this is when we discuss line bundles.[18] The following definition captures precisely what we mean by a 1-dimensional A-module:

`ex:inv-mod` **Exercise 4.9.1.** Let A be a commutative ring and L an A-module. Prove that the following conditions are equivalent:

`inv:1` **INV1** There are elements $f_1, \ldots, f_n \in A$ generating A as an ideal such that $L \otimes_A A[f_i^{-1}] \cong A[f_i^{-1}]$ for each i.

`inv:3` **INV2** For each prime ideal $\mathfrak{p}$ of A there is some $f \in A \smallsetminus \mathfrak{p}$ such that $L \otimes_A A[f^{-1}] \cong A[f^{-1}]$.

`inv:2` **INV3** The module L is finitely presented and for every prime ideal $\mathfrak{p}$ of A we have $L \otimes_A A_\mathfrak{p} \cong A_\mathfrak{p}$.

Here, $A_\mathfrak{p}$ denotes the *localization* of A at the prime ideal $\mathfrak{p}$.

[18] A 1-dimensional A-module should be an A-module that looks *locally* like A. We will see later that there are modules that look locally like A that are not isomorphic to A.

def:inv-mod

Definition 4.9.2. Let A be a commutative ring. An A-module L is called **locally free of rank** 1 or **invertible** if it satisfies the equivalent conditions of Exercise 4.9.1.

ex:mod-morph-loc-props

Exercise 4.9.3 (Very important, if you haven't done it before)**.** Let A be a commutative ring. Suppose L and M are A-modules. Show that a homomorphism $L \to M$ is

(i) injective,

(ii) surjective,

(iii) an isomorphism

if and only if the map $L \otimes_A A_\mathfrak{p} \to M \otimes_A A_\mathfrak{p}$ has the same property for *every* prime ideal $\mathfrak{p}$ of A.

Solution. By exercise 3.2.13, localization is exact.

$$0 \longrightarrow L \xrightarrow{\varphi} M$$

is exact, so is

$$0 \longrightarrow L \otimes_A A_\mathfrak{p} \xrightarrow{\varphi_\mathfrak{p}} M \otimes_A A_\mathfrak{p}$$

for any prime $\mathfrak{p}$.

On the other hand, if we start with the exact sequence

$$0 \longrightarrow L \otimes_A A_\mathfrak{p} \xrightarrow{\varphi_\mathfrak{p}} M \otimes_A A_\mathfrak{p},$$

we also know that

$$0 \longrightarrow \ker(\varphi) \longrightarrow L \xrightarrow{\varphi} M$$

and thus

$$0 \longrightarrow \ker(\varphi) \otimes_A A_\mathfrak{p} \longrightarrow L \otimes_A A_\mathfrak{p} \xrightarrow{\varphi_\mathfrak{p}} M \otimes_A A_\mathfrak{p}$$

must be exact, so $\ker(\varphi) \otimes_A A_\mathfrak{p} = 0$ for all primes $\mathfrak{p}$ of A. By proposition 3.2.15, $\ker(\varphi) = 0$.

To prove the surjective case, reverse the arrows everywhere above, and replace ker with coker. The case of isomorphism obviously follows. □

Exercise 4.9.4. Let L be an invertible A-module.

(i) Show that the ring of A-module endomorphisms $\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(L, L)$ is isomorphic to A.

(ii) Show that the group of A-module automorphisms $\mathrm{Aut}_{A\text{-}\mathsf{Mod}}(L)$ is A^*.

Of course, A is a locally free A-module of rank 1, but the following examples demonstrate that there can be other locally free A-modules of rank 1 as well.

ex:non-pid

Exercise 4.9.5. Let A be the ring $\mathbf{Z}[x]/(x^2+5)$. Show that the ideal $(2, 1+\sqrt{-5})$ is locally free of rank 1 but is not isomorphic to A as an A-module.

Solution. Let $M = (2, 1+\sqrt{-5})A$. Consider the short exact sequence

$$0 \to M \to A \to \mathbf{Z}/2\mathbf{Z} \to 0.$$

For any prime ideal $\mathfrak{p}$ of A not containing (2), we have

$$0 \to M_{\mathfrak{p}} \to A_{\mathfrak{p}} \to \mathbf{Z}_{\mathfrak{p}}/2\mathbf{Z}_{\mathfrak{p}} = 0 \to 0,$$

so that $M_{\mathfrak{p}} \cong A_{\mathfrak{p}}$.

On the other hand, if $\mathfrak{p} \supset (2)$, then $3 \notin \mathfrak{p}$, and so

$$6 \in (1+\sqrt{-5}) \Rightarrow 6 \cdot 3^{-1} = 2 \in (1+\sqrt{-5}),$$

and so the ideal $M_{\mathfrak{p}}$ is principal, and thus isomorphic to $A_{\mathfrak{p}}$.

Finally, if $(2, 1+\sqrt{-5})$ is not a principal ideal, it cannot be isomorphic to A as an A-module. To see that it is not principal, observe that we have a multiplicative norm function $N : A \to \mathbf{Z}$ where $N(a+b\sqrt{-5}) = a^2+5b^2$. Now, suppose that $(2, 1+\sqrt{-5}) = (\alpha)$, for some $\alpha \in A$. Then there must be some $\beta \in A$ such that $2 = \alpha\beta$. But then applying norms we must have

$$4 = N(\alpha)N(\beta),$$

and so $N(\alpha)$ must be 1, 2, or 4. On the other hand, there must be some $\gamma \in A$ such that $1+\sqrt{-5} = \alpha\gamma$. Again applying norms, we have

$$6 = N(\alpha)N(\gamma),$$

and so $N(\alpha)$ must be 1 or 2.

By the definition of N, it is immediately clear that if $N(\alpha) = 1$, we must have $\alpha = \pm 1$, which is a contradiction because $(2, 1+\sqrt{-5})$ is proper. Furthermore, it is also clear by the definition of N that no element of A can have norm 2: $a^2+5b^2 = 2$ has no solution in the integers. Thus, $(2, 1+\sqrt{-5})$ is not principal. □

ex:ell-curve-ideal

Exercise 4.9.6. Let $A = \mathbf{C}[x,y]/(y^2-x^3-x)$. Show that the ideal $(x,y)A$ is locally free of rank 1 but is not isomorphic to A. (Hint: drawing a picture may help.)

Solution. Consider the short exact sequence

$$0 \to (x,y)A \to A \to \mathbf{C} \to 0.$$

Observe that this makes $\mathbf{C}$ into an A-module where $x \cdot \lambda = 0$ for all $\lambda \in \mathbf{C}$ and $y \cdot \lambda = 0$ for all $\lambda \in \mathbf{C}$. Thus, for any prime ideal $\mathfrak{p}$ of A not containing x, we have $\mathbf{C} \otimes_A A_{\mathfrak{p}} = 0$, and so

$$(x,y)A_{\mathfrak{p}} \cong A_{\mathfrak{p}}.$$

The only alternative is $x \in \mathfrak{p}$, but then in $A_\mathfrak{p}$, $x \pm 1$ is a unit, and so

$$x = \frac{y^2}{x^2 - 1} \in (y),$$

and so $(x,y)A_\mathfrak{p}$ is principal, and isomorphic to $A_\mathfrak{p}$ as an $A_\mathfrak{p}$-module.

Finally, observe that if (x,y) is not principal, it is not isomorphic to A as an A-module.

By way of contradiction, suppose that $(x,y) = (\alpha)$ for some $\alpha \in A$. Then there is some $\beta \in A$ such that

$$x = \alpha\beta,$$

so that α divides x. Note that we may consider A as an extension of $\mathbf{C}[x]$, which comes equipped with a multiplicative norm:

$$N : \mathbf{C}[x][y]/(y^2 - x^3 - x) \to \mathbf{C}[x], \quad f(x) + g(x)y \mapsto f^2(x) - g^2(x)(x^3 - x).$$

It is immediate from the definition of N that no element of A has norm of degree 1 (as a polynomial in x). Now, the statement $x = \alpha\beta$ implies $x^2 = N(\alpha)N(\beta)$. Furthermore, $\mathbf{C}[x]$ is a unique factorization domain and so either $N(\alpha) = 1$ or $N(\alpha) = x^2$ (the argument above excludes $N(\alpha) = x$).

If $N(\alpha) = 1$, then α is a unit. The ideal (α) must be proper however, so this is a contradiction. If $N(\alpha) = x^2$, then β must be a unit. The ideal (x,y) is proper, so we can be certain that α is not a unit. If β is a unit, then $(x) = (\alpha)$, but by assumption $y \in (\alpha)$ and clearly $y \notin (x)$.[19] Thus, (x,y) is not principal. □

ex:extend-inv-mod **Exercise 4.9.7.** Suppose that L is a locally free A-module of rank 1 and B is an A-algebra. Show that $L \otimes_A B$ is a locally free B-module of rank 1.

Solution. Let $\varphi : A \longrightarrow B$ be the morphism of rings making B an A-algebra. By definition then, for any $\mathfrak{p} \in \operatorname{Spec}(B)$, the localization of the $B-$module $L \otimes_A B$ at $\mathfrak{p}$ is $L \otimes_A B_\mathfrak{p}$. We recall however, two things. The morphism φ induces a map $\operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$, so $B_\mathfrak{p}$ is is in fact an $A_{\varphi^\star(\mathfrak{p})}$-algebra. Second, for any algebra over a ring R, M we have that $R \otimes_R M \cong M$ [AM, Proposition 2.14] so $A_{\varphi^\star(\mathfrak{p})} \otimes_{A_{\varphi^\star(\mathfrak{p})}} B_\mathfrak{p} \cong B_\mathfrak{p}$ whence

$$L \otimes_A B_\mathfrak{p} \cong \underbrace{L \otimes_A A_{\varphi^\star(\mathfrak{p})}}_{L_{\varphi^\star(\mathfrak{p})}} \otimes_{A_{\varphi^\star(\mathfrak{p})}} B_\mathfrak{p}.$$

Since L invertible we've then that the above is isomorphic to $A_{\varphi^\star(\mathfrak{p})} \otimes_{A_{\varphi^\star(\mathfrak{p})}} B_\mathfrak{p}$, which by our second recollection is isomorphic to $B_\mathfrak{p}$; $L \otimes_A B$ is an invertible B-module. □

ex:mult-inv-mod **Exercise 4.9.8.** Suppose that L and L' are locally free A-modules of rank 1. Show that $L \otimes_A L'$ is also a locally free A-module of rank 1.

[19] One can see that $y \notin (x)$ by computing the quotient $\mathbf{C}[x,y]/(y^2 - x^3 - x, x) \cong \mathbf{C}[y]/(y^2)$, in which y represents a non-zero element; hence y is not contained in the ideal $(x)\mathbf{C}[x,y]/(y^2 - x^3 - x)$.

ex:inverse-inv-mod

Exercise 4.9.9. Suppose that L is a locally free A-module of rank 1. Show that the A-module $\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(L, A)$ is also locally free of rank 1.

Remark 4.9.10. The previous exercises demonstrate that for a commutative ring A, isomorphism classes of invertible A-modules form a group, called the Picard group, under tensor product. The inverse of an invertible A-module M, $\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(M, A)$ is often denoted M^{-1} or $-M$. Exercise 4.9.7 tells us that the Picard group is a covariant functor from the category of commutative rings to the category of groups.

Lemma 4.9.11. *Let A be an integral domain and M a finitely generated invertible A-module. Then M is isomorphic to an A-submodule of A. In particular, $M \subset A$ is isomorphic to an ideal of A.*

lem:domain-inv-mod

cor:pid-inv-mod

Corollary 4.9.11.1. *Let A be a principal ideal domain and M a finitely-generated invertible A-module. Then $M \cong A$ as A-modules.*

Definition 4.9.12. For each commutative ring A, let $\mathbf{P}^n(A)$ be the set of equivalence classes of pairs (L, p) where L is a locally free A-module of rank 1 and

$$p : A^{n+1} \to L$$

is a surjection of A-modules. We consider (L, p) to be equivalent to (L', p') if there is an A-module isomorphism $\varphi : L \to L'$ with $\varphi \circ p = p'$.

If $f : A \to B$ is a homomorphism of commutative rings, define $\mathbf{P}^n(f) : \mathbf{P}^n(A) \to \mathbf{P}^n(B)$ to be the map sending (L, p) to $(L \otimes_A B, p \otimes \mathrm{id}_B)$.

Exercise 4.9.13. Verify that the definition of $\mathbf{P}^n(f)$ above does indeed take objects of $\mathbf{P}^n(A)$ to $\mathbf{P}^n(B)$. Deduce that $\mathbf{P}^n$ is a covariant functor from ComRng to Sets.

Solution. In problem 4.9.7 we showed the B-module $L \otimes_A B$ to be invertible, thus to prove $\mathbf{P}^n : \mathsf{ComRing} \longrightarrow \mathsf{Sets}$ we need show that the map,

$$p \otimes \mathrm{id}_B : A^{n+1} \otimes_A B \longrightarrow L \otimes_A B,$$

does in fact define a surjection $B^{n+1} \longrightarrow L \otimes_A B$, and that the composition of morphisms is preserved.

To show that $p \otimes \mathrm{id}_B$ defines a surjection $B^{n+1} \longrightarrow L \otimes_A B$ we need only demonstrate an isomorphism of the rings $B^{n+1} \longrightarrow A^{n+1} \otimes_A B$. The reason being that tensor is a right exact, so the exactness of the sequence,

$$A^{n+1} \xrightarrow{p} L \longrightarrow 0,$$

forces,

$$A^{n+1} \otimes_A B \xrightarrow{p \otimes_A B} L \otimes_A B \longrightarrow 0,$$

exact. This very-same right-exactness however actually provides the isomorphism too!

The product ring A^{n+1}, by virtue of the finiteness of the product, is isomorphic to the direct sum $\bigoplus^{n+1} A$. Thus it suffices to show that $\left(\bigoplus^{n+1} A\right) \otimes_A B$ is isomorphic to $\bigoplus^{n+1} B$ to define $p \otimes_A \mathrm{id}_B$ on B^{n+1}. Right exactness is the preservation of finite co-limits though, and the direct sum of modules is just such a co-limit; here,

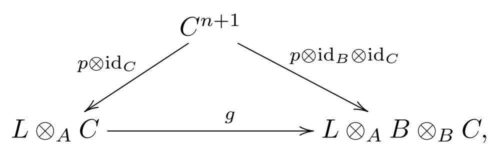
$$\left(\bigoplus^{n+1} A\right) \otimes_A B \cong \bigoplus^{n+1} (A \otimes_A B)\,.$$

The right-hand side is isomorphic to $\bigoplus^{n+1} B$ by [AM, Proposition 2.14], whence $p \otimes \mathrm{id}_B$ defines a surjection B^{n+1} onto $L \otimes_A B$.

These data comprise a covariant functor; we show they preserve composition. To that end, let

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

be a diagram in $\mathsf{ComRing}$. Observe that $\mathbf{P}^n\left(\psi \circ \varphi\right)(L, p) = (L \otimes_A C, p \otimes \mathrm{id}_C)$, and $\mathbf{P}^n\left(\psi\right) \circ \mathbf{P}^n\left(\varphi\right) = (L \otimes_A B \otimes_B C, p \otimes \mathrm{id}_B \otimes \mathrm{id}_C)$ so for the functoriality of $\mathbf{P}^n$ we need an isomorphism between the two. In this category an isomorphism is an isomorphism of modules such that the appropriate diagram commutes. Here, we require the existence of an isomorphism g such that the diagram,



commutes. Since $B \otimes_B C \cong C$, existence is clear; $\mathbf{P}^n$ is a covariant functor. $\square$

Exercise 4.9.14. When K is a field, we now have two definitions of $\mathbf{P}^n(K)$. Find a natural isomorphism between these two definitions.

Exercise 4.9.15. For each commutative ring A, define $U(A)$ to be the set of surjective A-module homomorphisms $A^{n+1} \to A$.

(i) Define a morphism $U \to \mathbf{A}^{n+1}$ by sending $f \in U(A)$ to $(f(e_1), \ldots, f(e_{n+1}))$.

(ii) Show that under this identification, $U(K) = K^{n+1} \setminus \{0\}$ for any field K.

(iii) Construct a map $U \to \mathbf{P}^n$ that gives the quotient map $K^{n+1} \setminus \{0\} \to \mathbf{P}^n(K)$ when evaluated on a field K.

Exercise 4.9.16 (The Veronese embedding)**.** (i) Let L and M be locally free A-modules of rank 1 and suppose that $x : A \to L$ and $y : A \to M$ are A-module homomorphisms. Let $xy : A \to L \otimes M$ be the map $(xy)(t) = x(t) \otimes y(t)$. Verify that xy is an A-module homomorphism.

(ii) Let n be a positive integer. Given a point $(L,(x_0,x_1)) \in \mathbf{P}^1(A)$, show that $(L^{\otimes n},(x_0^n, x_0^{n-1}x_1, x_0^{n-2}x_2, \ldots, x_0x_1^{n-1}, x_1^n))$ is a point of $\mathbf{P}^n(A)$. Show that this defines a natural transformation $\mathbf{P}^1 \to \mathbf{P}^n$. Try to visualize this map.

(iii) More generally, let n and m be positive integers and suppose $(L,(x_0,\ldots,x_n))$ is a point of $\mathbf{P}^n(A)$. Let $f_1,\ldots,f_N$ be the set of monomials of degree m in $n+1$ variables. Show that $(f_1(x_0,\ldots,x_n),\ldots,f_N(x_0,\ldots,x_n))$ is an element of $\mathbf{P}^N(A)$. Show that this defines a natural transformation $\mathbf{P}^n \to \mathbf{P}^N$. This is known as the **Veronese embedding**.

(iv) In the last part, what is N as a function of n and m?

(v) Show that the Veronese embedding is injective.

Exercise 4.9.17 (The Segre embedding)**.** Suppose that $(L,(x_0,\ldots,x_n)) \in \mathbf{P}^n(A)$ and $(M,(y_0,\ldots,y_m)) \in \mathbf{P}^m(A)$.

(i) Show that
$$(L \otimes M, \{x_i y_i \mid 0 \le i \le n,\ 0 \le j \le m\})$$
defines a point of $\mathbf{P}^{(n+1)(m+1)-1}(A)$.

(ii) Verify that this is a natural transformation
$$\mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^{(n+1)(m+1)-1}.$$
This is known as the **Segre embedding**.

(iii) Show that the Segre embedding is *injective*.

Exercise 4.9.18. Prove that $\mathbf{P}^n(A \times B) = \mathbf{P}^n(A) \times \mathbf{P}^n(B)$.

In the special cases $K = \mathbf{R}$ and $K = \mathbf{C}$ we get $\mathbf{P}^n(\mathbf{R}) = \mathbf{R}\mathrm{P}^n$ and $\mathbf{P}^n(\mathbf{C}) = \mathbf{C}\mathrm{P}^n$. We give these sets their usual topologies, namely the quotient topology induced from that of $K^{n+1} \smallsetminus \{0\}$.

As with affine space, we may also define projective space over a base. For any commutative ring k, we define
$$\mathbf{P}^n_k : k\text{-}\mathsf{Alg} \to \mathsf{Sets}$$
by $\mathbf{P}^n_k(A) = \mathbf{P}^n(A)$. The only difference between $\mathbf{P}^n$ and $\mathbf{P}^n_k$ is that $\mathbf{P}^n_k(A) \to \mathbf{P}^n_k(B)$ is defined only for a *k-algebra homomorphism* $A \to B$; it is not defined when $A \to B$ is only a homomorphism of commutative rings.

4.10 Homogeneous polynomials

A polynomial is called **homogeneous** of degree d if all of its monomial terms have total degree d.

Exercise 4.10.1. A polynomial $f \in k[t_0, \ldots, t_n]$ is homogeneous of degree d *if and only if* the following equation holds for every λ in every k-algebra K:

$$f(\lambda t_0, \ldots, \lambda t_n) = \lambda^d f(t_0, \ldots, t_n).$$

Let us suppose that f is a homogeneous polynomial of degree d in $n+1$ variables and $(L, (x_0, \ldots, x_n))$ is an element of $\mathbf{P}^n(A)$. Then we can regard $f(x_0, \ldots, x_n)$ as an A-module homomorphism

$$f(x_0, \ldots, x_n) : A \to L^{\otimes d}.$$

Indeed, if $a x_0^{k_0} \cdots x_n^{k_n}$ is a monomial of degree d, then it defines a map $A \to L^{\otimes d}$ and we can get the map $f(x_0, \ldots, x_n)$ mentioned above by summing the maps associated to its monomial terms.

Now, if f is a homogeneous polynomial in $k[t_0, \ldots, t_n]$ we can define $X = V(f)$ as follows:

$$X(A) = \bigl\{(L, (x_0, \ldots, x_n)) \in \mathbf{P}^n(A) \bigm| f(x_0, \ldots, x_n) = 0\bigr\}.$$

In other words, $X(A)$ is the set of $(L, (x_0, \ldots, x_n))$ such that $f(x_0, \ldots, x_n)$ is the zero homomorphism from A to L.

Exercise 4.10.2. Verify that this definition makes sense. Make sure to understand why the definition does not make sense if f is not homogeneous.

Warning 4.10.3. The notation $V(f)$ is ambiguous! Depending on the situation, it could refer to a subset of $\mathbf{A}^{n+1}$ or $\mathbf{P}^n$. We will have to rely on context to sort this out!

We also use $V(f_1, \ldots, f_m)$ to denote the intersection $V(f_1) \cap \cdots \cap V(f_m)$.

Proposition 4.10.4. *If $K = \mathbf{C}$ or $K = \mathbf{R}$, then for homogeneous polynomials $f_1, \ldots, f_m \in K[t_0, \ldots, t_n]$, the set $V(f_0, \ldots, f_m) \subset \mathbf{P}^n(K)$ is closed. Here $\mathbf{P}^n(K)$ has the quotient topology from $\mathbf{A}^n(K) \smallsetminus \{0\}$.*

4.11 Projective examples

Exercise 4.11.1. Show that $\mathbf{P}^0 \cong h^{\mathbf{Z}}$.

We sometimes refer to $\mathbf{P}^1$ as the **projective line**. It contains two special points $\mathbf{P}^0 \to \mathbf{P}^1$. In other words, for any commutative ring A, we can write down two special elements $0 = (A, (0,1)) \in \mathbf{P}^1(A)$ and $\infty = (A, (1,0)) \in \mathbf{P}^1(A)$.

If we have a map $f : \mathbf{P}^1 \to X$ for some functor $X : \mathsf{ComRng} \to \mathsf{Sets}$, then for any commutative ring A, we get a map $f_A : \mathbf{P}^1(A) \to X(A)$, and hence two elements $f(0), f(\infty) \in X(A)$.

We call a map $f : \mathbf{P}^n \to \mathbf{P}^m$ **linear** if it is equivalent to one of the form

$$f(L, p) = (L, p \circ F)$$

where F is a $(n+1) \times (m+1)$-matrix with coefficients in $\mathbf{Z}$. Of course, one can make an analogous definition for maps $\mathbf{P}^n_A \to \mathbf{P}^m_A$ by allowing the coefficients of F to lie in A.

Exercise 4.11.2 (This is a good exercise to do)**.** Not every $(n+1)\times(m+1)$ matrix F with values in $\mathbf{Z}$ defines a map $\mathbf{P}^n \to \mathbf{P}^m$.

(i) Show that a linear map $F : \mathbf{Z}^{m+1} \to \mathbf{Z}^{n+1}$ defines a map $\mathbf{P}^n \to \mathbf{P}^m$ if and only if F is surjective. (Hint: If F is not surjective, find a surjection $\mathrm{coker}(F) \to \mathbf{F}_q$ for some q and obtain an element of $\mathbf{P}^n(\mathbf{F}_q)$ whose image in $\mathbf{P}^m(\mathbf{F}_q)$ is not defined.)

(ii) Conclude that there is no map $\mathbf{P}^n \to \mathbf{P}^m$ if $n > m$.

(iii) Prove the analogous result for maps $\mathbf{P}^n_A \to \mathbf{P}^m_A$, where A is a commutative ring.

Exercise 4.11.3. Let k be a field and let Let $X \subset \mathbf{P}^2_k$ be defined by a homogeneous polynomial of degree 2 that is not the square of a linear polynomial. Assume that $X(k)$ has a point P. Use

Exercise 4.11.4. (i) Suppose that $\alpha, \beta \in \mathbf{P}^n(\mathbf{Z})$. Show that there is a linear map $f : \mathbf{P}^1 \to \mathbf{P}^n$ such that $f(0) = \alpha$ and $f(\infty) = \beta$.

(ii) Show that there are a total of 2 such maps (in bijection with the elements of $\mathbf{Z}^*$).

(iii) More generally, if $\alpha, \beta \in \mathbf{P}^n_A(A)$ for a *principal ideal domain* A, show that there is a unique map $f : \mathbf{P}^1_A \to \mathbf{P}^n_A$ such that $f(0) = \alpha$ and $f(\infty) = \beta$.

(iv) Show that such maps are in bijection with the elements of A^*.

Solution.

(i) Let $\alpha, \beta \in \mathbf{P}^n(\mathbf{Z})$. Denote $\alpha = (L, p)$, $\beta = (M, q)$. Note that by Corollary 4.9.11.1, we may assume that $L = M = \mathbf{Z}$. Define a map

$$\varphi : \mathbf{Z}^{n+1} \xrightarrow{\begin{pmatrix} q \\ p \end{pmatrix}} \mathbf{Z}^2.$$

Note that φ is surjective because p and q are. Thus, for any $(\mathbf{Z}, r) \in \mathbf{P}^1(\mathbf{Z})$,

$$r \circ \varphi : \mathbf{Z}^{n+1} \to \mathbf{Z}$$

is surjective, defining an element $(\mathbf{Z}, r \circ \varphi) \in \mathbf{P}^n(\mathbf{Z})$. Define

$$f : \mathbf{P}^1(\mathbf{Z}) \to \mathbf{P}^n(\mathbf{Z}), \quad (\mathbf{Z}, r) \mapsto (\mathbf{Z}, r \circ \varphi).$$

It is immediate that $f(\mathbf{Z}, (0,1)) = \alpha$ and $f(\mathbf{Z}, (1,0)) = \beta$.

(ii) Note that by the condition of linearity, every such map f must arise as a composition of a linear map $\mathbf{Z}^{n+1} \to \mathbf{Z}^2$. Fixing the image of $(0,1)$ and $(1,0)$, it follows that f is determined up to multiplying the rows of φ by a unit of $\mathbf{Z}$. This is equivalent to only multiplying the top row of φ by a unit, and so all such maps are parametrized by $\mathbf{Z}^*$.

□

Exercise 4.11.5. Call a polynomial $f(x_0, \ldots, x_m; y_0, \ldots, y_n)$ with coefficients in a commutative rings k *bihomogeneous of bidegree* (a, b) if every monomial term of f has degree a in the x-variables and degree b in the y-variables.

(i) Prove that f is bihomogeneous of bidegree (a, b) if and only if

$$f(\lambda x_0, \ldots, \lambda x_m; y_0, \ldots, y_n) = \lambda^a f(x_0, \ldots, x_m; y_0, \ldots, y_m)$$
$$f(x_0, \ldots, x_m; \lambda y_0, \ldots, \lambda y_n) = \lambda^b f(x_0, \ldots, x_m; y_0, \ldots, y_m)$$

for every λ in every k-algebra K.

(ii) Show that a bihomogeneous polynomial as above defines a natural subset of $\mathbf{P}^m \times \mathbf{P}^n$.

Exercise 4.11.6. Let $X = V(f) \subset \mathbf{P}^2$ where $f(x, y, z) = y^2 z - x^3 - xz^2$. This is an example of an *elliptic curve*.

(i) $X(\mathbf{C})$ is homeomorphic to a torus $S^1 \times S^1$.

(ii) Conclude that $H^1(X, \mathbf{C}) \cong \mathbf{C}^2$.

Exercise 4.11.7. For each commutative ring A and each $i = 0, \ldots, n$, let $U_i(A)$ be the subset of $\mathbf{P}^n(A)$ consisting of those points $(A, (t_0, \ldots, t_n))$ with $t_i \in A^*$.

(i) Prove that U_i is a "natural" subset of $\mathbf{P}^n$ for each i. Show, in other words, that whenever $\varphi : A \to B$ is a homomorphism of commutative rings

$$\begin{array}{ccc} U_i(A) & \longrightarrow & U_i(B) \\ \downarrow & & \downarrow \\ \mathbf{P}^n(A) & \longrightarrow & \mathbf{P}^n(B) \end{array}$$

(ii) Show that each U_i is naturally isomorphic to $\mathbf{A}^n$.

(iii) Suppose that K is a field. Show that every element of $\mathbf{P}^n(K)$ lies in $U_i(K)$ for some i.

(iv) Give an example of a commutative ring and an element of $\mathbf{P}^n(A)$ that does not lie in $U_i(A)$ for any i.

Solution. (i) Recall that if $f : A \to B$ is a homomorphism of rings, then $\mathbf{P}^n(f)(L, p) = (L \otimes_A B, p \otimes_A \mathrm{id}_B)$. Thus, we can define $U_i(f) = \mathbf{P}^n(f)\big|_{U_i(A)}$. Then we just need to check that the range of $U_i(f)$ lies in $U_i(B)$. Let $(A, (t_0, \ldots, t_n)) \in U_i(A)$ and denote $p = (t_0, \ldots, t_n)$. Then by definition, $U_i(f)((A, p)) = \mathbf{P}^n(f)((A, p)) = (B \otimes_A A, p \otimes_A \mathrm{id}_B)$. But $A \otimes_A B \cong B$ via the map $a \otimes_A b \mapsto f(a)b$, and $p \otimes_A \mathrm{id}_B$ is a map from $A^{n+1} \otimes_A B$, which is isomorphic to B^{n+1} via the sequence of isomorphisms: $A^{n+1} \otimes_A B \to$

$(A \otimes_A B)^{n+1} \to B^{n+1}$; $(a_0, \dots, a_n) \otimes_A b \mapsto (a_0 \otimes_A b, \dots, a_n \otimes_A b) \mapsto (f(a_0)b, \dots, f(a_n)b)$. Hence, if $(b_0, \dots, b_n) \in B^{n+1}$, then

$$\begin{aligned}(p \otimes_A \mathrm{id}_B)(b_0, \dots, b_n) &= (p \otimes_A \mathrm{id}_B)(1 \otimes_A b_0, \dots, 1 \otimes_A b_n) \\ &= (p \otimes_A \mathrm{id}_B)\left((1, 0, \dots, 0) \otimes_A b_0 + \dots + (0, \dots, 0, 1) \otimes_A b_n\right) \\ &= p(1, 0, \dots, 0) \otimes_A b_0 + \dots + p(0, \dots, 0, 1) \otimes_A b_n \\ &= t_0 \otimes_A b_0 + \dots + t_n \otimes_A b_n \\ &\mapsto f(t_0)b_0 + \dots + f(t_n)b_n.\end{aligned}$$

Thus, $p \otimes_A \mathrm{id}_B$ is represented by the row vector $(f(t_0), \dots, f(t_n))$. But since ring homomorphisms take units to units, $f(t_i) \in B^*$, hence $U_i(f) : U_i(A) \to U_i(B)$.

Now for each commutative ring A, define $\tau_A : U_i(A) \to \mathbf{P}^n(A)$ to be the natural inclusion map, and consider the above diagram. Starting with the top and right arrows, we have $\tau_B \circ U_i(f) = \tau_B \circ \mathbf{P}^n(f)\big|_{U_i(A)} = \mathbf{P}^n(f)\big|_{U_i(A)} = \mathbf{P}^n(f) \circ \tau_A$. Therefore, the diagram is commutative, so U_i is naturally a subset of $\mathbf{P}^n$.

(ii) For all commutative rings A, define $\tau_A : U_i(A) \to \mathbf{A}^n(A)$ by $\tau_A((A, (t_0, \dots, t_n)) = (t_0 t_i^{-1}, \dots, \widehat{t_i t_i^{-1}}, \dots, t_n t_i^{-1})$, where the hat over the i–th coordinate denotes omission of that coordinate.

We check that τ_A is well–defined. Suppose $(A, (t_0, \dots, t_n)) \sim (A, (s_0, \dots, s_n))$ and denote $p = (t_0, \dots, t_n)$ and $q = (s_0, \dots, s_n)$. Then there exists an isomorphism $\varphi : A \to A$ such that $\varphi p = q$. Hence if e_i is the i–th basis element of A^{n+1} (i.e., the element of A^{n+1} whose i–th term is 1 and all other terms are zero), then $(\varphi p)(e_i) = \varphi(t_i) = s_i$. So for all $a \in A$, $\varphi(a) = \varphi(a t_i^{-1} t_i) = a t_i^{-1} s_i$. Therefore, for each j, $s_j = q(e_j) = (\varphi p)(e_j) = t_j t_i^{-1} s_i$, whence $s_j s_i^{-1} = t_j t_i^{-1} s_i s_i^{-1} = t_j t_i^{-1}$, so τ_A is well–defined. Reversing the above argument yields that τ_A is injective, since if $s_j s_i^{-1} = t_j t_i^{-1}$ for all j, then we can define $\varphi(a) = a t_i^{-1} s_i$.

Next, we claim that τ_A is surjective. Let $(x_1, \dots, x_n) \in \mathbf{A}^n(A)$. Then $(A, (x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n))$ is a putative element of $U_i(A)$ which maps to $(x_1, \dots, x_n)$, provided it is indeed an element of $U_i(A)$, i.e., the row vector $(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n)$ defines a surjective A–module homomorphism $A^{n+1} \to A$. But for any $a \in A$, the element $(0, \dots, a, \dots, 0)$ - which consists of a in the i–th position and zero elsewhere - maps to a. Thus, $(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n)$ is surjective, so τ_A is a bijection.

Finally, let $f : A \to B$ be a ring homomorphism and consider the square

$$\begin{array}{ccc} U_i(A) & \longrightarrow & U_i(B) \\ \downarrow & & \downarrow \\ \mathbf{A}^n(A) & \longrightarrow & \mathbf{A}^n(B). \end{array}$$

As proven in (i), $U_i(f)((A,(t_0,\dots,t_n)) = (B,(f(t_0),\dots,f(t_n))$. Thus,

$$\begin{aligned}(\mathbf{A}^n(f)\tau_A)((A,(t_0,\dots,t_n))) &= \mathbf{A}^n(f)(t_0t_i^{-1},\dots,\widehat{t_it_i^{-1}},\dots,t_nt_i^{-1})\\ &= (f(t_0t_i^{-1}),\dots,\widehat{f(t_it_i^{-1})},\dots,f(t_nt_i^{-1}))\\ &= (f(t_0)f(t_i)^{-1},\dots,\widehat{f(t_i)f(t_i)}^{-1},\dots,f(t_n)f(t_i)^{-1})\\ &= \tau_B((B,(f(t_0),\dots,f(t_n)))\\ &= (\tau_BU_i(f))((A,(t_0,\dots,t_n))).\end{aligned}$$

Therefore, the square commutes and so U_i is naturally isomorphic to $\mathbf{A}^n$.

(iii) Let $(L,p)\in\mathbf{P}^n(K)$. Then L is locally free of rank one and $p : K^{n+1}\to L$ is a surjective linear transformation. Since K is a field, its only prime ideal is $\mathfrak{p}=0$. Moreover, $K_\mathfrak{p}\cong K$, so $L\otimes K_\mathfrak{p}\cong L\otimes K\cong L$, so $L\cong K$. Thus, there is an isomorphism $\varphi : L\to K$, so $(L,p)\sim(K,q)$, where $q=\varphi p$. But q is a surjective linear transformation $K^{n+1}\to K$, so q is represented by a row vector $(t_0,\dots,t_n)$. Thus, $(L,p)\sim(K,(t_0,\dots,t_n))$. Moreover, $(t_0,\dots,t_n)$ is surjective, so there exists some i such that t_i is nonzero. But this implies that $t_i\in K^*$, whence $(L,p)\in U_i(K)$.

(iv) Let $n=1$, $A=\mathbf{Z}$, and let $(A,(2,3))\in\mathbf{P}^1(A)$, i.e., $p=(2,3)$ is the map $p(x,y)=2x+3y$. Since 2 and 3 are coprime, p is surjective. However, neither 2 nor 3 is a unit of $\mathbf{Z}$, so $(A,(2,3))$ is not in U_1 or U_2.

□

An exercise was removed from this section and changed to an affine problem (Exercise 4.8.6).

Exercise 4.11.8. Compute the number of points of $\mathbf{P}^n(\mathbf{F}_q)$ for every finite field q.

4.12 The Grassmannian

Points of $\mathbf{P}^n$ correspond to rank 1 quotients of an $(n+1)$-dimensional vector space. We can also define a space of rank k quotients for each k.

Definition 4.12.1. Let A be a commutative ring. An A-module M is said to be **locally free of rank** n if, for every prime ideal $\mathfrak{p}$ of A, the $A_\mathfrak{p}$-module $M_\mathfrak{p}=A_\mathfrak{p}\otimes_A M$ is isomorphic to A^n (the direct sum of n copies of A).

Definition 4.12.2. For each commutative ring A, let $\mathrm{Grass}(k,n)(A)$ be the set of equivalence classes of pairs (V,p) where V is a locally free A-module of rank k and

$$p : A^n \to V$$

is a surjection of A-modules. We consider (V,p) to be equivalent to (W,q) if there is an A-module isomorphism $\varphi : V\to W$ such that $\varphi\circ p=q$.

Exercise 4.12.3. Check that $\mathbf{P}^n = \mathrm{Grass}(1, n+1)$.

Exercise 4.12.4. Let (V, p) be an element of $\mathrm{Grass}(k, n)(A)$ for some commutative ring A. Define $p^\vee : V^\vee \to A^n$ to be the dual map (where $V^\vee$ is the A-module $\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(V, A)$ and we identify A^n with $\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(A^n, A)$ in the standard way). Let $W = A^n / V^\vee$ and let $q : A^n \to W$ be the quotient map. Show that the transformation $\varphi(V, p) = (W, q)$ defines an (natural) isomorphism $\mathrm{Grass}(k, n) \cong \mathrm{Grass}(n-k, n)$.

Exercise 4.12.5. Compute the number of points of $\mathrm{Grass}(k, n)(\mathbf{F}_q)$ for each k and n and each finite field $\mathbf{F}_q$.

4.13 Open subsets

sec:open

Let S be subset of a commutative ring A. Define a functor $U \subset h^A$ by

$$U(B) = \{\varphi \in h^A(B) \mid \varphi(S)B = B\}.$$

In other words, $U(B)$ is the set of homomorphisms of commutative rings $\varphi : A \to B$ such that $\varphi(S)$ generates the unit ideal in B. We call subfunctors of this form **open**.

More generally, if A is a k-algebra one can make the same definition for open subfunctors of $h^A : k\text{-}\mathsf{Alg} \to \mathsf{Sets}$.

4.14 Points

sec:points

ex:field-opens

Exercise 4.14.1. Suppose that k is a field. Show that the functor $h^k : \mathsf{ComRng} \to \mathsf{Sets}$ has no open subfunctors other than itself and the empty set.

Exercise 4.14.1 show that functors represented by fields have something in common with points. We will therefore tend to think of these functors as the points in the world of algebraic geometry. [20]

Let us examine the points of some familiar spaces.

The points of h^A

A point of h^A corresponds to a map $h^k \to h^A$. By the Yoneda lemma, to give such a point is the same as to give an element of $h^A(k)$, or equivalently, a map of commutative rings $A \to k$.

However, we can quickly observe that there are several different kinds of points, distinguished by qualities of the ring homomorphism $\mathbf{Z} \to k$:

If $A \to k$ is surjective, we call the corresponding point a **closed point** of A. If I is the kernel of $A \to k$ then $h^k = V(I)$, so this is compatible with the ideal that the closed subfunctors of h^A are the $V(I)$.

[20] Take care, however, that this intuition only goes so far! One obvious difference is that there are many non-isomorphic fields, but only one point. So we must always remember that in algebraic geometry, our spaces can have many different kinds of points. Another important difference is that fields can have automorphisms (so by the Yoneda lemma, so do the functors h^k), while points cannot. We will see later that this implies there is a bit of geometry hidden inside of h^k, even though h^k has no open subsets!

ex:closed-pt-max

Exercise 4.14.2. Show that the closed points of h^A are in bijection with the maximal ideals of A.

Solution. A surjective morphism of commutative rings, a closed point of h^A, puts the target isomorphic to a quotient of the source. When the target is a field, this isomorphism forces the ideal involved in the quotient to be maximal. Conversely any maximal ideal describes a unique surjection onto a field; the closed points of h^A are in bijection with M-Spec(A). □

ex:points-inj

Exercise 4.14.3 (A hard, not necessarily important exercise)**.** Show that a map $h^k \to h^A$, corresponding to $\varphi : A \to k$, is injective if and only if every element of k is a ratio $\varphi(a)/\varphi(b)$ for some $a, b \in A$.

def:schematic-point

The exercise points to a second sort of map $h^k \to h^A$. We call such an injective natural transformation a **schematic point** of A. Thus, by the exercise, a schematic point of h^A is a map $\varphi : A \to k$ such that every element of k is a ratio of the images of two elements of A. Every closed point is a schematic point.

Two schematic points $P : h^K \to h^A$ and $Q : h^L \to h^A$ are said to be equivalent if there is an isomorphism $\varphi : h^K \to h^L$ with $Q \circ \varphi = P$. If we view P and Q as elements of $h^A(K) = \mathrm{Hom}_{\mathsf{ComRng}}(A, K)$ and $h^A(L) = \mathrm{Hom}_{\mathsf{ComRng}}(A, L)$ then this is saying that there is an isomorphism of fields $f : L \to K$ with $f_*(Q) = P$.

ex:sch-pt-prime

Exercise 4.14.4. Show that the schematic points of h^A are in bijection with the prime ideals of A.

Exercise 4.14.5. Suppose $j : h^K \to h^A$ is a point of A. Show that there is a unique schematic point $i : h^k \to h^A$ and a map $f : h^K \to h^k$ such that $i \circ f = j$.

Finally, we call a map $h^k \to A$ a **geometric point** if k is a separably closed field. We won't have use for this notion for a while, so feel free to ignore it for now.

The points of $h^{\mathbf{Z}}$

First we determine the closed points. They correspond to the maximal ideals of $\mathbf{Z}$, which are in bijection with the positive prime numbers.

There is one more prime ideal, $\{0\} \subset \mathbf{Z}$, which corresponds to the map $h^{\mathbf{Q}} \to h^{\mathbf{Z}}$.

ex:Q-gen-in-Z

Exercise 4.14.6. Show that $h^{\mathbf{Q}}$ is contained in every non-empty open subset of $h^{\mathbf{Z}}$.

Because of Exercise 4.14.6 we refer to $h^{\mathbf{Q}}$ as the the *generic point* of $h^{\mathbf{Z}}$.

The generic point of an integral domain

If A is an integral domain[21] then the ideal $\{0\} \subset A$ is prime. It is the kernel of the map from A to its field of fractions K, which corresponds to a schematic point $h^K \to h^A$. This is known as the **generic point** of h^A.

ex:generic

Exercise 4.14.7. Show that the generic point is contained in every open subfunctor of h^A.

The affine line over a field

Let $\mathbf{A}^1_k = h^{k[x]}$ where k is a field. The maximal ideals of $k[x]$ are in bijection with the monic irreducible polynomials with coefficients in k. If k is algebraically closed then the monic irreducible polynomials are the linear polynomials $x - \lambda$, $\lambda \in k$. Therefore the set of closed points of $\mathbf{A}^1_k$ is simply k in this case.

If k is not algebraically closed, the situation is a little more interesting. Now we get several kinds of closed points: one for each monic irreducible polynomial.

Exercise 4.14.8. Let $k = \mathbf{F}_p$ with $p \in \mathbf{Z}$ a prime number. For each $q = p^n$, compute the number of closed points $h^{\mathbf{F}_q} \to \mathbf{A}^1_k$ of $\mathbf{A}^1_k$.

The points of $\mathbf{A}^1$

The schematic points of $\mathbf{A}^1$ correspond to the prime ideals of the ring $\mathbf{Z}[x]$. We will first determine the closed points, which correspond to the maximal ideals.

Note first that if $\mathfrak{m}$ is a maximal ideal of $\mathbf{Z}[x]$ then $\mathfrak{m}$ must contain some prime number p. Indeed $\mathfrak{m} \cap \mathbf{Z}$ is a maximal ideal of $\mathbf{Z}$, hence equal to (p) for some prime p. Therefore $\mathfrak{m}$ is the pre-image of some maximal ideal of $\mathbf{Z}[x]/p = \mathbf{F}_p[x]$. We have just seen that the maximal ideals in $\mathbf{F}_p[x]$ correspond to the monic irreducible polynomials of $\mathbf{F}_p[x]$, so the closed points of $\mathbf{A}^1$ correspond to pairs (p, f) where p is a positive prime number of $\mathbf{Z}$ and f is a monic irreducible polynomial of $\mathbf{F}_p[x]$.

We have additional schematic points corresponding to prime ideals that are not maximal. There is a generic point, corresponding to the ideal $\{0\} \subset \mathbf{Z}[x]$.

There are also prime ideals corresponding to the irreducible polynomials $P \in \mathbf{Z}[x]$. Each of these gives a closed embedding $V(P) \to \mathbf{A}^1$. The ideal $(P) \subset \mathbf{Z}[x]$ corresponds to the ideal $\{0\} \subset \mathbf{Z}[x]/(P)$, which is the generic point of $V(P)$.

Notice that there is a map $q : \mathbf{A}^1 \to h^{\mathbf{Z}}$ corresponding to the ring homomorphism $\mathbf{Z} \to \mathbf{Z}[x]$. For each closed point $V(p) \cong h^{\mathbf{F}_p} \to h^{\mathbf{Z}}$ we can look at the fiber F defined by

$$F(A) = \{\varphi \in \mathbf{A}^1(A) \mid q(\varphi) \in V(p)\}.$$

Then $F \cong h^{\mathbf{F}_p[x]}$. Thus we can think of $\mathbf{A}^1$ as being a family of lines over $h^{\mathbf{Z}}$.

[21] meaning A has no non-zero divisors of zero

The points of a discrete valuation ring

The ring $\mathbf{Z}_{(p)}$ of integers localized at a prime p has 2 prime ideals. One is $p\mathbf{Z}_{(p)}$, which corresponds to the unique closed point of $h^{\mathbf{Z}_{(p)}}$ and the other is $\{0\}$, which corresponds to the generic point $h^{\mathbf{Q}} \to h^{\mathbf{Z}_{(p)}}$.

In general, a discrete valuation ring will have exactly two points: one closed point and one open point. Other examples of discrete valuation rings include $k[[t]]$ where k is a field, and $k[t]_{(t)}$.

ex:dvr-generic

Exercise 4.14.9. Let R be a discrete valuation ring and $h^K \subset h^R$ its generic point. Show that h^K is an *open* subfunctor of h^R.

Solution. Recall the relevant facts about discrete valuation rings, namely R is a local principal ideal domain with maximal ideal $\mathfrak{m} = (m)$. Every element of R that is not contained in this maximal ideal is a unit.

We wish to show that h^K is naturally isomorphic to the open functor U given by

$$U(A) = \{\varphi \in h^R(A) | \varphi(m)A = A\}.$$

Note that for any $\varphi \in U(A)$, $\varphi(r)$ is a unit in A for any nonzero $r \in R$, and therefore φ factors through the field of fractions K:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & A \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle g} & \\ & K. & \end{array}$$

Furthermore, any element $g \in h^K(A)$ yields an element $gi \in U(A)$. I claim that this defines a natural isomorphism η from h^K to U given by

$$\eta_A(g) = gi$$

(and $\eta_A^{-1}(gi) = g$.)

Let $f \in \mathrm{Hom}(A, B)$. The following diagram commutes:

$$\begin{array}{ccc} h^K(A) & \xrightarrow{\eta_A} & U(A) \\ {\scriptstyle f-}\downarrow & & \downarrow{\scriptstyle f-} \\ h^K(B) & \xrightarrow[\eta_B]{} & U(B), \end{array} \tag{5}$$

and it is clear that η is a natural isomophism. □

5 Flat descent

5.1 An example of the descent problem: Gluing modules

We consider the following situation: A is a commutative ring, and $A \to B$ and $A \to C$ are epimorphisms, with $D = B \otimes_A C$. Let u and v denote the maps

$B \to D$ and $C \to D$, respectively. Let $X = h^A$, $U = h^B$, $V = h^C$, and $W = h^D$.

example:descent-1

Example 5.1.1. (i) A useful example to have in mind is the following:

$$\begin{aligned} A &= \mathbf{Z}[\sqrt{-5}] \\ B &= \mathbf{Z}\left[\frac{1}{2}, \sqrt{-5}\right] \\ C &= \mathbf{Z}\left[\frac{1}{3}, \sqrt{-5}\right] \\ D &= \mathbf{Z}\left[\frac{1}{6}, \sqrt{-5}\right] \end{aligned}$$

example:descent-2

(ii) Another example that is useful mainly as a counterexample is

$$\begin{aligned} A &= \mathbf{Z} \\ B &= \mathbf{Z}/2\mathbf{Z} \\ C &= \mathbf{Z}\left[\frac{1}{2}\right] \\ D &= 0. \end{aligned}$$

example:descent-3

(iii) And still another example is

$$\begin{aligned} A &= \mathbf{Z} \\ B &= \mathbf{Z}_{(2)} && \text{the localization at the prime } 2\mathbf{Z} \\ C &= \mathbf{Z}\left[\frac{1}{2}\right] \\ D &= \mathbf{Q}. \end{aligned}$$

Exercise 5.1.2. Do the following for each of the examples above:

(i) Check that the maps $A \to B$, $A \to C$, and $A \to D$ make U, V, and W into subfunctors of X.

(ii) Verify that $U \cap V = W$.

Suppose that M is an A-module. We can produce a B-module $\widetilde{M}_B = B \otimes_A M$, a C-module $\widetilde{M}_C = C \otimes_A M$, and a D-module $\widetilde{M}_D = D \otimes_A M$, as well as isomorphisms

$$D \underset{B}{\otimes} \widetilde{M}_B = D \underset{B}{\otimes} B \underset{A}{\otimes} M \xrightarrow{\sim} D \underset{A}{\otimes} M \xleftarrow{\sim} D \underset{C}{\otimes} C \underset{A}{\otimes} M = D \underset{C}{\otimes} \widetilde{M}_C.$$

We may ask, conversely, if we are given a B-module $\mathscr{M}_B$, a C-module $\mathscr{M}_C$, and isomorphisms

$$D \underset{B}{\otimes} \mathscr{M}_B \xrightarrow{\sim} \mathscr{M}_D \xleftarrow{\sim} D \underset{C}{\otimes} \mathscr{M}_C$$

if this arises from an A-module M, in the sense that there is a commutative diagram

$$\begin{array}{ccccc} \widetilde{M}_B & \longrightarrow & \widetilde{M}_D & \longleftarrow & \widetilde{M}_C \\ \wr\downarrow & & \downarrow\wr & & \downarrow\wr \\ \mathscr{M}_B & \longrightarrow & \mathscr{M}_D & \longleftarrow & \mathscr{M}_C \end{array}$$

in which the vertical arrows are isomorphisms.

To make this question more precise, consider the category $\mathscr{C}$ of tuples

$$\mathscr{M} = (\mathscr{M}_B, \mathscr{M}_C, \mathscr{M}_D, \mathscr{M}_u, \mathscr{M}_v)$$

where for $Q \in \{B, C, D\}$, the object $\mathscr{M}_Q$ is a Q-module and

$$\mathscr{M}_u : \mathscr{M}_B \to \mathscr{M}_D$$
$$\mathscr{M}_v : \mathscr{M}_C \to \mathscr{M}_D$$

are additive functions satisfying

$$\mathscr{M}_u(bx) = u(b)\mathscr{M}_u(x)$$
$$\mathscr{M}_v(cy) = v(c)\mathscr{M}_v(y)$$

for all $b \in B$, $c \in C$, $x \in \mathscr{M}_B$, and $y \in \mathscr{M}_C$. These are required to satisfy the additional condition that the maps

$$D \underset{B}{\otimes} \mathscr{M}_B \to \mathscr{M}_D : d \otimes x \mapsto d\mathscr{M}_u(x)$$
$$D \underset{C}{\otimes} \mathscr{M}_C \to \mathscr{M}_D : d \otimes y \mapsto d\mathscr{M}_v(y)$$

are isomorphisms.

We have a functor $\Phi : A\text{-}\mathsf{Mod} \to \mathscr{C}$ sending a module M to $\widetilde{M}$, where

$$\widetilde{M}_B = B \underset{A}{\otimes} M$$
$$\widetilde{M}_C = C \underset{A}{\otimes} M$$
$$\widetilde{M}_D = D \underset{A}{\otimes} M$$
$$\widetilde{M}_u(b \otimes x) = u(b) \otimes x$$
$$\widetilde{M}_v(c \otimes y) = v(c) \otimes y.$$

We will refer to objects of the category $\mathscr{C}$ as **descent data**. Given $\mathscr{M} \in \mathscr{C}$, is it possible to find an A-module M and an isomorphism $\mathscr{M} \cong \widetilde{M}$? If so then $\mathscr{M}$ is called an **effective** descent datum. Are there properties of B and C that guarantee that every descent datum is effective? If one has a morphism of effective descent data, must it come from a morphism of A-modules? Must that morphism be unique? In other words, is Φ and equivalence of categories?

Exercise 5.1.3. Consider the descent datum for Example 5.1.1 (i)

$$\begin{aligned} \mathscr{M}_B &= B \\ \mathscr{M}_C &= C \\ \mathscr{M}_D &= D \\ \mathscr{M}_u(b) &= b \\ \mathscr{M}_v(c) &= \frac{1-\sqrt{-5}}{3}c. \end{aligned}$$

(i) Verify that $\mathscr{M}$ is a descent datum. (Hint: In verifying the map $D \otimes_C \mathscr{M}_C \to \mathscr{M}_D$ is an isomoprhism you will need to use the fact that $\frac{1-\sqrt{-5}}{3}$ is a unit in D.)

(ii) Let $M = \{(b, c) \in \mathscr{M}_B \times \mathscr{M}_C \mid \mathscr{M}_u(b) = \mathscr{M}_v(c)\}$. Show that M is an A-module and construct a morphism $\widetilde{M} \to \mathscr{M}$.

(iii) Construct an isomorphism between $\mathscr{M}$ and $\mathscr{M}'$, where $\mathscr{M}'$ is the descent datum defined below:

$$\begin{aligned} \mathscr{M}'_B &= (2, 1+\sqrt{-5})B = B \\ \mathscr{M}'_C &= (2, 1+\sqrt{-5})C = 2C \\ \mathscr{M}'_D &= (2, 1+\sqrt{-5})D = D \\ \mathscr{M}'_u(b) &= b \\ \mathscr{M}'_v(c) &= c. \end{aligned}$$

(iv) Define M' by repeating the definition of M with $\mathscr{M}$ replaced by $\mathscr{M}'$.

(v) Show that $M' \cong (2, 1+\sqrt{-5})A$ and deduce that $\mathscr{M}'$ is effective.

(vi) Conclude that $M \cong M'$ and thus that $\mathscr{M}$ is effective as well.

Notice that we have glued together two principal ideals in B and C to obtain a non-principal ideal in A.

Exercise 5.1.4. Consider Example 5.1.1 (ii) and define $\mathbf{Z}$-modules

$$\begin{aligned} M &= \mathbf{Z} \\ N &= \mathbf{Z}\Big[\frac{1}{2}\Big] \times \mathbf{Z}/2\mathbf{Z}. \end{aligned}$$

(i) Show that M and N are not isomorphic.

(ii) Show that $\widetilde{M}$ and $\widetilde{N}$ are isomorphic.

Before trying to answer this question of which descent data are effective, we first note that the descent data we have considered here are just one kind of descent data among many. For example, we might want to try to descend a module that was initially defined on a larger collection of A-algebras. Or we might try to descend a module that was defined on an A-algebra R such that the map $A \to R$ was not an epimorphism. How can we treat all of these kinds of descent at once?

Another way to motivate the construction that will appear in a moment is to consider the following: If we have a descent datum $\mathscr{M} \in \mathscr{C}$ that is isomorphic to $\widetilde{M}$ then we should be able to define $\mathscr{M}_E = E \otimes_A M$ in an unambiguous way, for *any* A-algebra E. Of course, doing this for all A-algebras at once amounts to solving the descent problem, but as a first step, we can try to extend the definition of $\mathscr{M}$ to all A-algebras that at least *admit a map* from B or C. This will lead us to the notion of a *quasi-coherent module*.

Recall that we are writing $X = h^A$, $U = h^B$, $V = h^C$, and $W = h^D$. We view U and V as subfunctors of X since the maps $A \to B$ and $A \to C$ are epimorphisms. Define Y to be the union of the subfunctors U and V of X. That is,

$$Y(R) = U(R) \cup V(R)$$

for any A-algebra R. We will say that an A-algebra R is in Y if the structural map $A \to R$, which is an element of $h^A(R) = X(R)$, lies in $Y(R) \subset X(R)$.

Thinking geometrically, we may imagine U and V as open subsets of X. Taking their union includes all of the points of X, of course, but it will only contain those *maps* $Z \to X$ that factor through at least one of U or V.

Exercise 5.1.5 (Sanity check)**.** To make sure you understand the definition, show that $Y(A)$ does not contain the identity map $\mathrm{id}_A \in X(A)$ in any of the examples introduced at the beginning of this section.

Now, we define another category of descent data. Let $\mathscr{C}'$ be the category whose objects consists of an E-module $\mathscr{M}_E$ for each A-algebra E inside Y, together with maps $\mathscr{M}_f : \mathscr{M}_E \to \mathscr{M}_F$ whenever there is an A-algebra homomorphism $f : E \to F$. These data are required to satisfy two conditions:

(i) $\mathscr{M}_{f \circ g} = \mathscr{M}_f \circ \mathscr{M}_g$ when such a composition makes sense, and

(ii) the map $F \otimes_E \mathscr{M}_E \to \mathscr{M}_F$ sending $\lambda \otimes x$ to $f(\lambda) \otimes \mathscr{M}_f(x)$ is an isomorphism for any A-algebra homomorphism f.

A morphism in $\mathscr{C}'$ from $\mathscr{M}$ to $\mathscr{N}$ consists of giving, for each A-algebra E in Y, an E-module homomorphism $\varphi_E : \mathscr{M}_E \to \mathscr{N}_E$, subject to the condition that the diagram

$$\begin{array}{ccc} \mathscr{M}_E & \xrightarrow{\mathscr{M}_f} & \mathscr{M}_F \\ {\scriptstyle \varphi_E}\downarrow & & \downarrow{\scriptstyle \varphi_F} \\ \mathscr{N}_E & \longrightarrow & \mathscr{N}_F \end{array}$$

be commutative.[22]

[22] You may recognize this as the same diagram that appears in the definition of a natural

It should be obvious that there is a functor from $\mathscr{C}'$ to $\mathscr{C}$ that forgets $\mathscr{M}_E$ for every $E \notin \{B, C, D\}$. It may appear therefore that defining an object of $\mathscr{C}'$ requires a great deal more data than defining an object of $\mathscr{C}$. In fact, this is not the case: The functor described above is an *equivalence of categories*.

We will sketch the description of an inverse functor $\mathscr{C} \to \mathscr{C}'$ and leave the verification that it actually is an inverse as an exercise.

Suppose $\mathscr{M} \in \mathscr{C}$ and E is an A-algebra in Y. Then there is an A-algebra map $B \to E$ or a map $C \to E$. If there is just one such map then let us write $Q_E = B$ or $Q_E = C$, as the case warrants, and g_E for the map $Q_E \to E$ for the A-algebra homomorphism (which is unique because $A \to Q_E$ is an epimorphism!). It is possible that there are A-algebra maps $B \to E$ *and* $C \to E$, but in this case there is also a map $D = B \otimes_A C \to E$. In this case we take $Q_E = D$ and g_E to be the unique A-algebra map $D \to E$ (uniqueness again coming from the fact that $A \to D$ is an epimorphism). Now we define $\mathscr{M}'_E = E \otimes_{Q_E} \mathscr{M}_{Q_E}$.

This is not the end of the definition. We also need the maps $\mathscr{M}'_f : \mathscr{M}'_E \to \mathscr{M}'_F$ associated to each A-algebra map $f : E \to F$. If $Q_E = Q_F$ then the diagram

$$\begin{array}{ccc} Q_E & & \\ {\scriptstyle g_E}\downarrow & \searrow^{g_F} & \\ E & \longrightarrow & F \end{array}$$

commutes (because $A \to Q_E$ is an epimorphism). Therefore, by the universal property of the tensor product, we obtain a map $\mathscr{M}'_f : \mathscr{M}'_E \to \mathscr{M}'_F$ by $\mathscr{M}'_f(\lambda \otimes x) = f(\lambda) \otimes x$.

Matters become more subtle when $Q_E \neq Q_F$. Note, however, that in this case we must have $Q_F = D$ because F admits A-algebra homomorphisms from *both* B and C. In this case we have a commutative diagram

$$\begin{array}{ccc} Q_E & \xrightarrow{g_E} & E \\ \downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{g_F} & F. \end{array}$$

Thus we may take $\mathscr{M}'_f$ to be the composition

$$\begin{array}{ccccccccc} \mathscr{M}'_E = E \otimes_{Q_E} \mathscr{M}_{Q_E} & \to & F \otimes_E \otimes_{Q_E} \mathscr{M}_{Q_E} & \xrightarrow{\sim} & F \otimes_D D \otimes_{Q_E} \mathscr{M}_{Q_E} & \to & F \otimes_D \mathscr{M}_D & = & \mathscr{M}'_F \\ \lambda \otimes x & \longmapsto & 1 \otimes \lambda \otimes x & & \lambda \otimes \mu \otimes x & \longmapsto & \lambda \otimes \mu \mathscr{M}_w(x) & & \end{array}$$

where $w = u$ if $Q_E = B$ and $w = v$ if $Q_E = C$.

---

transformation. You may be tempted therefore to try to define $\mathscr{C}'$ as the category of functors from the category of A-algebras in Y to some target category. The trouble in making this definition is saying exactly what the target category should be! The if E is an A-algebra then $\mathscr{M}_E$ must be an E-module. But if F is another A-algebra then $\mathscr{M}_F$ will be an F-module. Thus the values of the "functor" $\mathscr{M}$ live in different places for different inputs. In order to make sense of such functors, one is led to the notion of a *fibered category*: see [Gro3], [Gro2, Exposé VI], or [FGI$^+$, Chapter 3], among many others.

Exercise 5.1.6. (i) Verify that $\mathscr{M}'_f \circ \mathscr{M}'_g = \mathscr{M}'_{f\circ g}$ when the composition is defined and conclude that $\mathscr{M}'$ is an object of $\mathscr{C}'$.

(ii) Show that the construction $\mathscr{M} \mapsto \mathscr{M}'$ described above determines a functor $\Psi : \mathscr{C} \to \mathscr{C}'$.

(iii) Check that Φ and Ψ are inverse equivalences of categories.

Objects of $\mathscr{C}'$ will be called **quasi-coherent modules** on Y. In the remaineder of this section we will discuss sufficient conditions on Y to ensure that the functor $A\text{-}\mathsf{Mod} \to \mathscr{C}'$ is an equivalence of categories.

5.2 A second example: Galois descent

Question 5.2.1. Given a complex vector space V, is there a real vector space W such that $V = \mathbf{C} \otimes_{\mathbf{R}} W$?

This may not sound like the most interesting question since we know that vector spaces are classified up to isomorphism by the non-negative integers and therefore every complex vector space is induced up to isomorphism from a real vector space. However, we can ask a more precise question:

Question 5.2.2. What additional data does a complex vector space have if it comes from a real vector space? Are these data enough to characterize the real vector space it is induced from in an essentially unique way?

If we are able to answer this question completely, it will give us an alternate characterization of the category of real vector spaces. In particular, it will give us another way of thinking about algebraic objects that involve giving extra structure to real vector spaces. We will see some interesting examples of this below.

Let W be a real vector space. Then $\widetilde{W}_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} W$ is a complex vector space. Note, however, that complex conjugation gives an automorphism of $\mathbf{C}$ as a $\mathbf{R}$-algebra:

$$\sigma : \mathbf{C} \to \mathbf{C} : z \mapsto \overline{z}.$$

This induces a function

$$\widetilde{W}_\sigma : \widetilde{W}_{\mathbf{C}} \to \widetilde{W}_{\mathbf{C}} : (z \otimes x) \mapsto \overline{z} \otimes x.$$

This function is $\mathbf{R}$-linear but not $\mathbf{C}$-linear. In fact, it satisfies the following relation for all $z \in \mathbf{C}$ and $x \in \widetilde{W}$:

$$\widetilde{W}_\sigma(zx) = \sigma(z)\widetilde{W}_\sigma(x).$$

Notice finally that $\widetilde{W}_\sigma^2 = \mathrm{id}_{\widetilde{W}_{\mathbf{C}}}$. We may therefore interpret $\widetilde{W}_\sigma$ as specifying an extension of the action of the Galois group $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ on $\mathbf{C}$ to a compatible action on $\widetilde{W}_{\mathbf{C}}$.

Let $\mathscr{C}$ be the category of all pairs $\mathscr{W} = (\mathscr{W}_{\mathbf{C}}, \mathscr{W}_\sigma)$ where $\mathscr{W}_{\mathbf{C}}$ is a complex vector space and $\mathscr{W}_\sigma : \mathscr{W}_{\mathbf{C}} \to \mathscr{W}_{\mathbf{C}}$ is an $\mathbf{R}$-linear isomorphism satisfying

$$\mathscr{W}_\sigma^2 = \mathrm{id}_{\mathscr{W}_{\mathbf{C}}} \mathscr{W}_\sigma(zx) = \sigma(z)\mathscr{W}_\sigma(x).$$

A morphism from $\mathscr{W} \to \mathscr{W}'$ consists of a $\mathbf{C}$-linear function $f : \mathscr{W}_{\mathbf{C}} \to \mathscr{W}'_{\mathbf{C}}$ such that the diagram below commutes:

$$\begin{array}{ccc} \mathscr{W}_{\mathbf{C}} & \xrightarrow{f} & \mathscr{W}'_{\mathbf{C}} \\ {\scriptstyle \mathscr{W}_\sigma}\downarrow & & \downarrow{\scriptstyle \mathscr{W}'_\sigma} \\ \mathscr{W}_{\mathbf{C}} & \xrightarrow{f} & \mathscr{W}'_{\mathbf{C}}. \end{array}$$

ex:C-R-descent-pullback

Exercise 5.2.3. Show that the construction $\Phi(W) = \widetilde{W}$ determines a functor $\Phi : \mathbf{R}\text{-}\mathsf{Mod} \to \mathscr{C}$.

It is not hard to show that Φ is an equivalence of categories directly, as the following exercise illustrates. However, in order to demonstrate the generalization we have in mind we will need to work with the more general language of quasi-coherent modules.

ex:C-R-descent-sections

Exercise 5.2.4. Let $\mathscr{W}$ be an object of $\mathscr{C}$. Define

$$\Psi(\mathscr{W}) = \{x \in \mathscr{W}_{\mathbf{C}} \mid \mathscr{W}_\sigma(x) = x\}.$$

(i) Let $W = \Psi(\mathscr{W})$. Show that the inclusion map $W \to \mathscr{W}_{\mathbf{C}}$ extends to a morphism of descent data $\widetilde{W} \to \mathscr{W}$.

(ii) Consider a basis z_i, $i \in I$ for $\mathscr{W}_{\mathbf{C}}$ as a complex vector space. Show that the elements z_i and $\overline{z}_i$ form a basis for $\mathscr{W}_{\mathbf{C}}$ as a *real* vector space.

(iii) Deduce that the elements $z_i + \overline{z}_i$, $i \in I$ form a basis for W as a real vector space.

(iv) Conclude that the map $\mathbf{C} \otimes_{\mathbf{R}} W \to \mathscr{W}_{\mathbf{C}}$ is an isomorphism and therefore that $\widetilde{W} \to \mathscr{W}$ is an isomorphism of descent data.

Exercise 5.2.5. Using the fundamental theorem of Galois theory, generalize Exercises 5.2.3 and 5.2.4 to the situation where $\mathbf{R}$ is replaced by an arbitrary field K and $\mathbf{C}$ is replaced by a Galois extension of K.

ex:C-R-descent-operations

Exercise 5.2.6. Define

$$\mathscr{V} \times \mathscr{W} = (\mathscr{V}_{\mathbf{C}} \times \mathscr{W}_{\mathbf{C}}, \mathscr{V}_\sigma \times \mathscr{W}_\sigma) \mathscr{V} \otimes \mathscr{W} = (\mathscr{V}_{\mathbf{C}} \otimes \mathscr{W}_{\mathbf{C}}, \mathscr{V}_\sigma \otimes \mathscr{W}_\sigma)$$

where

$$(\widetilde{V}_\sigma \times \widetilde{W}_\sigma)(v, w) = (\widetilde{V}_\sigma(v), \widetilde{W}_\sigma(w))$$
$$(\widetilde{V}_\sigma \otimes \widetilde{W}_\sigma)(v \otimes w) = \widetilde{V}_\sigma(v) \otimes \widetilde{W}_\sigma(w).$$

Let $\mathscr{V}$ and $\mathscr{W}$ be objects of $\mathscr{C}$.

(i) Show that $(V \times W)^{\sim} \cong \widetilde{V} \times \widetilde{W}$.

(ii) Show that $(V \otimes W)^{\sim} \cong \widetilde{V} \otimes \widetilde{W}$.

ex:C-R-descent-algebra

Exercise 5.2.7. (i) Check for yourself that to specify an associative $\mathbf{R}$-algebra structure[23] on an $\mathbf{R}$-vector space V is the same as to give a $\mathbf{R}$-linear map

$$\mu : V \underset{\mathbf{R}}{\otimes} V \to V$$

satisfying $\mu(u \otimes \mu(v \otimes w)) = \mu(\mu(u \otimes v) \otimes w)$.

(ii) Let $\mathscr{D}$ be the category of pairs $(\mathscr{W}_{\mathbf{C}}, \mathscr{W}_\sigma)$ where $\mathscr{W}_{\mathbf{C}}$ is a $\mathbf{C}$-*algebra* and $\mathscr{W}_\sigma : \mathscr{W}_{\mathbf{C}} \to \mathscr{W}_{\mathbf{C}}$ is a $\mathbf{R}$-*algebra homomoprhism* satisfying $\mathscr{W}_\sigma(zx) = \overline{z}\mathscr{W}_\sigma(x)$ for all $z \in \mathbf{C}$ and $x \in \mathscr{W}_{\mathbf{C}}$. Show that $\mathscr{D}$ is equivalent to the category Ass-$\mathbf{R}$-Alg of associative $\mathbf{R}$-algebras.

(iii) Let $\mathscr{Q}$ be the object of $\mathscr{D}$ where $\mathscr{Q}_{\mathbf{C}} = \mathsf{Mat}_{2\times 2}(\mathbf{C})$ is the ring of 2×2 complex matrices and

$$\mathscr{Q}_\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix}.$$

Verify that this is indeed an object of $\mathscr{D}$ and determine what the corresponding associative $\mathbf{R}$-algebra is. (Hint: It has a name.)

5.3 Notational conventions

Before embarking on the statement and proof of faithfully flat descent, we introduce some notation conventions. Let X : ComRng $\to$ Sets be a functor. We write $(A, \xi) \in X$ to mean that A is a commutative ring and $\xi \in X(A)$. We define an X-morphism $\varphi : (A, \xi) \to (B, \eta)$ to be a homomorphism of commutative rings $\varphi : A \to B$ such that $X(\varphi)(\xi) = \eta$. Composition is defined by composing the underlying homomorphisms of commutative rings.

Exercise 5.3.1. Verify that the above definitions yield a category.

We'll also reuse the letter X for this category.

Exercise 5.3.2. Show that if $X = h^R$ then the category associated to X as above is the category of R-algebras.

We'll usually refer to an object (A, ξ) of X by the single letter A, in much the way we refer to an R-algebra by the same letter as its underlying commutative ring. Thus saying $A \in h^R$ is another way of saying A is an R-algebra, and $A \in h^{\mathbf{Z}}$ means simply that A is a commutative ring.

[23] To say an $\mathbf{R}$-algebra is "associative" means that it is associative but not necessarily commutative. I didn't make the terminology up.

5.4 Quasi-coherent modules

Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor. A **quasi-coherent module** on X consists of the following data:

qcoh:1 **QCOH1** for each $A \in X$, an A-module $\mathscr{M}_A$;

qcoh:2 **QCOH2** for each morphism $\varphi : A \to B$ in X a function $\mathscr{M}_A \to \mathscr{M}_B$ such that

(i) $\mathscr{M}_\varphi(ax) = \varphi(a)\mathscr{M}_\varphi(x)$ for all $a \in A$ and $x \in \mathscr{M}_A$, and

(ii) $\mathscr{M}_\varphi(x + y) = \mathscr{M}_\varphi(x) + \mathscr{M}_\varphi(y)$.

These are required to satisfy the following conditions:

qcoh:3 **QCOH3** if $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ is a sequence of homomorphisms of commutative rings then $\mathscr{M}_\psi \circ \mathscr{M}_\varphi = \mathscr{M}_{\psi\circ\varphi}$, and

qcoh:4 **QCOH4** if $\varphi : A \to B$ is an X-morphism then the map

$$B \underset{A}{\otimes} \mathscr{M}_A \to \mathscr{M}_B : b \otimes x \mapsto b\mathscr{M}_\varphi(x)$$

is an isomorphism.

A morphism $f : \mathscr{M} \to \mathscr{M}'$ of quasi-coherent modules on X is a collection of maps $f_A : \mathscr{M}_A \to \mathscr{M}'_A$ for each $A \in X$ such that for every X-morphism $\varphi : A \to B$, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{M}_A & \xrightarrow{f_A} & \mathscr{M}'_A \\ {\scriptstyle \mathscr{M}_\varphi}\downarrow & & \downarrow{\scriptstyle \mathscr{M}'_\varphi} \\ \mathscr{M}_B & \xrightarrow{f_B} & \mathscr{M}'_B \end{array} \qquad (6)$$

eqn:4

Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor and suppose $\mathscr{M} \in \mathsf{QCoh}(X)$. Define $\Gamma(X, \mathscr{M})$ to be the collection of all tuples $(x_A)_{A\in X}$ where $x_A \in \mathscr{M}_A$ for each $A \in X$ and $\mathscr{M}_\varphi(x_A) = x_B$ whenever $\varphi : A \to B$ is an X-morphism.

ex:sec-lim **Exercise 5.4.1** (If you know about inverse limits)**.** Show that

$$\Gamma(X, \mathscr{M}) = \varprojlim_{A\in X} \mathscr{M}_A.$$

ex:qcoh-amod **Exercise 5.4.2.** Suppose that X is a subfunctor of h^A. Verify that $\Gamma(X, \mathscr{M})$ is an A-module and that the above defines a functor $\mathsf{QCoh}(X) \to A\text{-}\mathsf{Mod}$.[24]

[24] More generally, any morphism $X \to h^A$ gives $\Gamma(X, \mathscr{M})$ the structure of an A-module; see Exercise 5.4.3.

Solution. First, we define addition and A–multiplication pointwise on the elements of $\Gamma(X,\mathscr{M})$, i.e., for $(x_\xi)_{\substack{\xi:A\to B\\ B\in\mathsf{ComRng}}}, (y_\xi)_{\substack{\xi:A\to B\\ B\in\mathsf{ComRng}}} \in \Gamma(X,\mathscr{M})$, and $a\in A$, define

$$\begin{aligned}(x_\xi)_{\substack{\xi:A\to B\\ B\in\mathsf{ComRng}}} + (y_\xi)_{\substack{\xi:A\to B\\ B\in\mathsf{ComRng}}} &= (x_\xi + y_\xi)_{\substack{\xi:A\to B\\ B\in\mathsf{ComRng}}} \quad \text{and}\\ a\cdot (x_\xi)_{\substack{\xi:A\to B\\ B\in\mathsf{ComRng}}} &= (\xi(a)\cdot x_\xi)_{\substack{\xi:A\to B\\ B\in\mathsf{ComRng}}}.\end{aligned}$$

Then we claim that this makes $\Gamma(X,\mathscr{M})$ into an A–module. First, note that for any $\xi : A\to B$ in ComRng, ξ gives B the structure of an A–module through multiplication defined by $a\cdot b := \xi(a)b$. Thus, $\Gamma(X,\mathscr{M})$ is a subset of the product of a collection of A–modules, whence it suffices to check that $\Gamma(X,\mathscr{M})$ is closed under the above operations, i.e., for any $(x_\xi)_{\substack{\xi:A\to B\\ B\in\mathsf{ComRng}}}, (y_\xi)_{\substack{\xi:A\to B\\ B\in\mathsf{ComRng}}} \in \Gamma(X,\mathscr{M})$, and $a\in A$,

$$(x_\xi)_{\substack{\xi:A\to B\\ B\in\mathsf{ComRng}}} + a\cdot (y_\xi)_{\substack{\xi:A\to B\\ B\in\mathsf{ComRng}}} \in \Gamma(X,\mathscr{M}).$$

But this is equivalent to checking that for each $\xi : A\to B$, $\eta : A\to C$, and $\varphi : B\to C$ with $\varphi_*(\xi)=\eta$,

$$\mathscr{M}_\varphi\left(x_\xi + \xi(a)y_\xi\right) = x_\eta + \eta(a)y_\eta.$$

But, since $\mathscr{M}_\varphi$ is a B–module homomorphism ($\mathscr{M}_\xi$ is a B–module, and $\mathscr{M}_\eta$ carries the structure of a B–module via $b\cdot x := \varphi(b)x$), we have that

$$\begin{aligned}\mathscr{M}_\varphi\left(x_\xi + \xi(a)y_\xi\right) &= \mathscr{M}_\varphi\left(x_\xi\right) + \varphi(\xi(a))\cdot \mathscr{M}_\varphi\left(y_\xi\right)\\ &= x_\eta + \varphi(\xi(a))y_\eta\\ &= x_\eta + \eta(a)y_\eta.\end{aligned}$$

Thus, $\Gamma(X,\mathscr{M})$ is closed under these operations, so it is a submodule of the aforementioned product, and therefore an A–module. □

`ex:sec-A-mod` **Exercise 5.4.3.** This exercise generalizes Exercise 5.4.2.

Suppose that X is equipped with a morphism $f : X\to h^A$.[25] Verify the following construction gives $\Gamma(X,\mathscr{M})$ the structure of an A-module: For each $B\in X$ note that $f(B)\in h^A$ so $f(B)$ is an A-algebra. For each $B\in X$, let us denote by $\xi_B : A\to B$ the homomorphism of commutative rings giving rise to this A-algebra structure. Then for each $a\in A$ and each $(x_B)_{B\in X}, (y_B)_{B\in X}\in \Gamma(X,\mathscr{M})$, define

$$\begin{aligned}a.(x_B)_{B\in X} &= (\xi_B(a)x_B)_{B\in X}\\ (x_B)_{B\in X} + (y_B)_{B\in X} &= (x_B + y_B)_{B\in X}.\end{aligned}$$

Show that this gives $\Gamma(X,\mathscr{M})$ the structure of an A-module and defines a functor from $\mathsf{QCoh}(X)$ to A-Mod.

This exercise can also be done using Exercise 5.4.1.

[25] We will have use only for the case where X is a sub-functor of h^A below.

We may generalize the definition of Γ slightly. Suppose that $f : Y \to X$ is a morphism of functors and $\mathscr{M}$ is a quasi-coherent module on X. Define $\Gamma(Y, \mathscr{M})$ to be the collection of all tuples $(x_A)_{A \in Y}$ where

(i) $x_A \in \mathscr{M}_{f(A)}$ for each $A \in X$, and

(ii) for each Y-morphism $\varphi : A \to B$ we have $\mathscr{M}_{f(\varphi)}(x_A) = x_B$.

ex:sec-A-mod2

Exercise 5.4.4. Generalize Exercise 5.4.3 to show that if $\mathscr{M}$ is a quasi-coherent module on X, and $f : Y \to X$ is a morphism of functors, and $g : Y \to h^A$ is a second morphism of functors then $\Gamma(Y, \mathscr{M})$ is naturally equipped with an A-module structure.

ex:qcoh-assoc-to-mod

Exercise 5.4.5. Construct a functor $A\text{-}\mathsf{Mod} \to \mathsf{QCoh}(h^A)$ by sending an A-module M to the quasi-coherent module $\widetilde{M}$ with $\widetilde{M}_B = M \otimes_A B$ for each A-algebra B. The map $\widetilde{M}_B \to \widetilde{M}_C$ associated to an A-algebra homomorphism $\varphi : B \to C$ sends $x \otimes b$ to $x \otimes \varphi(b)$.

ex:qcoh-unit

Exercise 5.4.6. Let M be an A-module and $U : \mathsf{ComRng} \to \mathsf{Sets}$ a subfunctor of h^A.[26] Let $\widetilde{M}$ be the quasi-coherent module on U defined in Exercise 5.4.5. Construct a morphism $M \to \Gamma(U, \widetilde{M})$ induced by the canonical maps $M \to \widetilde{M}_B = B \otimes_A M$ and show that it is natural in M.

ex:qcoh-counit

Exercise 5.4.7. Let $\mathscr{M}$ be a quasi-coherent module on U, where $U \subset h^A$ (or, more generally, U is equipped with a morphism to h^A). Construct a map $\Gamma(U, \mathscr{M})^\sim \to \mathscr{M}$ given by the following rule:

$$\begin{array}{ccc} \Gamma(U, \mathscr{M})^\sim_B & \longrightarrow & \mathscr{M}_B \\ \| & & \| \\ B \otimes_A \Gamma(U, \mathscr{M}) & \longrightarrow & \mathscr{M}_B \\ & & \\ b \otimes (x_C)_{C \in U} & \longmapsto & b x_B \end{array}$$

Verify that this is a morphism of quasi-coherent modules.

ex:qcoh-pullback

Exercise 5.4.8. Let $f : X \to Y$ be a morphism of functors $\mathsf{ComRng} \to \mathsf{Sets}$. If $\mathscr{M}$ is a quasi-coherent module on Y, define $f^*(\mathscr{M})_\xi = \mathscr{M}_{f(\xi)}$. Show that $f^*\mathscr{M}$ is a quasi-coherent module on X.

lem:triv-descent

Lemma 5.4.9. *The functors $\mathscr{M} \mapsto \Gamma(h^A, \mathscr{M})$ and $M \mapsto \widetilde{M}$ define mutually inverse equivalences of categories between $\mathsf{QCoh}(h^A)$ and A-Mod.*

Proof. This is actually a version of the Yoneda lemma, and the proof is, unsurprisingly, very similar. Note that if $\mathscr{M}$ is quasi-coherent on h^A then an element $(x_B)_{B \in A\text{-}\mathsf{Alg}}$ of $\Gamma(h^A, \mathscr{M})$ is uniquely determined by $x_A \in \mathscr{M}_A$. Hence the projection $\Gamma(h^A, \mathscr{M}) \to \mathscr{M}_{\mathrm{id}_A}$ is an isomorphism of A-modules. Applying this to $\mathscr{M} = \widetilde{M}$ we get

$$\Gamma(h^A, \widetilde{M}) = \widetilde{M}_A = M \underset{A}{\otimes} A = M.$$

[26] More generally, you may wish to consider any morphism of functors $U \to h^A$.

Conversely, suppose that $\mathscr{M}$ is a quasi-coherent module on h^A. Then for all A-algebras B, the map $\mathscr{M}_A \otimes_A B \to \mathscr{M}_B$ is an isomorphism, by the definition of quasi-coherence. But $\mathscr{M}_A \otimes_A B = \big(\Gamma(h^A, \mathscr{M})^{\sim}\big)_B$ by definition. This gives an isomorphism $\Gamma(h^A, \mathscr{M})^{\sim} \to \mathscr{M}$. □

ex:pullback-subfunc

Exercise 5.4.10. Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor and $U \subset X$ a subfunctor. Suppose that $\xi \in U(A)$ corresponds to a morphism $h^A \to X$, which we denote by the same letter. Show that the subfunctor $\xi^{-1}U \subset h^A$ is in fact equal to h^A.

5.5 Faithfully flat descent (affine case)

Definition 5.5.1. Let A be a commutative ring and B_i a family of A-modules, with i drawn from an indexing set I. We say that the B_i are **faithfully flat** a sequence of A-modules

$$M' \to M \to M''$$

is exact if and only if the sequences

$$B_i \underset{A}{\otimes} M' \to B_i \underset{A}{\otimes} M \to B_i \underset{A}{\otimes} B''$$

are exact for *all* i. We call a family of A-algebras faithfully flat if they are faithfully flat as a family of A-modules.

Exercise 5.5.2. Show that a collection of A-algebras B_i is faithfully flat if and only if the A-module $\bigoplus B_i$ is faithfully flat.

lem:sec-calc2

Lemma 5.5.3. *Let B_i, $i \in I$ be a collection of A-algebras. Take $X = h^A$ and, for each i in I, let $U_i \subset X$ be the image of the map $h^{B_i} \to h^A$. Define $U = \bigcup U_i$. Let $\mathscr{M}$ be a quasi-coherent module on U and define $M = \Gamma(U, \mathscr{M})$ and*

$$M' = \big\{(x_{I_i})_{i\in I} \,\big|\, v_0(x_{B_i}) = v_1(x_{B_j}) \in \mathscr{M}(B_i \underset{A}{\otimes} B_j)\big\}.$$

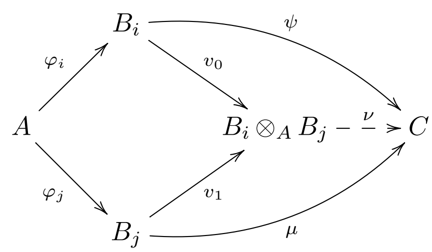
Then the restriction map

$$\begin{array}{ccc} M & \longrightarrow & M' \\ (x_C)_{C\in U} & \longmapsto & (x_{B_i})_{i\in I} \end{array}$$

is an isomorphism of A-modules.

Proof. The point is to show that any $(x_{\varphi_i})_{i\in I} \in M'$ can be extended in a unique way to an $(x_\xi)_{\xi\in U(C)} \in M$. First suppose that $(x_\xi)_{\xi\in U(C)}$ and $(y_\xi)_{\xi\in U(C)}$ have the same image in M'. We wish to show that $x_\xi = y_\xi$ for all ξ. Assume $\xi \in U(C)$ corresponds to a map $A \to C$. Then by definition, ξ lies in $U_i(C)$ for some i, so there is a factorization $A \xrightarrow{\varphi_i} B_i \xrightarrow{\psi} C$ for some map $\psi : B_i \to C$. Then $\xi = \psi_*(\varphi_i)$ so that $x_\xi = \psi_*(x_{\varphi_i})$ and $y_\xi = \psi_*(y_{\varphi_i})$. Since $x_{\varphi_i} = y_{\varphi_i}$, it follows that $x_\xi = y_\xi$.

The above shows that the map $M \to M'$ is an injection. We check now that it is surjective. Suppose that $(x_{\varphi_i})_{i \in I} \in M'$. We wish to extend this to $(x_\xi)_{\xi \in U(C)}$ for every commutative ring C. To that end, let ξ be an element of $U(C) \subset h^A(C)$. By definition of U, we can factor ξ as $A \xrightarrow{\varphi_i} B_i \xrightarrow{\psi} C$ so that $\xi = \psi_*(\varphi_i)$. We would like to define $x_\xi = \psi_*(\xi_{\varphi_i})$. However, we must verify that this definition does not depend on the choice of ψ: Suppose we had another factorization of ξ as $A \xrightarrow{\varphi_j} B_j \xrightarrow{\mu} C$. Then we can find a diagram



where the map ν is guaranteed by the universal property of the tensor product. Then we have

$$\psi_*(\xi_{\varphi_i}) = \nu_* v_0 \xi_{\varphi_i} = \nu_* v_1 \xi_{\varphi_j} = \mu_*(\xi_{\varphi_j})$$

because $v_0(\xi_{\varphi_i}) = v_1(\xi_{\varphi_j})$ by the definition of M'. This proves that $(x_{\varphi_i})_{i \in I} \in M$ can be extended, in a unique way, to $(x_\xi)_{\xi \in U(C)} \in M$ and therefore that $M' \to M$ is an isomorphism. $\square$

Corollary 5.5.3.1. *With notation as in the lemma, the sequence*

$$\Gamma(U, \mathscr{M}) \longrightarrow \prod_{i \in I} \mathscr{M}(B_i) \rightrightarrows \prod_{i,j \in I} \mathscr{M}(B_i \underset{A}{\otimes} B_j),$$

is exact.

cor:flat-base-change

Corollary 5.5.3.2. *Let $X = h^A$ and $X' = h^A$ and suppose $A \to A'$ is a* flat *homomorphism of commutative rings corresponding to a map $f : X' \to X$. Suppose that $U \subset X$ is the union of the images of maps $h^{B_i} \to X$, $i \in I$, as in the lemma, and take $U' = f^{-1}U \subset X'$. Let $\mathscr{M}$ be a module on U. If the indexing set I is finite then the map*

$$A' \underset{A}{\otimes} \Gamma(U, \mathscr{M}) \to \Gamma(U', \mathscr{M})$$

is an isomorphism.

Proof. Let us put $B_i' = A' \otimes_A B_i$. We have a commutative diagram of A-modules

$$\begin{array}{ccccc}
\Gamma(U, \mathscr{M}) & \longrightarrow & \mathscr{M}(B_i) & \rightrightarrows & \mathscr{M}(B_i \otimes_A B_j) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(U', \mathscr{M}) & \longrightarrow & \mathscr{M}(B_i') & \rightrightarrows & \mathscr{M}(B_i' \otimes_{A'} B_j').
\end{array}$$

The bottom row is a diagrma of A'-modules so we may apply the universal property of the tensor product and obtain a diagram

$$\begin{array}{ccccc} A' \otimes_A \Gamma(U, \mathscr{M}) & \longrightarrow & A' \otimes_A \mathscr{M}(B_i) & \rightrightarrows & A' \otimes_A \mathscr{M}(B_i \otimes_A B_j) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(U', \mathscr{M}) & \longrightarrow & \mathscr{M}(B'_i) & \rightrightarrows & \mathscr{M}(B'_i \otimes_{A'} B'_j). \end{array}$$

Note that the upper row remains exact because A' is flat. For notational simplicity, write $B_{ij} = B_i \otimes_A B_j$ and $B'_{ij} = B'_i \otimes_{A'} B'_j = A' \otimes_A B_{ij}$. Noting now that

$$A' \underset{A}{\otimes} \mathscr{M}(B_i) = A' \underset{A}{\otimes} B_i \underset{B_i}{\otimes} \mathscr{M}(B_i) = B'_i \underset{B_i}{\otimes} \mathscr{M}(B_i)$$
$$A' \underset{A}{\otimes} \mathscr{M}(B_{ij}) = A' \underset{A}{\otimes} B_{ij} \underset{B_{ij}}{\otimes} \mathscr{M}(B_{ij}) = B'_{ij} \underset{A}{\otimes} \mathscr{M}(B_{ij})$$

we deduce that the quasi-coherence of $\mathscr{M}$ implies the two vertical arrows on the right are isomorphisms. Therefore the leftmost vertical arrow is an isomorphism as well. □

Let A be a commutative ring and let $U \subset h^A$ be a subfunctor. We will say that U is a **quasi-compact subfunctor** if there is a *finite* collection of A-algebras B_i such that U is the union of the images of the maps $h^{B_i} \to h^A$.[27] Using this terminology, we get the following restatement of the last corollary:

cor:qc-flat-base-change

Corollary 5.5.3.3. *Let $U \subset h^A$ be a quasi-compact sub-functor and A' a flat A-algebra. Let $f : h^{A'} \to h^A$ be the associated morphism of functors and define $U' = f^{-1}U$. Then for any module $\mathscr{M}$ on h^A, the map*

$$A' \underset{A}{\otimes} \Gamma(U, \mathscr{M}) \to \Gamma(U', \mathscr{M})$$

is an isomorphism.

In order to state the faithfully flat descent theorem we make one more definition. We call a subfunctor $U \subset h^A$ a **faithfully flat subfunctor** if it contains a faithfully flat collection of A-algebras. If a subfunctor is *both* faithfully flat and quasi-compact we say it is an **fpqc cover**.

r:qc-flat-base-change-in-U

Corollary 5.5.3.4. *Let $U \subset h^A$ be a quasi-compact subfunctor and B an A-algebra contained in U. Then for any quasi-coherence module $\mathscr{M}$ on U, the morphism*

$$B \otimes_A \Gamma(U, \mathscr{M}) \longrightarrow \mathscr{M}_B$$
$$b \otimes (x_C)_{C \in U} \longmapsto bx_B$$

is an isomorphism.

[27] Later we will generalize this definition to any morphism of functors.

Proof. Let $f : h^B \to h^A$ denote the morphism of functors associated to the A-algebra structure on B. Take $V = f^{-1}U$. Then we have a factorization

$$B \underset{A}{\otimes} \Gamma(U, \mathscr{M}) \to \Gamma(V, \mathscr{M}) \to \mathscr{M}_B.$$

The first of these maps is an isomorphism by Corollary 5.5.3.3 and the second is an isomorphism by Lemma 5.4.9. ☐

thm:fpqc-descent-affine

Theorem 5.5.4. *Let $X = h^A$ for a commutative ring A and suppose that $U \subset h^A$ is an fpqc cover. Then the functors*

$$\Phi : A\text{-}\mathsf{Mod} \to \mathsf{QCoh}(U) : M \mapsto \widetilde{M}$$
$$\Psi : \mathsf{QCoh}(U) \to A\text{-}\mathsf{Mod} : \mathscr{M} \mapsto \Gamma(U, \mathscr{M})$$

are equivalences.

Proof. First we show that that the morphism $\Psi \circ \Phi(M) \to M$ from Exercise 5.4.6 is an isomorphism. To verify that this map is an isomorphism, it is enough to show that the induced maps

$$B \underset{A}{\otimes} M \to B \underset{A}{\otimes} \Psi \circ \Phi(M) \tag{7}$$ eqn:6

are isomorphisms for all B in a faithfully flat family of A-algebras. By assumption U contains a faithfully flat collection of A-algebras so it will be sufficient to demonstrate that (7) *is an isomorphism for all flat A-algebras B in U.*

Let us suppose then that B is a flat A-algebra in U. Denote the map $h^B \to h^A$ by f and note that $f^{-1}U = h^B$. We have a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & \Gamma(U, \widetilde{M}) \\ \downarrow & & \downarrow \\ B \otimes_A M & \longrightarrow & \Gamma(U', \widetilde{M}) \end{array}$$

inducing a commutative diagram

$$\begin{array}{ccc} B \otimes_A M & \longrightarrow & B \otimes_A \Gamma(U, \widetilde{M}) \\ \| & & \downarrow \\ B \otimes_A M & \longrightarrow & \Gamma(U', \widetilde{M}). \end{array}$$

Now, B is in U so $U' = h^B$ by Exercise 5.4.10 so the lower horizontal arrow is an isomorphism by Lemma 5.4.9. Furthermore, Corollary 5.5.3.2 implies that the vertical arrow on the right is an isomorphism. Therefore the upper horizontal arrow is an isomorphism as well, as we wanted.

To complete the proof of the theorem, [28] we will also have to show that the map $\Phi \circ \Psi(\mathscr{M}) \to \mathscr{M}$ from Exercise 5.4.7 is an isomorphism for any quasi-coherent module $\mathscr{M}$ on U. Recalling the definition of the map $\Phi \circ \Psi(\mathscr{M}) =$

[28] A simpler proof of the second half of the theorem is possible when U is generated by a faithfully flat collection of A-algebras that is finite in number. In that case, to demonstrate

$\Gamma(U,\mathscr{M})^\sim \to \mathscr{M}$ and the notion of isomorphism of quasi-coherent modules we deduce that it is sufficient to show that the maps

$$C \underset{A}{\otimes} \Gamma(U,\mathscr{M}) = \Gamma(U,\mathscr{M})^\sim_C \to \mathscr{M}_C \tag{8}$$ eqn:3

are isomorphisms for all $C \in U$.

As before, it will be sufficient by faithful flatness to demonstrate this assertion after tensoring with a faithfully flat collection of A-algebras. Since U contains a faithfully flat collection of A-algebras, it will be enough to show that *the maps*

$$B \underset{A}{\otimes} C \underset{A}{\otimes} \Gamma(U,\mathscr{M}) \to B \underset{A}{\otimes} \mathscr{M}_C$$

are isomorphisms for all $B \in U$ that are flat over A. If B is such an A-algebra, let us define $D = B \otimes_A C$ and write V for $f^{-1}U$ where $f : h^B \to h^A$ is the associated morphism of functors. Noting that

$$D \underset{A}{\otimes} \Gamma(U,\mathscr{M}) = D \underset{B}{\otimes} B \underset{A}{\otimes} \Gamma(U,\mathscr{M})$$
$$B \underset{A}{\otimes} \mathscr{M}_C = B \underset{A}{\otimes} C \underset{C}{\otimes} \mathscr{M}_C = D \underset{C}{\otimes} \mathscr{M}_C$$

we realize that our task is to show that the map

$$D \underset{B}{\otimes} B \underset{A}{\otimes} \Gamma(U,\mathscr{M}) \to D \underset{C}{\otimes} \mathscr{M}_C$$

is an isomorphism. But we have a commutative diagram

$$\begin{array}{ccc} D \otimes_B B \otimes_A \Gamma(U,\mathscr{M}) & \longrightarrow & D \otimes_C \mathscr{M}_C \\ \downarrow & & \downarrow \\ D \otimes_B \Gamma(V,\mathscr{M}) & \longrightarrow & \mathscr{M}_D \end{array}$$

The vertical arrows are isomorphisms: on the left, by Corollary 5.5.3.2, and on the right by the definition of quasi-coherence. But the lower horizontal arrow is an isomorphism by Lemma 5.4.9, since $V = h^B$. $\square$

---

that (8) is an isomorphism, we observe that since $C \in U$ there is an A-algebra map $B \to C$ where $B \in U$ is a flat A-algebra. But the map

$$B \underset{A}{\otimes} \Gamma(U,\mathscr{M}) \to \mathscr{M}_B$$

is an isomorphism by Corollary 5.5.3.4 so the upper horizontal arrow in the diagram below is an isomorphism:

$$\begin{array}{ccc} C \otimes_B B \otimes_A \Gamma(U,\mathscr{M}) & \longrightarrow & C \otimes_B \mathscr{M}_B \\ \downarrow & & \downarrow \\ C \otimes_A \Gamma(U,\mathscr{M}) & \longrightarrow & \mathscr{M}_C \end{array}$$

But the left vertical arrow is an isomorphism by standard properties of the tensor product and the right vertical arrow is an isomorphism by the definition of quasi-coherence. Therefore the lower horizontal arrow is an isomorphism.

ex:qcoh-alg **Exercise 5.5.5.** Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor. A **quasi-coherent algebra** $\mathscr{A}$ on X consists of the following data:

(i) for each $\xi \in X(B)$ a B-algebra $\mathscr{A}_\xi$,

(ii) for each homomorphism of commutative rings $\varphi : B \to C$ and each $\xi \in X(B)$, an isomorphism of commutative rings $\mathscr{A}_\varphi : C \otimes_B \mathscr{A}_\xi \to \mathscr{A}_{\varphi_* \xi}$

such that $\mathscr{A}_\varphi \circ \mathscr{A}_\psi = \mathscr{A}_{\varphi \circ \psi}$ when the composition makes sense.

(i) Let $U \to h^B$ be a morphism of functors and A a B-algebra. Construct a quasi-coherent algebra on U whose underlying quasi-coherent module is $\widetilde{A}$.

(ii) With the situation being as in the last part, show that for any quasi-coherent algebra $\mathscr{A}$ on U, the B-module $\Gamma(U, \mathscr{A})$ has the structure of a B-algebra.

(iii) Prove, using faithfully flat descent that if $X = h^B$ and $U \subset X$ is generated by a finite, faithfully flat collection of A-algebras then the functors $\mathscr{A} \mapsto \Gamma(U, \mathscr{A})$ and $A \mapsto \widetilde{A}$ are inverse equivalences between the categories of quasi-coherent algebras on U and B-algebras.

ex:desc-submod **Exercise 5.5.6** (Descent for submodules)**.** Let M an A-module. For each A-algebra B let $F(B)$ be the set of sub-modules of $B \otimes_A M$. If $\varphi : B \to C$ is an A-algebra homomorphism, let $F(\varphi) : F(B) \to F(C)$ be the function that sends $N \subset B \otimes_A M$ to the image of $C \otimes_A N \to C \otimes_B B \otimes_A M \cong C \otimes_A M$.

(i) Show that F is a covariant functor from A-Alg to Sets.

(ii) Suppose that B is a faithfully flat A-algebra. Show that $F(A) \to F(B)$ is *injective*.

(iii) Suppose that B is a faithfully flat A-algebra and $N \subset B \otimes_A M$ is a submodule such that $F(v_0)(N) = F(v_1)(N)$, where $v_i : B \to B \otimes_A B$ are the two canonical maps. Show that $N = F(L)$ for some A-submodule L of M.

5.6 Galois theory

Let L be a finite dimensional Galois extension of a field K. Define $X = h^K$ and let $U \subset X$ be the image of $h^L \to h^K$. We would like to understand quasi-coherent modules on U in a more concrete way. This will eventually lead to the fundamental theorem of Galois theory.

If $\mathscr{M}$ is a quasi-coherent module on U then we can obtain an L-vector space by evaluating $\mathscr{M}(L)$. Let us call this vector space M.

Each element of the Galois group $G = \mathrm{Gal}(L/K)$ is a K-algebra map $g : L \to L$ and therefore gives a function $\mathscr{M}(g) : \mathscr{M}(L) \to \mathscr{M}(L)$. It is important to understand this this function is *not* L-linear. In fact, the condition that $\mathscr{M}(g)$ satisfies is the following

$$\mathscr{M}_g(\lambda x) = \lambda^g \mathscr{M}_g(x)$$

where we have written $\mathscr{M}_g$ in place of $\mathscr{M}(g)$ to make the equation more readable. We also have the compatibility of $\mathscr{M}$ with compositions of K-algebra homomorphisms. That is, $\mathscr{M}_g \circ \mathscr{M}_h = \mathscr{M}_{gh}$.

Let $\mathscr{C}$ be the category of pairs (M, φ) where M is an L-vector space and φ is a system of maps $\varphi_g : M \to M$, one for each $g \in \mathrm{Gal}(L/K)$, satisfying

(i) $\varphi_{gh} = \varphi_g \varphi_h$ for all $g, h \in G$, and

(ii) $\varphi_g(\lambda x) = \lambda^g \varphi_g(x)$.

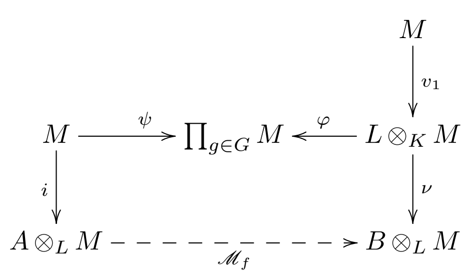
A morphism from (M, φ) to (N, ψ) is a function $f : M \to N$ such that $\varphi_g(f(x)) = f(\varphi_g(x))$ for all $x \in M$.

Exercise 5.6.1. Verify that the discussion above gives a functor $\Psi : \mathsf{QCoh}(U) \to \mathscr{C}$.

We will now construct a functor $\mathscr{C} \to \mathsf{QCoh}(U)$ that we will eventually see is inverse to the one above. Given $(M, \varphi) \in \mathscr{C}$ we would like to obtain a quasi-coherent module $\mathscr{M} \in \mathsf{QCoh}(U)$ inducing (M, φ). If A is a K-algebra that is in U then there is a K-algebra homomorphism $L \to A$. Of course, this homomorphism is not unique, so we will have to choose one, which we denote ξ_A. Now define $\mathscr{M}_A = A \otimes_L M$ with ξ_A being the map $L \to A$ used in the definition.

We must also give a definition of $\mathscr{M}_f : \mathscr{M}_A \to \mathscr{M}_B$ for any K-algebra homomorphism $f : A \to B$. Here we run into a problem, as there is no guarantee that $f \circ \xi_A = \xi_B$. Indeed, without the maps φ_g there would be no way to construct $\mathscr{M}_f$.

We construct $\mathscr{M}_f$ using a commutative diagram



(9) eqn:5

where the labelled morphisms are defined in the following ways:

$$v_1(x) = 1 \otimes x$$
$$\nu(\lambda \otimes x) = f \circ \xi_A(\lambda) \otimes x$$
$$\varphi(\lambda \otimes x) = (\lambda^g x)_{g \in G}$$
$$\psi(\lambda x) = (\lambda^g \varphi_g(x))_{g \in G}$$
$$i(x) = 1 \otimes x$$

Exercise 5.6.2. Show that φ is an isomorphism.

Lemma 5.6.3. *The map $\varphi^{-1} \circ \psi$ in diagram (9) satisfies*

$$\varphi^{-1} \circ \psi(\lambda x) = (\lambda \otimes 1)\varphi^{-1} \circ \psi(x)$$

Proof. Fix some $h \in G$ and let $(x_g)_{g \in G}$ be an element of $\prod_{g \in G} M$ such that $x_g = 0$ for $g \neq h$. Then there is some $y \in L \otimes_K M$ with $\varphi(y) = x$. Thus $\varphi((\lambda \otimes 1)y) = \lambda^h x$. We conclude that $\varphi^{-1}(\lambda x) = \lambda^{g^{-1}} y$. Moreover, if we write a general element $x \in \prod_{g \in G} M$ as

$$x = \sum_{g \in G} x_g e_g$$

then we have

$$\varphi^{-1}(\lambda x) = \sum_{g \in G} \lambda^{g^{-1}} \varphi^{-1}(x_g e_g).$$

Thus

$$\varphi^{-1}\psi(\lambda x) = \varphi^{-1}(\sum_{g \in G} \lambda^g \varphi_g(x)) = \sum_{g \in G} (\lambda^g)^{g^{-1}} \varphi^{-1}(x_g e_g) = \lambda \sum_{g \in G} \varphi^{-1}(x_g e_g)$$

$$\lambda \varphi^{-1}\psi(x) = \lambda \sum_{g \in G} \varphi^{-1}(x_g e_g)$$

□

Using the lemma, we therefore have

$$\nu \circ \varphi^{-1} \circ \psi(\lambda x) = f \circ \xi_A(\lambda)\nu \circ \varphi^{-1} \circ \psi(x).$$

Therefore the map

$$A \times M \to B \times M : (a, x) \mapsto f(a) \otimes \nu \circ \varphi^{-1} \circ \psi(x)$$

is L-bilinear (with respect to the map $\xi_A : L \to A$). It follows by the universal property of the tensor product that there is therefore a map

$$\mathscr{M}_f : A \underset{L}{\otimes} M \to B \underset{L}{\otimes} M : a \otimes x \mapsto f(a) \otimes \nu \circ \varphi^{-1} \circ \psi(x)$$

Exercise 5.6.4. (i) Verify that $\mathscr{M}_f \circ \mathscr{M}_{f'} = \mathscr{M}_{f \circ f'}$ when the composition makes sense.

(ii) Verify that this gives a functor $\Phi : \mathscr{C} \to \mathsf{QCoh}(U)$.

(iii) Check that the two functors $\Phi : \mathscr{C} \to \mathsf{QCoh}(U)$ and $\Psi : \mathsf{QCoh}(U) \to \mathscr{C}$ defined above are inverse to one another.

Exercise 5.6.5. Deduce, using faithfully flat descent, that the following categories are equivalent:

(i) K-Mod and $\mathscr{C}$;

(ii) K-Alg and the category of pairs (A, φ) where A is an L-algebra and φ is an action of $\mathrm{Gal}(L/K)$ on A compatible with the action on L.[29]

We refer to the objects of $\mathscr{C}$ as quasi-coherent Galois modules and the pairs (A, φ) in the second part as quasi-coherent Galois algebras.

Exercise 5.6.6. (i) Let $K = \mathbf{R}$ and $L = \mathbf{C}$. Find the $\mathbf{R}$-algebra corresponding to $\mathbf{C} \times \mathbf{C}$ with the nontrivial element $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{R}) \cong \mathbf{Z}/2\mathbf{Z}$ acting by $(x, y)^\sigma = (x^\sigma, y^\sigma)$.

(ii) Let $K = \mathbf{R}$ and $L = \mathbf{C}$. Find the $\mathbf{R}$-algebra corresponding to $\mathbf{C} \times \mathbf{C}$ with the nontrivial element $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{R}) \cong \mathbf{Z}/2\mathbf{Z}$ acting by $(x, y)^\sigma = (y^\sigma, x^\sigma)$.

(iii) Find the Galois algebra corresponding to the K-algebra K.

(iv) Find the Galois algebra corresponding to the K-algebra L.

(v) Find the K-algebra corresponding to the Galois algebra $L \times L$ with $\mathrm{Gal}(L/K)$ acting individually on the two factors.

(Hint: For the latter three parts of the exercise, you can characterize the algebras in question in the categories where they are defined by universal properties; then find the objects satisfying the corresponding universal properties in the other category.)

ex:split-over-L

Exercise 5.6.7. Let $\mathscr{D}$ be the category of K-algebras A such that $L \otimes_K A$ is isomorphic as an L-algebra to a product of copies of L. Let $\mathscr{C}_0$ be the category of quasi-coherent Galois algebras (B, φ) in $\mathscr{C}$ such that B is isomorphic to a product of copies of L. Show that $\mathscr{C}_0$ and $\mathscr{D}$ are equivalent categories.

ex:grothendieck-galois

Exercise 5.6.8. Let $\mathscr{C}_0$ be the category from Exercise 5.6.7. Let $\mathscr{B}$ be the category of right $\mathrm{Gal}(L/K)$-sets.

(i) Show that $A \mapsto \mathrm{Spec}\, A$ determines a functor $\Xi : \mathscr{C}_0^\circ \to \mathscr{B}$.

(ii) Show that Ξ is an equivalence of categories.

(iii) Deduce that the category of finite dimensional K-algebras that split over L is (contravariantly) equivalent to the category of finite $\mathrm{Gal}(L/K)$-sets. This is Grothendieck's formulation of the fundamental theorem of Galois theory.

ex:galois

Exercise 5.6.9. Here we will deduce the usual formulation of Galois theory from Grothendieck's formulation (Exercise 5.6.8), as well as a few other things. Let $G = \mathrm{Gal}(L/K)$.

(i) Show that the G-set corresponding to the K-algebra K is the trivial action of G on a set with one element.

[29] This means that φ consists of homomorphisms $\varphi_g : A \to A$ of commutative rings such that $\varphi_g(\lambda x) = \lambda^g \varphi_g(x)$ for all $\lambda \in L$ and $x \in A$.

(ii) Show that the G-set corresponding to the K-algebra L is the action of G on itself.

(iii) Show that products of K-algebras correspond to disjoint unions of G-sets.

(iv) Show that tensor products of K-algebras correspond to products of G-sets.

(v) Show that if a K-algebra A corresponds to a G-set X then sub-algebras of A correspond to surjections of G-sets $X \to Y$.

(vi) Show that surjections of G-sets from the G-set G correspond to subgroups of G. (This is just a group theory question.) The G-set corresponding to a subgroup H is the set of right cosets of H in G.

(vii) Show that the surjection of G-sets $G \to H\backslash G$ (the set of right cosets of H in G) corresponds to the K algebra L^H. (Hint: consider the universal properties of $H\backslash G$ and L^H.)

(viii) Deduce that the map sending a subgroup H of G to L^H determines a bijection between subgroups of G and subfields of L containing K.

5.7 Faithfully flat descent (general case)

def:fpqc-cover

Definition 5.7.1. Let $X = h^A$ for a commutative ring A and let $U \subset X$ be a subfunctor. We say that U is **faithfully flat** if U contains a faithfully flat collection of A-algebras. We say that U is **quasi-compact** if it is generated by a finite collection of A-algebras. If both conditions hold we say that U is an **fpqc cover** of X.[30]

For a general functor $X : \mathsf{ComRng} \to \mathsf{Sets}$ we say that $U \subset X$ is an fpqc cover if, for each $\xi : h^A \to X$, the pre-image $\xi^{-1}U \subset h^A$ is an fpqc cover.

ex:fpqc-examples

Exercise 5.7.2. (i) Let A be a commutative ring and $f_1, \ldots, f_n$ elements of A that generate A as an ideal. Show that $D(f_1) \cup \cdots \cup D(f_n) \subset h^A$ is an fpqc cover.

(ii) Let A be a commutative ring and $\mathfrak{p} \subset A$ a prime ideal generated by $f_1, \ldots, f_n$. Let $U \subset h^A$ be the union of $h^{A_\mathfrak{p}}$ and $D(f_1) \cup \cdots \cup D(f_n)$. Show that U is an fpqc cover.

(iii) Let A be a commutative ring, L an invertible A-module, and $f_0, \ldots, f_n \in L$ elements that generate L as a module. Define $U_i(B) \subset h^A(B)$ to be the set of all $\varphi : A \to B$ such that $1 \otimes \varphi(f_i)$ generates $B \otimes_A L$ as a B-module. Show that $\bigcup_{i=0}^n U_i$ is an fpqc cover of h^A.

(iv) Let $U_i \subset \mathbf{P}^n$ be the collection of all $(L, (x_0, \ldots, x_n)) \in \mathbf{P}^n$ such that x_i generates L. Deduce from the last part that $\bigcup_{i=0}^n U_i \subset \mathbf{P}^n$ is an fpqc cover.

[30]The origin of the letters "fpqc" is the French term "fidelèment plat et quasi-compacte", or "faithfully flat and quasi-compact". The latter term, "quasi-compact", refers to the finiteness of the family of B_i.

(v) Let K be a field and $U \subset h^K$ the image of any map $h^A \to h^K$ where A is a non-zero K-algebra. Show that U is an fpqc cover of h^K.

`ex:fpqc-cover-pullback`

Exercise 5.7.3 (Important)**.** Let $f : X \to Y$ be a morphism of functors and assume that $U \subset Y$ is an fpqc cover. Show that $f^{-1}U \subset X$ is also an fpqc cover.

`def:qcoh-mod`

Definition 5.7.4. Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor. A collection of data satisfying **QCOH1**, **QCOH2**, and **QCOH3** is called a **module** on X.[31] If it also satisfies **QCOH4** then it is called a **quasi-coherent module**.

5.8 Gluing maps

Definition 5.8.1. We will say that a functor $F : \mathsf{ComRng} \to \mathsf{Sets}$ **satisfies fpqc descent** if, whenever $U \subset h^A$ is an fpqc cover, a map $U \to F$ extends uniquely to $h^A \to F$.

`ex:fpqc-descent-product`

Exercise 5.8.2. Suppose that X and Y are functors satisfying fpqc descent. Show that $X \times Y$ satisfies fpqc descent.

5.8.1 Descent for morphisms to representable functors

Let $U \subset h^A$ be an fpqc cover and suppose we are given a map $U \to h^B$ for some commutative ring B. That is, for every A-algebra $C \in U$ we have given a map of commutative rings $B \to C$. Let us define $\mathscr{B}$ to be the quasi-coherent module $(A \otimes B)^\sim$ on U and let us write $\mathscr{A} = \widetilde{A}$. Then we get a morphism of quasi-coherent algebras $\mathscr{B} \to \mathscr{A}$ (where the map $\mathscr{B}_C \to \mathscr{A}_C$ is the given map $C \otimes B \to C$). By faithfully flat descent for quasi-coherent algebras, this map must come from a map of A-modules $A \otimes B \to A$, which, by the universal property of the tensor product, is the same as a map $B \to A$.

`thm:fpqc-descent-rep`

Theorem 5.8.3. *Let B be a commutative ring. Then h^B satisfies fpqc descent.*

`ex:fpqc-descent-V`

Exercise 5.8.4 (This should be easy)**.** Suppose that B is a commutative ring and $I \subset B$ is an ideal. Show that $V(I)$ satisfies fpqc descent.

5.8.2 Descent for morphisms to projective space

Now let us suppose that $U \subset h^A$ is an fpqc cover as before and we are given a morphism $U \to \mathbf{P}^n$ for some n. This means that we are to give an object of $\mathbf{P}^n(C)$ for every $C \in U$, which amounts to giving an invertible C-module $\mathscr{L}_C$ and a surjection of C-modules $p_C : C^{n+1} \to \mathscr{L}_C$.

By definition, if $\varphi : C \to D$ is a morphism in U then we have $\varphi_*(\mathscr{L}_C, p_C) = (D \otimes_C \mathscr{L}_C, \mathrm{id}_D \otimes_C p_C)$, which means that $\mathscr{L}_C$ is a quasi-coherent module on U. By faithfully flat descent for quasi-coherent modules, there is an A-module L and an isomorphism $\widetilde{L} \cong \mathscr{L}$.

[31] This might also be called a presheaf of $\mathcal{O}_X$-modules elsewhere.

Now, notice that the maps $p_C : C^{n+1} \to \mathscr{L}$ determine a morphism of quasi-coherent modules $\widetilde{A}^{n+1} \to \mathscr{L}$. Composing with the isomorphism $\mathscr{L} \cong \widetilde{L}$ we get a morphism of quasi-coherent modules $\widetilde{A}^{n+1} \to \widetilde{L}$. By faithfully flat descent, this must be induced from a morphism of A-modules $p : A^{n+1} \to L$. We therefore obtain an object $(L, p) \in \mathbf{P}^n(A)$.

thm:fpqc-descent-Pn

Theorem 5.8.5. *If $U \subset h^A$ is an fpqc cover then a morphism $U \to \mathbf{P}^n$ extends uniquely to $h^A \to \mathbf{P}^n$.*

Exercise 5.8.6. Complete the sketch of the proof of Theorem 5.8.5.

Exercise 5.8.7. Let A be a commutative ring. Recall that $\mathbf{P}^n_A(B)$ is the collection of triples (L, p, φ) where $(L, p) \in \mathbf{P}^n(B)$ and $\varphi \in h^A(B)$. (Thus $\mathbf{P}^n_A = \mathbf{P}^n \times h^A$.) Show that $\mathbf{P}^n_A$ satisfies fpqc descent.

x:fpqc-descent-proj-closed

Exercise 5.8.8. Let I be a collection of homogeneous polynomials in the variables $t_0, \ldots, t_n$ with coefficients in a commutative ring A. Define $X = V(I) \subset \mathbf{P}^n_A$. Show that X satisfies fpqc descent.

Exercise 5.8.9.

$$\begin{array}{llll} A = \mathbf{Z}[\sqrt{-5}] & B = A\left[\frac{1}{2}\right] & C = A\left[\frac{1}{3}\right] & D = A\left[\frac{1}{6}\right] \\ X = h^A & U = h^B & V = h^C & W = h^D \end{array}$$

Let $Y \subset X$ be the subfunctor consisting of all A-algebras that admit a map from B or from C. In other words, $Y = U \cup V$.

(i) Show that Y is an fpqc cover of X.

For each $E \in Y$, define an element of $\mathbf{P}^1(E)$: if $E \in U$, take $f_E = (E, (2, 1 + \sqrt{-5}))$; if $E \in V$, take $f_E = (E, (1 - \sqrt{-5}, 3))$.

(ii) Verify f is well-defined. (Check, in other words, that if $E \in U$ and $E \in V$ then the two definitions above coincide.)

(iii) Show that this defines a morphism $f : Y \to \mathbf{P}^1$. (This requires verifying that the definition of f is natural.)

(iv) Conclude using Theorem 5.8.5 that there is a unique extension of f to a map $X \to \mathbf{P}^1$.

(v) What $(L, p) \in \mathbf{P}^1(A)$ restricts to f on U?

ex:projection-by-descent

Exercise 5.8.10. Consider $X = V(y^2 z - x^3 + xz^2) \subset \mathbf{P}^2$. In this exercise we will construct a morphism $X \to \mathbf{P}^1$.

Let $(L, (x, y, z))$ be an element of $X(A)$ for some commutative ring A. Let $U \subset h^A$ be the collection of all $\varphi : A \to B$ such that one of the maps $\mathrm{id}_B \otimes_A y$ or $\mathrm{id}_B \otimes_A z$ is an isomorphism.

(i) Show that if $\mathrm{id}_B \otimes_A y$ is an isomorphism if and only if

$$\varphi_*(L,(x,y,z)) = (B,((\mathrm{id}_B \otimes y)^{-1}(\mathrm{id}_B \otimes x), 1, (\mathrm{id}_B \otimes y)^{-1}(\mathrm{id}_B \otimes z)))$$

as an object of $\mathbf{P}^1(B)$. Thus $\varphi_*(L,(x,y,z))$ is isomorphic to an object of the form $(B,(x',1,z'))$. Verify a similar remark if $\mathrm{id}_B \otimes_A z$ is an isomorphism.

(ii) Show that U is an fpqc cover of h^A. (Hint: To show that U is quasi-compact, we have to show that U contains a finite, faithfully flat collection of A-algebras. By definition, there is a finite collection of elements $f_i \in A$ such that the f_i generate A as an ideal and, for each i, there is an isomorphism $\varphi_i : A[f_i^{-1}] \otimes_A L \to A[f_i^{-1}]$. Then U contains the A-algebras $A[f_i^{-1}, \varphi_i(y)^{-1}]$ and $A[f_i^{-1}, \varphi_i(z)^{-1}]$ for all i. Show that these are a finite, faithfully flat collection of A-algebras.)

(iii) Show that the following definitions are well-posed:

(a) If $(C,(x,y,1)) \in U(C)$, define $f(C,(x,y,1)) = (C,(x,1))$.

(b) If $(C,(x,1,z)) \in U(C)$, define $f(C,(x,1,z)) = (C,(1,x^2-z^2))$.

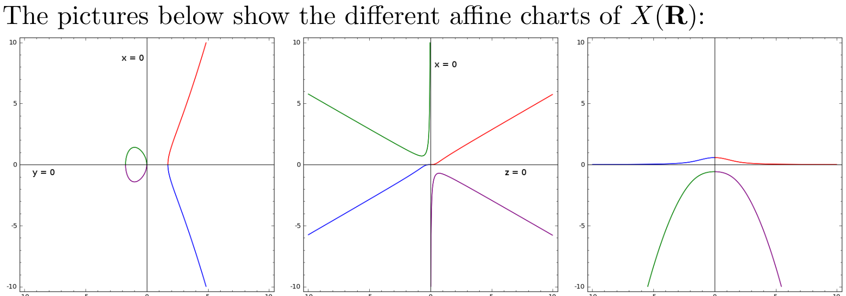
What you must show here is that if $(L,(x,y,z))$ can be writen in two of the above forms then the definition of $f(L,(x,y,z))$ does not depend on which form is chosen. (Hint: Note that if both y and z generate L, so does x.)

(iv) Show that this definition is natural in C and therefore defines a morphism $X \to \mathbf{P}^1$.

(v) Using Theorem 5.8.5, conclude that, for each C and each $(L,p) \in X(C)$, there is a unique map $h^C \to \mathbf{P}^1$ that restricts to f on U.

(vi) Let $\overline{f}(L_C, p_C)$ denote the object of $\mathbf{P}^1(C)$ associated to $(L,p) \in \mathbf{P}^1(C)$ in the last part. Show that $\overline{f}$ is natural in C and therefore defines a morphism $X \to \mathbf{P}^1$.

The pictures below show the different affine charts of $X(\mathbf{R})$:



The coloring indicates how the different pieces are glued.

6 An introduction to the Weil conjectures

6.1 References

The original article [Wei] where Weil posed his conjectures is fairly readable. If you're of a number theoretic bent, and you enjoy things like Gauss sums, you may want to check it out. Weil's argument is also given, in greater detail, in [IR, Chapter 11, §3].

6.2 Counting in categories

Suppose that we want to count the number of objects of a category. For example, suppose that X is a finite set and we want to count the number of injections $i : Y \to X$, where Y is allowed to be any set. On one hand, we know that the number of subsets of X is $2^{\#X}$; but the number of injections is clearly infinite![32] Indeed, there are infinitely many different sets with one element, and (at least provided X is non-empty) we can find an injection from each of these singleton sets into X.

You may be tempted to say, "Well, these are different counting problems. It's no surprise they have different answers." But consider the following categories: let $\mathscr{C}$ be the category whose objects are pairs (Y, i) where Y is a set and $i : Y \to X$ is an injection; a morphism from (Y, i) to (Z, j) is a function $f : Y \to Z$ such that $jf = i$. Define $\mathscr{C}'$ be the full subcategory of pairs (Y, i) in $\mathscr{C}$ where $i : Y \to X$ is the *inclusion of a subset.*

Exercise 6.2.1. (i) Show that $\mathscr{C}'$ has exactly $2^{\#X}$ objects.

(ii) Show that $\mathscr{C}$ has infinitely many objects.

(iii) Show that the inclusion functor $\mathscr{C}' \to \mathscr{C}$ is an equivalence of categories.

Solution. (i) In $\mathscr{C}'$, the objects (Y, i) are uniquely determined by the subset $Y \subseteq X$, of which there are $2^{\#X}$.

(ii) Let $\{a\} = Y_a$ be any one element set, x a specified element of X, and define the maps $i_a : Y_a \to X$ by $a \mapsto x$. We have (Y_a, i_a) are objects in $\mathscr{C}$. Since there are infinitely many choices for elements a, there are infinitely many objects in $\mathscr{C}$.

(iii) Let $T : \mathscr{C}' \to \mathscr{C}$ be the inclusion functor. Define $S : \mathscr{C} \to \mathscr{C}'$ to be the functor taking a pair (Y, i) to $(i(Y), j)$ where j is the inclusion map of $i(Y)$ into X, and taking morphisms $f : (Y, i_Y) \to (Z, i_Z)$ to the inclusion map $i_Y(Y) \hookrightarrow i_Z(Z)$.

For $Y \subseteq X$ and inclusion map $j_Y : Y \hookrightarrow X$,

$$ST(Y, j_Y) = S(Y, j_Y) = (j(Y), j_Y) = (Y, j_Y),$$

hence $ST = I_{\mathscr{C}'}$ is the identity functor on $\mathscr{C}'$. On the other hand, for set Y and inclusion $i_Y : Y \to X$,

$$TS(Y, i_Y) = T\left(i_Y(Y), j_{i_Y(Y)}\right) = \left(i_Y(Y), j_{i_Y(Y)}\right)$$

[32] if X in non-empty, anyway

where $j_{i_Y(Y)}$ is the inclusion $i_Y(Y) \hookrightarrow X$. To show there is a natural transformation from the identity on $\mathscr{C}$ to ST, one must show that for any morphism of objects $f : (Y, i_Y) \to (Z, i_Z)$, the diagram

$$\begin{array}{ccc} (Y, i_Y) & \xrightarrow{i_Y} & (i_Y(Y), j_{i_Y(Y)}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j_{YZ}} \\ (Z, i_Z) & \xrightarrow{i_Z} & (i_Z(Z), j_{i_Z(Z)}) \end{array}$$

commutes, where j_{YZ} is the inclusion $i(Y) \hookrightarrow i(Z)$. It suffices to show that $i_Z \circ f = j_{YZ} \circ i_Y$ as set maps. But if y is an element of Y, then

$$i_Z \circ f(y) = i_Y(y) = j_{YZ} \circ i_Y(y)$$

where the first equality holds because of the definition of a morphism in $\mathscr{C}$ and the second equality holds because j_{YZ} is an inclusion of sets. Hence there is a natural transformation from $I_{\mathscr{C}} \to TS$.

Furthermore, the map $i_Y : (Y, i_Y) \to (i_Y(Y), j_{i_Y(Y)})$ is an isomorphism. The set map i_Y is injective, hence invertible on its image, giving a map $i_Y^{-1} : i_Y(Y) \to Y$ where $i_Y \circ i_Y^{-1}$ is exactly inclusion of $i(Y)$ into X, so i_Y^{-1} induces a map on pairs $(i_Y(Y), j_{i_Y(Y)}) \hookrightarrow (Y, i_Y)$. Since the maps on sets $i_Y \circ i_Y^{-1}$ and $i_Y^{-1} \circ i_Y$ are both identity maps, they induce identity maps on pairs, so the maps induced by i_Y and i_Y^{-1} on pairs are inverses. Hence the natural transformation from $I_{\mathscr{C}} \to TS$ is actually a natural isomorphism and T is an equivalence of categories. □

The above example shows that counting objects in categories is not a good idea: it doesn't even give the same number for equivalent categories!

Exercise 6.2.2. Let $\mathscr{C}$ be the category with exactly one object and one morphism. For each positive integer n, find a category equivalent to $\mathscr{C}$ having n objects.

Exercise 6.2.3. Let $\mathscr{C}$ be the category of n-element sets, where n is a fixed positive integer. Let $\mathscr{C}'$ be the category with a single object having automorphism group the symmetric group on n letters.

1. Show that $\mathscr{C}$ and $\mathscr{C}'$ are equivalent categories.

Let $\mathscr{D}$ be the category of *ordered* n-element sets and let $\mathscr{D}'$ be the category with a single object and a single morphism.

2. Show that $\mathscr{D}$ and $\mathscr{D}'$ are equivalent categories and deduce that $\#\mathscr{D} = 1$.

Let $F : \mathscr{D} \to \mathscr{C}$ be the functor that sends an ordered set to its underlying set.

3. Remark that the number of orderings of a fixed finite set is $n!$. Conclude that $\#\mathscr{D}$ should be equal to $n!\#\mathscr{C}$.

4. Conclude that the size of $\#\mathscr{C}$ must be $\frac{1}{n!}$.

We may formalize the above discussion as follows. We wish to define a function

$$\# : \{\text{groupoids}\} \to \mathbf{Z}_{\geq 0} \cup \{\infty\}$$

with the following properties:

1. If $\mathscr{C} \simeq \mathscr{D}$ then $\#\mathscr{C} = \#\mathscr{D}$.

2. $\#(\mathscr{C} \amalg \mathscr{D}) = \#\mathscr{C} + \#\mathscr{D}$.

3. If $\mathscr{C}$ is the category with one object and one morphism then $\#\mathscr{C} = 1$.

4. Suppose that there is a functor $\mathscr{C} \to \mathscr{D}$ such that every object and every morphism of $\mathscr{D}$ has exactly n pre-images. Then $\#\mathscr{C} = n\#\mathscr{D}$.

Proposition 6.2.4. *The only function # as above is defined as follows: for each isomorphism class c of $\mathscr{C}$, choose an object X_c in c. Then*

$$\#\mathscr{C} = \sum_c \frac{1}{\# \operatorname{Aut} X_c}$$

where $\# \operatorname{Aut} X_c$ is the size of the automorphism group of X_c.

Exercise 6.2.5. Prove the proposition.

Exercise 6.2.6. How many finite sets are there?

Exercise 6.2.7. Let X be a finite set and define $\mathscr{C}$ to be the category whose objects are pairs (Y, f) where Y is a finite set and $f : Y \to X$ is a function. A morphism $(Y, f) \to (Z, g)$ is a function $h : Y \to Z$ such that $gh = f$. What is the size of $\mathscr{C}$?

6.3 Background definitions

Recall that a field k is said to be of **characterstic** p if the kernel of the canonical homomorphism $\mathbf{Z} \to k$ is (p).[33]

A commutative ring is said to be **reduced** if it has no nilpotent elements.

6.4 The ζ function

Exercise 6.4.1. (i) Let R be a *finite* reduced commutative ring. Show that R is a product of finite fields. (Hint: show that the intersection of any two distinct maximal ideals is the zero ideal and use the Chinese remainder theorem.)

[33] Note that this terminology should generally not be applied when k is a commutative ring that is not a field. For example, it is more appropriate to say $\mathbf{Z}$ is *of mixed characteristic* than to say it has characteristic zero.

(ii) Let R be a finite reduced k-algebra, where k is a field with q elements. Show that R is isomorphic to a finite product $\prod_i \mathbf{F}_{q^{\nu_i}}$ with the ν_i being integers ≥ 1.

(iii) Compute the automorphism group of the ring $\prod_{i=1}^{n} \mathbf{F}_{q^{\nu_i}}$. Note that your answer will depend on the ν_i.

Fix a finite field k, let I be an ideal in $k[t_1, \ldots, t_n]$. Let $A = k[t_1, \ldots, t_n]/I$. For each non-negative integer ν, let $\mathscr{C}(X, \nu)$ be the category of all *finite, reduced, commutative A-algebras B* such that $\dim_k B = \nu$. Morphisms in $\mathscr{C}(X, \nu)$ are isomorphisms of A-algebras.

Example 6.4.2. Let $k = \mathbf{F}_q$ and let $A = k$. Then, up to isomorphism,

(i) the only element of $\mathscr{C}(X, 0)$ is the zero ring,

(ii) the only element of $\mathscr{C}(X, 1)$ is $\mathbf{F}_q$,

(iii) $\mathscr{C}(X, 2)$ consists of $\mathbf{F}_{q^2}$ and $\mathbf{F}_q \times \mathbf{F}_q$,

(iv) $\mathscr{C}(X, 3)$ consists of $\mathbf{F}_{q^3}$, $\mathbf{F}_{q^2} \times \mathbf{F}_q$, and $\mathbf{F}_q \times \mathbf{F}_q \times \mathbf{F}_q$,

(v) $\mathscr{C}(X, 4)$ consists of $\mathbf{F}_{q^4}$, $\mathbf{F}_{q^3} \times \mathbf{F}_q$, $\mathbf{F}_{q^2} \times \mathbf{F}_{q^2}$, $\mathbf{F}_{q^2} \times \mathbf{F}_q \times \mathbf{F}_q$, and $\mathbf{F}_q \times \mathbf{F}_q \times \mathbf{F}_q \times \mathbf{F}_q$.

We define

$$Z(X, t) = \sum_{\nu=0}^{\infty} \#\mathscr{C}(X, \nu) t^{\nu}.$$

Example 6.4.3. Let $A = k = \mathbf{F}_q$. Then

$$\begin{aligned}
\#\mathscr{C}(X, 0) &= 1 \\
\#\mathscr{C}(X, 1) &= 1 \\
\#\mathscr{C}(X, 2) &= \frac{1}{\mathrm{Aut}_{\mathbf{F}_q} \mathbf{F}_{q^2}} + \frac{1}{\mathrm{Aut}_{\mathbf{F}_q}(\mathbf{F}_q \times \mathbf{F}_q)} = \frac{1}{2} + \frac{1}{2} = 1 \\
\#\mathscr{C}(X, 3) &= \frac{1}{\mathrm{Aut}_{\mathbf{F}_q} \mathbf{F}_{q^3}} + \frac{1}{\mathrm{Aut}_{\mathbf{F}_q}(\mathbf{F}_{q^2} \times \mathbf{F}_q)} + \frac{1}{\mathrm{Aut}_{\mathbf{F}_q}(\mathbf{F}_q \times \mathbf{F}_q \times \mathbf{F}_q)} \\
&= \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1 \\
\#\mathscr{C}(X, 4) &= \frac{1}{\mathrm{Aut}_{\mathbf{F}_q} \mathbf{F}_{q^4}} + \frac{1}{\mathrm{Aut}_{\mathbf{F}_q}(\mathbf{F}_{q^3} \times \mathbf{F}_q)} + \frac{1}{\mathrm{Aut}_{\mathbf{F}_q}(\mathbf{F}_{q^2} \times \mathbf{F}_{q^2})} \\
&\quad + \frac{1}{\mathrm{Aut}_{\mathbf{F}_q}(\mathbf{F}_{q^2} \times \mathbf{F}_q \times \mathbf{F}_q)} + \frac{1}{\mathrm{Aut}_{\mathbf{F}_q} {\mathbf{F}_q}^4} \\
&= \frac{1}{4} + \frac{1}{3} + \frac{1}{8} + \frac{1}{4} + \frac{1}{24} = 1.
\end{aligned}$$

Thus we may expect that $Z(X, t) = 1 + t + t^2 + \cdots = \frac{1}{1-t}$. We will see in a moment that this is indeed the case.

Once again, let $X = V(I)$ and $A = k[t_1, \dots, t_n]/I$. For each non-negative integer ν, let $\mathscr{D}(X, \nu)$ be the category of finite, commutative A-algebras B such that (1) B is a field,[34] and (2) $\dim_k B = \nu$. Note that $\mathscr{D}(X, \nu)$ is a full subcategory of $\mathscr{C}(X, \nu)$. Let

$$W(X,t) = \sum_{\nu=0}^{\infty} \#\mathscr{D}(X,\nu) t^{\nu}.$$

Exercise 6.4.4. Prove that

$$Z(X,t) = \exp W(X,t).$$

Assume $k = \mathbf{F}_q$. For each positive integer ν, let $N(X, \nu) = \#X(\mathbf{F}_{q^\nu})$ be the number of solutions to the equations defining X in $\mathbf{F}_{q^\nu}$. Define

$$V(X,t) = \sum_{\nu=1}^{\infty} N(X,\nu) t^{\nu-1}.$$

Exercise 6.4.5. Prove that $V(X,t) = \frac{d}{dt} W(X,t)$.

Exercise 6.4.6. Let $A = k = \mathbf{F}_q$ and let $X(B) = \mathrm{Hom}_k(A, B)$. Compute $V(X,t)$, $W(X,t)$, and $Z(X,t)$.

Exercise 6.4.7. Let $k = \mathbf{F}_q$. Compute $V(\mathbf{A}^n_k, t)$, $W(\mathbf{A}^n_k, t)$, and $Z(\mathbf{A}^n_k, t)$.

Exercise 6.4.8. Let $k = \mathbf{F}_q$. Compute $V(\mathbf{P}^n_k, t)$, $W(\mathbf{P}^n_k, t)$, and $Z(\mathbf{P}^n_k, t)$.

Exercise 6.4.9 (Just to make sure you understand the definitions)**.** Show that if A and B are commutative k-algebras, where k is a finite field with q elements, and $X(C) = \mathrm{Hom}_k(A, C)$ and $Y(C) = \mathrm{Hom}_k(B, C)$, then $Z(X,t) = Z(Y,t)$.

Exercise 6.4.10. Let k be a finite field with q elements and let $f \in k[x,y]$ be the polynomial $y^2 - x^3 - 1$. Let $X = V(f)$.

1. Show that if k has characteristic 2 or 3 then $Z(X,t) = Z(\mathbf{A}^1, t)$.

2. Use a computer to compute the first few terms[35] of $Z(X,t)/Z(\mathbf{A}^1,t)$ with $k = \mathbf{F}_5$. Use this to conjecture a general formula for $Z(X,t)$. Using your answer from above, conjecture a formula for $N(X,\nu)$ for all values of ν.

3. Repeat the above with $k = \mathbf{F}_q$ for various other values of q. You won't be able to compute as many terms, but you should still be able to observe a number of patterns. Record as many of these as you can discern.

[34]Note: the zero ring is *not* a field.
[35]I was able to compute up to 7 terms quickly using the CU sage server.

7 Sheaves

7.1 References

Your main reference here should be [Vak, Chapter 3]. The treatment there is just excellent. You may also want to look at [Har, II.1], which will be a suitable introduction for most of the applications in this class. However, if you subsequently do much with sheaves that aren't coherent, you will find Hartshorne's introduction lacking.

If you are particularly enthusiastic, you can try [Bre], [Ive], or [God] and [Gro1]. The latter paper is the one that gave sheaf cohomology the firm foundations it needed for algebraic geometry. However, the notation is a bit dated.

And if you want a really challenging reference that goes well beyond the scope of this course (in the direction of the Weil conjectures), consult [AGV].

7.2 The definition(s) of a sheaf

The espace étalé

`def:sheaf-etale`

Definition 7.2.1. Recall that a **local homeomorphism** of topological spaces is an open map $p : F \to X$ such that F has a cover by open subsets U with the property that $p\big|_U : U \to p(U)$ is a homeomorphism.

An **étale space** over a topological space X is a pair (F, p) where F is a topological space and $p : F \to X$ is a local homeomorphism. A morphism of étale spaces $(F, p) \to (G, q)$ is a morphism of topological spaces $u : F \to G$ such that $q \circ u = p$. As is customary, we frequently leave p tacit in the notation and refer to an étale space $F \to X$.

We denote by $\mathrm{Et}(X)$ the collection of all étale spaces over X and, for Y and Z in $\mathrm{Et}(X)$, we write $\mathrm{Hom}_{\mathrm{Et}}(Y, Z)$ for the set of maps of étale spaces from Y to Z.

The functorial definition (version 1)

Exercise 7.2.2. Let X be a topological space. Verify that with the definitions above, $\mathrm{Et}(X)$ is a category.

`def:sheaf-func`

Definition 7.2.3. A **presheaf** F on X is a contravariant functor from the category of open subsets of X to the category of sets. That is, a presheaf consists of

PSH1 for each open subset U of X, a set $F(U)$, and

PSH2 for each inclusion of open subsets $V \subset U$ of X, a function $\rho_{UV} : F(U) \to F(V)$ called restriction

such that

PSH3 if $W \subset V \subset U$ are open subsets of X then the restriction map $\rho_{UW} : F(U) \to F(W)$ agrees with the composition of restriction maps $F(U) \xrightarrow{\rho_{UV}} F(V) \xrightarrow{\rho_{VW}} F(W)$.

It is conventional to write $x\big|_V = \rho_{UV}(x)$ for $x \in F(U)$.

Suppose that F and G are presheaves on X. Define $\mathrm{Hom}_X(F, G)$ to be the set of natural transformations from F to G. That is,

PSH4 to give $\varphi \in \mathrm{Hom}_X(F, G)$ is to give $\varphi_U : F(U) \to G(U)$ for each open $U \subset X$, such that

PSH5 whenever $V \subset U$ are open subsets of X, the diagram

$$\begin{array}{ccc} F(U) & \xrightarrow{\varphi_U} & G(U) \\ {\scriptstyle \rho^F_{UV}}\downarrow & & \downarrow{\scriptstyle \rho^G_{UV}} \\ F(V) & \xrightarrow{\varphi_V} & G(V) \end{array}$$

is commutative.

A presheaf is called a **sheaf** if

`sheaf:1` **SH1** sections are determined locally: if $x, y \in F(U)$ and $x\big|_{U_i} = y\big|_{U_i}$ for all U_i in an open cover of U then $x = y$, and

`sheaf:2` **SH2** compatible local sections glue: if $x_i \in F(U_i)$ for all U_i in an open cover of U and $x_i\big|_{U_i \cap U_j} = x_j\big|_{U_i \cap U_j}$ for all i, j then there is an $x \in F(U)$ such that $x\big|_{U_i} = x_i$.

Given that the second definition is so much more complicated than the first, one may naturally wonder why one would ever want to use it. One reason is that the second turns out to be much more convenient to use in practice. For example, one is sometimes interested in sheaves of things that are not sets. It is unclear how to generalize Definition 7.2.1 to work in that setting. Even if one is interested in sheaves of things that have an underlying set—like abelian groups, or rings—it can be confusing to try to understand the algebraic information using that definition.

The second reason we prefer the second definition to the first is that there is an important generalization of the notion of a topological space, known as a **site** (or, really, a **topos**), in which the second definition becomes much simpler than the first. In fact, once one has enough categorical background to talk about topoi, it is possible to make Definition 7.2.3 as efficient as Definition 7.2.1.

Both points of view will be useful for us becuase different constructions are easier in one or in the other. For example, pushforward of sheaves is simple to define using Definition 7.2.3, but difficult with 7.2.1. Exactly the reverse is true for pullback.

The functorial definition (version 2)

`ex:presheaf-func` **Exercise 7.2.4.** Let $\mathrm{Open}(X)$ denote the category whose objects are the open subsets of X. We define

$$\mathrm{Hom}_{\mathrm{Open}(X)}(U, V) = \mathrm{Hom}_X(U, V).$$

That is $\mathrm{Hom}_{\mathrm{Open}(X)}(U,V)$ is empty unless $U \subset V$, in which case it consists of a single element.

Verify that the category of presheaves on X is the category of functors $\mathrm{Hom}(\mathrm{Open}(X)^\circ, \mathsf{Sets})$. (Check in other words, that the definition of a presheaf is exactly the same as the definition of a functor and that morphisms of presheaves are the same as natural transformations.)

`ex:rep-func-1`

Exercise 7.2.5. Let X be a topological space. For each $U \in \mathrm{Open}(X)$, we write h_U for the functor

$$h_U(V) = \mathrm{Hom}_{\mathrm{Open}(X)}(V,U)$$

(i) Show that h_U is a subfunctor of h_X.

Let $\mathcal{U}$ be an collection of open subsets of a topological space X. Construct a functor $\mathscr{U} : \mathrm{Open}(X)^\circ \to \mathsf{Sets}$ as follows: If $V \subset X$ is an open set and $V \subset U$ for some $U \in \mathcal{U}$ then $\mathscr{U}(V)$ consists of a single element; otherwise $\mathscr{U}(V)$ is empty.

(ii) Show that $\mathscr{U}$ is a subfunctor of h_X.

(iii) Show that $\mathscr{U} = \bigcup_{U\in\mathcal{U}} h_U$.

(iv) Show that $\mathscr{U}$ is a sheaf if and only if $\mathscr{U} = h_V$ for some open $V \subset X$.

If $\mathscr{U} \subset h_W$ we will say it is a **cover** if the collection of all $V \subset X$ with $\mathscr{U}(V) \neq \varnothing$ covers V.

Solution. (i) Let V be an open set in X. Then

$$h_U(V) = \mathrm{Hom}_{\mathrm{Open}(X)}(V,U) = \begin{cases} 1 & : \quad V \subset U \\ \emptyset & : \quad \text{otherwise} \end{cases},$$

where $1 = \{x\}$ is a fixed one element set. For each open V in X, define $h_U(V) \to h_X(V)$ to be either the unique map $\emptyset \to h_X(V)$, or if $h_U(V) = 1$, the map which takes the single element to the inclusion $V \hookrightarrow X$. Now suppose $W \subset V$ and consider the following diagram:

$$\begin{array}{ccc} h_U(V) & \longrightarrow & h_X(V) \\ \downarrow & & \downarrow \\ h_U(W) & \longrightarrow & h_X(W), \end{array}$$

where the horizontal arrows are the inclusion maps. If $h_U(V) = \emptyset$, then there is a unique map $h_U(V) \to h_X(W)$, whence the square is commutative. Suppose $h_U(V) = \{U \hookrightarrow X\}$. Then following the top and right arrows, we have

$$\begin{array}{cccc} x & \longmapsto & V \hookrightarrow X & \\ & & \downarrow & \\ & & W \hookrightarrow V \hookrightarrow X & = W \hookrightarrow X, \end{array}$$

while the left and bottom arrows yield

$$\begin{array}{ccc} x & & \\ \big\downarrow & & \\ W \hookrightarrow X & \longmapsto & W \hookrightarrow X, \end{array}$$

so the diagram is commutative in both cases and therefore $h_U \subset h_X$.

(ii) As before, for each open V in X, $\mathscr{U}(V)$ is either a one element set or the empty set. Thus, we can define the map $\mathscr{U}(V) \to h_X(V)$ to be either the unique map $\emptyset \to h_X(V)$, or if $\mathscr{U}(V) = 1$, the map which takes the single element to the inclusion $V \hookrightarrow X$. Then it follows by an identical argument to part (i) that $\mathscr{U}$ is a subfunctor of h_X.

(iii) Let V be an open set in X. First, suppose $V \not\subset U$ for all $U \in \mathcal{U}$. Then for all $U \in \mathcal{U}$, $h_U(V) = \emptyset$, so $\cup_{U \in \mathcal{U}} h_U(V) = \emptyset$. Conversely, if $V \subset U$ for some $U \in \mathcal{U}$, then $\cup_{U \in \mathcal{U}} h_U(V) = 1$. Thus in either case, it follows that there is a unique map $\mathscr{U}(V) \to \cup_{U \in \mathcal{U}} h_U(V)$ and this map is a bijection. Moreover, if $W \subset V$ and $\mathscr{U}(V) = 1$, then it is easy to see that $\mathscr{U}(V)$ (and hence $\cup_{U \in \mathscr{U}} h_U(V)$) is also equal to 1. Suppose $W \subset V$ and consider the following square:

$$\begin{array}{ccc} \mathscr{U}(V) & \longleftrightarrow & \bigcup_{U \in \mathcal{U}} h_U(V) \\ \big\downarrow & & \big\downarrow \\ \mathscr{U}(W) & \longleftrightarrow & \bigcup_{U \in \mathcal{U}} h_U(W). \end{array}$$

We are presented with two cases: If $\cup_{U \in \mathscr{U}} h_U(W) = 1$, then it is final and so the diagram is cummutative. If $\cup_{U \in \mathscr{U}} h_U(W) = \emptyset$, then by the previous paragraph, $\mathscr{U}(V) = \emptyset$, so $\mathscr{U}(V)$ is initial and therefore the diagram is commutative. Hence, the two functors are naturally isomorphic.

(iv) Suppose $\mathscr{U}$ is a sheaf, and let $V = \cup_{U \in \mathcal{U}} U$. Then we claim that $\mathscr{U} = h_V$. Note that for each $U, W \in \mathcal{U}$, $\mathscr{U}(U) = \mathscr{U}(W) = \mathscr{U}(U \cap W) = 1$. For each $U \in \mathcal{U}$, let x_U denote the element of $\mathscr{U}(U)$. Since there is a unique map $1 \to 1$, it follows by the gluing axiom that there exists $x \in \mathscr{U}(\cup_{U \in \mathcal{U}} U)$ such that $x|_U = x_U$ for all U. In particular, this implies that $\mathscr{U}(\cup_{U \in \mathcal{U}} U) = 1$, so $\cup_{U \in \mathcal{U}} U \in \mathcal{U}$. Consequently, it is easy to see that for all $U \subset X$, we have the equivalent formula

$$\mathscr{U}(U) = \begin{cases} 1 & : \quad W \subset V \\ \emptyset & : \quad \text{otherwise} \end{cases},$$

so $\mathscr{U} = h_V$.

Conversely, suppose $\mathscr{U} = h_V$ for some open $V \subset X$. For each open $U \subset X$, $h_V(U)$ contains at most one element, so global sections are determined locally either trivially (if $h_V(U) = 1$) or vacuously (if $h_V(U) = \emptyset$). Let $U \subset$

X be open and let $\mathcal{U} = \{U_i | i \in I\}$ be an open cover of U. Furthermore, suppose $\{x_i | x_i \in U_i\}$ is a collection of sections which agree on overlaps. In particular, this implies that for each i, $h_V(U_i)$ is nonempty, so $U_i \subset V$. Hence, $U \subset \cup_i U_i \subset V$, so $h_V(U) = 1$. Thus, for each i, there is a restriction map $h_V(U) \to h_V(U_i)$ and because each set has one element, it follows that $x \mapsto x_i$, so the gluing axiom is satisfied. Therefore, $\mathscr{U} = h_V$ is a sheaf.

□

ex:sheaf-yoneda

Exercise 7.2.6. Show, using Yoneda's lemma, that for any presheaf F on X, and any open $V \subset X$, the function

$$\mathrm{Hom}_{\mathsf{Psh}(X)}(h_V, F) \to F(V)$$

is a bijection.

ex:sheaf-func

Exercise 7.2.7. Show that a presheaf $F : \mathrm{Open}(X)^\circ \to \mathsf{Sets}$ is a sheaf if and only if, for any open $W \subset X$ and any cover $\mathscr{U} \subset h_W$, the restriction map

$$\mathrm{Hom}_{\mathsf{Psh}(X)}(h_W, F) \to \mathrm{Hom}_{\mathsf{Psh}(X)}(\mathscr{U}, F)$$

is a bijection.

Conclude, using Exercise 7.2.6, that a presheaf is a sheaf if and only if every element of $\mathrm{Hom}(\mathscr{U}, F)$ can be extended *uniquely* to an element of $F(W)$.

Using Exercises 7.2.7, we arrive at a more economical version of Definition 7.2.7:

def:sheaf-func-2

Definition 7.2.8. Let X be a topological space. A **presheaf** is a functor $F : \mathrm{Open}(X)^\circ \to \mathsf{Sets}$. Morphisms of presheaves are natural transformations.

A presheaf is called a **sheaf** if, whenever $\mathscr{U} \subset h_X$ is a cover, the function

$$\mathrm{Hom}_{\mathsf{Psh}(X)}(h_X, F) \to \mathrm{Hom}_{\mathsf{Psh}(X)}(\mathscr{U}, F)$$

is a bijection. A morphism of sheaves is a morphism of the underlying presheaves:

$$\mathrm{Hom}_{\mathsf{Sh}(X)}(F, G) = \mathrm{Hom}_{\mathsf{Psh}(X)}(F, G).$$

The functorial definition (version 3)

At least for the moment, the following version of Definition 7.2.8 will be used mainly for technical purposes. However, this generalization actually illustrates the essential idea behind the concept of a *Grothendieck topology*, which itself is the technical ingredient making the *étale topology*—and with it, the proof of the Weil conjectures—possible.

def:sheaf-func-3

Definition 7.2.9. Let X be a topological space. A presheaf on $\mathrm{Et}(X)$ is a functor $F : \mathrm{Et}(X)^\circ \to \mathsf{Sets}$. Morphisms of presheaves are natural transformations.

A presheaf is called a sheaf if, whenever $Y \in \mathrm{Et}(X)$ and $\mathscr{U} \subset h_Y$ is a cover of Y the function

$$\mathrm{Hom}_{\mathsf{Psh}(\mathrm{Et}(X))}(h_Y, F) \to \mathrm{Hom}_{\mathsf{Psh}(\mathrm{Et}(X))}(\mathscr{U}, F)$$

is a bijection.

Note that we might naturally interpret Definition 7.2.8 as describing presheaves and sheaves on $\mathrm{Open}(X)$. It is therefore reasonable to write $\mathsf{Psh}(\mathrm{Open}(X))$ and $\mathsf{Sh}(\mathrm{Open}(X))$ for what were called in $\mathsf{Psh}(X)$ and $\mathsf{Sh}(X)$ in Definition 7.2.9. This notation will be useful in the next few exercises, where we will compare $\mathsf{Sh}(\mathrm{Open}(X))$ and $\mathsf{Sh}(\mathrm{Et}(X))$: these turn out to be equivalent categories. Take care, however, that $\mathsf{Psh}(\mathrm{Et}(X))$ and $\mathsf{Psh}(\mathrm{Open}(X))$ are *not* equivalent!

`ex:et-open-rest` **Exercise 7.2.10.** Suppose that F is a presheaf on $\mathrm{Et}(X)$. Define $G = F\big|_{\mathrm{Open}(X)}$ to be the functor obtained by restricting F to the open subsets of X. That is $G(U) = F(U)$ whenever $U \subset X$ is open.

(i) Show that $F\big|_{\mathrm{Open}(X)}$ is a presheaf on $\mathrm{Open}(X)$ and that $F \mapsto F\big|_{\mathrm{Open}(X)}$ determines a functor from $\mathsf{Psh}(\mathrm{Et}(X))$ to $\mathsf{Psh}(\mathrm{Open}(X))$. (Hint: the most efficient way to do this may be to make use of a functor $\mathrm{Open}(X) \to \mathrm{Et}(X)$.)

(ii) Show that if F is a sheaf on $\mathrm{Et}(X)$ then $F\big|_{\mathrm{Open}(X)}$ is a sheaf on $\mathrm{Open}(X)$. Deduce a functor $\Psi : \mathsf{Sh}(\mathrm{Et}(X)) \to \mathsf{Sh}(\mathrm{Open}(X))$.

7.3 Stalks

Definition 7.3.1. Let F be a presheaf on X in the sense of Definition 7.2.3 (or Definition 7.2.8 or Definition 7.2.9) and let x be a point of X. The **stalk** of F at x is the set

$$\begin{aligned} F_x &= \varinjlim_{\substack{x \in U \\ U \subset X \text{ open}}} F(U) \\ &= \coprod_{\substack{x \in U \\ U \subset X \text{ open}}} F(U) \Big/ \sim \end{aligned}$$

where $(U, s) \sim (V, t)$ if there is some open $W \subset U \cap V$ containing x with $s\big|_W = t\big|_W$.

`ex:sheaf-sections` **Exercise 7.3.2.** (i) Let X and Z be topological spaces. For each open $U \subset X$, let $F(U)$ be the set of continuous functions from U to Z. Show that F is a sheaf on X in the sense of Definition 7.2.3.

(ii) [Vak, Exercise 3.2.G] Let $p : Y \to X$ be a continuous function between topological spaces. For each open subset $U \subset X$, define $F(U)$ to be the set of *sections* of p over U. This means that $F(U)$ is the set of continuous maps $s : U \to Y$ such that ps coincides with the inclusion map from U to X. Verify that F is a sheaf in the sense of Definition 7.2.3.

(iii) Obtain from the above a functor from $\mathrm{Et}(X)$ to $\mathsf{Sh}(\mathrm{Et}(X))$ (as well as a functor $\mathrm{Et}(X) \to \mathsf{Sh}(\mathrm{Open}(X))$).

Exercise 7.3.3. Here we construct an inverse to the functor constructed in Exercise 7.3.2. Assume that $\mathscr{F}$ is a sheaf, in the sense of Definition 7.2.3 (or Definition 7.2.8 or Definition 7.2.9), over a topological space X. Let $F = \mathscr{F}^{\text{ét}}$ be the set

$$F = \coprod_{x \in X} \mathscr{F}_x.$$

We will give F a topology: For each pair (U, s) where $U \subset X$ is open and $s \in \mathscr{F}(U)$, let $V(U, s)$ be the set of all $(x, s_x) \in F$ such that $x \in U$. Define a map $p : F \to X$ sending (x, s) to x. Call the sets $V(U, s)$ open.

(i) Show that the $V(U, s)$ form a basis for a topology.

(ii) Show that p is a local homeomorphism.

(iii) Show that $\mathscr{F} \mapsto \mathscr{F}^{\text{ét}}$ defines a functor from the category of sheaves in the sense of Definition 7.2.3 (or Definition 7.2.8 or Definition 7.2.9) to $\mathrm{Et}(X)$.

If $\mathscr{F}$ is a presheaf then $\mathscr{F}^{\text{ét}}$ is known as the **espace étalé** of $\mathscr{F}$.

Exercise 7.3.4. Verify that the functors constructed in the last two exercises are mutually inverse equivalences. Deduce that $\mathrm{Et}(X)$, $\mathsf{Sh}(\mathrm{Open}(X))$ and $\mathsf{Sh}(\mathrm{Et}(X))$ are equivalent categories.

From now on, we will just write $\mathsf{Sh}(X)$ to mean $\mathsf{Sh}(\mathrm{Et}(X))$ or $\mathsf{Sh}(\mathrm{Open}(X))$. This notation is technically ambiguous, since $\mathsf{Sh}(\mathrm{Et}(X))$ and $\mathsf{Sh}(\mathrm{Open}(X))$ are technically different categories (that are equivalent but not equal). However, in practice this won't cause any confusion.

Exercise 7.3.5. Let F be a sheaf on X and U an open subset of X.

(i) Let $p : E \to X$ be the espace étalé of F. Let $E' = p^{-1}(U)$. Show that $p\big|_{E'} : E' \to U$ is a local isomorphism and therefore determines a sheaf on U.

(ii) For an open subset $V \subset U$, define $F'(V) = F(V)$. Show that F' is a sheaf on U.

(iii) Show that E' is isomorphic to the espace étalé of F'.

This operation is known as *restriction* of a sheaf to an open subset. We will see later that it is an instance of the operation known as *pullback*.

Sheafification

Let $\mathscr{F}$ be a presheaf on a topological space X. Using the constructions outlined in the exercises above, we can form a local homeomorphism $p : \mathscr{F}^{\text{ét}} \to X$. Then, we can define $\mathscr{F}^{\text{sh}}$ to be the sheaf of sections of p.

Exercise 7.3.6. (i) Construct a map $i : \mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$.

(ii) Verify that $\mathscr{F}^{\mathrm{sh}}$ has the following universal property: for any sheaf $\mathscr{G}$, the map

$$\mathrm{Hom}_{\mathsf{Sh}(X)}(\mathscr{F}^{\mathrm{sh}}, \mathscr{G}) \to \mathrm{Hom}_{\mathsf{Psh}(X)}(\mathscr{F}, \mathscr{G})$$

is a bijection.

The sheaf $\mathscr{F}^{\mathrm{sh}}$ is known as the **sheafification** of the presheaf $\mathscr{F}$.

7.4 Some sheaf examples

ex:sheaf-final

Exercise 7.4.1. Let X be a topological space and define $F(U)$ to be a one-element set for each open $U \subset X$.

(i) Show that F is a sheaf.

(ii) Show that F is the *final* sheaf.

ex:sheaf-initial

Exercise 7.4.2. Let X be a topological space and define $F(U) = \varnothing$ for all $U \subset X$.

(i) Show that F is a sheaf.

(ii) Show that F is the *initial* sheaf.

ex:sheaf-empty

Exercise 7.4.3. Show that if F is a sheaf on a topological space X then $F(\varnothing)$ is a 1-element set.

ex:sheaf-disjoint-union

Exercise 7.4.4. Show that if F is a sheaf on X and $U, V \subset X$ are open subsets with $U \cap V = \varnothing$ then the map

$$F(U \cup V) \to F(U) \times F(V)$$

that sends x to $(x\big|_U, x\big|_V)$ is an isomorphism.

ex:sheaf-opens

Exercise 7.4.5. Let X be a topological space. For each open $U \subset X$[36] let $F(U)$ be the set of open subsets of U. Show that F is a sheaf.

ex:sheaf-closeds

Exercise 7.4.6. Let X be a topological space. For each open $U \subset X$, let $F(U)$ be the set of closed subsets of U. Show that F is a sheaf.

ex:sheaf-open-coarser

Exercise 7.4.7. Let X be a topological space and X' a coarser topology on the same space. For each open $U \subset X$, let $F(U)$ be the set of open subsets of U *in the topology on U induced from X'*. Show that F is a presheaf on X but it is *not* a sheaf.

ex:sheaf-local-prop

Exercise 7.4.8. Suppose $F \subset h_X$ is a subsheaf. Show that $F \cong h_U$ for some open $U \subset X$.

Subsheaves of the final sheaf are known as **local properties**. Can you see why?

[36] or for each local homeomorphism $U \to X$ if you prefer Definition 7.2.9

ex:sheaf-coincide **Exercise 7.4.9.** Let F be a sheaf on a topological space X. Let α, β be two elements of $F(X)$. Define U to be the set of points $x \in X$ where $\alpha_x = \beta_x$. Show that U is open in X.

eaf-disjoint-union-example **Exercise 7.4.10.** Let X be the topological space with two points and the discrete topology. Let $F = h_X$ be the final sheaf. Define $G(U) = F(U) \amalg F(U)$.

(i) Show that G is not a sheaf.

(ii) What is the espace étalé of G?

(iii) What is the sheafification of G?

ex:sheaf-covering-space **Exercise 7.4.11.** Let $p : F \to X$ be a covering space. Show that p is a local homeomorphism.

ex:skyscraper-sheaf **Exercise 7.4.12.** Let S be a set and x a point of a topological space X. If U is an open set of X, define

$$F(U) = \begin{cases} S & x \in U \\ 1 & x \notin U \end{cases}$$

where 1 denotes a 1-element set. The restriction map $F(U) \to F(V)$ is the unique map $F(U) \to 1$ if $x \notin V$ and is the identity $S \to S$ if $x \in V$.

(i) Show that F is a sheaf.

(ii) What is the espace étalé of F?

ex:sheaves-point **Exercise 7.4.13.** Show that the category of sheaves on a point is equivalent to the category of sets.

ex:sheaf-open-ext-by-zero **Exercise 7.4.14.** Let $U \subset X$ be an open subset of a topological space X. Define a sheaf F by the rule

$$F(V) = \begin{cases} 1 & V \subset U \\ \varnothing & \text{else} \end{cases}$$

where 1 denotes a 1-element set.

(i) Verify that F is a sheaf. It is known as the **extension of U by the void**.

(ii) What is the espace étalé of F?

(iii) Suppose that G is a sheaf on X. Show that $\mathrm{Hom}_{\mathsf{Sh}(X)}(F, G)$ is naturally in bijection with $G(U)$.

ex:subsheaves **Exercise 7.4.15.** Suppose that F is a sheaf on X and $F'(U) \subset F(U)$ is a natural collection of subsets. Such a collection is known as a **subpresheaf** of F. Verify that F' is a sheaf if and only if the following property holds: A section $x \in F(U)$ lies in $F'(U)$ if and only if there exists an open cover of U by subsets V such that $x\big|_V \in F'(V)$.

If this condition holds, we say that F' is a **subsheaf** of F.

ex:sheaf-rest

Exercise 7.4.16. Let F be a sheaf on X and $Y \subset X$ an open subset. Define a presheaf G on Y by setting $G(U) = F(U)$, noting that U is open in X because Y is open in X. Show that G is a sheaf on Y. It is denoted $F\big|_Y$ and called the restriction of F to Y.

Limits of sheaves

ex:sheaf-prod

Exercise 7.4.17. Let F_i, $i \in I$ be a collection of presheaves. Define $F(U) = \prod_{i \in I} F_i(U)$.

(i) Show that F is a presheaf.

(ii) Show that F has the universal property of a product of presheaves.

(iii) Prove that if each F_i is a sheaf then so is F.

(iv) Prove that if each F_i is a sheaf then $\prod F_i$ has the universal property of a product.

(v) What does this correspond to in terms of the espace étalé?

We call F the **product** of the F_i and write $F = \prod_{i \in I} F_i$.

ex:stalk-prod

Exercise 7.4.18. Let F_i, $i \in I$ be a collection of sheaves on a topological space X and let $F = \prod_{i \in I} F_i$ as in Exercise 7.4.17.

(i) Assume that I is finite. Show that, for all $x \in X$, the map on stalks $F_x \to \prod_{i \in I} (F_i)_x$ is an isomorphism.

(ii) Give an example of an infinite set I, a collection F_i of sheaves, and a point x of a topological space X such that the map on stalks $F_x \to \prod_{i \in I} (F_i)_x$ is *not* an isomorphism.

ex:sheaf-eq

Exercise 7.4.19. Let $\varphi, \psi : F \to G$ be two morphisms of presheaves. Define $E(U)$ to be the set of all $x \in F(U)$ such that $\varphi(x) = \psi(x)$.

(a) Show that E is a presheaf.

(b) Show that E is the universal example of a presheaf with a map $\alpha : E \to F$ such that $\varphi\alpha = \psi\alpha$.

(c) Supposing that F and G are sheaves, show that E is a sheaf.[37]

(d) Show that if F and G are sheaves then E is the universal example of sheaf with a map $\alpha : E \to F$ such that $\psi\alpha = \varphi\alpha$.

(e) What does this correspond to in terms of the espace étalé?

ex:sheaf-fp

Exercise 7.4.20. (i) Let $\varphi : E \to G$ and $\psi : F \to G$ be two maps of presheaves. Construct their fiber product.

[37] In fact, it is sufficient for this exercise that F be a sheaf and G be a *separated presheaf*.

(ii) Show that your construction yields a sheaf if E, F, and G are all sheaves.[38]

(iii) What does this correspond to in terms of the espace étalé?

ex:sheaf-lim

Exercise 7.4.21. Let F_i, $i \in I$ be a diagram of presheaves.

(i) Construct $\varprojlim F_i$.

(ii) Show that, if all the F_i are sheaves then $\varprojlim F_i$ is a sheaf.

(iii) What does this correspond to in terms of the espace étalé?

Colimits of sheaves

ex:sheaf-coproduct

Exercise 7.4.22. (i) Let F_i, $i \in I$ be a collection of etale spaces over X. Let $F = \coprod_{i \in I} F_i$. Show that the map $F \to X$ is a local homeomorphism.

(ii) Show that F is the coproduct of the F_i in $\mathrm{Et}(X)$.

(iii) Conclude that the category of sheaves on X has arbitrary coproducts.

(iv) Let $\mathscr{F}_i$ be the sheaves of sections of the étale spaces F_i. Define $\mathscr{F}(U) = \coprod_{i \in I} \mathscr{F}_i(U)$. Show that this is the coproduct of the $\mathscr{F}_i$ in the category of presheaves.

(v) Observe that, as constructed in the last part, $\mathscr{F}$ is not a sheaf in general. Can you find conditions on X under which $\mathscr{F}$ is guaranteed to be a sheaf?

(vi) Show that $\mathscr{F}^{\text{ét}} = F$.

(vii) Show that $\mathscr{F}^{\text{sh}}$ is the coproduct of the $\mathscr{F}_i$ in $\mathsf{Sh}(X)$.

7.5 Morphisms of sheaves and their properties

One can take almost any property of sets and get a corresponding property of sheaves by inserting the word "locally" at appropriate places in the definition. In this section we will play that game with morphisms of sheaves.

Definition 7.5.1. A morphism of sheaves $\varphi : F \to G$ is called

injective if, whenever $x, y \in F(U)$ and $\varphi(x) = \varphi(y)$, there is a cover of U by open sets V such that $x\big|_V = y\big|_V$;

surjective if, whenever $y \in G(U)$ there is an cover of U by open sets V such that $y\big|_V = \varphi(x)$ for some $x \in F(V)$;

bijective if it is both injective and surjective.

I strongly recommend doing some part of the following exercises.

[38] In fact, it suffices for G to be a separated presheaf.

ex:sheaf-inj

Exercise 7.5.2. Let $\varphi : F \to G$ be a morphism of sheaves on a topological space X. Prove that the following conditions are equivalent:

(i) φ is injective;

(ii) $\varphi_U : F(U) \to G(U)$ is injective for all open $U \subset X$;

(iii) $\varphi_x : F_x \to G_x$ is injective for all $x \in X$;

(iv) $\varphi^{\text{ét}} : F^{\text{ét}} \to G^{\text{ét}}$ is injective;

(v) φ is a monomorphism;

(vi) $(\varphi, \varphi) : F \to F \times_G F$ is surjective.

ex:sheaf-surj

Exercise 7.5.3. Let $\varphi : F \to G$ be a morphism of sheaves on a topological space X. Prove that the following conditions are equivalent:

(i) φ is surjective;

(ii) $\varphi_x : F_x \to G_x$ is surjective for all $x \in X$;

(iii) $\varphi^{\text{ét}} : F^{\text{ét}} \to G^{\text{ét}}$ is surjective;

(iv) φ is an epimorphism;

Give an example of a surjection of sheaves such that $\varphi_U : F(U) \to G(U)$ is *not* surjective for all open $U \subset X$.

ex:sheaf-bij

Exercise 7.5.4. Let $\varphi : F \to G$ be a morphism of sheaves on a topological space X. Prove that the following conditions are equivalent:

(i) φ is bijective;

(ii) $\varphi_U : F(U) \to G(U)$ is bijective for all open $U \subset X$;

(iii) $\varphi_x : F_x \to G_x$ is bijective for all $x \in X$;

(iv) $\varphi^{\text{ét}} : F^{\text{ét}} \to G^{\text{ét}}$ is bijective;

(v) φ is an isomorphism.

ex:sheaf-hom

Exercise 7.5.5. Let F and G be sheaves on a topological space X. For each open $U \subset X$, let $H(U) = \mathrm{Hom}_{\mathsf{Sh}(U)}(F\big|_U, G\big|_U)$. Show that H is a sheaf.

This is called the **sheaf of morphisms**, or "sheaf hom", from F to G. It is frequently denoted $\underline{\mathrm{Hom}}(F, G)$.

Solution. We will verify that H is a presheaf $\mathrm{Open}(X)^\circ \to \mathsf{Sets}$. For each $U \in \mathrm{Open}(X)$, we have defined

$$H(U) = \mathrm{Hom}_{\mathsf{Sh}(U)}(F\big|_U, G\big|_U).$$

For any $W \subset U$, we have a restriction map $H(U) \to H(W)$ which takes a natural transformation $\eta \in \mathrm{Hom}_{\mathsf{Sh}(U)}(F\big|_U, G\big|_U)$ and maps it to the corresponding $\eta\big|_W \in \mathrm{Hom}_{\mathsf{Sh}(W)}(F\big|_W, G\big|_W)$, where

$$(\eta\big|_W)_V = \eta_V$$

for all $V \subset W$. Finally, for open inclusions $V \subset W \subset U$, the map $H(U) \to H(V)$ is the same as the composition $H(U) \to H(W) \to H(V)$, since for any $O \subset V$,

$$\eta_O = (\eta\big|_V)_O = ((\eta\big|_W)\big|_V)_O.$$

We now must verify this is in fact a sheaf. Let $\alpha, \gamma \in H(U)$ for some $U \in \mathrm{Open}(X)$, and let $\mathscr{U}$ be an open cover of U such that $\alpha\big|_V = \gamma\big|_V$ for all $V \in \mathscr{U}$. We wish to show that $\alpha = \gamma$. Since α and γ are natural transformations, for any $V \in \mathscr{U}$ we have commutative diagrams

$$\begin{array}{ccc} F\big|_U(W) & \xrightarrow{\alpha_W} & G\big|_U(W) \\ \downarrow & & \downarrow \\ F\big|_U(W \cap V) & \xrightarrow{\alpha_{W\cap V}} & G\big|_U(W \cap V) \end{array} \qquad \begin{array}{ccc} F\big|_U(W) & \xrightarrow{\gamma_W} & G\big|_U(W) \\ \downarrow & & \downarrow \\ F\big|_U(W \cap V) & \xrightarrow{\gamma_{W\cap V}} & G\big|_U(W \cap V), \end{array}$$

so in particular for any $f \in F\big|_U(W)$,

$$(\alpha_W(f))\big|_{W\cap V} = \alpha_{W\cap V}(f\big|_{W\cap V}) = \gamma_{W\cap V}(f\big|_{W\cap V}) = (\gamma_W(f))\big|_{W\cap V}.$$

Thus $\alpha_W(f)$, $\gamma_W(f)$ are two elements of $G\big|_U(W)$ that agree on an open cover of of W, and thus $\alpha_W = \gamma_W$ for any $W \in \mathrm{Open}(U)$, that is $\alpha = \gamma$.

Finally, suppose we have $\mathscr{U}$ an open cover of U, $\alpha_V \in H(V)$ for all $V \in \mathscr{U}$, and $\alpha_V\big|_{V\cap W} = \alpha_W\big|_{V\cap W}$ for all $V, W \in \mathscr{U}$. Then for any $V, W \in \mathscr{U}$, we have the commutative diagram

$$\begin{array}{ccccc} & & F(V \cup W) & & \\ & \swarrow & & \searrow & \\ F(V) & \longrightarrow & F(V \cup W) & \longleftarrow & F(W) \\ {\scriptstyle \alpha_V}\downarrow & & {\scriptstyle \alpha_{V\cap W}}\downarrow & & {\scriptstyle \alpha_W}\downarrow \\ G(V) & \longrightarrow & G(V \cup W) & \longleftarrow & G(W) \\ & \nwarrow & & \nearrow & \\ & & G(V \cup W) & & \end{array}$$

Finally, since G is a sheaf on X, this diagram induces a map

$$F(V \cup W) \to G(U \cup W)$$

that takes $f \in F(V \cup W)$ to the unique element of $G(V \cup W)$ who's restriction to V (resp. W) is $\alpha_V(f\big|_V)$ (resp. $\alpha_W(f\big|_W)$). Hence to define a natural transformation $\alpha \in \mathrm{Hom}(F\big|_U, G\big|_U)$, it is enough to define compatible $\alpha_V \in \mathrm{Hom}(F\big|_V, G\big|_V)$ for all V in an open cover of U, that is there must be a unique $\alpha \in \mathrm{Hom}(F\big|_U, G\big|_U)$ such that $\alpha\big|_V = \alpha_V$ for all $V \in \mathscr{U}$, and thus H is a sheaf. □

7.6 Equivalence relations and quotients

ex:sheaf-fprod

Exercise 7.6.1. Let $\mathscr{F}$ be a sheaf on a topological space X.

(i) Define $\mathscr{G}(U) = \mathscr{F}(U) \times \mathscr{F}(U)$. Show that $\mathscr{G}$ is a sheaf in the sense of Defintion 7.2.3 (or Definition 7.2.8 or Definition 7.2.9).

(ii) Let $F = \mathscr{F}^{\text{ét}}$ be the espace étalé of F. Define a topological space $G = F \times_X F$. Show that the map $H \to X$ is a local homeomoprhism so that H is an étale space over X.

(iii) Construct an isomorphism between $\mathscr{G}^{\text{ét}}$ and G.

We write $\mathscr{F} \times \mathscr{F}$ for this sheaf. Note that $(\mathscr{F} \times \mathscr{F})^{\text{ét}} \neq \mathscr{F}^{\text{ét}} \times \mathscr{F}^{\text{ét}}$ unless X is a point!

In analogy with the definition of an equivalence relation on a set, we define an equivalence relation on a sheaf.

def:sheaf-eqrel

Definition 7.6.2. Let $\mathscr{F}$ be a sheaf on a topological space X. An equivalence relation on $\mathscr{F}$ is a subsheaf $\mathscr{R} \subset \mathscr{F} \times \mathscr{F}$ such that

EQ1 (reflexivity) if $x \in \mathscr{F}(U)$ then $(x, x) \in \mathscr{R}(U)$,

EQ2 (symmetry) if $(x, y) \in \mathscr{R}(U)$ then $(y, x) \in \mathscr{R}(U)$, and

EQ3 (transitivity) if $(x, y) \in \mathscr{R}(U)$ and $(y, z) \in \mathscr{R}(U)$ then $(x, z) \in \mathscr{R}(U)$.

In other words, $\mathscr{R}$ is an equivalence relation on $\mathscr{F}$ if and only if $\mathscr{R}$ is a subsheaf of $\mathscr{F} \times \mathscr{F}$ and $\mathscr{R}(U)$ is an equivalence relation on $\mathscr{F}(U)$ for all open $U \subset X$.

ex:eqreal-subsheaf-sig

Exercise 7.6.3. Make sure you understand the significance of the condition that $\mathscr{R}$ be a sub*sheaf*, as opposed to merely a sub*presheaf*, in the definition above. You will know you have succceded when the following sentence makes sense: "Sections that are locally related must be globally related."

ex:fprod-eq-rel

Exercise 7.6.4. Let $u : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Define $\mathscr{R} = \mathscr{F} \times_{\mathscr{G}} \mathscr{F}$. That is, $\mathscr{R}(U)$ is the set of pairs $a, b \in \mathscr{F}(U)$ such that $u(a) = u(b) \in \mathscr{G}(U)$. Show that $\mathscr{R}$ is an equivalence relation on $\mathscr{F}$.

def:sheaf-quotient

Definition 7.6.5. If $\mathscr{R}$ is an equivalence relation on $\mathscr{F}$ we define the **quotient presheaf** to be the presheaf $\mathscr{G}$ where $\mathscr{G}(U)$ be the quotient of $\mathscr{F}(U)$ by the equivalence relation $\mathscr{R}(U)$. The **quotient sheaf** is defined to be the sheafification $\mathscr{G}^{\text{sh}}$ of the quotient presheaf.

Exercise 7.6.6. Consider the map $p : \mathbf{R} \to S^1$. This is a local isomorphism so its sections form a sheaf F. Define an equivalence relation: $R = F \times F$ (note the espace étalé of $F \times F$ is $\mathbf{R} \times_{S^1} \mathbf{R}$). That is, R is the equivalence relation in which all sections of F are declared to be equivalent. Let Q be the quotient presheaf of F by R.

(i) Show that $\Gamma(S^1, Q) = \varnothing$.

(ii) Show that if $U \subset S^1$ is an open subset other than S^1 itself then $Q(U)$ consists of exactly one element.

(iii) Show that the quotient *sheaf* of F by R is represented by S^1.

7.7 Pushforward and pullback

Let F be a sheaf over a topological space X and $p : X \to Y$ a continuous function. For $U \subset Y$ open, define $p_* F(U) = F(p^{-1}U)$. This is known as the pushforward of F.

Exercise 7.7.1. Verify that $p_* F$ is a sheaf on Y.

Now let G be a sheaf on Y and $p : X \to Y$ a continuous function. Viewing G as an espace étalé over Y, define $p^{-1}G$ to be the fiber product $G \times_Y X$.

Exercise 7.7.2. Verify that the projection $p^{-1}G \to X$ is a local isomorphism and that $p^{-1}G$ is therefore a sheaf over X.

Exercise 7.7.3. Let $p : X \to Y$ be a continuous function between topological spaces. Assume that F is a sheaf on X and G is a sheaf on Y. Construct a natural bijection between $\mathrm{Hom}(p^* G, F)$ and $\mathrm{Hom}(G, p_* F)$ using the following steps:

(i) Construct a map $u : G \to p_* p^* G$:

$$G(U) = \mathrm{Hom}_Y(U, G) \to \mathrm{Hom}_Y(p^{-1}U, G) = \Gamma(p^{-1}U, p^* G) = \Gamma(U, p_* p^* G).$$

Obtain from this a map $\varphi(F, G) : \mathrm{Hom}(p^* G, F) \to \mathrm{Hom}(G, p_* F)$ as the composition

$$\mathrm{Hom}(p^* G, F) \xrightarrow{p_*} \mathrm{Hom}(p_* p^* G, p_* F) \xrightarrow{\mathrm{Hom}(u, p_* F)} \mathrm{Hom}(G, p_* F).$$

This is the map we will demonstrate to be an isomorphism.

(ii) Show that $\varphi(F, G)$ is an isomorphism if G is representable by an open subset of Y.

(iii) Show that $\varphi(F, G)$ is an isomorphism if G is representable by a disjoint union of open subsets of Y.

(iv) Show that if there is a surjection $G' \to G$ and $\varphi(F, G')$ is an injection then $\varphi(F, G)$ is an injection.

(v) Show that every sheaf G has a surjection from some $G' \cong \coprod U_i$ where each $U_i \subset Y$ is open. Deduce from the previous part that $\varphi(F, G)$ is injective for all F and G.

(vi) Suppose that G is the quotient of a sheaf G' by an equivalence relation G'' and $\varphi(F, G')$ is bijective and $\varphi(F, G'')$ is injective. Show that $\varphi(F, G)$ is bijective.

(vii) Deduce that $\varphi(F, G)$ is a bijection for all F and G.

sec:sheaf-constant

7.8 Constant sheaves

ex:const-presheaf

Exercise 7.8.1. Let X be a topological space and let S be a set. For each $U \subset X$, define $F(U) = S$. Give an example of X and S where this construction *does not* give a sheaf.

Solution. Let $X = \{a, b\}$ with the discrete topology. Let $S = \{c, d\}$. Consider the sections $c \in F(\{a\})$ and $d \in F(\{b\})$. Obviously, $\{a\} \cap \{b\} = \emptyset$, and so if F were a sheaf, there would exist an element of $S = F(X)$ which restricts to both c and d. This is impossible, so F cannot be a sheaf. □

Let X be a topological space and S a set. A function $f : X \to S$ is called locally constant if X can be covered by subsets on which f is constant.

Let S_X be the presheaf whose value on $U \subset X$ is the set of locally constant functions from U to S.

Exercise 7.8.2. (i) Verify that the presheaf S_X defined above is a sheaf.

(ii) Show that the espace étalé of S_X is $S \times X$, with S given the discrete topology.

def:sheaf-const

Definition 7.8.3. A sheaf on X is called **constant** if it is isomorphic to S_X for some set S.

7.9 Locally constant sheaves

One familiar source of local isomorphisms is covering spaces. Recall that a continuous map $p : E \to X$ is called a covering space if there is a cover of X by open subsets U such that $p^{-1}(U)$ is homeomorphic to $U \times S$ for some discrete topological space S.

Exercise 7.9.1. Show that a covering space $p : E \to X$ is a local isomorphism and conclude that a covering space is a sheaf by Definition 7.2.1.

Exercise 7.9.2. Verify that the following conditions are equivalent for a sheaf F on a topological space X:

(i) The espace étalé of F is a covering space of X.

(ii) There is an open cover of X by subsets $U \subset X$ such that $F\big|_U$ is a constant sheaf.

7.10 Further exercises

ex:sheaf-basis **Exercise 7.10.1.** To specify a sheaf, it is sufficient to specify its values on a basis of open subsets.

Exercise 7.10.2. Let X be the topological spaces $\{0, 1\}$ in which the open sets are $\varnothing$, $\{0\}$, and X. Let $i : U \to X$ be the inclusion of the open set $\{0\}$ and let $j : Z \to X$ be the inclusion of the closed set $\{1\}$. Let p be the projection from X to a point. Let S be any set, regarded as a sheaf on a point.

(i) Determine i^*p^*S and j^*p^*S.

(ii) Determine i^*j_*S and j^*j_*S.

(iii) Determine i^*i_*S and j^*i_*S.

Exercise 7.10.3. Let X be a topological space with an open subset $i : U \to X$ and complementary closed subset $j : Z \to X$. Suppose that F is a topological space on X.

(i) Construct a morphism $j^*F \to j^*i_*i^*F$ of sheaves on Z.

Exercise 7.10.4. In a topological space X, a point $x \in X$ is said to be a **specialization** of $y \in X$ if x is in the closure of $\{y\}$. We write $y \rightsquigarrow x$ in this case.

(i) Let F be a sheaf on a topological space X and suppose that $y \rightsquigarrow x$. Construct a morphism $F_x \to F_y$ as in the last exercise.

(ii) Show that to give a sheaf of sets on a finite topological space, it is equivalent to give a set F_x for each $x \in X$ and a function $F_x \to F_y$ whenever $y \rightsquigarrow x$ such that if $z \rightsquigarrow y \rightsquigarrow x$ then the map $F_z \to F_x$ agrees with the composition $F_z \to F_y \to F_x$.

Exercise 7.10.5. Show that a sheaf is locally constant if and only if its espace étalé is a covering space.

Exercise 7.10.6. Let F_i, $i \in I$ be sheaves on X. Let $F = \prod_i F_i$ be their product. Demonstrate that $F_x \neq \prod_i (F_i)_x$ in general.

Exercise 7.10.7. [Vak, Exercise 3.3.C] Given an example of a topological space X with a point x and a pair of sheaves F and G on X such that $\underline{\mathrm{Hom}}(F, G)_x \neq \mathrm{Hom}(F_x, G_x)$.

7.11 Sheaves of groups

sec:sheaves-groups

def:sheaf-of-groups **Definition 7.11.1.** Let X be a topological space. A functor $G : \mathrm{Open}(X)^\circ \to \mathsf{Grp}$ is called a **sheaf of groups** on X if the induced functor $\mathrm{Open}(X)^\circ \xrightarrow{G} \mathsf{Grp} \xrightarrow{\text{forget}} \mathsf{Sets}$ is a sheaf of sets.

Exercise 7.11.2. Show that if G is a group then the functor $\mathscr{G}(U) = \mathrm{Cont}(U, G)$, in which G is given the discrete topology, defines a sheaf of groups. The group structure on $\mathscr{G}(U)$ is given by pointwise multiplication of functions.

Exercise 7.11.3. Suppose that G is a sheaf of groups on X. Show that $\Gamma(X, G) \neq \varnothing$. (Hint: the identity.)

Recall that a left action of a group G on a set F is a map $\alpha : G \times F \to F$ such that

$$\alpha(g, \alpha(h, x)) = \alpha(gh, x)$$

for all $g, h \in G$ and $x \in F$. A right action is a map of the same kind such that

$$\alpha(g, \alpha(h, x)) = \alpha(hg, x).$$

When it is not necessary to distinguish one action of G on F from another, we usually we write $g.x$ instead of $\alpha(g, x)$ when α is a left action (and we write $x.g$ when α is a right action).

A morphism of G-sets from F to F' is a function $\varphi : F \to F'$ such that $\varphi(\alpha(g, x)) = \alpha(g, \varphi(x))$ for all $g \in G$ and $x \in F$..

def:sheaf-group-action

Definition 7.11.4. Let X be a topological space, let G be a sheaf of groups on X, and let F be a sheaf of sets. A (left or right) **action** of G on F is a map of sheaves of sets

$$G \times F \to F$$

such that for each open $U \subset X$ the map

$$G(U) \times F(U) \to F(U)$$

is a (left or right) action of the group $G(U)$ on the set $F(U)$.

A sheaf of sets equipped with a (left or right) action of G is known as a **sheaf of (left or right) *G*-sets**, or sometimes as just a ***G*-set**. A morphism of sheaves of G-sets $F \to F'$ is a morphism of sheaves of sets such that for each open $U \subset X$ the map $F(U) \to F'(U)$ is a morphism of $G(U)$-sets.

ex:sheaf-G-sets-category

Exercise 7.11.5. Let G be a sheaf of groups on a topological space X. Verify that sheaves of (left or right) G-sets form a category.

x:sheaf-groups-self-action

Exercise 7.11.6. Let G be a sheaf of groups on X. Define a function

$$\begin{aligned} G(U) \times G(U) &\longrightarrow G(U) \\ (g, h) &\longmapsto gh. \end{aligned}$$

Show that this is a morphism of sheaves $G \times G \to G$ and gives an action of G on itself.

af-groups-torsor-on-circle **Exercise 7.11.7.** Let $X = S^1$, let $G = \mathbf{Z}_X$ and let F be the sheaf of sections of the projection

$$p : \mathbf{R} \to S^1 : t \mapsto (\cos(2\pi t), \sin(2\pi t)).$$

Recall that this means

$$\begin{aligned} G(U) &= \mathrm{Cont}(U, \mathbf{Z}) \\ F(U) &= \Gamma(U, (\mathbf{R}, p)) = \big\{ s \in \mathrm{Cont}(U, \mathbf{R}) \,\big|\, ps = \mathrm{id}_U \big\}. \end{aligned}$$

Define

$$\begin{array}{rcl} G(U) \times F(U) & \longrightarrow & F(U) \\ (g, f) & \longmapsto & g + f \end{array}$$

where $g + f : U \to \mathbf{R}$ is the function $(g + f)(x) = g(x) + f(x)$.

Show that this is an action of G on F.

sheaf-maps-to-Pn-quotient **Exercise 7.11.8.** Let X be a topological space.

(i) For each open $U \subset X$, let $F(U) = \mathrm{Cont}(U, \mathbf{C}\mathrm{P}^n)$ be the set of continuous functions from U to $\mathbf{C}\mathrm{P}^n$. Show that F is a sheaf.

(ii) For each open $U \subset X$, let $H(U) = \mathrm{Cont}(U, \mathbf{C}^{n+1} \smallsetminus \{0\})$. Show that H is a sheaf.

(iii) For each open $U \subset X$, let $G(U) = \mathrm{Cont}(U, \mathbf{C}^*)$. Show that G is a sheaf of groups.

(iv) Define a function

$$G(U) \times H(U) \to H(U) : (\lambda, (x_0, \ldots, x_n)) \mapsto (\lambda x_0, \ldots, \lambda x_n).$$

Show that this determines an action of the sheaf of groups G on H.

(v) Show that F is isomorphic to the quotient sheaf G/H.

7.12 Sheaves of rings and their modules

sec:sheaves-rings

def:sheaf-of-rings **Definition 7.12.1.** Let X be a topological space. A functor $\mathscr{A} : \mathrm{Open}(X)^\circ \to \mathsf{ComRng}$ is called a **sheaf of commutative rings** on X if the induced functor $\mathrm{Open}(X)^\circ \xrightarrow{\mathscr{A}} \mathsf{ComRng} \xrightarrow{\text{forget}} \mathsf{Sets}$ is a sheaf of sets.

Let A be a ring. Recall that a left A-module is an abelian group equipped with a bilinear map $\alpha : A \times M \to M$ such that $\alpha(a, \alpha(b, x)) = \alpha(ab, x)$. In view of the analogy with the definition of an action of a group on a set, we view this as a (linear) action of a ring on an abelian group.

Right actions are of course defined analogously: one should have $\alpha(a, \alpha(b, x)) = \alpha(ba, x)$.

def:sheaf-rings-action **Definition 7.12.2.** Let $\mathscr{A}$ be a sheaf of rings on a topological space X and let $\mathscr{M}$ be a sheaf of abelian groups on X. A (left or right) **action** of $\mathscr{A}$ on $\mathscr{M}$ is a morphism of sheaves $\mathscr{A} \times \mathscr{M} \to \mathscr{M}$ such that, for each open $U \subset X$, the function $\mathscr{A}(U) \times \mathscr{M}(U) \to \mathscr{M}(U)$ defines a (left or right) action of the ring $\mathscr{A}(U)$ on the abelian group $\mathscr{M}$.

A sheaf of abelian groups equipped with an action of $\mathscr{A}$ is called a **sheaf of $\mathscr{A}$-modules**, or an **$\mathscr{A}$-module**. A morphism of $\mathscr{A}$-modules $\mathscr{M} \to \mathscr{M}'$ is a morphism of sheaves of sets such that $\mathscr{M}(U) \to \mathscr{M}'(U)$ is a morphism of $\mathscr{A}(U)$-modules for all open $U \subset X$.

ex:sheaf-ring-self-module **Exercise 7.12.3.** Show that every sheaf of rings $\mathscr{A}$ has a canonical structure as both a left and right $\mathscr{A}$-module coming from the canonical left and right $\mathscr{A}(U)$-module structure on $\mathscr{A}(U)$.

Exercise 7.12.4. Let $\mathcal{O}$ be the sheaf of continuous functions valued in $\mathbf{C}$ on $\mathbf{C}\mathrm{P}^n$. That is $\Gamma(U, \mathcal{O}) = \mathrm{Cont}(U, \mathbf{C})$.[39] Recall that each point x of $\mathbf{C}\mathrm{P}^n$ corresponds to a 1-dimensional linear subspace L_x of $\mathbf{C}^{n+1}$.

(i) Define $\Gamma(U, \mathcal{O}^{n+1}) = \mathrm{Cont}(U, \mathbf{C}^{n+1})$. Verify that this is a sheaf.

(ii) If $f \in \Gamma(U, \mathcal{O})$ and $g \in \Gamma(U, \mathcal{O}^{n+1})$, define $f.g(x) = f(x)g(x)$ where the multiplication $f(x)g(x)$ takes place coordinatewise. Show that this makes $\mathcal{O}^{n+1}$ a sheaf of $\mathcal{O}$-modules.

(iii) Define $\Gamma(U, \mathcal{O}(-1)) \subset \Gamma(U, \mathcal{O}^{n+1})$ to be the set of all $f \in \mathrm{Cont}(U, \mathbf{C}^{n+1})$ such that $f(x) \in L_x$. Show that $\mathcal{O}(-1)$ is a subsheaf of $\mathcal{O}^{n+1}$.[40]

(iv) Show that $\mathcal{O}(-1)$ is a sheaf of submodules of $\mathcal{O}^{n+1}$.

8 Sheaf cohomology

8.1 Torsors

Let G be a sheaf of groups on a topological space X. Suppose that F is a sheaf of sets on X. An **left action** of G on F is a map $G \times F \to F$ of sheaves on X such that for each $U \subset X$ the map

$$G(U) \times F(U) = (G \times F)(U) \to F(U)$$

defines a left action of the group $G(U)$ on the set $F(U)$.

Definition 8.1.1. Let G be a sheaf of groups on a topological space X. A **left G-torsor** is a sheaf F with a left action of G satisfying the following two properties:

[39] Variants of this exercise are possible with continuous functions replaced by C^∞ functions, holomorphic functions, polynomial functions, etc.

[40] The definition of $\mathcal{O}(-1)$ depends heavily on context! Sometimes it may be used to denote the sheaf of holomorphic sections of the tautological line bundle, sometimes the sheaf of algebraic sections, and sometimes the tautological line bundle itself.

1. if $a, b \in F(U)$ for some open $U \subset X$ then there exists a unique $g \in G(U)$ such that $g.a = b$, and
2. if $x \in X$ there is an open neighborhood $U \subset X$ of x such that $F(U) \neq \varnothing$.

Should F satisfy only the first condition, it is called a **pseudo-torsor**. If F and F' are G-torsors, a morphism of G-torsors $\varphi : F \to F'$ is a morphism of sheaves such that, for all open $U \subset X$, the map $\varphi_U : F(U) \to F'(U)$ is $G(U)$-equivariant.[41]

One similarly has the notion of a **right G-torsor** where the action of G on F is from the right. Unless otherwise specified, all torsors will be left torsors.

ex:tor-1 **Exercise 8.1.2.** Let F be a sheaf on X with a left action of G. Prove that the following conditions are equivalent:

(i) F is a G-torsor.

(ii) For each $x \in X$, the action of G_x on F_x is simply transitive.

(iii) For each $x \in X$, the G_x-set F_x is isomorphic, as a G_x-set, to G_x.

(iv) F is a G-pseudo-torsor and the projection $F^{\text{ét}} \to X$ is surjective.

Example 8.1.3. If G is a sheaf of groups on X then G can be made to act on itself using its multiplication law. It may therefore be regarded as a left or right G-torsor. This is known as the **trivial G-torsor**.

ex:tor-2 **Exercise 8.1.4.** Let F be a G-torsor. Prove that F is isomorphic to the trivial G-torsor if and only if $\Gamma(X, F) \neq \varnothing$.

ex:tor-3 **Exercise 8.1.5.** Let $X = \mathbf{C}^*$ and let F be the sheaf of sections of the map $p : \mathbf{C} \to \mathbf{C}^*$ defined by $p(z) = e^{2\pi i z}$. Let G be the constant sheaf $G(U) = \mathrm{Hom}(U, \mathbf{Z})$ on X. Define an action of G on F by the following rule: If $n : U \to \mathbf{Z}$ is a section of G over U and z is a section of F over U (that is, a continuous section $z : U \to \mathbf{C}$ such that $p \circ z = \mathrm{id}_U$), define

$$n.z = z + n.$$

In other words, $n.z(x) = z(x) + n(x)$.

(i) Verify that this is an action of G on F and makes F into a G-torsor. (Make sure to verify that $p \circ (n.z) = \mathrm{id}_U$.)

(ii) Show that this G-torsor is non-trivial.

Solution.

[41] Recall that this means that $\varphi_U(g.a) = g.\varphi_U(a)$ for all $a \in F(U)$ and $g \in G(U)$.

(i) Let $U \subset X$ be open. Let $n, m \in G(U)$ and $z \in F(U)$. First, note that the map is well-defined:

$$p \circ (n \cdot z) = e^{2\pi i(z+n)} = e^{2\pi iz} = p \circ z.$$

Furthermore,

$$(n+m) \cdot z = z + (n+m) = z + n + m = n \cdot (m \cdot z).$$

Thus, G acts on F. It is clear that F satisfies the second torsor condition. All that remains is to check that the action of G on F is simply transitive. Again, fix an open $U \subset X$ and $z, w \in F(U)$. Then $e^{2\pi iz} = e^{2\pi iw}$, and so $e^{2\pi i(z-w)} = 0$. Thus, $z - w \in \mathbf{Z}$. It follows that the action of G is simply transitive.

(ii) The G-torsor F is non-trivial because it admits no global sections. This follows from the observation that $\mathbf{C}^*$ admits no global logarithm.

□

Suppose that $G' \to G$ is a homomorphism of sheaves of groups and F' is a G'-torsor. We will construct a G-torsor from F'.

ex:torsor-covar **Exercise 8.1.6.** Let G' act on $G \times F'$ by the rule $h.(g, f') = (gh^{-1}, hf')$ and define $F = (G \times F')/G'$.

(i) Let G act on $G \times F'$ by the rule $g'.(g, f') = (g'g, f')$. Show that this induces a well-defined action on F.

(ii) Show that F is a G-torsor. This G-torsor is sometimes denoted $G \times^{G'} F'$ or $G \otimes_{G'} F'$.

(iii) Show that the transformation $F' \mapsto G \otimes_{G'} F'$ is a functor from the category of G'-torsors to the category of G-torsors.

(iv) Show that if F' is a trivial G'-torsor then F is a trivial G-torsor.

ex:torsor-fiber **Exercise 8.1.7.** Let $\varphi : G \to H$ be a homomorphism of sheaves of groups on X. For each $h \in \Gamma(X, H)$, define a presheaf G_h by

$$G_h(U) = \big\{ g \in G(U) \bigm| \varphi(g) = h\big|_U \big\}.$$

Let K be the kernel of φ.

(i) Check that G_h is a sheaf for each $h \in \Gamma(X, H)$.

(ii) For each $h \in \Gamma(X, H)$, describe a natural left action of K on G_h making G_h into a K-pseudo-torsor.

(iii) Show that if φ is surjective then G_h is a K-torsor for all $h \in \Gamma(X, H)$.

Exercise 8.1.8. Let $f : X \to Y$ be a continuous map of topological spaces. Suppose that G is a sheaf of groups on Y.

(i) Check that $f^{-1}G$ is a sheaf of groups on X.

(ii) Let F be a G-torsor. Show that $f^{-1}F$ is naturally a $f^{-1}G$-torsor.

Exercise 8.1.9. Let $f : X \to Y$ be a continuous map of topological spaces and suppose that G is a sheaf of grups on X.

(i) Check that f_*G is a sheaf of groups on Y.

(ii) Show that if F is a G-torsor on X then f_*F is a f_*G-pseudo-torsor on Y.

(iii) Give an example of $f : X \to Y$, G, and F where f_*F is not a f_*G-torsor.

Torsors and extensions

`def:extension` **Definition 8.1.10.** Let $\mathcal{O}$ be a sheaf of rings on a topological space and let F and G be sheaves of left $\mathcal{O}$-modules. An **extension** of F by G is an exact sequence

$$0 \to G \to E \to F \to 0.$$

A morphism of extensions is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & G & \longrightarrow & E' & \longrightarrow & F & \longrightarrow & 0. \end{array}$$

Composition of morphisms is defined in the evident way. The category of all extensions of F by G is denotes $\mathrm{Ext}(F, G)$. The set of isomorphism classes in $\mathrm{Ext}(F, G)$ is denoted $\mathrm{Ext}^1(F, G)$.

`ex:ext-aut` **Exercise 8.1.11.** Let E be an extension of F by G. Construct an isomorphism of groups $\mathrm{Aut}_{\mathrm{Ext}(F,G)}(E) = \mathrm{Hom}(F, G)$.

Consider an extension of $\mathcal{O}$ by F

$$0 \to F \to E \xrightarrow{p} \mathcal{O} \to 0.$$

For each $x \in \Gamma(X, \mathcal{O})$, we may define a sheaf E_x by

$$E_x(U) = \left\{ y \in E(U) \,\middle|\, p(y) = x\big|_U \right\}$$

By Exercise 8.1.7, E_x has the structure of an F-torsor.

`ex:ext-torsors` **Exercise 8.1.12.** (i) Verify that the assignment $E \mapsto E_1$, where 1 here denotes the section of $\mathcal{O}$ corresponding to the multiplicative identity, defines a functor from $\mathrm{Ext}(\mathcal{O}, F)$ to the category of F-torsors.

(ii) Prove that this functor is an equivalence of categories. (This part may be tricky.)

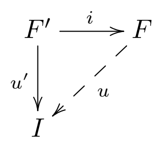
8.2 Acyclic sheaves

8.2.1 Injective sheaves

prop:inj **Proposition 8.2.1.** *Let $\mathcal{O}$ be a sheaf of rings on X and I a sheaf of $\mathcal{O}$-modules. The following properties are equivalent:*

INJ1 $\mathrm{Hom}_{\mathcal{O}\text{-}\mathsf{Mod}}(F, I)$ *is an exact functor in the variable F.*

INJ2 *If $i : F' \to F$ is an injection of $\mathcal{O}$-modules and $u' : F' \to I$ is any morphism of $\mathcal{O}$-modules then there is a $\mathcal{O}$-module morphism $u : F \to I$ such that $u \circ i = u'$. That is, any diagram of solid lines indicating morphisms of $\mathcal{O}$-modules*



with i injective can be completed by a dashed arrow u rendering the whole diagram commutative.

INJ3 *For any injective $\mathcal{O}$-module homomorphism $i : I \to F$ there exists a $\mathcal{O}$-module homomorphism $p : F \to I$ with $pi = \mathrm{id}_I$.*

ex:inj **Exercise 8.2.2.** Prove Proposition 8.2.1.

def:inj **Definition 8.2.3.** A $\mathcal{O}$-module satisfying the equivalent conditions of Proposition 8.2.1 is called **injective**.

ex:inj-Z-mod **Exercise 8.2.4.** Show that an abelian group I is injective if and only if it is **divisible**: if $x \in I$ and n is a nonzero integer then there is some $y \in I$ with $x = ny$. Conclude that $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are injective $\mathbf{Z}$-modules.

ex:inj-torsor **Exercise 8.2.5.** Let I be an injective sheaf of $\mathcal{O}$-modules. Show that every I-torsor is trivial. (Hint: first translate this into a statement about extensions using Exercise 8.1.12.)

Exercise 8.2.6. Prove that if I is an injective sheaf and one has an exact sequence

$$0 \to I \to F \to G \to 0$$

of sheaves on a space X then the sequence

$$0 \to \Gamma(X, I) \to \Gamma(X, F) \to \Gamma(X, G) \to 0$$

is exact. (Hint: Split the exact sequence and use the commutation of Γ with products.)

8.3 Non-abelian cohomology

Let F be a sheaf of sets on a topological space X. We define $H^0(X, F) = \Gamma(X, F)$. Higher cohomology is not defined in this case.

Let G be a sheaf of groups on X. Define $H^1(X, G)$ to be the set of isomorphism classes of G-torsors on X. Note that $H^1(X, G)$ has the structure of a *pointed set*: it is a set with a distinguished element corresponding to the isomorphism class of the trivial G-torsor.

Recall from Exercise 8.1.6 that if $G' \to G$ is a homomorphism of sheaves of groups, it induces a functor from the category of G'-torsors to the category of G-torsors, hence determines a function $H^1(X, G') \to H^1(X, G)$. By Exercise 8.1.6, this function takes the distinguished element of $H^1(X, G')$ corresponding to the trivial G'-torsor to the element of $H^1(X, G)$ corresponding to the trivial G-torsor. It is therefore a morphism of *pointed sets*.

8.3.1 The short exact sequence

Suppose that we have two morphisms of sheaves $p, q : G \to G''$ on a topological space X. Define a presheaf G' by

$$G'(U) = \{ g \in G(U) \mid p(g) = q(g) \}$$

This sheaf is called the **equalizer** of the arrows p and q.

Exercise 8.3.1. Check that G' is a sheaf.

Proposition 8.3.2. *Let $p, q : G \to G''$ be two maps of sheaves of X and let G' be the equalizer of p and q. Then the sequence*

$$1 \to H^0(X, G') \to H^0(X, G) \rightrightarrows H^0(X, G'')$$

is exact, in the sense that $H^0(X, G')$ is the equalizer of $H^0(X, p), H^0(X, q) : H^0(X, G) \to H^0(X, G'')$.

One particular, important case of the above is when G and G'' are sheaves of groups and $p : G \to G''$ is a sheaf homomorphism. We take q to be the trivial homomorphism that sends all of G to the identity in G''. Then the equalizer of p and q is simply the $\ker(p)$. We therefore get the following corollary:

Corollary 8.3.2.1. *Suppose that*

$$1 \to G' \to G \to G''$$

is an exact sequence of sheaves of groups on X. Then

$$1 \to H^0(X, G') \to H^0(X, G) \to H^0(X, G'')$$

is an exact sequence of sheaves of groups.

8.3.2 The (slightly) long(er) exact sequence

Let $G' \to G$ be a homomorphism of sheaves of groups and let $G'' = G' \backslash G$ be the quotient sheaf (by the left action of G' on G). Note that G'' is not a sheaf of groups; it is merely a sheaf of sets.

Recall from Exercise 8.1.7 that if G' is a sheaf of subgroups of G and $G'' = G' \backslash G$ then for each section $h \in \Gamma(X, G'')$ we get a G'-torsor G_h. This gives a map $H^0(X, G'') \to H^1(X, G')$.

Definition 8.3.3. Suppose $(A, a) \xrightarrow{f} (B, b) \xrightarrow{g} (C, c)$ is a sequence of morphisms of pointed sets. The sequence is said to be **exact** if $g^{-1}(c) = f(A)$.

Proposition 8.3.4. *Let G' be a sheaf of subgroups of G and let $G'' = G' \backslash G$. Then the sequence*

$$1 \to H^0(X, G') \to H^0(X, G) \to H^0(X, G'') \to H^1(X, G') \to H^1(X, G)$$

is exact as a sequence of pointed sets.

Proposition 8.3.5. *Suppose that G' is a sheaf of normal subgroups of G and let $G'' = G/G'$. Then the sequence*

$$H^1(X, G') \to H^1(X, G) \to H^1(X, G'')$$

is exact as a sequence of pointed sets.

8.3.3 Flaccid sheaves

Definition 8.3.6. Let F be a sheaf on a topological space X. We say that F is **flaccid** if $F(U) \to F(V)$ is surjective whenever $V \subset U$ are open subsets of X.

Exercise 8.3.7. Suppose that P is a torsor under a flaccid sheaf of groups G on a topological space X. Show that P is isomorphic to G as a G-torsor:

(i) Suppose that P is a G-torsor on $U \cup V$ and that $P(U)$ and $P(V)$ are both non-empty. Show that $P(U \cup V)$ is non-empty.

(ii) Suppose that $U = \bigcup_{i \in S} U_i$ is an ascending union of open subsets $U_i \subset U_j$ (for $i \leq j$ indexed by a totally ordered set S). Assume that $P(U_i) \neq \varnothing$. Show that $P(U) \neq \varnothing$.

(iii) Conclude that $P(X) \neq \varnothing$ and that therefore $P \simeq G$.

Exercise 8.3.8. Suppose that G is a sheaf of groups on X and G' is a flaccid sheaf of groups containing G. Using the previous exercise, show that there is an exact sequence (of pointed sets):

$$0 \to H^0(X, G) \to H^0(X, G') \to H^0(X, G'/G) \to H^1(X, G) \to 0.$$

8.4 Derived functors

Before undertaking the definition of higher cohomology groups we take a brief detour into the theory of derived functors. Here we will consider an *additive functor* $F : \mathscr{C} \to \mathscr{D}$ where $\mathscr{C}$ and $\mathscr{D}$ are *abelian categories.* [42]

We will assume in addition that F is a **left exact functor**. This means that for any exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathscr{C}$, the sequence

$$0 \to FA \to FB \to FC$$

in $\mathscr{D}$ is exact. Our goal in this section is to extend this latter sequence to a long exact sequence

$$0 \to FA \to FB \to FC \to R^1FA \to R^1FB \to \cdots .$$

In order to make this definition, we require an **adapted class** of objects of $\mathscr{C}$. [43] By this we will mean a collection of objects $\mathscr{C}_0 \subset \mathscr{C}$ satisfying the following propeties [GM, III.3]:

adap:0 **AC1** If $A, B \in \mathscr{C}_0$ then $A \oplus B \in \mathscr{C}_0$. [44]

adap:1 **AC2** If $A \in \mathscr{C}_0$ and

$$0 \to A \to B \to C \to 0$$

is an exact sequence in $\mathscr{C}$ then the sequence

$$0 \to FA \to FB \to FC \to 0$$

is exact in $\mathscr{D}$.

adap:3 **AC3** If $A, B \in \mathscr{C}_0$ and

$$0 \to A \to B \to C \to 0$$

is exact then $C \in \mathscr{C}_0$.

adap:2 **AC4** For every $A \in \mathscr{C}$ there is some $I \in \mathscr{C}_0$ and an injection $A \to I$.

[42] If you are unfamiliar with the definition of an abelian category, imagine a category in which one has finite direct sums, finite direct products, kernels, and cokernels, and these constructions behave in the same way as one is accustomed to expect from the category of R-modules. Examples include sheaves of abelian groups, sheaves of modules under a sheaf of rings, and quasi-coherent modules.

Additivity of the functor F means that F preserves direct sums: the natural maps $F(A \oplus B) \to F(A) \oplus F(B)$ and $F(A) \oplus F(B) \to F(A \oplus B)$ are inverse isomorphisms.

[43] Note that the definition of an adapted class depends on the functor F. It would therefore be more accurate to refer to a class of objects of $\mathscr{C}$ that are adapted *to* F.

[44] This condition is for convenience and doesn't play an essential role.

Exercise 8.4.1. Assuming condition **AC4**, verify that conditions **AC2** and **AC3** are together equivalent to the following: [45] For any exact sequence

$$0 \to I^0 \to I^1 \to I^2 \to \cdots$$

in $\mathscr{C}$ with all $I^j \in \mathscr{C}_0$, the sequence

$$0 \to FI^0 \to FI^1 \to FI^2 \to \cdots$$

is exact in $\mathscr{D}$.

Assuming an adapted class has been selected, we define derived functors R^pFA for each $p \geq 0$ and each $A \in \mathscr{C}$. We make the definition by induction, with $p = 0$ and $p = 1$ as base cases. Define $R^0FA = FA$. To define R^1FA, select an injection $A \subset I$ with $I \in \mathscr{C}_0$ and define R^1FA to make the sequence

$$0 \to FA \to FI \to F(I/A) \to R^1FA \to 0$$

exact. To define R^pFA, assume by induction that R^iFA has already been defined for $i \leq p-1$. Choose an injection $A \subset I$ with $I \in \mathscr{C}_0$ and define

$$R^pFA = R^{p-1}F(I/A).$$

Exercise 8.4.2. Verify that the definition above is equivalent to the following: Select a *resolution* of A by objects in $\mathscr{C}_0$, i.e., an exact sequence

$$0 \to A \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots$$

with all $I^j \in \mathscr{C}_0$. Then define R^pFA to be the p-th homology of the complex $FI^\bullet$. More precisely,

$$R^pFA = \frac{\ker F(d^p)}{\operatorname{image} F(d^{p-1})}.$$

Exercise 8.4.3. (i) Show that, up to canonical isomorphism, the definition of R^1FA does not depend on the choice of the inclusion $A \subset I$. (Hint: Consider two choices of inclusions $A \subset I$ and $A \subset J$ with $I, J \in \mathscr{C}_0$. Denote the two potential definitions of R^1FA by R^1_IFA and R^1_JFA. Reduce to the case where $I \subset J$ by considering $J' = I \oplus J$. Under this assumption, construct a map of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & I & \longrightarrow & I/A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & J & \longrightarrow & J/A & \longrightarrow & 0 \end{array}$$

and apply F to obtain a map $R^1_IFA \to R^1_JFA$. Make use of the snake lemma to show this is an isomorphism.)

[45] This condition is the one given in [GM].

(ii) Show that the isomorphism $R^1_I FA \to R^1_J FA$ does not depend on the choice of inclusion $I \subset J$.

(iii) Conclude that $R^p FA$ is well-defined up to canonical isomorphism for all $p \geq 0$.

Exercise 8.4.4. Consider an exact sequence

$$0 \to A \to B \to C \to 0.$$

(i) Construct a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & J & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

with $I, J \in \mathscr{C}_0$ and the vertical arrows injective.

(ii) Apply the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & FI & \longrightarrow & FJ & \longrightarrow & FK & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F(I/A) & \longrightarrow & F(J/B) & \longrightarrow & F(K/C) & \longrightarrow & 0 \end{array}$$

to obtain an exact sequence

$$0 \to FA \to FB \to FC \to R^1 FA \to R^1 FB$$

(iii) Show that this sequence does not depend on the choice of the inclusions $A \subset I$ and $B \subset J$ from the first part.

(iv) Show that the sequence

$$0 \to I/A \to J/B \to K/C \to 0$$

is exact and use it to extend the sequence above to a long exact sequence

$$R^p FA \to R^p FB \to R^p FC \to R^{p+1} FA \to R^{p+1} FB$$

for all p.

(v) Show that this sequence does not depend on any of the choices made above.

Injective objects

def:injective-object

Definition 8.4.5. An object I of an abelian category $\mathscr{C}$ is called **injective** if it satisfies the following equivalent conditions:

INJ1 $\mathrm{Hom}_{\mathscr{C}}(-, I)$ is an exact functor.

INJ2 If $M \to N$ is an injection in $\mathscr{C}$ then $\mathrm{Hom}_{\mathscr{C}}(N, I) \to \mathrm{Hom}_{\mathscr{C}}(M, I)$ is surjective.

INJ3 If $I \to M$ is an injection in $\mathscr{C}$ then it identifies I with a direct summand of M.

INJ4 Every exact sequence

$$0 \to I \to M \to N \to 0$$

in $\mathscr{C}$ can be split.

Exercise 8.4.6. Show that the conditions in Definition 8.4.5 are equivalent.

Definition 8.4.7. An abelian category $\mathscr{C}$ is said to have **enough injectives** if every object of $\mathscr{C}$ admits an injection into an injective object.

Exercise 8.4.8. Assume that $\mathscr{C}$ is an abelian category with enough injectives. Show that if $\mathscr{C}_0$ is defined to be the class of injective objects of $\mathscr{C}$ then $\mathscr{C}_0$ is adapted to *any* left exact additive functor $F : \mathscr{C} \to \mathscr{D}$.

Grothendieck's criteria for the existence of enough injectives

8.5 Abelian cohomology of sheaves

Definition 8.5.1. An object I of an abelian category[46] $\mathscr{A}$ is called **injective** if the functor $M \mapsto \mathrm{Hom}_{\mathscr{A}}(M, I)$ is exact.

prop:sh-inj

Proposition 8.5.2 ([Gro1, Théorème 1.10.1])**.** *Let $\mathcal{O}$ be a sheaf of rings on X and let F be a sheaf of left $\mathcal{O}$-modules. Then there is an injection $F \to I$ where I is an injective sheaf of $\mathcal{O}$-modules.*

Recall that the functor

$$\Gamma : \mathcal{O}\text{-}\mathsf{Mod} \to \Gamma(X, \mathcal{O})\text{-}\mathsf{Mod}$$

is left exact. Since the proposition guarantees there are enough injective $\mathcal{O}$-modules, we can use it to define the right derived functors of Γ, denoted $R^p\Gamma$.

[46]Examples of abelian categories include the category of left or right R-modules, where R is a ring, or the category of sheaves of left or right R modules, where R is a sheaf of rings.

Exercise 8.5.3. (i) Suppose that U is an open subset of a topological space X and $\mathcal{O}$ is a sheaf of rings on X. Let F be a sheaf of $\mathcal{O}$-modules. Define $F_U(V)$ to be the collection of all $\sigma \in F(V)$ such that $\sigma_x = 0$ for all $p \in X \smallsetminus U$.[47] Show that F_U is a sheaf on X. [48]

(ii) Construct a natural bijection $\mathrm{Hom}_{\mathcal{O}_X\text{-}\mathsf{Mod}}(\mathcal{O}_U, F) = \Gamma(U, F)$.

(iii) Show that $\mathcal{O}_U \to \mathcal{O}$ is injective.

(iv) Show that an injective sheaf on a topological space X is flaccid.

(v) Conclude that $R^1\Gamma(X, F) \simeq H^1(X, F)$ where $H^1(X, F)$ is defined to be the set of isomorphism classes of F-torsors on X.

Using this we can define the higher cohomology groups of sheaves of abelian groups. Let F be a sheaf of $\mathcal{O}$-modules on X. Choose an injection $F \to I$ where I is an injective sheaf of $\mathcal{O}$-modules. Define

$$H^p(X, F) = H^{p-1}(X, I/F)$$

for $p \geq 2$.

9 Schemes

9.1 Open subfunctors and open covers

Let A be a commutative ring. Let S be a subset of A. Recall that the functor

$$F(B) = \big\{ \varphi \in h^A(B) \;\big|\; \varphi(S)B = B \big\}$$

is said to be an *open subfunctor* of A.[49] We use the notation $D(S)$ for the functor defined above.

Recall that if $X : \mathsf{ComRng} \to \mathsf{Sets}$ is a functor, $U \subset X$ is a subfunctor, A is a commutative ring, and $\xi \in X(A)$ then we define a subfunctor $U_{(A,\xi)} \subset h^A$ by

$$U_{(A,\xi)}(B) = \big\{ \varphi \in h^A(B) \;\big|\; \varphi_*(\xi) \in U(B) \big\}.$$

Exercise 9.1.1. Verify that $U_{(A,\xi)} \cong h^A \times_X U$ where the map $h^A \to X$ is the one associated by Yoneda's lemma to $\xi \in X(A)$.

Solution. $h^A \times_X U$ is the limit of the diagram

$$\begin{array}{ccc} & & U \\ & & \downarrow \\ h^A & \longrightarrow & X, \end{array}$$

[47] Here σ_x denotes the image of σ in the stalk of F at x.

[48] The sheaf F_U is often denoted $i_! i^* F$, with i denoting the inclusion map $i : U \to X$.

[49] Note that $\varphi(S)B$ is the ideal of B generated by B; in particular it always contains 0, even if $S = \varnothing$.

where arrows are natural transformations. Thus for each commutative ring B, we have that $(h^A \times_X U)(B)$ is $h^A(B) \times_{X(B)} U(B)$, the limit in Sets of the diagram

$$\begin{array}{ccc} & & U(B) \\ & & \downarrow \\ h^A(B) & \longrightarrow & X(B). \end{array}$$

However, since the morphism $U(B) \to X(B)$ is just an inclusion of sets, and the morphism $h^A(B) \to X(B)$ is just $\varphi \mapsto \varphi_*(\xi)$, we know that

$$h^A(B) \underset{X(B)}{\times} U(B) = \{\varphi \in h^A(B) \mid \varphi_*(\xi) \in U(B)\},$$

and the isomorphism is obvious. □

Now, if $X : \mathsf{ComRng} \to \mathsf{Sets}$ is any functor, we call a subfunctor $U \subset X$ open if, for any commutative ring A and any $\xi \in X(A)$, the subfunctor $U_{(A,\xi)} \subset h^A$ is open.

Definition 9.1.2. We call a subfunctor $U \subset X$ an **open cover** if both of the following conditions hold:

(i) U is a union of open subfunctors, and

(ii) for every field k, we have $U(k) = X(k)$.

We call it an **open covering sieve** if it contains an open cover.

Exercise 9.1.3. Suppose that $U \subset X$ is an open covering sieve. Show that for any $\xi \in X(A)$, the subfunctor $U_{(A,\xi)} \subset h^A$ is an open covering sieve.

Exercise 9.1.4. Let A be a commutative ring and $f, g \in A$. Show that

$$(D(f) \cup D(g))(B) = \{\varphi \in h^A(B) \mid \varphi(f)B = B \text{ or } \varphi(g)B = B\}$$
$$D(\{f, g\}) = \{\varphi \in h^A(B) \mid (\varphi(f), \varphi(g))B = B\}.$$

Demonstrate by example that $D(f) \cup D(g) \neq D(\{f, g\})$.

Exercise 9.1.5. (i) Show that for any $S \subset A$,

$$D(S) \supset \bigcup_{f \in S} D(f).$$

(ii) Show that if k is a field then

$$D(S)(k) = \bigcup_{f \in S} D(f)(k).$$

(iii) Let S_i, $i \in I$ be a collection of subsets of A and for each i. Show that $\bigcup_{i\in I} D(S_i)$ is an open covering sieve of h^A if and only if $\bigcup_{i\in I}\bigcup_{f\in S_i} D(f)$ is an open covering sieve of A.

(iv) Conclude that every open covering sieve of h^A contains a subcover generated by open sets of the form $D(f)$. Thus the $D(f)$ form a *basis* for the topology of h^A.

pen-cover-characterization **Exercise 9.1.6.** Show that $U = \bigcup_{f\in S} D(f)$ covers h^A if and only if $D(S) = h^A$. (Hint: Note that $D(S) = h^A$ if and only if $SA = A$ and that this is the case only if there is no maximal ideal $\mathfrak{m} \subset A$ that contains S. Then show $U(A/\mathfrak{m}) = h^A(A/\mathfrak{m})$ if and only if $\mathfrak{m}$ does not contain S.)

ex:affine-quasi-compact **Exercise 9.1.7.** Suppose that $U = \bigcup_{i\in I} U_i$ is an open covering sieve of U. Show that there is a finite subset $I_0 \subset I$ such that $U_0 = \bigcup_{i\in I_0} U_i$ is also an open covering sieve of U. Thus h^A is **quasi-compact**.

Recall that a subfunctor $U \subset h^A$ is said to be an fpqc cover if it is the union of the images of a finite number of maps $h^B \to h^A$ and there is a faithfully flat collection of homomorphisms $\varphi : A \to C$ such that $\varphi \in U(C)$. We say $U \subset h^A$ is an fpqc covering sieve if it contains an fpqc cover, and we say that $U \subset X$ is an fpqc covering sieve if $U_{(A,\xi)}$ is an fpqc covering sieve for all commutative rings A and all $\xi \in X(A)$.

Proposition 9.1.8. *An open covering sieve is an fpqc covering sieve.*

Proof. To show that $U \subset X$ is an fpqc covering sieve it is sufficient to show that $U_{(A,\xi)} \subset h^A$ is an fpqc covering sieve for all $\xi \in X(A)$. We can therefore assume $X = h^A$.

Suppose that $U \subset h^A$ is an open covering sieve. By the exercises above, we can find a subset $S \subset A$ such that $U \supset \bigcup_{f\in S} D(f)$ and $\bigcup_{f\in S} D(f)$ covers h^A. By Exercise 9.1.7, we can assume that S is finite. That is, we can assume

$$U = D(f_1) \cup \cdots \cup D(f_n).$$

But each $A[f_i^{-1}]$ is flat over A, so U is an fpqc cover. Thus every open covering sieve $U \subset h^A$ contains an fpqc cover. □

9.2 Gluing morphisms to affine schemes

Definition 9.2.1. An **affine scheme** is a representable functor $\mathsf{ComRng} \to \mathsf{Sets}$.

Suppose that $X : \mathsf{ComRng} \to \mathsf{Sets}$ is a functor. Define a quasi-coherent module

$$\mathscr{A}_{(A,\xi)} = A$$

for all commutative rings A and all $\xi \in X$. In fact, each $\mathscr{A}_{(A,\xi)}$ is not just an A-module; it is an A-*algebra*. Therefore we say that $\mathscr{A}$ is a **quasi-coherent**

algebra. We will use the notation $\mathcal{O}_X$ for this quasi-coherent algebra. Our convention of writing $\mathscr{A}_{(A,\xi)}$ for the value of the quasi-coherent algebra $\mathscr{A}$ on $\xi \in X(A)$ becomes rather cumbersome when $\mathscr{A} = \mathcal{O}_X$, so we will instead write $\mathcal{O}_X(A, \xi)$.

Let B be any commutative ring. We can construct a quasi-coherent algebra $\mathscr{B} = B \otimes_{\mathbf{Z}} \mathcal{O}_X$ by defining

$$\mathscr{B}_{(A,\xi)} = B \underset{\mathbf{Z}}{\otimes} A.$$

Lemma 9.2.2. *There is a unique bijection*

$$\mathrm{Hom}(X, h^B) \to \mathrm{Hom}_{\mathsf{QCohAlg}(X)}(B \underset{\mathbf{Z}}{\otimes} \mathcal{O}_X, \mathcal{O}_X)$$

that is natural in both X and B and coincides, when $X = h^A$, via the map

$$\mathrm{Hom}_{\mathsf{QCohAlg}(X)}(B \otimes \mathcal{O}_X, \mathcal{O}_X) \to \mathrm{Hom}_{A\text{-}\mathsf{Alg}}(B \underset{A}{\otimes}, A) = \mathrm{Hom}_{\mathsf{ComRng}}(B, A)$$

with the canonical map $\mathrm{Hom}(h^A, h^B) \to \mathrm{Hom}_{\mathsf{ComRng}}(B, A)$.

Proof. To give a map $f : X \to h^B$ is, by definition, the same as to give, for every commutative ring A and every $\xi \in X(A)$, a ring homomorphism $f(\xi) : B \to A$, subject to the familiar naturality condition. Note, however, that to give $f(\xi) : B \to A$ is the same as to give an A-algebra homomorphism $B \otimes A \to A$. In other words, it is the same as to give a map $\varphi_{(A,\xi)} : (B \otimes \mathcal{O}_X)(A, \xi) \to (\mathcal{O}_X)(A, \xi)$. We leave it as an exercise to verify that the naturality enjoyed by f implies the naturality of $\varphi_{(A,\xi)}$ and therefore that to give f is the same as to give a map of quasi-coherent algebras $\varphi : B \otimes \mathcal{O}_X \to \mathcal{O}_X$. □

affine-schemes-are-sheaves

Theorem 9.2.3. *Let $U \subset X$ be an open covering sieve and let A be a commutative ring. Then the function*

$$\mathrm{Hom}(X, A) \to \mathrm{Hom}(U, A)$$

induced from the inclusion $U \subset X$ is a bijection.

Proof. By the lemma, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(X, A) & \longrightarrow & \mathrm{Hom}(U, A) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathsf{QCohAlg}(X)}(A \otimes \mathcal{O}_X, \mathcal{O}_X) & \longrightarrow & \mathrm{Hom}_{\mathsf{QCohAlg}(U)}(A \otimes \mathcal{O}_U, \mathcal{O}_U) \end{array}$$

in which the vertical arrows are isomorphisms. But by fpqc descent, the lower horizontal arrow is an isomorphism as well: if i is used to denote the inclusion of U in X then $A \otimes \mathcal{O}_U = i^*(A \otimes \mathcal{O}_X)$ and $i^*\mathcal{O}_X = \mathcal{O}_U$ and fpqc descent guarantees that

$$\mathrm{Hom}_{\mathsf{QCohAlg}(X)}(\mathscr{A}, \mathscr{B}) \to \mathrm{Hom}_{\mathsf{QCohAlg}(U)}(i^*\mathscr{A}, i^*\mathscr{B})$$

is an isomorphism for all quasi-coherent algebras $\mathscr{A}$ and $\mathscr{B}$. □

9.3 The definition of a scheme

def:Zariski-sheaf

Definition 9.3.1. Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor. We say that X is a **Zariski sheaf** if, whenever $U \subset Y$ is a open covering sieve, the map

$$\mathrm{Hom}(Y, X) \to \mathrm{Hom}(U, X)$$

is a bijection.

We may interpret Theorem 9.2.3 as saying that h^A is a Zariski sheaf whenever A is a commutative ring.

def:scheme

Definition 9.3.2. A **scheme** is a functor $X : \mathsf{ComRng} \to \mathsf{Sets}$ such that

SCH1 X is a Zariski sheaf, and

SCH2 there is an open cover of X by subfunctors U, each of which is isomorphic to h^A for some commutative ring A.

9.4 Projective space

Recall that if A is a commutative ring, $\mathbf{P}^n(A)$ is the set of equivalence classes of tuples $(L, x_0, \ldots, x_n)$ where

(i) L is an invertible A-module,

(ii) $x_0, \ldots, x_n$ are elements of L that generate it as an A-module.

Proposition 9.4.1. *For all n, the functor $\mathbf{P}^n$ is a scheme.*

:maps-from-functors-to-Pn

Exercise 9.4.2. For each functor $X : \mathsf{ComRng} \to \mathsf{Sets}$, define $F(X)$ to be the set of pairs $(\mathscr{L}, p)$ where

(i) $\mathscr{L}$ is a quasi-coherent module on X, and

(ii) $p : \mathscr{O}_X^{n+1} \to \mathscr{L}$ is a surjection of quasi-coherent modules.[50]

Construct a bijection $F(X) \cong \mathrm{Hom}(X, \mathbf{P}^n)$ that is natural in X.

ex:Pn-fpqc-sheaf

Exercise 9.4.3. Suppose that $U \subset X$ is an fpqc covering sieve. Show that the map

$$\mathrm{Hom}(X, \mathbf{P}^n) \to \mathrm{Hom}(U, X)$$

is a bijection. (Hint: Let F be as in Exercise 9.4.2 and show that $F(X) \to F(U)$ is a bijection.) Conclude that the same property holds if $U \subset X$ is an open covering sieve.

ex:Pn-scheme

Exercise 9.4.4. Let $U_i \subset \mathbf{P}^n$ where $U_i(A)$ is the set of all $(L, x_0, \ldots, x_n) \in \mathbf{P}^n(A)$ such that $Ax_i = L$.

[50] We say that a morphism $f : \mathscr{M} \to \mathscr{N}$ of quasi-coherent modules is **surjective** if $f_{(A,\xi)} : \mathscr{M}_{(A,\xi)} \to \mathscr{N}_{(A,\xi)}$ is a surjection for all commutative rings A and all $\xi \in X(A)$. Note that injectivity of quasi-coherent modules is not defined the same way.

(i) Recall that because L is invertible, there is an A-module $L^\vee$ such that $L \otimes L^\vee \cong A$. Show that the map $A \to L : a \mapsto ax_i$ induces a map $L^\vee \to A$ for which we also use the letter x_i.

(ii) Show that $B \otimes x_i = B \otimes_A L$ if and only if $(\mathrm{id}_B \otimes x_i)(B \otimes L^\vee) = B$.

(iii) Deduce that if $\xi = (L, x_0, \ldots, x_n)$ then $(U_i)_{(A,\xi)} = D(x_i(L^\vee))$.

(iv) Conclude that $U_i \subset \mathbf{P}^n$ is open.

(v) Show that $U_i \cong \mathbf{A}^n$ for all i.

(vi) Conclude that $\mathbf{P}^n$ is a scheme.

9.5 Open and closed subschemes

sec:subschemes

def:closed-subfunctor

Definition 9.5.1. Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor. A subfunctor $Z \subset X$ is called **closed** if $Z_{(A,\xi)} \subset h^A$ is a closed subfunctor for each commutative ring A and each $\xi \in X(A)$.[51]

ex:closed-subscheme

Exercise 9.5.2. Let X be a scheme and $Z \subset X$ a closed subfunctor. Show that Z is a scheme.

ex:open-subscheme

Exercise 9.5.3. Let X be a scheme and $U \subset X$ an open subfunctor. Show that U is a scheme.

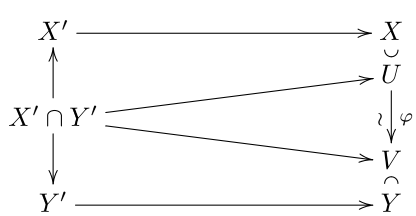
9.6 Gluing along open subschemes

sec:gluing-open

Let X and Y be schemes with $U \subset X$ and $V \subset Y$ open subschemes. Assume given an isomorphism $\varphi : U \xrightarrow{\sim} V$. Define a functor $Z : \mathsf{ComRng} \to \mathsf{Sets}$ by

$$Z(A) = \big\{(X', Y', \alpha, \beta)\big\}$$

where $X', Y' \subset h^A$ are open subfunctors covering h^A and $\alpha : X' \to X$ and $\beta : Y' \to Y$ are natural transformations making the diagram below commute:



Exercise 9.6.1. Show that Z, as defined above, is a scheme.

Exercise 9.6.2. Generalize the above construction to glue more than two pieces together.

[51] Recall that $W \subset h^A$ is called closed if $W = V(I)$ for some ideal I of A.

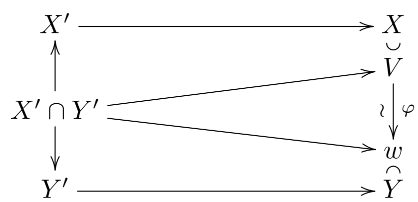
Exercise 9.6.3. Let $X = Y = \mathbf{A}^1 = h^{\mathbf{Z}[t]}$ and let $U = V = D(t) \subset \mathbf{A}^1$. Note that $U = V = h^{\mathbf{Z}[t,t^{-1}]}$. Define $\varphi : U \to V$ to be the map induced from the isomorphism of rings

$$\mathbf{Z}[t, t^{-1}] \to \mathbf{Z}[t, t^{-1}] : t \mapsto t^{-1}.$$

Show that the scheme obtained by gluing according to the above construction is $\mathbf{P}^1$.

9.7 Gluing along closed subschemes

Let $X, Y : \mathsf{ComRng} \to \mathsf{Sets}$ be functors containing closed subfunctors $V \subset X$ and $W \subset Y$. Assume given an isomorphism $\varphi : V \simeq W$. Define a functor $Z : \mathsf{ComRng} \to \mathsf{Sets}$ almost exactly as in Section 9.6: $Z(A)$ is the collection of diagrams



where $X', Y' \subset h^A$ are *closed* and h^A is the smallest closed subfunctor of h^A containing both X' and Y'.

`ex:closed-glue-sheaf` **Exercise 9.7.1.** Assume that X and Y satisfy fpqc descent. Show that Z does as well.

Exercise 9.7.2. Show that $Z(k) = (X(k) \amalg Y(k))/\sim$ where ξ is considered equivalent to η if $\xi \in V(k)$ and $\eta \in W(k)$ and $\varphi(\xi) = \eta$.

Exercise 9.7.3. Let $U \subset X$ be the open complement of V. Show that U is an open subfunctor of Z.

Now assume that both X and Y are schemes. We will show that Z is a scheme as well. We have to produce an open cover of Z by affine schemes. That is, for every field k and every $\xi \in Z(k)$ we must find an open $U \subset Z$ and with $\xi \in U(k)$ and with $U \simeq h^A$ for some commutative ring A.

If ξ is contained in $X(k) \smallsetminus V(k)$ or in $Y(k) \smallsetminus U(k)$ then it is certainly contained in an open subscheme of Z. Therefore we can assume that ξ is contained in $X \cap Y \subset Z$. Note that $X \cap Y \simeq V \simeq W$.

Choose open affine subschemes $P \subset X$ and $Q \subset Y$ containing ξ. Then $V \cap P$ is a closed subscheme of P and in particular is affine. The same applies to $W \cap Q \subset Y$. Let us write

$$P \simeq h^A \qquad\qquad Q \simeq h^B$$
$$P \cap V \simeq h^{A'} \qquad\qquad Q \cap W \simeq h^{B'}.$$

Note that $Q \cap W$ and $\varphi^{-1}(P \cap V)$ are both open affine neighborhoods of ξ in W. There is therefore an open affine $W \subset (P \cap V) \cap \varphi^{-1}(P \cap V)$ containing ξ. We can assume, moreover, that $W = h^{A'[\overline{f}^{-1}]} = h^{B'[\overline{g}^{-1}]}$ for some $\overline{f} \in A'$ and some $\overline{g} \in B'$. Then choose lifts $f \in A$ for $f' \in A'$ and $g \in B$ for $g \in B'$. Let $P_1 = D(f) \subset P$, let $Q_1 = D(g) \subset Q$, and note that $\varphi(P_1 \cap W) = Q_1 \cap V$.

Exercise 9.7.4. Let R_1 be the functor obtained by gluing P_1 to Q_1 along $\varphi\big|_{P_1 \cap V} : P_1 \cap V \to Q_1 \cap V$. Show that R_1 is an open subfunctor of Z.

In view of the exercise it will be enough to show that R_1 is a scheme since every $\xi \in Z(k)$ will then have an open neighborhood that is a scheme. This reduces our problem to the situation where X and Y are *affine* schemes.

The situation is now the following: we have $X \simeq h^A$ and $Y \simeq h^B$ and a diagram

$$\begin{array}{ccc} h^C & \longrightarrow & h^A \\ \downarrow & & \\ h^B & & \end{array}$$

where $h^C \simeq V \simeq W$ corresponding to a diagram of surjections of commutative rings

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ B & \longrightarrow & C. \end{array}$$

Let $D = A \times_C B$. Letting Z denote, as before, the functor obtained by gluing X and Y along $V \simeq W$, we show that $Z \simeq h^D$.

Certainly we have a map $h^D \to Z$ because h^D contains h^A and h^B as closed subfunctors with intersection h^C. We will show that this map is an isomorphism.

Exercise 9.7.5. Verify that the intersection of h^A and h^B in h^D is indeed h^C.

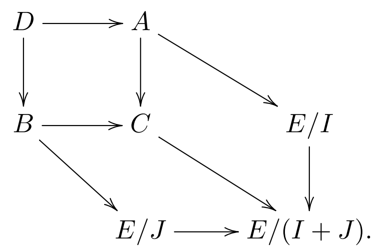
Suppose that E is any commutative ring containing and we have an element of $Z(E)$. This corresponds to ideals I and J in E and maps $V(I) \to h^A$ (i.e., $A \to E$) and $V(J) \to h^B$ (i.e., $B \to E$) such that

1. there is no ideal of E contained in the intersection of I and J (there is no closed subscheme of h^E containing both $V(I)$ and $V(J)$ other than h^E itself),

The first of these conditions guarantees that $I \cap J = 0$ so that I is in fact an ideal of E/J and J is an ideal of E/I. Thus the kernel of $E \to E/I$ is the same as the kernel of $E/J \to E/(I+J)$, which implies that the square

$$\begin{array}{ccc} E & \longrightarrow & E/I \\ \downarrow & & \downarrow \\ E/J & \longrightarrow & E/(I+J) \end{array} \qquad (10) \quad \boxed{\texttt{eqn:7}}$$

is cartesian. But by assumption our element of $Z(E)$ gives a commutative diagram



As diagram (10) is cartesian, we obtain a map $D \to E$. This works for any map $h^E \to Z$ and therefore gives a map $Z \to h^D$.

Exercise 9.7.6. Verify that the maps $h^D \to Z$ and $Z \to h^D$ constructed above are mutually inverse.

This completes the proof of the first of the following:

Theorem 9.7.7. *Let X and Y be schemes containing closed subschemes $V \subset X$ and $W \subset Y$ and let $\varphi : V \simeq W$. Then the functor defined above is a scheme and it has the universal property of a pushout. Moreover, when $X = h^A$, $Y = h^B$, and $W \simeq V = h^C$ we may take $Z = h^{A \times_C B}$.*

Proof. What is left is to check the universal property. Suppose we have compatible maps $X \to S$ and $Y \to S$ that agree along $V \simeq W$. We must construct a unique map $Z \to S$. Because the map we are constructing is unique, it will be sufficient to treat the case where $X = h^A$, $Y = h^B$, and $V \simeq W = h^C$.

If S is also affine, the universal property of the fiber product of rings yields the theorem. In general, we may select a cover of S by open affine subschemes U. Let X', Y', V', W', and Z' be the pre-images of such an open affine subscheme in X, Y, V, W, and Z respectively. Note that Z' is the scheme obtained by gluing X' to Y' along $V' \simeq W'$. Therefore, because U is affine, there exists a unique map $Z' \to U$ extending the maps $X' \to U$ and $Y' \to U$ and the proof is complete. □

9.8 The topological space of a functor

There is a second, more conventional, definition of a scheme as a topological space equipped with a sheaf of rings. In this section we will see how to extract a topological space and a sheaf of rings from any functor. We will eventually use this to prove that the two definitions of a scheme are equivalent.

Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor. Let $|X|$ be the set of *schematic points* of X (see 4.14 for the definition). If $U \subset X$ is an open subfunctor then $|U|$ is a subset of $|X|$.

Exercise 9.8.1. Show that the collection of all $|U|$, as U ranges among the open subfunctors of X, defines a topology on $|X|$.

Let $\mathscr{M}$ be a quasi-coherent module on X. For each open $U \subset X$, define $M(U) = \Gamma(U, \mathscr{M})$.

Exercise 9.8.2. Show that M, as defined above, is a sheaf on $|X|$.

In particular, the quasi-coherent algebra $\mathcal{O}_X$ induces a sheaf of algebras $\mathcal{O}_{|X|}$ on $|X|$.[52]

9.8.1 Locally ringed spaces

Exercise 9.8.3. Let X be a topological space and $\mathscr{A}$ a sheaf of rings. Let f be an element of $\mathscr{A}(U)$. Show that there is a biggest open subset $V \subset U$ such that $f\big|_V$ is invertible. We denote this subset $D(f)$.

Proposition 9.8.4. *Let X be a topological space and $\mathscr{A}$ a sheaf of commutative rings on X. The following are equivalent conditions:*

LR1 *For any $x \in X$ the stalk $\mathscr{A}_x$ is a local ring.*

LR2 *For any $U \subset X$ and any pair of sections $f, g \in \mathscr{A}(U)$ that generated $\mathscr{A}(U)$ as an ideal, the open subsets $D(f)$ and $D(g)$ cover U.*

LR3 *For any $U \subset X$ and any finite collection of sections $f_1, \ldots, f_n \in \mathscr{A}(U)$ that generate $\mathscr{A}(U)$ as an ideal, the open subsets $D(f_1), \ldots, D(f_n)$ cover U.*

*We call a sheaf of rings $\mathscr{A}$ satisfying the above conditions a **sheaf of local rings**.*

Definition 9.8.5. A **locally ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of *local* rings on X.

Exercise 9.8.6. Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor. Show that $\mathcal{O}_{|X|}$ is a sheaf of local rings on X.

9.8.2 The spectrum of a ring

Let A be a commutative ring and $I \subset A$ a subset of A. We write $V(I)$ for the set of all prime ideals of A containing I.

Exercise 9.8.7. Let S be a subset of A and denote by $(S)A$ the ideal generated by S. Then $V((S)A) = V(S)$.

Exercise 9.8.8. Let I be a subset of a commutative ring A. Then I is a prime ideal of A if and only if there is a homomorphism of rings $\varphi : A \to F$, with F a field, and $I = \ker(\varphi)$.

[52] Usually the same notation $\mathcal{O}_X$ is used both for the quasi-coherent module on the functor X and the sheaf on $|X|$. This notational imprecision is so universal that we will be unable to avoid it in these notes.

ex:top **Exercise 9.8.9.** Let A be a commutative ring.

(i) $V(A) = \varnothing$.

(ii) $V(0)$ contains all prime ideals of A.

(iii) Let I_k, $k \in K$ be ideals in A. Then $V(\sum_{k\in K} I_k) = \bigcap V(I_k)$.

(iv) For ideals I and J of A we have $V(I \cap J) = V(I) \cup V(J)$.

Definition 9.8.10. Let A be a commutative ring. Its **spectrum**, denoted $\operatorname{Spec} A$, is the topological space whose underlying set is the set of prime ideals of A. The closed subsets of $\operatorname{Spec} A$ are the sets $V(I)$ where I is an ideal of A.

By Exercise 9.8.9, this is indeed a topology.

Exercise 9.8.11. Show that $\operatorname{Spec} A$ coincides with $|h^A|$ and that the topologies are the same.

If $\varphi : A \to B$ is a homomorphism of rings, it induces a function $\operatorname{Spec}(\varphi) : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ by the rule

$$\operatorname{Spec}(\varphi)(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p}).$$

Exercise 9.8.12. With notation as above, $\operatorname{Spec}(\varphi)(\mathfrak{p})$ is a prime ideal, so $\operatorname{Spec}(\varphi)$ is a well-defined function.

ex:spec-basis **Exercise 9.8.13.** (i) For each $f \in A$, let $A[f^{-1}] = A[x]/(xf = 1)$. Show that the homomorphism $A \to A[f^{-1}]$ induces an *injection* $\operatorname{Spec} A[f^{-1}] \to \operatorname{Spec} A$.

(ii) Show that the injection from the last part is *open*.

(iii) Verify that if $g = f^n$ then $A[g^{-1}] = A[f^{-1}]$.

(iv) Verify that $D(fg) = D(f) \cap D(g)$.

(v) Show that the subsets $\operatorname{Spec} A[f^{-1}]$ of $\operatorname{Spec} A$ form a basis of open subsets for the topology of $\operatorname{Spec} A$.

(vi) Show that $\operatorname{Spec} A[f_j^{-1}]$, $j \in J$ covers $\operatorname{Spec} A$ if and only if it is possible to write $1 = \sum_{j\in J} a_j f_j$ for some $a_j \in A$, with all but finitely many of the $a_j = 0$.

Exercise 9.8.14. Prove that $\operatorname{Spec} A$ is quasi-compact for any commutative ring A.

Exercise 9.8.15. Prove that the natural map $\operatorname{Spec}(A \times B) \leftarrow \operatorname{Spec}(A) \amalg \operatorname{Spec}(B)$ is a homeomorphism.

9.8.3 Morphisms of locally ringed spaces

Exercise 9.8.16. Let $\mathscr{A}$ and $\mathscr{B}$ be sheaves of local rings on a topological space X and let $\varphi : \mathscr{A} \to \mathscr{B}$ be a homomorphism of sheaves of rings. For each point $x \in X$, let $\mathfrak{m}_x \subset \mathscr{A}_x$ and $\mathfrak{n}_x \subset \mathscr{B}_x$ be the maximal ideals. Show that the following conditions are equivalent:

(i) For each point x of X, we have $\varphi(\mathfrak{m}_x) \subset \mathfrak{n}_x$.

(ii) For each point x of X, we have $\varphi^{-1}(\mathfrak{n}_x) = \mathfrak{m}_x$.

(iii) For each point $x \in X$ and each $f \in \mathscr{A}_x$, we have $f \in \mathscr{A}_x^*$ if and only if $\varphi_x(f) \in \mathscr{B}_x^*$.

(iv) For each $f \in \mathscr{A}(U)$ we have $D_{\mathscr{A}}(f) = D_{\mathscr{B}}(\varphi(f))$.

A homomorphism satisfying these equivalent conditions is called a **local homomorphism** or a **homomorphism of local rings**.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces. A morphism

$$(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

consists of a continuous map $f : X \to Y$ and a *local* homomorphism of sheaves of commutative rings

$$\varphi : f^* \mathcal{O}_Y \to \mathcal{O}_X.$$

Exercise 9.8.17. Verify that with morphisms defined as above, the locally ringed spaces form a category, $\mathsf{LocRngSpc}$.

Let $(X, \mathcal{O}_X)$ be a locally ringed space. Recall that if A is a commutative ring, we can form a locally ringed space $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

Exercise 9.8.18. Verify that the assignment $A \mapsto (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ gives a contravariant functor from ComRng to $\mathsf{LocRngSpc}$.

Suppose that $(X, \mathcal{O}_X)$ is a locally ringed space. For any commutative ring A, define

$$F(A) = \operatorname{Hom}_{\mathsf{LocRngSpc}}\Big((\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}), (X, \mathcal{O}_X)\Big).$$

Exercise 9.8.19. Verify that the construction above gives a covariant functor $F : \mathsf{ComRng} \to \mathsf{Sets}$.

53 ←53

[53] TODO: Describe the equivalence of categories between schemes as functors and schemes as locally ringed spaces

9.9 Examples and exercises

ex:scheme-fiber-product

Exercise 9.9.1 (Important!)**.** Let $f : X \to Z$ and $g : Y \to Z$ be morphisms of covariant functors from ComRng to Sets. Let W be the functor

$$W(A) = \{(x, y) \in X(A) \times Y(A) \mid f(x) = g(y)\}.$$

Note that W comes with projection maps

$$p : W \to X : (x, y) \mapsto x$$
$$p : W \to Y : (x, y) \mapsto y.$$

(i) Show that W, together with the maps p and q, represents the fiber product of X with Y over Z.

(ii) Suppose that X, Y, and Z are all schemes. Show that W is a scheme.

ex:relative-spec

Exercise 9.9.2. Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor and $\mathscr{A}$ a quasi-coherent algebra on X. Define a functor $F : \mathsf{ComRng} \to \mathsf{Sets}$ by

$$F(B) = \{(\xi, \varphi) \mid \xi \in X(B), \varphi \in \mathrm{Hom}_{B\text{-}\mathsf{Alg}}(\mathscr{A}_{(B,\xi)}, B)\}.$$

Show that if X is a scheme then F is a scheme. We use the notation $F = \mathrm{Spec}_X \mathscr{A}$ for this construction.[54]

ex:closed-subscheme-reprise

Exercise 9.9.3. Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor and let $Z \subset X$ be a closed subfunctor. For each commutative ring A and each $\xi \in h^A$ we therefor obtain a closed subfunctor $Z_{(A,\xi)} \subset h^A$. By definition of closed subfunctors of h^A, this means that $Z_{(A,\xi)}$ is isomorphic to $h^{A/I}$ for some ideal $I \subset A$. Define $\mathscr{B}_{(A,\xi)} = A/I$.

(i) Show that $\mathscr{B}$ is a quasi-coherent algebra on X.

(ii) Show that $Z = \mathrm{Spec}_X \mathscr{B}$.

(iii) Combine the above with Exercise 9.9.2 to obtain another proof that a closed subscheme of a scheme is a scheme.

ex:symmetric-algebra

Exercise 9.9.4. Let A be a commutative ring and M an A-module. The n-th **symmetric power** of M is the A-module generated by products $m_1 \cdots m_n$ with each $m_i \in M$ and the following relations:

SYM1 for any permuation $\sigma \in S_n$, we have $m_1 \cdots m_n = m_{\sigma(1)} \cdots m_{\sigma(n)}$,

SYM2 for any $m_1, m_1', m_2, \ldots, m_n \in M$ we have

$$(m_1 + m_1')m_2 \cdots m_n = m_1 m_2 \cdots m_n + m_1' m_2 \cdots m_n,$$

and

[54] Note that this construction makes sense even if X is just a functor and is not a scheme!

SYM3 for any $\lambda \in A$ and $m_1, \dots, m_n \in M$ we have

$$\lambda(m_1 m_2 \cdots m_n) = (\lambda m_1) m_2 \cdots m_n.$$

The n-th symmetric power of M is denoted $\operatorname{Sym}^n M$.

(i) Let $F : A\text{-}\mathsf{Mod} \to \mathsf{Sets}$ be the functor sending an A-module N to the set of n-linear symmetric functions [55]

$$f : M^n \to N.$$

Show that $\operatorname{Sym}^n M$ represents F.

Let $\operatorname{Sym} M$ denote the direct sum $\sum_{n=0}^{\infty} \operatorname{Sym}^n M$.

(ii) Show that $\operatorname{Sym} M$ has the structure of an A-algebra with

$$(m_1 \cdots m_n)(m'_1 \cdots m'_p) = m_1 \cdots m_n m'_1 \cdots m'_p.$$

(iii) Show that if B is an A-algebra then there is a bijection

$$\operatorname{Hom}_{A\text{-}\mathsf{Alg}}(\operatorname{Sym} M, B) \xrightarrow{\sim} \operatorname{Hom}_{A\text{-}\mathsf{Mod}}(M, B)$$

that is natural in B.

ex:abelian-cone **Exercise 9.9.5.** Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor and $\mathscr{M}$ a quasi-coherent module on X. For each commutative ring A and each $\xi \in X(A)$, define

$$F(A) = \big\{(\xi, \varphi) \,\big|\, \xi \in X(A),\ \varphi \in \operatorname{Hom}_{A\text{-}\mathsf{Mod}}(\mathscr{M}_{(A,\xi)}, A)\big\}.$$

(i) Show that F gives a functor $\mathsf{ComRng} \to \mathsf{Sets}$. Note that as part of this exercise, you will have to supply the map $F(A) \to F(B)$ associated to a homomorphism of commutative rings $A \to B$.

(ii) Suppose that X is a scheme. Use Exercises 9.9.4 and 9.9.2 to show that F is also a scheme.

ex:open-not-affine **Exercise 9.9.6.** Let $X = \mathbf{A}^2 = h^{\mathbf{Z}[x,y]}$. Show that $U = D(\{x, y\}) \subset X$ is a scheme but is not affine. (Hint: If U were isomorphic to h^B, what would h^B be?)

def:loc-free **Definition 9.9.7.** An A-module M is called **locally free** if there is a finite collection $S \subset A$ such that

LF1 $SA = A$ and

LF2 $A[f^{-1}] \otimes_A M$ is a free $A[f^{-1}]$-module for each $f \in S$.

[55] For f to be n-linear means that for each i, the expression $f(m_1, \dots, m_n)$ is a linear function of m_i (the remaining variables are held fixed). For f to be symmetric means that $f(m_1, \dots, m_n)$ is invariant under permutation of the indices.

We say that M is **locally free of rank** n if, in the definition above, $A[f^{-1}] \otimes_A M$ is free of rank n for all $f \in S$ above.

A quasi-coherent module $\mathscr{M}$ on a functor X : $\mathsf{ComRng} \to \mathsf{Sets}$ is called locally free if $\mathscr{M}_{(A,\xi)}$ is a locally free A-module for each commutative ring A and each $\xi \in X(A)$.

ex:loc-free **Exercise 9.9.8.** (i) Show that a locally free A-module is flat.

(ii) Show that M is locally free of rank n if and only if M is locally free and $k \otimes_A M$ is an n-dimensional k-vector space for each field k and each homomorphism $A \to k$.

(iii) Let $f : M \to N$ be a homomorphism of locally free A-modules of the same rank. Show that f is an isomorphism if and only if it is surjective.

10 Algebraic curves

sec:curves

10.1 The Fermat cubic

sec:fermat-cubic

Let $X = V(x^3 + y^3 + z^3) \subset \mathbf{P}^2$. Define a projection

$$p : X \to \mathbf{P}^1 : (L, (x, y, z)) \mapsto (L, (x, y)).$$

Exercise 10.1.1. Show that this is a well-defined map.

Our main goal in this section will be to understand the topology of $X(\mathbf{C})$.

Exercise 10.1.2. Show that for every $\xi \in \mathbf{P}^1(\mathbf{C})$, the fiber $p^{-1}(\xi)$ consists of either 1 or 3 points. The former occurs when $\xi \in \{\alpha, \beta, \gamma\}$ where

$$\alpha = (-1, 1) \qquad \beta = (-\rho, 1) \qquad \gamma = (-\rho^2, 1)$$

and $\rho = e^{2\pi i/3}$.

Exercise 10.1.3. Show that the map

$$p : X(\mathbf{C}) \smallsetminus p^{-1}\{\alpha, \beta, \gamma\} \to \mathbf{P}^1(\mathbf{C}) \smallsetminus \{\alpha, \beta, \gamma\}$$

is a covering space.

Recall the classification of covering spaces:

Theorem 10.1.4. *If $p : E \to B$ is a covering space and $b \in B$ is a basepoint, there is an action of $\pi_1(B, b)$ on the set $p^{-1}(b)$ by $\ell.x = \widetilde{\ell}(1)$ where $\widetilde{\ell}$ is the unique lift of $\ell : [0, 1] \to B$ to a continuous map $\widetilde{\ell} : [0, 1] \to E$ with $\widetilde{\ell}(0) = x$. This group action is called **monodromy** and it determines E up to unique isomorphism.*

We can therefore determine which covering space we are dealing with by studying the mondromy action on $p^{-1}(b)$, where $b \in \mathbf{P}^1(\mathbf{C})$ is a basepoint not coinciding with α, β, or γ. For simplicity, we may take $b = (0,1) \in U_1(\mathbf{C}) \subset \mathbf{P}^1(\mathbf{C})$.

Note that $\pi_1(\mathbf{P}^1(\mathbf{C}) \smallsetminus \{\alpha, \beta, \gamma\})$ is generated by small loops ℓ_α, ℓ_β, and ℓ_γ around each of the points α, β, and γ. We make the convention that these loops are all oriented counterclockwise. We note also that $\ell_\alpha \ell_\beta \ell_\gamma = 1$; this will be a convenient check on our calculations below.

We can choose the loops ℓ_α, ℓ_β, and ℓ_γ so that they are all contained entirely inside $U_1(\mathbf{C}) \cong \mathbf{C}$. This is convenient, because we can identify $p^{-1}U_1(\mathbf{C})$ quite concretely:

Exercise 10.1.5. Show that $p^{-1}U_1(\mathbf{C})$ is isomorphic to the subset of $(x, z) \in \mathbf{C}^2$ such that $x^3 + 1 + z^3 = 0$.

Let z_0, z_1, and z_2 be the three points of $p^{-1}(b)$. Using the identification from the previous exercise, we can see that we may take $z_0 = -1$, $z_1 = -\rho$, and $z_2 = -\rho^2$.

Exercise 10.1.6. Show that if b is any point of $\mathbf{C}$ and z_0 is any point of $p^{-1}(b)$ and ℓ is any loop in $\mathbf{C}$ based at b then

$$\ell . z_0 = e^{\int_\ell d \log z} z_0.$$

Using the exercise we see that we can compute the monodromy by integrating $d \log z$ around each of the loops ℓ_α, ℓ_β, and ℓ_γ. Now, we have

$$(x - \alpha)(x - \beta)(x - \gamma) + z^3 = 0$$

so

$$d \log(x - \alpha) + d \log(x - \beta) + d \log(x - \gamma) + 3 d \log z = 0.$$

Thus we can integrate $d \log z$ by integrating the differentials $d \log(x - \alpha)$, $d \log(x - \beta)$, and $d \log(x - \gamma)$. We will use Cauchy's integral formula:

Theorem 10.1.7. *Suppose that f is a holomorphic function on the interior of a simple closed loop $\ell \subset \mathbf{C}$.*[56]

1. $\int_\ell f(x) dx = 0$.
2. $\int_\ell \frac{f(x) dx}{x - x_0} = 2\pi i f(x_0)$ *if x_0 is in the interior of ℓ.*

Exercise 10.1.8. (i) Prove the first part of Cauchy's integral formula using Green's theorem. (Hint: View $f(x+iy) = g(x+iy) + ih(x+iy)$ as a vector field $(g(x, y), h(x, y))$ and show that the differentiability of f implies that $\frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}$.[57]

[56]This means that f is a differentiable function of a complex variable. In other words, f is holomorphic at x if the limit $\lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$ exists; here δ is a complex number. Another formulation of differentiability at a point x_0 is that $f(x) = f(x_0) + (x - x_0) g(x)$ for some continuous function g.

[57]This is one of the **Cauchy–Riemann equations**; the other is $\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y}$.

(ii) Use the answer from the first part to show that the second part will follow in general once it is proved for a *single* simple closed loop ℓ enclosing x_0.

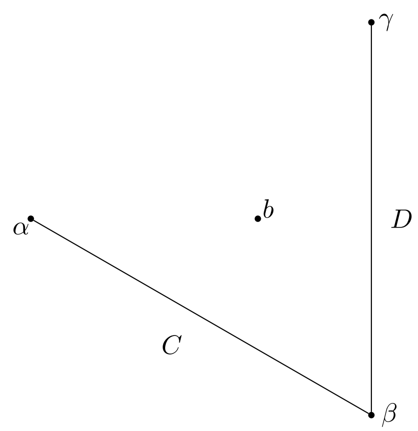
(iii) Show that the integral $\int_\ell \frac{f(x)-f(x_0)}{x-x_0}dx$ vanishes. (Hint: By the previous part, one can choose a convenient simple closed loop around x_0, for example a circle of arbitrarily small radius r. Choose r small enough so that $\frac{f(x)-f(x_0)}{x-x_0} < \epsilon$ and estimate the integral.)

We may now compute the monodromy around the three loops ℓ_α, ℓ_β, and ℓ_γ.

Exercise 10.1.9. (i) Use Cauchy's integral formula to prove that $\int_\ell d\log z = \rho$ when ℓ is any one of the loops ℓ_α, ℓ_β, or ℓ_γ.

(ii) Verify that this is consistent with the fact that $\ell_\alpha \ell_\beta \ell_\gamma = 1$.

We will now determine the topological surface underlying $X(\mathbf{C})$. Let us choose curves C connecting α to β and D connecting β to γ.

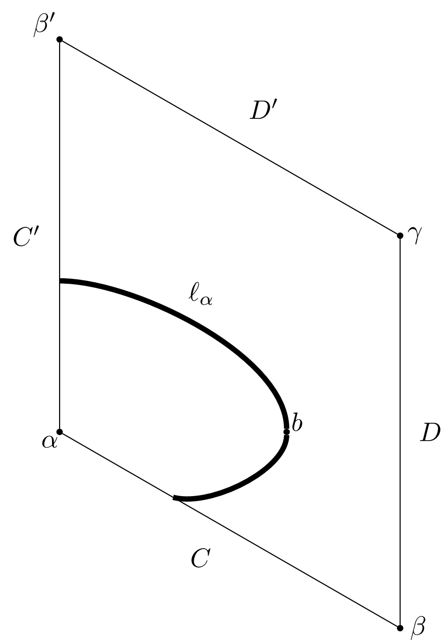


[58] ←[58]

Note that $V = \mathbf{P}^1(\mathbf{C}) \smallsetminus C \smallsetminus D$ is a contractible space. In particular, $\pi_1(V, b) = 1$ so $p^{-1}V$ is a *trivial* covering space. That is, it is isomorphic to 3 copies of V that we will label V_0, V_1, and V_2. Understanding $X(\mathbf{C})$ is now a matter of understanding how these three pieces are glued together.

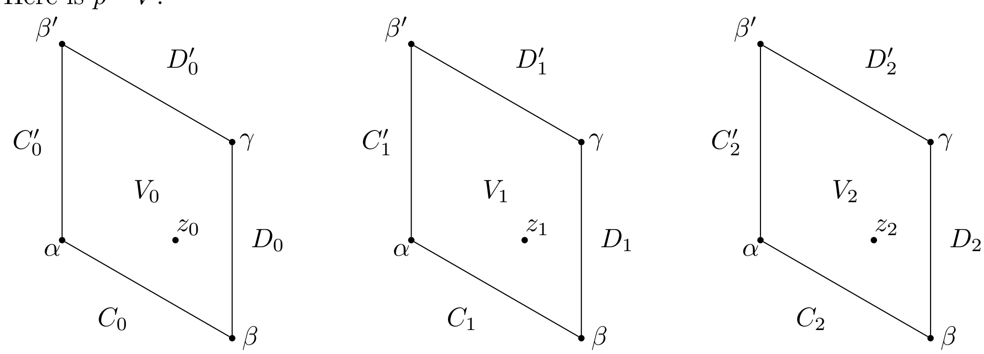
Here is a picture of V with the path ℓ_α illustrated:

[58]TODO: Insert a picture of a sphere here with C and D indicated.



The points labelled β and β' are to be glued together, as is the path C to the path C' and the path D to the path D'.

Here is $p^{-1}V$:



Exercise 10.1.10. Show that the only way to achieve the right monodromy is to identify:

(i) $C'_0 = C_1$

(ii) $C'_1 = C_2$

(iii) $C'_2 = C_0$

(iv) $D_0 = D'_1$

(v) $D_1 = D'_2$

(vi) $D_2 = D_0'$.

Illustrate that the resulting surface is a torus.

Exercise 10.1.11. Repeat the steps above for the Fermat quartic $x^4+y^4+z^4 = 0$. What is the genus of this surface?

Exercise 10.1.12. Repeat the steps above for the Fermat curves $x^n+y^n+z^n = 0$ for all n. What are their genera?

sec:hyperelliptic

10.2 A hyperelliptic curve

Exercise 10.2.1. Let $Y(A)$ be the set of (L, x, y, z) such that $(L, (x, y)) \in \mathbf{P}^1_{\mathbf{C}}(A)$ and $z \in L^{\otimes 3}$. Show that Y is a scheme.

Exercise 10.2.2. Let $f(x, y)$ be a separable homogeneous polynomial of degree 6 with coefficients in $\mathbf{C}$. Let $X(A) \subset Y(A)$ be the set of all $(L, x, y, z) \in Y(A)$ such that $z^2 = f(x, y)$. Show that X is a *closed* subscheme of Y.

Exercise 10.2.3. Let $p : X \to \mathbf{P}^1_{\mathbf{C}}$ be the projection $p(L, x, y, z) = (L, (x, y))$.

(i) Show that this is a morphism of schemes.

(ii) Show that if ξ is a point of $\mathbf{P}^1(\mathbf{C})$ then $p^{-1}(\xi)$ consists of either 1 or 2 points, with the former occurring only when ξ is in $V(f)$.

Exercise 10.2.4. Let $\xi_1, \ldots, \xi_6$ be the six points where $\#p^{-1}(\xi_i) = 1$ and let $\Xi = \{\xi_1, \ldots, \xi_6\}$. Show that $X(\mathbf{C}) \smallsetminus p^{-1}\Xi \to \mathbf{P}^1(\mathbf{C}) \smallsetminus \Xi$ is a covering space.

Exercise 10.2.5. Compute the monodromy of X around each of the ξ_i and use the result to assemble $X(\mathbf{C})$ as a topological surface. What is the genus of $X(\mathbf{C})$?

Exercise 10.2.6. Repeat the steps above with $Y(A)$ replaced by the collection of all (L, x, y, z) such that $(L, x, y) \in \mathbf{P}^1(A)$ and $z \in L^{\otimes n}$ and f replaced by a separable polynomial of degree $2n$.

Exercise 10.2.7. How do the calculations above change when f is not assumed to be separable? What happens if f has one repeated root, for example?

sec:riemann-hurwitz-1

10.3 Riemann–Hurwitz theorem (over C)

def:euler-char

Definition 10.3.1. Let S be a topological space that is obtained by gluing together finitely many cells. The **Euler characteristic** of S is[59]

$$\chi(S) = \sum_{n=0}^{\infty} \#(n\text{-cells of } S).$$

[59]Note that with this definition, it is not obvious that $\chi(S)$ only depends on S and not how it was assembled from cells. One may prove this independence using homology or cohomology; see Exercise 10.3.4.

Exercise 10.3.2. Let S be a surface of genus g (a torus with g handles). Show that $\chi(S) = 2 - 2g$.

Exercise 10.3.3. Compute $\chi(S^n)$ for all n.

ex:euler-char-homology

Exercise 10.3.4. (i) Prove that $\chi(S) = \sum_{n=0}^{\infty} (-1)^n \dim_{\mathbf{Q}} H_n(S, \mathbf{Q})$ where $H_n(S, \mathbf{Q})$ is the n-th homology group of S with $\mathbf{Q}$-coefficients.

(ii) Deduce that homotopy equivalent spaces have the same Euler characteristics. Conclude in particular that the Euler characteristic of a space does not depend on how the space was constructed.

The exercise indicates how we can extend the definition of Euler characteristic to any topological space whose homology that may not have infinitely many non-zero homology groups:

$$\chi(S) = \sum_{n=0}^{\infty} (-1)^n \dim_{\mathbf{Q}} H_n(S, \mathbf{Q}).$$

Exercise 10.3.5. Suppose that $f : S \to S'$ is a covering space of degree n. Show that $\chi(S) = n\chi(S')$.

Exercise 10.3.6. Let $f : S \to S'$ be a continuous map such that there is a finite collection of points $\Xi \subset S'$ such that $f\big|_{S \smallsetminus f^{-1}(\Xi)}$ is a covering space of degree n. For each $\xi \in \Xi$, let

$$e(\xi) = \#(f^{-1}(\xi) - 1).$$

Show that

$$\chi(S) = n\chi(S') + \sum_{\xi \in \Xi} e(\xi).$$

11 Quasi-coherent sheaves

sec:qcoh-sheaves

This section is in progress and still lacks important definitions and statements.

11.1 The adjoint functor theorem

Suppose C is a category and $F : C^\circ \to \mathsf{Sets}$ is a functor. Construct a category C/F whose objects are pairs (X, ξ) with $X \in C$ and $\xi \in F(X)$. We define

$$\mathrm{Hom}_{C/F}((X, \xi), (Y, \eta)) = \big\{ f \in \mathrm{Hom}_C(X, Y) \bigm| f^*\eta = \xi \big\}.$$

Exercise 11.1.1. Verify that C/F is a category.

Suppose X_i, $i \in I$ is a diagram in C with colimit X. We have compatible maps $X_i \to X$ for all i and therefore compatible maps $F(X) \to F(X_i)$ for every i. From this we obtain a map

$$F(\varinjlim X_i) \to \varprojlim F(X_i).$$

Definition 11.1.2. Assume that C is a diagram with all colimits and $F : C^\circ \to D$ is a functor. We say that F **carries colimits to limits** if $F(\varinjlim X_i)$ represents the limit $\varprojlim F(X_i)$ for all diagrams X_i, $i \in I$ in C.

Exercise 11.1.3. Let $\mathscr{C}$ be a small category that admits all colimits. Show that $\mathscr{C}$ has a final object. (Hint: Since $\mathscr{C}$ is small, it is a diagram in itself. Take the colimit of the diagram.)

Theorem 11.1.4. *Suppose that C admits all small colimits*[60]*, that F carries colimits to limits, and that C/F is equivalent to a small category. Then F is representable.*

Proof. First note that to show F is representable it is equivalent to show that C/F has a final object. The hypotheses ensure that C/F is equivalent to a small category and therefore admits all colimits. By the exercise, it has a final object. □

prop:func-gens-rep

Proposition 11.1.5. *Let C be a category with a small collection of generators and $F : C^\circ \to \mathsf{Sets}$ a functor that takes colimits to limits. Then F is representable.*

Corollary 11.1.5.1. *Let C be a category with a small collection of generators and $G : C^\circ \to D$ a functor that takes colimits to limits. Then G has a right adjoint.*

Proof. For each $Y \in D$, apply 11.1.5 to the functor

$$F : C^\circ \to \mathsf{Sets} : X \mapsto \mathrm{Hom}_D(GX, Y).$$

□

11.2 Quasi-cohesion

Let $\mathscr{D}$ be a diagram in ComRng. A **quasi-coherent module** on $\mathscr{D}$ consists of the following data:

1. For each $A \in \mathscr{D}$, an A-module $\mathscr{M}_A$, and
2. for each homomorphism $u : A \to B$ in $\mathscr{D}$ a function $\mathscr{M}_u : \mathscr{M}_A \to \mathscr{M}_B$

with the following properties

3. $\mathscr{M}_u(x + y) = \mathscr{M}_u(x) + \mathscr{M}_u(y)$ for all $x, y \in \mathscr{M}_A$ and all $u : A \to B$ in $\mathscr{D}$,
4. $\mathscr{M}_u(\lambda x) = u(\lambda)\mathscr{M}_u(x)$ for all $x \in \mathscr{M}_A$ and $\lambda \in A$ and $u : A \to B$ in $\mathscr{D}$,
5. $\mathscr{M}_u \circ \mathscr{M}_u = \mathscr{M}_{u \circ v}$ when the composition makes sense, and
6. the map $B \otimes_A \mathscr{M}_A \to \mathscr{M}_B : (b \otimes x) \mapsto b\mathscr{M}_u(x)$ is an isomorphism for all $u : A \to B$ in $\mathscr{D}$.

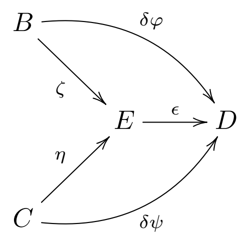
[60] A small colimit is a colimit of a diagram whose shape is a set.

Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor. We say that X is **essentially small** if there is a diagram $\mathscr{D}$ of commutative rings A and $\xi_A \in X(A)$ such that

1. for any morphism $u : A \to B$ in $\mathscr{D}$ we have $u_*(\xi_A) = \xi_B$, and

2. if A is a commutative ring and $\eta \in X(A)$ then there is an fpqc cover $U \subset X$ having the following property: for every $\varphi \in U(B) \subset h^A(B)$ there is a homomorphism $\psi : C \to B$ with $C \in \mathscr{D}$ and $\varphi_*(\eta) = \psi_*(\xi_C)$.[61]

3. Suppose A is a commutative ring, $\eta \in X(A)$, and we have two homomorphisms $\varphi : B \to A$ with $\eta = \varphi_*(\beta)$ for some $\beta \in X(B)$ and $\psi : C \to A$ with $\eta = \psi_*(\gamma)$ for some $\gamma \in X(C)$. Then there is an fpqc cover $U \subset h^A$ with the property that, for every commutative ring D and every $\delta \in U(D)$ there is a commutative ring E in $\mathscr{D}$ and a common factorization of the maps $\delta \circ \varphi : B \to D$ and $\delta \circ \psi : C \to D$ through maps $\zeta : B \to D$ and $\eta : C \to D$:



Lemma 11.2.1. *Suppose that $\mathscr{D}$ is a diagram of commutative rings witnessing the essential smallness of a functor $X : \mathsf{ComRng} \to \mathsf{Sets}$. Then the restriction functor*

$$\mathsf{QCoh}(X) \to \mathsf{QCoh}(\mathscr{D})$$

is an equivalence of categories.

Proof. Let $U \subset X$ be the subfunctor consisting of all $\eta \in X(A)$ such that there exists some $B \in \mathscr{D}$ and some homomorphism $\varphi : B \to A$ with $\varphi_*(\xi_B) = \eta$. This is an fpqc covering sieve of X and we have a factorization

$$\mathsf{QCoh}(X) \to \mathsf{QCoh}(U) \to \mathsf{QCoh}(\mathscr{D}).$$

Since we already know that the first functor is an equivalence, it is sufficient to study the second. We will be content to indicate the construction of an inverse functor.

Let $\mathscr{M} \in \mathsf{QCoh}(\mathscr{D})$. For each commutative ring A and each $\eta \in U(A)$, choose a map $\varphi : B \to A$ with $B \in \mathscr{D}$ and $\varphi_*(\xi_B) = \eta$. Define $\mathscr{M}_{(A,\eta)} = A \otimes_B \mathscr{M}_B$.

We must construct maps $\mathscr{M}_\varphi : \mathscr{M}_{(A,\eta)} \to \mathscr{M}_{(A',\eta')}$ for each homomorphism of commutative rings $\varphi : A \to A'$ such that $\varphi_*\eta = \eta'$. This amounts to the construction of a canonical isomorphism $A \otimes_B \mathscr{M}_B \simeq A \otimes_C \mathscr{M}_C$ when there are maps $\beta : B \to A$ and $\gamma : C \to A$ with $B, C \in \mathscr{D}$ and $\beta_*(\xi_B) = \eta = \gamma_*(\xi_C)$.

[61]The essential property we want here is for the functor $\mathsf{QCoh}(h_A) \to \mathsf{QCoh}(U)$ to be an equivalence. We could use that instead of the requirement that U be an fpqc cover.

If there is a common factorization of β and γ through some $D \in \mathscr{D}$ we will be done. This is not guaranteed to exist by our hypotheses, but it does at least exist over an fpqc cover of h^A, which is enough to prove that the map exists.

Finally we have to check that $\mathscr{M}_\psi \circ \mathscr{M}_\varphi = \mathscr{M}_{\psi\circ\varphi}$ when the composition makes sense. One proceeds once again by fpqc descent. $\square$

`lem:QCohD-small-gen`

Lemma 11.2.2. *Suppose that $\mathscr{D}$ is a diagram of commutative rings of cardinality κ. Then any non-zero quasi-coherent module $\mathscr{M}$ on $\mathscr{D}$ contains a quasi-coherent submodule $\mathscr{M}'$ whose cardinality is bounded by κ. Thus $\mathsf{QCoh}(\mathscr{D})$ is generated by the κ-small quasi-coherent modules.*

Proof. The following proof is adapted from [Sta, Lemma 077N].

We describe a process by which to construct $\mathscr{N}$. Begin by choosing some $A \in \mathscr{D}$ and some $x \in \mathscr{M}_A$ that is not zero. Let $\mathscr{N}^0_B$ be the sub-module of $\mathscr{M}_B$ generated by $\mathscr{M}_u(x)$ as u ranges over all homomorphisms $u : A \to B$ in $\mathscr{D}$. Then $\mathscr{N}^0 \subset \mathscr{M}$ is a $\mathcal{O}_X$-submodule of $\mathscr{M}$ but is unlikely to be quasi-coherent.

Assume now that $\mathscr{N}^j$ has already been defined and let $i = j + 1$. Note that $\mathscr{M}_A$ is the filtered union of the finitely generated submodules containing $\mathscr{N}^j_A$. We write $\mathscr{M}_A = \bigcup_{V\in\mathscr{V}} V$ where $\mathscr{V}$ is the collection of all such submodules. Therefore $B \otimes_A \mathscr{M}_A = \varinjlim_{V\in\mathscr{V}}(B \otimes_A V)$ and, by the exactness of filtered colimits,

$$\varinjlim_{V\in\mathscr{V}} \ker\Big(B \underset{A}{\otimes} V \to \mathscr{M}_B\Big) = 0$$

$$\varinjlim_{V\in\mathscr{V}} \operatorname{coker}\Big(B \underset{A}{\otimes} V \to \mathscr{M}_B\Big) = 0.$$

In particular, if $z \in B \otimes_A \mathscr{N}^j$ then there is some $V \in \mathscr{V}$ such that the image of z in $B \otimes_A V$ vanishes. For each u and z as above, choose $V(u, z) \in \mathscr{V}$ to be a submodule such that the image of z in $B \otimes_A V(u, z) = 0$. Similarly if $y \in \mathscr{N}^j_B$ then there is some $V \in \mathscr{V}$ such that y lies in the image of $B \otimes_A V$ under $\mathscr{M}_u$ and we take $W(u, y)$ to be an element of $\mathscr{V}$ such that y lies in the image of $B \otimes_A W(u, y)$.

Now we can define $\mathscr{N}^i_C$, for each $C \in \mathscr{D}$, to be the C-submodule of $\mathscr{M}_C$ generated by $\mathscr{N}^j_C$ and the images of $V(u, z)$ and $W(u, y)$ under $\mathscr{M}_v$ for all

1. homomorphisms $v : A \to C$ in $\mathscr{D}$,
2. homomorphisms $u : A \to B$ in $\mathscr{D}$,
3. elements $y \in \mathscr{N}^j_B$, and
4. elements $z \in B \otimes_A \mathscr{N}^j_A$ that induce zero in $B \otimes_A \mathscr{M}_A$.

It is immediate by induction that $\mathscr{N}^i \subset \mathscr{M}$ is an $\mathcal{O}_X$-submodule for all i and therefore that $\mathscr{N} = \bigcup_{i=0}^\infty \mathscr{N}^i$ is a $\mathcal{O}_X$-submodule of $\mathscr{M}$. We verify that $\mathscr{N}$ is quasi-coherent and produce a bound on the cardinalities of the submodules $\mathscr{N}_A \subset \mathscr{M}_A$ for all A.

First we verify that $\mathscr{N}$ is quasi-coherent. Suppose that $u : A \to B$ is a homomorphism in $\mathscr{D}$ and let φ denote the B-module homomorphism $B \otimes_A \mathscr{N}_A \to \mathscr{N}_B$. If $y \in \mathscr{N}_B$ then $y \in \mathscr{N}_B^j$ for some j and therefore y lies in the image of the map $B \otimes_A \mathscr{N}_A^{j+1} \to \mathscr{N}_B$. Of course, this is contained in the image of φ and we conclude that φ is surjective.

Now suppose that $z \in B \otimes_A \mathscr{N}_A$ lies in the kernel of the φ. Then because $\mathscr{N}$ is a filtered union, we have $B \otimes_A \mathscr{N}_A = \varinjlim_i (B \otimes_A \mathscr{N}_A^i)$ and therefore z is the image of some $z' \in B \otimes_A \mathscr{N}_A^i$. But the image of z' in $B \otimes_A \mathscr{M}_A \cong \mathscr{M}_B$ vanishes by assumption, to therefore the image of z' in $B \otimes_A \mathscr{N}_A^{i+1}$ vanishes as well, by the construction of $\mathscr{N}_A^{i+1}$. This completes the proof that $\mathscr{N}$ is quasi-coherent.

It remains to bound the cardinality of $\mathscr{N}_A$. It is sufficient to bound the cardinality of each $\mathscr{N}_A^i$ as $\mathscr{N}_A$ is the countable union of the $\mathscr{N}_A^i$. We do this by induction. We know that $\mathscr{N}_A^0$ is generated by a set S whose cardinality is bounded by the cardinality of the set of morphisms of $\mathscr{D}$, which is bounded by κ. Therefore $\#\mathscr{N}_A^0$ is bounded by $(1 + \#S + \#S^2 + \cdots)^{\#A}$. Taking an upper bound for this cardinal as A ranges in $\mathscr{D}$ we obtain an upper bound λ_0 for the size of $\mathscr{N}_A^0$ for all $A \in \mathscr{D}$.

Now we proceed by induction. We note that for each homomorphism $u : A \to B$ in $\mathscr{D}$ and each $y \in \mathscr{N}_B^j$ and each $z \in B \otimes_A \mathscr{N}_A^j$ the submodules $V(u, z)$ and $W(u, y)$ of $\mathscr{M}_A$ are generated by $\mathscr{N}_A^j$ and a finite additional collection of elements so

$$\#V(u, z) \leq \#\mathscr{N}_A^j \times \#A^{\aleph_0} \leq \lambda_j \times \#A^{\aleph_0}$$

and the same bound holds for $\#W(u, y)$. We obtain $\mathscr{N}_C^{j+1}$ by taking the submodule generated by all $V(u, z)$ and $W(u, y)$ under all maps $\mathscr{M}_v : \mathscr{M}_A \to \mathscr{M}_C$. The cardinality of $\mathscr{N}_C^{j+1}$ is therefore bounded by $\sum_{u,v,z} \#C \times \#V(u, z) + \sum_{u,v,y} \#C \times \#W(u, y)$. As the numbers of u and v are bounded by κ and of y and z are bounded by λ_j we can take the supremum over all $C \in \mathscr{D}$ and obtain an absolute bound λ_{j+1} on the size of $\mathscr{N}^{j+1}$ depending only on $\mathscr{D}$. Now taking the supremum over all j, we get an absolute bound λ on the size of $\mathscr{N}_A$ for all $A \in \mathscr{D}$ depending only on $\mathscr{D}$. □

Corollary 11.2.2.1. *Let X : $\mathsf{ComRng} \to \mathsf{Sets}$ be an essentially small functor. Then $\mathsf{QCoh}(X)$ is generated by a small subcategory.*

Proof. Since $\mathsf{QCoh}(X) \simeq \mathsf{QCoh}(\mathscr{D})$ for a diagram $\mathscr{D}$, it is sufficient to prove the assertion for $\mathsf{QCoh}(\mathscr{D})$, which is Lemma 11.2.2. □

Theorem 11.2.3. *Let X : $\mathsf{ComRng} \to \mathsf{Sets}$ be an essentially small functor. Then the functor $\mathsf{QCoh}(X) \to \mathsf{Mod}(X)$ has a right adjoint called the **coherator**.*

Proof. Apply the adjoint functor theorem, taking into account that $\mathsf{QCoh}(X)$ has a small collection of generators. □

11.3 Sheaves on functors

If $X : \mathsf{ComRng} \to \mathsf{Sets}$ is a functor, we have a collection of subfunctors $U \subset X$ that are said to be open. By definition, this means that for any commutative ring A and any $\xi \in X(A)$, the subfunctor $U_{(A,\xi)} \subset h^A$ is open, which, again by definition, means that $U_{(A,\xi)} = D(S)$ for some $S \subset A$, i.e.,

$$U_{(A,\xi)}(B) = \left\{\varphi \in h^A(B) \,\middle|\, \varphi(S)B = B\right\}.$$

We also know what it means for a collection of open subfunctors $U_i \subset X$ indexed by a set I to cover X: we should have

$$\bigcup_{i \in I} U_i(k) = X(k)$$

for any field k. A notion of open sets and knowledge of when open sets cover one another are all that we need to make sense of sheaves.

def:small-Zariski-sheaf

Definition 11.3.1. Let $\mathrm{Open}(X)$ denote the category of open subfunctors of X, where the morphisms are inclusions. A **presheaf** on (the small Zariski site of) X is a functor $F : \mathrm{Open}(X)^\circ \to \mathsf{Sets}$. It is said to be a sheaf if it satisfies the axioms **SH1** and **SH2**.

11.4 Quasi-coherent sheaves and quasi-coherent modules

Let $\mathscr{M}$ be a quasi-coherent module on a functor $X : \mathsf{ComRng} \to \mathsf{Sets}$. For each open $U \subset X$, define $M(U) = \Gamma(U, \mathscr{M})$. This is evidently a presheaf on X.

Exercise 11.4.1. Show that M, as defined above, is a sheaf on X.

Recall that we can define a quasi-coherent module $\mathscr{A}$ on X by

$$\mathscr{A}_{(A,\xi)} = A$$

for any commutative ring A and any $\xi \in X(A)$. Applying the above construction, we get a sheaf $\mathcal{O}_X$ on X with

$$\mathcal{O}_X(U) = \Gamma(U, \mathscr{A}).$$

Recall that sheaves of $\mathcal{O}_X$-modules form an *abelian category*. This means that we can form kernels and quotients and these satisfy all the familiar properties we expect from the category of modules under a commutative ring. Moreover, we can form arbitrary direct sums and direct products of sheaves of $\mathcal{O}_X$-modules as well.

def:qcoh-sheaf

Definition 11.4.2. We say that a sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules on X is **quasi-coherent** if there is an open cover of X by subfunctors U such that $\mathscr{F}\big|_U$ is isomorphic to the cokernel of a map $\mathcal{O}_U^{\oplus J} \to \mathcal{O}_U^{\oplus I}$ for some sets I and J.

Note that the definition of a quasi-coherent *sheaf* that we use here is different from the definition of a quasi-coherent *module*. Fortunately, the following theorem prevents this terminology from becoming unnecessarily confusing:

thm:qcoh-qcoh

Theorem 11.4.3. *Let X be a scheme and let $\mathscr{C}$ denote the category of quasi-coherent modules on X. Let $\mathscr{D}$ denote the category of quasi-coherent sheaves. Let Φ be the assignment that sends a quasi-coherent module $\mathscr{M}$ to the sheaf $M = \Phi(\mathscr{M})$ with*

$$M(U) = \Gamma(U, \mathscr{M}).$$

Then $\Phi(\mathscr{M})$ is a quasi-coherent sheaf and Φ is an equivalence of categories.

11.5 Cohomology of quasi-coherent sheaves on affine schemes

sec:cohom-qcoh-affine

The fundamental theorem concerning the cohomology of quasi-coherent sheaves is the following, due to Serre:

Theorem 11.5.1 (Serre)**.** *Let X be an affine scheme and F a quasi-coherent sheaf on X. Then $H^p(X, F) = 0$ for $p > 0$.*

While Serre only proved this for noetherian schemes, it is important to know that the cohomology of quasi-coherent sheaves is trivial for *all* affine schemes. The theorem in this generality is due to Grothendieck, but relies on spectral sequences, familiarity with which we don't assume. Kempf later showed that the spectral sequences in Grothendieck's argument could be hidden, and it is Kempf's proof that we will follow here.

The proof of the theorem relies essentially on the following lemma:

Lemma 11.5.2. *If $X = \operatorname{Spec} A$ is an affine scheme then the functor*

$$\Gamma : \mathsf{QCoh}(X) \to A\text{-}\mathsf{Mod}$$

is exact.

Proof. As a consequence of faithfully flat descent, we have seen that Γ is an equivalence of categories! ☐

For any open subset $U \subset X$, we denote by UF the sheaf $i_* i^* F$, where $i : U \to X$ is the inclusion. In concrete terms

$$UF(V) = F(U \cap V)$$

for any open $V \subset X$.

ex:U-left-exact

Exercise 11.5.3. Show that $F \mapsto UF$ is a left exact functor. That is, if

$$0 \to F \to G \to H \to 0$$

is an exact sequence of sheaves on X then the sequence

$$0 \to UF \to UG \to UH$$

is exact as well.

Lemma 11.5.4. *The map* $H^1(X, UF) \to H^1(U, F)$ *is injective.* [62]

Proof. Choose an inclusion $F \subset G$ with G flaccid. Let $H = G/F$ be the quotient sheaf.

Exercise 11.5.5. Suppose that G is a flaccid sheaf on X.

(i) Show that UG is a flaccid sheaf on X.

(ii) Show that $G\big|_U$ is flaccid sheaf on U.

This gives us an exact sequence

$$0 \to F\big|_U \to G\big|_U \to H\big|_U \to 0$$

and therefore a long exact sequence

$$0 \to \Gamma(U, F) \to \Gamma(U, G) \to \Gamma(U, H) \to H^1(U, F) \to 0,$$

taking into account that $H^1(U, G) = 0$ because $G\big|_U$ is flaccid.

Exercise 11.5.3 also gives us an exact sequence

$$0 \to UF \to UG \to UH.$$

If we denote by UH' the quotient UG/UF we get another long exact sequence

$$0 \to \Gamma(X, UF) \to \Gamma(X, UG) \to \Gamma(X, UH') \to H^1(X, UF) \to 0,$$

once again making use of the vanishing of $H^1(X, UG)$ because UG is flaccid.

Exercise 11.5.6. Show that for any sheaf on a topological space X there are *unique* maps

$$H^p(X, F) \to H^p(X, UF) \to H^p(U, F)$$

for all p that satisfy the following properties:

(i) they are natural in F;

(ii) for $p = 0$ they coincide with the restriction maps $F(X) \to F(U) \xrightarrow{F} (U)$;

(iii) if $0 \to F \to G \to H \to 0$ is exact then the diagram

$$\begin{array}{ccccc} H^p(X, H) & \longrightarrow & H^p(X, UH) & \longrightarrow & H^p(U, H) \\ \downarrow & & \downarrow & & \downarrow \\ H^{p+1}(X, F) & \longrightarrow & H^{p+1}(X, UF) & \longrightarrow & H^{p+1}(U, F) \end{array}$$

commutes.

We denote the image of $\alpha \in H^p(X, F)$ in $H^p(X, UF)$ by $U\alpha$. For the image in $H^p(U, F)$ we write $\alpha\big|_U$.

[62] This lemma is true without the assumption that F be quasi-cohrent.

Now, we consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(X, UF) & \longrightarrow & \Gamma(X, UG) & \longrightarrow & \Gamma(X, UH') & \longrightarrow & H^1(X, UF) & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(U, F) & \longrightarrow & \Gamma(U, G) & \longrightarrow & \Gamma(U, H) & \longrightarrow & H^1(U, F) & \longrightarrow & 0 \end{array}$$

Note that the vertical arrows on the left are isomorphisms by definition. The map $\Gamma(X, UH') \to \Gamma(U, F)$ factors as

$$\Gamma(X, UH') \to \Gamma(X, UH) \to \Gamma(U, H),$$

which is an injection followed by a bijection, whence $\Gamma(X, UH') \to \Gamma(U, H)$ is a bijection. It follows by the 5-lemma that $H^1(X, UF) \to H^1(U, F)$ is an injection. $\square$

ex:cycles-loc-trivial

Exercise 11.5.7. Let F be a sheaf on a topological space X and $\alpha \in H^p(X, F)$.

(i) Show that there is a cover of X by open sets U with $\alpha\big|_U = 0$.

(ii) Conclude that if $p = 1$ then there is a cover of X by open sets U with $U\alpha = 0$.

We are now in a position to show that $H^1(X, F) = 0$ for F quasi-coherent on an affine scheme X. Fix $\alpha \in H^1(X, F)$. Choose an open cover of X by open affine subsets U_i with $U_i\alpha = 0$.

Exercise 11.5.8. Show that an affine scheme is quasi-compact.

Exercise 11.5.9. Suppose that F_i, $i \in I$ is a collection of sheaves on a topological space X. Show that the natural map

$$H^p(X, \prod F_i) \to \prod H^p(X, F_i)$$

is a bijection.

Since X is quasi-compact, we may assume that the U_i are finite in number. Now consider the map $F \to \prod_i U_i F$ and let F' be the quotient. This gives us an exact sequence (using the exercise above)

$$0 \to \Gamma(X, F) \to \prod_i \Gamma(X, U_i F) \xrightarrow{\varphi} \Gamma(X, F') \xrightarrow{\theta} H^1(X, F) \xrightarrow{\psi} \prod_i \Gamma(X, U_i F).$$

Note that $\prod_i U_i F$ and F' are quasi-coherent sheaves on X, so by the exactness of Γ, we know that φ is surjective so θ is the zero map. On the other hand, the U_i were selected so that $\psi(\alpha) = 0$. We deduce therefore that α lies in the image of the zero map θ, hence is zero.

This proves the theorem in the case $p = 1$.

Corollary 11.5.9.1. *The sequence*

$$0 \to UF \to UG \to UH \to 0$$

is exact.

Proof. We will prove the stronger statement that $UG(V) \to UH(V)$ is surjective whenever $V = D(f) \subset X$. This implies the surjectivity of the map of sheaves because the open sets $D(f)$ form a basis for X.

In this case, $UG(V) = G(U \cap V)$ and $UH(V) = H(U \cap V)$ and we have an exact sequence

$$G(U \cap V) \to H(U \cap V) \to H^1(U \cap V, F).$$

But $U \cap V = U \cap D(f)$ is affine: if $U = \operatorname{Spec} A$ then it is isomorphism to $\operatorname{Spec} A[f\big|_U^{-1}]$. Therefore $H^1(U \cap V, F) = 0$ and the surjectivity follows. □

Corollary 11.5.9.2. *For any open $U \subset X$, the map $H^2(X, UF) \to H^2(U, F)$ is injective.*

Proof. Note that we have a commutative diagram

$$\begin{array}{ccc} H^1(X, UH) & \longrightarrow & H^1(U, H) \\ \downarrow & & \downarrow \\ H^2(X, UF) & \longrightarrow & H^2(U, F) \end{array}$$

in which the vertical arrows are isomorphisms because UG and $G\big|_U$ are flaccid. As the upper horizontal arrow is known to be injective it follows that the lower horizontal arrow is injective as well. □

Corollary 11.5.9.3. *We have $H^2(X, F) = 0$.*

Proof. Consider $\alpha \in H^2(X, F)$ and choose an open cover of X by U_i such that $\alpha\big|_{U_i} = 0$ for all i. Then $U_i\alpha = 0$ for all i. Let F' be the quotient of the map $F \to \prod U_i F$ so that we have an exact sequence

$$H^1(X, F') \to H^2(X, F) \xrightarrow{\psi} \prod_i H^2(X, U_i F).$$

Since F' is quasi-coherent, we know that $H^1(X, F') = 0$. Therefore ψ is injective. But $\psi(\alpha) = 0$ by construction, so $\alpha = 0$. □

Corollary 11.5.9.4. *For any open $U \subset X$, the map $H^3(X, UF) \to H^3(U, F)$ is injective.*

Proof. Choose an injection $F \to G$ with G flaccid and let $H = G/F$. As $H^1(V, F) = 0$ for every affine open subscheme V of X, we find as above that the sequence

$$0 \to UF \to UG \to UH \to 0$$

is exact and therefore that $H^3(X, UF) = H^2(X, UH)$ (using the fact that $H^2(X, UG) = H^3(X, UG) = 0$ since UG is flaccid). Now, if we select an injection $H \to G'$ with G' flaccid and let H' be the cokernel then we have an exact sequence

$$0 \to UH \to UG' \to UH' \to 0$$

since $H^1(V, UH) = H^2(V, UF) = 0$ for every affine open subscheme $V \subset X$. It follows therefore that $H^2(X, UH) = H^1(X, UH')$. Furthermore, we have a commutative diagram

$$\begin{array}{ccccc} H^1(X, UH') & \longrightarrow & H^2(X, UH) & \longrightarrow & H^3(X, UF) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(U, H') & \longrightarrow & H^2(U, H) & \longrightarrow & H^3(U, F) \end{array}$$

in which all of the horizontal arrows are isomorphisms. As the vertical arrow on the left is an injection, it follows that so too is the vertical arrow on the right. □

Corollary 11.5.9.5. *We have $H^3(X, F) = 0$ for any quasi-coherent sheaf F on the affine scheme X.*

Proof. The proof proceeds as before: Fix $\alpha \in H^3(X, F)$ and choose a cover of X by a finite collection of open subsets U_i such that $\alpha\big|_{U_i} = 0$. Then $U_i\alpha = 0$. Consider the exact sequence

$$0 \to F \to \prod U_i F \to F' \to 0$$

and therefrom the long exact sequence

$$H^2(X, F') \to H^3(X, F) \xrightarrow{\psi} \prod H^3(X, U_i F).$$

Note now that $H^2(X, F') = 0$ since F' is quasi-coherent and $\psi(\alpha) = 0$ by construction of the U_i. □

Exercise 11.5.10. Turn the arguments above into an inductive argument demonstrating the theorem.

11.6 Serre's theorem

Theorem 11.6.1 (Serre)**.** *Suppose that X is a quasi-compact scheme and $H^1(X, F) = 0$ for every quasi-coherent subsheaf of the structure sheaf $\mathcal{O}_X$. Then X is affine.*

Corollary 11.6.1.1. *For a quasi-compact scheme, the following properties are equivalent:*

(i) X is an affine scheme;

(ii) $H^p(X, F) = 0$ for all quasi-coherent sheaves F and all $p > 0$.

Let A be the ring $\Gamma(X, \mathcal{O}_X)$.

Exercise 11.6.2. Show that there is a canonical morphism of schemes $X \to \operatorname{Spec} A$.

Let x be a schematic point of X. Choose some open affine $U \simeq \operatorname{Spec} B \subset X$ with $x \in |U| \subset |X|$.

Exercise 11.6.3. Show that for any scheme X and any open subset $U \subset X$ there is a quasi-coherent sheaf of ideals $I \subset \mathcal{O}_X$ such that $U = D(I)$. (Hint: On each open affine $V = \operatorname{Spec} A \subset X$, note that there is some *radical* ideal $J \subset A$ such that $U \cap V = D(J)$. Define $I(U) = J$. Show that this glues to give a quasi-coherent sheaf of ideals.)

Exercise 11.6.4. (i) Show that we can choose quasi-coherent sheaves of ideals $J \subset I \subset X$ with $D(J) = U \setminus \{x\}$ and $D(I) = U$.

(ii) Show that I/J is a skyscraper sheaf at the point x.

(iii) Conclude that $H^0(X, I/J) \neq 0$.

Now consider the exact sequence

$$0 \to J \to I \to J/I \to 0$$

and its associated long exact sequence in cohomology

$$0 \to H^0(X, J) \to H^0(X, I) \to H^0(X, J/I) \to H^1(X, J) = 0.$$

Since $H^0(X, I/J) \neq 0$ there is some $f \in H^0(X, I)$ that is not contained in $H^0(X, J)$. Note that $H^0(X, I) \subset H^0(X, \mathcal{O}_X)$ so that we can regard f as an element of $\Gamma(X, \mathcal{O}_X)$.

Exercise 11.6.5. (i) Show that $D(f) \subset X$ is contained in U.

(ii) Show that x is contained in $D(f)$.

(iii) Show that $D(f)$ is affine. (Hint: Show $D(f) \subset X$ coincides with $D(f\big|_U) \subset U$ and use the fact that U is affine.)

Definition 11.6.6. A morphism of schemes $g : X \to Y$ is said to be **affine** if there is an open cover of Y by affine subschemes $U \subset Y$ such that $g^{-1}U \subset X$ is also affine.

ex:affine-morphism **Exercise 11.6.7.** Let $g : X \to Y$ be a morphism of schemes. Prove that the following properties are equivalent:

AFF1 There exists a cover $\mathcal{U}$ of Y by affine open subschemes $U \subset Y$ such that $g^{-1}U$ is affine for each $U \in \mathcal{U}$.

AFF2 For *any* affine subscheme $U \subset Y$, the scheme $g^{-1}U$ is affine.

def:affine-morphism

Definition 11.6.8. A morphism of schemes satisfying the equivalent conditions of Exercise 11.6.7 is called an **affine morphism**.

Exercise 11.6.9. (i) Show that the map $X \to \operatorname{Spec} A$ described above is affine.

(ii) Conclude that X is affine.

11.7 Čech cohomology

Let X be a topological space and $U \subset X$ an open subset. Let $\mathbf{Z}_U$ be the sheaf associated to the étale space $(\mathbf{Z} \times U) \cup (0 \times X)$. The stalks of $\mathbf{Z}_U$ at points of U are $\mathbf{Z}$; outside of U the stalks are zero. Note that $\mathbf{Z}_U$ is a sheaf of abelian groups.

Exercise 11.7.1. Show that for any sheaf of abelian groups F on X we have a natural identification $\operatorname{Hom}(\mathbf{Z}_U, F) \simeq F(U)$.

If $U \subset V$ then we have a map $\mathbf{Z}_U \to \mathbf{Z}_V$. We denote this map by $\iota_{U,V}$.

Let X be a topological space, $\{U_i\}_{i \in I}$ a collection of open subsets of X indexed by a set I, and F a sheaf of abelian groups on X. We fix the notation $U_{i_1 \cdots i_n} = U_{i_1} \cap \cdots \cap U_{i_n}$.

Construct a sequence of homomorphisms

$$\cdots \to \bigoplus_{i,j,k \in I} \mathbf{Z}_{U_{ijk}} \to \bigoplus_{i,j \in I} \mathbf{Z}_{ij} \to \bigoplus_{i \in I} \mathbf{Z}_{U_i} \to \mathbf{Z}_X.$$

The morphisms are given by

$$\sum_{i=0}^{n} (-1)^i \iota_{U_0 \cap \cdots \cap U_n, U_0 \cap \cdots \cap \hat{U}_i \cap \cdots U_n}.$$

Exercise 11.7.2. Let X be a scheme. Prove that the following properties are equivalent:

(i) The diagonal map $X \to X \times X$ is a closed embedding.

(ii) If $f, g : Z \to X$ are two morphisms then the locus in Z where f and g agree is a closed subfunctor of Z.

(iii) If U and V are affine open subschemes of X then $U \cap V$ is also affine.

Definition 11.7.3. A functor $X : \mathsf{ComRng} \to \mathsf{Sets}$ whose diagonal map $X \to X \times X$ is a closed embedding is said to be **separated**.

11.8 Spectral sequences

Suppose that $p : X \to Y$ is a morphism of topological spaces and F is a sheaf of abelian groups on X. Recall that p_*F is defined to be the presheaf on Y with $p_*F(U) = F(p^{-1}U)$.

Exercise 11.8.1. (i) Show that if F is a sheaf on X then p_*F is a sheaf on Y.

(ii) Show that p_* is a left exact functor. That is, for any exact sequence

$$0 \to F \to G \to H \to 0$$

of sheaves of abelian groups on X, the sequence

$$0 \to p_*F \to p_*G \to p_*H$$

is exact.

(iii) Show that if Y is a point then p_* coincides with the functor $\Gamma(X, -)$.

Since the category of sheaves of abelian groups (or $\mathcal{O}_X$-modules) has enough injectives, and injective objects form an adapted class to *any* left exact functor, provided enough injectives exist, we may define derived functors $R^i p_*$ for all $i \geq 0$. We have $R^0 p_* = p_*$ and for any exact sequence

$$0 \to F \to G \to H \to 0$$

of sheaves of abelian groups on X, we have an exact sequence

$$R^i p_* F \to R^i p_* G \to R^i p_* H \to R^{i+1} p_* F$$

on Y. Note that if F, G and H are sheaves of $\mathcal{O}_X$-modules on X then the long exact sequence lives in the category of $p_*\mathcal{O}_X$-modules. If $p : X \to Y$ is a morphism of schemes then we have a morphism of sheaves $\mathcal{O}_Y \to p_*\mathcal{O}_X$, whence the long exact sequence may also be viewed as a sequence of $\mathcal{O}_Y$-modules.[63]

Exercise 11.8.2. Show that $R^i p_* F$ is the sheafification of the presheaf $U \mapsto H^i(p^{-1}U, F)$.

The main question we will address in this section is how to recover the cohomology of F from the sheaves $R^i p_* F$.

[63] This map of sheaves is obvious from the definition of schemes through ringed spaces. From the functorial perspective we have adopted here it might be less obvious why it exists. To construct it, show that for any scheme X and any open $U \subset X$ we have $\Gamma(U, \mathcal{O}_X) = \mathrm{Hom}_{\mathsf{Sch}}(U, \mathbf{A}^1)$. Verify that if $p : X \to Y$ is a morphism of schemes then for any open $U \subset Y$ we have $\Gamma(U, p_*\mathcal{O}_X) = \mathrm{Hom}_{\mathsf{Sch}}(p^{-1}U, \mathbf{A}^1)$. Then define the map $\mathcal{O}_Y \to p_*\mathcal{O}_X$ on $U \subset Y$ to be the map sending $f : U \to \mathbf{A}^1$ to the composition $p^{-1}U \xrightarrow{p|_{p^{-1}U}} U \xrightarrow{f} \mathbf{A}^1$.

Edge homomorphisms

As a first step, we can construct *edge homomorphisms*

$$H^i(Y, p_*F) \to H^i(X, F)$$
$$H^i(X, F) \to H^0(Y, R^i p_*F).$$

To make these constructions, we will need the following result:

`ex:pushforward-acyclic` **Exercise 11.8.3.** Suppose that $p : X \to Y$ is a continuous map and I is a sheaf on X.

(i Suppose that I is flaccid. Show that p_*I is a flaccid sheaf on Y.

(ii Suppose that I is an injective sheaf of abelian groups. Show that p_*I is an injective sheaf of abelian groups on Y. (Hint: p_* has a left exact left adjoint p^*.)

We define the edge homomorphisms by induction. Note in the first case that the groups $H^0(Y, p_*F)$ and $H^0(X, F)$ are canonically identified; in this case, the edge homomorphisms are the two isomorphisms.

For $i = 1$, choose an embedding of F in a flaccid sheaf I. We get an exact sequence

$$0 \to p_*F \to p_*I \to p_*(I/F) \to R^1p_*F \to 0$$

of sheaves on Y, as well as isomorphisms $R^ip_*(I/F) \xrightarrow{\sim} R^{i+1}p_*F$. Apply $\Gamma(Y, -)$ to this sequence and we get

$$\begin{array}{ccccccccc}
0 \longrightarrow & \Gamma(X,F) & \longrightarrow & \Gamma(X,I) & \longrightarrow & \Gamma(X,I/F) & \longrightarrow & H^1(X,F) & \longrightarrow 0 \\
& \| & & \| & & \| & & \downarrow & \\
0 \longrightarrow & \Gamma(Y,p_*F) & \longrightarrow & \Gamma(Y,p_*I) & \longrightarrow & \Gamma(Y,p_*(I/F)) & \longrightarrow & \Gamma(Y,R^1p_*F) &
\end{array}$$

in which the upper horizontal line is exact. The dashed arrow is induced from the exactness of the upper line and the commutativity. Now assume that the map $H^j(X, F) \to \Gamma(Y, R^jp_*F)$ has been defined for $j = i - 1$ and all F. Then we get a map $H^{i-1}(X, I/F) \to \Gamma(Y, R^{i-1}p_*(I/F))$. Taking into account the isomorphisms $H^{i-1}(X, I/F) \simeq H^i(X, F)$ and $R^{i-1}p_*(I/F) \simeq R^ip_*F$, we get the desired map.

For the other edge homomorphism, let G be the image of the map $p_*I \to p_*(I/F)$ so that we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & p_*F & \longrightarrow & p_*I & \longrightarrow & G & \longrightarrow 0 \\
& \| & & \| & & \downarrow & \\
0 \longrightarrow & p_*F & \longrightarrow & p_*I & \longrightarrow & p_*(I/F). &
\end{array}$$

Apply $\Gamma(Y, -)$ to this diagram to obtain

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(Y, p_*F) & \longrightarrow & \Gamma(Y, p_*I) & \longrightarrow & \Gamma(Y, G) & \longrightarrow & H^1(Y, p_*F) & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(X, F) & \longrightarrow & \Gamma(X, I) & \longrightarrow & \Gamma(X, I/F) & \longrightarrow & H^1(X, F) & \longrightarrow & 0
\end{array}
$$

with the dashed arrow induced by the commutativity of the diagram and the exactness of the rows. Assume now that the map $H^j(Y, p_*F) \to H^j(X, F)$ has been defined for $j = i-1$ and all F. We therefore have a map $H^{i-1}(Y, p_*(I/F)) \to H^{i-1}(X, I/F)$ and by composing the sequence of maps

$$H^i(Y, p_*F) = H^{i-1}(Y, G) \to H^{i-1}(Y, p_*(I/F)) \to H^{i-1}(X, I/F) = H^i(X, F)$$

we get the map $H^i(Y, p_*F) \to H^i(X, F)$.

ex:5-term-start **Exercise 11.8.4.** Show that the composition of edge homomorphisms gives an exact sequence

$$0 \to H^1(Y, p_*F) \to H^1(X, F) \to H^0(Y, R^1p_*F).$$

(Hint: Construct a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Gamma(Y, p_*I) & = & \Gamma(Y, p_*I) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Gamma(Y, G) & \longrightarrow & \Gamma(Y, p_*I/F) & \longrightarrow & \Gamma(Y, R^1p_*F) & = & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(Y, p_*F) & \longrightarrow & H^1(X, F) & \longrightarrow & \Gamma(Y, R^1p_*F) & &
\end{array}
$$

in which the middle row is exact by construction.)

ex:5-term-torsors **Exercise 11.8.5.** Reinterpret Exercise 11.8.4 in terms of torsors.

How close are the edge morphisms to being isomorphisms?

Exercise 11.8.4 gives the beginning of the answer to this question.

Continuing to use the notation introduced in the last section, we consider the exact sequences

$$0 \to p_*F \to p_*I \to G \to 0$$
$$0 \to G \to p_*(I/F) \to R^1p_*F \to 0$$

From these we obtain morphisms

$$H^i(Y, R^1p_*F) \to H^{i+1}(Y, p_*G) \to H^{i+2}(Y, p_*F).$$

In fact, we can construct a map

$$H^i(Y, R^jp_*F) \to H^{i+2}(Y, R^{j-1}p_*F)$$

for all i and j. Indeed, assuming the map has been constructed for $j' = j - 1$ and taking into account the maps $R^{j'}p_*(I/F) \to R^jp_*F$, we also get maps

$$H^i(Y, R^jp_*F) = H^i(Y, R^{j'}p_*(I/F)) \to H^{i+2}(Y, R^{j'-1}p_*(I/F)) \to H^{i+2}(Y, R^{j-1}p_*F).$$

Exercise 11.8.6. Show that for all i and j, the composition

$$H^{i-2}(Y, R^{j+1}p_*F) \to H^i(Y, R^jp_*F) \to H^{i+2}(Y, R^{j-1}, p_*F)$$

is zero.

Exercise 11.8.7. Show that for all i and j, the sequences

$$H^i(X, F) \to H^0(Y, R^ip_*F) \to H^2(Y, R^{i-1}p_*F)$$
$$H^{i-2}(Y, R^1p_*F) \to H^i(Y, p_*F) \to H^i(X, F)$$

are exact.

ex:5-term **Exercise 11.8.8.** Conclude from the previous exercise that there is an exact sequence

$$0 \to H^1(Y, p_*F) \to H^1(X, F) \to H^0(X, R^1p_*F) \to H^2(Y, p_*F) \to H^2(X, F).$$

This is known as the 5**-term exact sequence** of the Leray spectral sequence.

11.9 The cohomology of quasi-coherent sheaves on affine schemes (reprise)

11.10 Quasi-coherent sheaves on projective space

The results of this section are essentially due to Serre [Ser]; many other treatments are availabe: see [Vak, §16.4] or [Har, §II.5].

In this section we'll fix a positive integer n and study quasi-coherent sheaves on a fixed projective space $\mathbf{P}^n$. Our strategy will be to relate quasi-coherent sheaves on $\mathbf{P}$ to quasi-coherent sheaves on the open subscheme $U = D(t_0, \dots, t_n) \subset \mathbf{A}^{n+1}$ and then to relate quasi-coherent sheaves on U to quasi-coherent sheaves on $\mathbf{A}^{n+1}$.

Note first that there is a map $\pi : U \to \mathbf{P}^n$ defined by

$$\pi(x_0, \dots, x_n) = (A, x_0, \dots, x_n)$$

whenever $(x_0, \dots, x_n) \in U(A)$.

ex:U-basics **Exercise 11.10.1.** (i) Verify that the map defined above is actually well-defined. Make sure, in other words, that $\pi(x_0, \dots, x_n) \in \mathbf{P}^n(A)$ when $(x_0, \dots, x_n) \in U(A)$.

(ii) Show that the inclusion of the image of π in $\mathbf{P}^n$ is an fpqc cover.

Modules on open subsets

Let $i : U \to \mathbf{A}^{n+1}$ denote the inclusion. Recall that we can define functors

$$i_* : \mathsf{QCoh}(U) \to \mathsf{QCoh}(\mathbf{A}^{n+1})$$
$$i^* : \mathsf{QCoh}(\mathbf{A}^{n+1}) \to \mathsf{QCoh}(U)$$

as follows:

$$i_* F(V) = F(U \cap V) i^* F(V) = F(i(V)).$$

ex:U-A-sheaves **Exercise 11.10.2.** (i) Verify that if F is a sheaf on U then $i_* F$ is a sheaf on $\mathbf{A}^{n+1}$.

(ii) Verify that if F is a sheaf on $\mathbf{A}^{n+1}$ then $i^* F$ is a sheaf on U.

(iii) Show that $i^* i_* : \mathsf{Sh}(U) \to \mathsf{Sh}(U)$ is the identity functor.

U-A-sheaves-fully-faithful (iv) Show that the functor i_* is full faithful.

(v) Construct a canonical map $F \to i_* i^* F$ for any sheaf F on U sending $x \in F(V)$ to $x\big|_{U \cap V} \in i_* i^* F(V) = F(U \cap V)$.

(vi) Show that the functor $i_* i^*$ is left exact.

Exercise 11.10.2 (iv) tells us that $\mathsf{QCoh}(U)$ can be viewed as a full subcategory of $\mathsf{QCoh}(\mathbf{A}^{n+1})$. The following exercise allows us to specify this functor precisely:

Exercise 11.10.3. Let I denote the ideal $(t_0, \ldots, t_n) \subset \mathbf{Z}[t_0, \ldots, t_n]$.

1. Let F be a quasi-coherent module on $\mathbf{A}^{n+1}$ corresponding to a $\mathbf{Z}[t_0, \ldots, t_n]$-module M. Show that $i^* F = 0$ if and only if, for every $x \in M$ and every index $0 \leq i \leq n$, there is some exponent k such that $t^k x = 0$.

2. Let M be a $\mathbf{Z}[t_0, \ldots, t_n]$-module. For each non-negative integer m, construct a map
$$M \to \mathrm{Hom}(I^m, M)$$
sending $x \in M$ to the map f with $f(a_1 \cdots a_m) = a_1 \cdots a_m x$.

3. Let F be a quasi-coherent module on $\mathbf{A}^{n+1}$ and let F' be the kernel of the map $F \to i_* i^* F$. Show that $i^* F' = 0$.

4. Conclude that $F \to i_* i^* F$ is injective if and only if the map $M \to \mathrm{Hom}(I^m, M)$ is injective for all integers $m \geq 0$.

5. Assume that $F \to i_* i^* F$ is injective suppose that $x \in i_* i^* F$. Let (x) be the sub-module of $i_* i^* F$ generated by x. Show that $(x) \cap F$ contains $I^m x$ for some $m \geq 0$.

6. Conclude that $F \to i_* i^* F$ is a bijection if and only if the map $M \to \mathrm{Hom}(I^m, M)$ is a bijection for all integers $m \geq 0$.

The exercise gives us the following corollary, which we separate for the sake of later reference:

cor:qcoh-U

Corollary 11.10.3.1. *The category* $\mathsf{QCoh}(U)$ *is equivalent, via the functor* $i_* : \mathsf{QCoh}(U) \to \mathsf{QCoh}(\mathbf{A}^{n+1})$*, to the full subcategory of quasi-coherent modules on* $\mathbf{A}^{n+1}$ *such that, for every* $m \geq 0$*, the map*

$$M \to \mathrm{Hom}(I^m, M)$$

is a bijection.

Equivariant modules

Let V be the image of the map of functors $U \to \mathbf{P}^n$. By Exercise 11.10.1, the inclusion of V in $\mathbf{P}^n$ is an fpqc cover. We can therefore conclude that $\mathsf{QCoh}(\mathbf{P}^n) \simeq \mathsf{QCoh}(V)$. In this section we will relate $\mathsf{QCoh}(V)$ to $\mathsf{QCoh}(U)$.

By definition a quasi-coherent module $\mathscr{M}$ on V consists of an A-module $\mathscr{M}_{(A,\xi)}$ for each commutative ring A and each $\xi \in V(A)$.

By definition, $\xi \in V(A)$ means that there is some $\eta \in U(A)$ with $\pi(\eta) = \xi$. If η' is another lift of ξ then there is some $\lambda \in A^*$ with $\eta' = \lambda\eta$. Thus we can specify a quasi-coherent module on V by giving a quasi-coherent module $\mathscr{M}$ on U with the stipulation that $\mathscr{M}_{(A,\eta)} = \mathscr{M}_{(A,\lambda\eta)}$ whenever $\eta \in A^*$.

More generally, if $\mathscr{M}$ is a quasi-coherent module on $\mathbf{A}^{n+1}$ such that $\mathscr{M}_{(A,\eta)} = \mathscr{M}_{(A,\lambda\eta)}$ for every $\eta \in \mathbf{A}^{n+1}(A)$ and $\lambda \in A^*$ then we will say that $\mathscr{M}$ is an *equivariant* quasi-coherent module. Equivariant quasi-coherent modules on $\mathbf{A}^{n+1}$ form a full subcategory of $\mathsf{QCoh}(\mathbf{A}^{n+1})$ that we will denote $\mathsf{QCoh}([\mathbf{A}^{n+1}/\mathbf{G}_m])$.[64] Then we can summarize the above discussion in the following proposition:

Proposition 11.10.4. $\mathsf{QCoh}(\mathbf{P}^n)$ *is equivalent to the category of equivariant quasi-coherent modules on* $\mathbf{A}^{n+1}$ *that satisfy the condition of Corollary 11.10.3.1.*

Now we will try to get a more concrete sense of what equivariant modules are like on $\mathbf{A}^{n+1}$. Of course, a quasi-coherent module on $\mathbf{A}^{n+1}$ is nothing but a $\mathbf{Z}[x_0, \ldots, x_n]$-module M^0. Let us abbreviate $A = \mathbf{Z}[x_0, \ldots, x_n]$ and consider the two maps $A \to A[t, t^{-1}]$ defined below:

$$\eta : x_i \mapsto x_i$$
$$t\eta : x_i \mapsto tx_i.$$

Note that t is a unit in $A[t, t^{-1}]$ so that if $\mathscr{M}$ is equivariant and $\eta' = t\eta$, we will have $\mathscr{M}_{(A[t,t^{-1}],\eta)} = \mathscr{M}_{(A[t,t^{-1}],\eta')}$; we introduce the notation M^1 for this

[64] The reason for this notation is that the modules are equivariant with respect to the action of $\mathbf{G}_m$ on $\mathbf{A}^{n+1}$ given by $\lambda.(x_0, \ldots, x_n) = (\lambda x_0, \ldots, \lambda x_n)$. The notation $[\mathbf{A}^{n+1}/\mathbf{G}_m]$ stands for the quotient *stack* of this action, whose quasi-coherent modules are naturally identified—essentially by definition—with the equivariant quasi-coherent modules on $\mathbf{A}^{n+1}$.

common $A[t,t^{-1}]$-module. Now, if ξ denotes the identity map $A \to A$, we get two maps

$$M^0 = \mathscr{M}_{(A,\xi)} \xrightarrow{\mathscr{M}_\eta} \mathscr{M}_{(A[t,t^{-1}],\eta)} = M^1$$
$$M^0 = \mathscr{M}_{(A,\xi)} \xrightarrow{\mathscr{M}_{\eta'}} \mathscr{M}_{(A[t,t^{-1}],\eta')} = M^1.$$

In other words, we have two maps $\varphi, \psi : M^0 \to M^1$ satisfying

$$\varphi(az) = \eta(a)\varphi(z)\psi(az) = \eta'(a)\varphi(z).$$

Recall that by the definition of quasi-coherence, the map

$$A[t,t^{-1}] \underset{A}{\otimes} M^0 \to M^1 : (a,x) \mapsto a\varphi(x)$$

must be an isomorphism.[65] We can see explicitly that

$$A[t,t^{-1}] \underset{A}{\otimes} M^0 \simeq \sum_{j=-\infty}^{\infty} t^j M^0$$

and that we may therefore identify $M^1 = \sum_{j=-\infty}^{\infty} t^j \varphi(M^0)$.[66] For each integer j, let $M^0_j = \psi^{-1}(t^j M^0)$. This makes M^0 into a *graded A-module*:

Definition 11.10.5. Let A be a commutative ring. A **grading** on A is a collection of subgroups $A_j \in A$ for each $j \in \mathbf{Z}$ such that

GR1 $A = \sum_{j\in\mathbf{Z}} A_j$ and $A_j \cap A_k = 0$ for $j \neq k$[67] and

GR2 $A_j A_k \subset A_{j+k}$ for all $j,k \in \mathbf{Z}$.

A graded morphism of graded rings $\varphi : A \to B$ is a homomorphism of commutative rings that satisfies $\varphi(A_j) \subset B_j$ for all $j \in \mathbf{Z}$.

If A is a graded ring, an A-module M is said to be a **graded module** if it is equipped with a collection of subgroups $M_j \subset M$ for each $j \in \mathbf{Z}$ satisfying

GM1 $M = \sum_{j\in\mathbf{Z}} M_j$ and $M_j \cap M_k = 0$ if $j \neq k$, and

GM2 $A_j M_k \subset M_{j+k}$ for all $j,k \in \mathbf{Z}$.

A morphism of graded A-modules $\varphi : M \to M'$ is an A-module homomorphism that satisfies $\varphi(M_j) \subset M'_j$. If A is a graded commutative ring, we write $\mathsf{Gr}\text{-}A\text{-}\mathsf{Mod}$ for the category of graded A-modules.

Exercise 11.10.6. (i) Suppose that A is a graded commutative ring. Formulate the definition of a graded A-algebra.

[65] Here we are using the map $\eta : A \to A[t,t^{-1}]$ to form the tensor product. Of course, if we used η' and the map $(a,x) \mapsto a\psi(x)$ then we would obtain another map that must be an isomorphism.

[66] And similarly, $M^1 = \sum_{j=-\infty}^{\infty} t^j \psi(M^0)$, though we won't use this directly.

[67] That is, A is isomorphic to the direct sum of the A_j as an abelian group.

(ii) Suppose that $\mathbf{Z}$ is given the grading where $\mathbf{Z}_0 = \mathbf{Z}$ and $\mathbf{Z}_j = 0$ for $j \neq 0$. Show that a graded $\mathbf{Z}$-algebra is the same thing as a graded commutative ring.

Note that the discussion above applies in particular to the quasi-coherent module $\mathcal{O}$ having $\mathcal{O}_{(A,\xi)} = A$. It follows that if we put $A_j = {\eta'}^{-1}(t^j \eta(A))$ then A has the structure of a graded commutative ring.

Exercise 11.10.7. Show that $A_j \subset A = \mathbf{Z}[x_0, \dots, x_n]$ consists of all homogeneous polynomials in the variables $x_0, \dots, x_n$ of total degree j.

Exercise 11.10.8. With the definition of M_j^0 as above, show that M_j^0 is a graded A-module.

Proposition 11.10.9. *Let $A = \mathbf{Z}[x_0, \dots, x_n]$. The construction above gives an equivalence from the category of equivariant A-modules to the category of graded A-modules.*

Proof. We will be content to sketch the inverse construction. Let M be a graded A-module and let $\mathscr{M}$ be the associated quasi-coherent module on $\mathbf{A}^{n+1}$. For each $\eta : A \to B$ we therefore have $\mathscr{M}_{(B,\eta)} = B \otimes_A M$. Suppose that $\eta, \eta' : A \to B$ are equivalent maps, meaning that there is some $\lambda \in B^*$ such that $\eta'(x_i) = \lambda \eta(x_i)$ for every $i = 0, \dots, n$. Define an isomorphism

$$\mathscr{M}_{(B,\eta)} \to \mathscr{M}_{(B,\eta')}$$

by sending $b \otimes z$ to $b \otimes \lambda^j z$ when z is contained in the graded piece $M_j \subset M$.

Exercise 11.10.10. (i) Verify that the construction above gives an isomorphism between $\alpha : \mathscr{M}_{(B,\eta)} \to \mathscr{M}_{(B,\eta')}$ satisfying $\alpha(\eta(x)z) = \eta'(x)\alpha(z)$.

(ii) Show that if $\eta' = \lambda\eta$ and $\eta'' = \mu\eta'$ then the composition of the bijections

$$\mathscr{M}_{(B,\eta)} \to \mathscr{M}_{(B,\eta')} \to \mathscr{M}_{(B,\eta'')}$$

constructed above coincides with the map $\mathscr{M}_{(B,\eta)} \to \mathscr{M}_{(B,\eta'')}$ obtained by the same construction.

(iii) Deduce that we may identify all of the $\mathscr{M}_{(B,\eta)}$ for equivalent homomorphisms $\eta : A \to B$ and still have a quasi-coherent module on $\mathbf{A}^{n+1}$.[68] Thus we obtain an equivariant quasi-coherent module on $\mathbf{A}^{n+1}$.

(iv) Verify that this construction gives a functor from $\mathsf{Gr}\text{-}A\text{-}\mathsf{Mod}$ to $\mathsf{QCoh}([\mathbf{A}^{n+1}/\mathbf{G}_m])$ and show that this functor is inverse to the functor $\mathsf{QCoh}([\mathbf{A}^{n+1}/\mathbf{G}_m]) \to \mathsf{Gr}\text{-}A\text{-}\mathsf{Mod}$ described earlier.

□

[68] The intuitive meaning here is that you can treat all of the $\mathscr{M}_{(B,\eta)}$ as being equal. The slickest way I know to make that precise is to replace each $\mathscr{M}_{(B,\eta)}$ with $\varprojlim_{\eta'} \mathscr{M}_{(B,\eta')}$ where the limit is taken over the diagram of all η' equivalent to η. A less slick way (requiring the axiom of choice) is to choose on η in each equivalent class and replace $\mathscr{M}_{(B,\eta')}$ with $\mathscr{M}_{(B,\eta)}$ for all η' equivalent to η.

Quasi-coherent modules on projective space

Combining the results of the last two sections, we can characterize the quasi-coherent modules on projective space:

Theorem 11.10.11. *Let A denote the commutative ring $\mathbf{Z}[x_0, \ldots, x_n]$, graded by total degree in the variable $x_0, \ldots, x_n$. Let I denote the ideal $(x_0, \ldots, x_n) \subset A$. There is an equivalence of categories between $\mathsf{QCoh}(\mathbf{P}^n)$ and the category of graded equivariant A-modules M such that, for any non-negative integer m, the map*

$$M \to \mathrm{Hom}_{A\text{-}\mathsf{Mod}}(I^m, M)$$

is a bijection.

Invertible sheaves on $\mathbf{P}^n$

Definition 11.10.12. Let A be a graded commutative ring and M a graded A-module with graded pieces $M_j \subset M$. Denote by $M(p)$ the graded A-module with $M(p)_j = M_{j+p}$.

Definition 11.10.13. Let A be a graded commutative ring. We call a graded A-module invertible if its underlying A-module is invertible.

Exercise 11.10.14. Show that a quasi-coherent sheaf $\mathscr{L}$ on $\mathbf{P}^n$ is invertible if and only if the corresponding graded $\mathbf{Z}[x_0, \ldots, x_n]$-module is invertible. (Hint: Use the fact that $\mathscr{L}$ is invertible if and only if $\pi^*\mathscr{L}$ is invertible, where $\pi : U \to \mathbf{P}^n$ is the projection defined earlier in this section.)

Exercise 11.10.15. Let A be the commutative ring $\mathbf{Z}[x_0, \ldots, x_n]$, graded by total degree. Show that a graded A-module L is invertible if and only if L is isomorphic to $A(p)$ for some integer p.

Exercise 11.10.16. Let A be the graded commutative ring $\mathbf{Z}[x_0, \ldots, x_n]$, as above. Show that the graded A-module $A(p)$ corresponds, under the equivalence of the last section, to the invertible $\mathcal{O}_{\mathbf{P}^n}$-module $\mathcal{O}_{\mathbf{P}^n}(p)$.

Corollary 11.10.16.1. *We have $\mathsf{Pic}(\mathbf{P}^n) \simeq \mathbf{Z}$ with $p \in \mathbf{Z}$ corresponding to $\mathcal{O}_{\mathbf{P}^n}(p)$.*

11.11 The cohomology of quasi-coherent invertible sheaves on projective space

12 Infinitesimal properties

sec:infinitesimal

12.1 The dual numbers and the tangent space

Let A be a commutative ring. Consider the ring

$$A[\epsilon]/(\epsilon^2).$$

Elements of this ring are symbols $a + b\epsilon$ where $a, b \in A$. We have

$$(a + b\epsilon)(a' + b'\epsilon) = aa' + (ab' + a'b)\epsilon.$$

When $A = \mathbf{Z}$ this is known as the ring of **dual numbers**. Frequently we simply write $A[\epsilon]$ rather than $A[\epsilon]/(\epsilon^2)$ for the sake of brevity.

Note that there is unique a homomorphism of commutative rings $A[\epsilon] \to A$ sending ϵ to 0 and restricting to the identity on A.

When $A = k$ is a field, we think of $\operatorname{Spec} k[\epsilon]$ as a point with a tangent vector sticking out of it. In general, we think of $\operatorname{Spec} A[\epsilon]$ as a family of tangent vectors at each point of $\operatorname{Spec} A$. This motivates the following definition:

`def:tangent-space`

Definition 12.1.1. Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor and let A be a commutative ring. Suppose that $\xi \in X(A)$. Let $\pi : A[\epsilon] \to A$ denote the unique A-algebra homomorphism wth $\pi(\epsilon) = 0$. We define

$$T_\xi X = \{\eta \in X(A[\epsilon]) \mid \pi_*(\eta) = \xi\}.$$

When A is a field, we call $T_\xi X$ the **tangent space** to X at ξ.

`ex:zero-tangent-vector`

Exercise 12.1.2. Let $i : A \to A[\epsilon]$ be the homomorphism sending a to $i(a) = a + 0\epsilon$. Show that this is a homomorphism of commutative rings and verify that $i_*\xi \in T_\xi X$ for any functor $X : \mathsf{ComRng} \to \mathsf{Sets}$ and any $\xi \in X(A)$. Deduce that for any $\xi \in X(A)$ the tangent space $T_\xi X$ has the structure of a *pointed set*.[69] We call the distinguished element the **zero tangent vector**.

Solution. First, we deduce that i is in fact a ring homomorphism. Let $a, b \in A$. Then

$$i(a + b) := a + b + 0\epsilon = a + 0\epsilon + b + 0\epsilon = i(a) + i(b),$$

and

$$i(a \cdot b) = a \cdot b + 0\epsilon = (a + 0\epsilon) \cdot (b + 0\epsilon) = i(a) \cdot i(b).$$

The second part is immediate: $\pi \circ i = \mathrm{id}_A$, and so we have

$$\pi_* \circ i_* = \mathrm{id}_{X(A)}.$$

Thus, the tangent space $T_\xi X$ has a canonical choice of element, giving it the structure of a pointed set. □

Warning 12.1.3. Intuition from geometry may lead you to expect that the tangent space is a vector space and that the zero tangent vector is the zero element of the vector space. This is true when X is a scheme, but is not true for all functors $X : \mathsf{ComRng} \to \mathsf{Sets}$.

`def:derivation`

Definition 12.1.4. Let A be a commutative ring, B an A-algebra, and M a B-module. An A-**derivation** from B into M is a function $\delta : B \to M$ such that

DER1 $\delta(A) = 0$ and

[69] A pointed set is a set with a distinguished element.

DER2 $\delta(xy) = x\delta(y) + y\delta(x)$.

ex:tangent-rep-func **Exercise 12.1.5.** Let A and B be commutative rings and $\xi \in h^A(B)$. Construct a natural (in both A and B) bijection

$$T_\xi h^A \simeq \mathrm{Der}_{\mathbf{Z}}(A, B)$$

where B is given an A-module structure by means of the homomorphism $\xi : A \to B$.

Solution. We begin by selecting commutative rings A and B, where B is equipped with the structure of an A-module via $\xi : A \to B$.

We will first define the desired map, show that it is a bijection, and then verify its naturality in both A and B.

Let $\nu \in T_\xi h^A$. By definition, $\nu : A \to B[\epsilon]$ is a lift of ξ via the projection map π. Denote by τ the trivial lift: $\tau : a \mapsto \xi(a) + 0\epsilon$. Then $\delta := \nu - \tau$ induces a map $A \to \epsilon B$. But ϵB is isomorphic to B as a $\mathbf{Z}$-module, and so we may consider δ as a map $A \to B$.

Our claim is now two-fold: $\delta \in \mathrm{Der}_{\mathbf{Z}}(A, B)$, and that every element in $\mathrm{Der}_{\mathbf{Z}}(A, B)$ arises in this manner.

First we demonstrate that $\delta(1) = 0$. Being a ring homomorphism $\xi(1) = 1$, and so $\tau(1) = 1$. Similarly, ν is a ring homomorphism, and so $\nu(1) = 1$, implying that $\delta(1) = 0$. It follows that $\delta(n) = 0$ for all $n \in \mathbf{Z}$.

Next, the Leibniz rule:
Let $a, b \in A$. Denote $\nu(a) = a + \alpha\epsilon$ and $\nu(b) = b + \beta\epsilon$. Then

$$\delta(ab) = \nu(ab) - \tau(ab) = (a+\alpha\epsilon)(b+\beta\epsilon) - ab = ab + (b\alpha + a\beta)\epsilon - ab = a\delta(b) + b\delta(a).$$

Thus, $\delta \in \mathrm{Der}_{\mathbf{Z}}(A, B)$.

On the other hand, if $D \in \mathrm{Der}_{\mathbf{Z}}(A, B)$, then $\tau + D$ defines a lift of ξ. We conclude that there is a bijection between $T_\xi h^A$ and $\mathrm{Der}_{\mathbf{Z}}(A, B)$.

Next, we consider the naturality of the bijection in A and B. First A: let $\phi : C \to A$ be a ring homomorphism. Then ϕ induces a map $h^A \to h^C$ by pulling back, and thus grants us a map

$$\phi_* : T_\xi h^A \to T_{\phi^* \xi} h^C,$$

where $\nu : A \to B[\epsilon]$ is sent to $\nu \circ \phi : C \to B[\epsilon]$.

On the other hand, ϕ also induces a map

$$\mathrm{Der}_{\mathbf{Z}}(A, B) \to \mathrm{Der}_{\mathbf{Z}}(C, B),$$

where a derivation δ is sent to $\delta \circ \phi$.

We verify that $\delta \circ \phi \in \mathrm{Der}_{\mathbf{Z}}(C, B)$. First, $\phi(1) = 1$, and so $\delta(\phi(n)) = 0$ for any $n \in \mathbf{Z}$. Second, if $x, y \in C$,

$$\delta(\phi(xy)) = \delta(\phi(x)\phi(y)) = \phi(x)\delta(\phi(y)) + \phi(y)\delta(\phi(x)).$$

Now we are ready to check naturality in A. Let τ_A be the trivial lift of ξ. Observe that $\tau_C = \tau_A \circ \phi$.

Let $\nu \in T_\xi h^A$. The bijection between tangent spaces and derivations sends ν to $\delta_A := \nu - \tau_A$. Pulling back via ϕ sends δ_A to a $\mathbf{Z}$-derivation of C into B:

$$\phi^*\delta_A = \delta_A \circ \phi = \nu \circ \phi - \tau_A \circ \phi = \nu \circ \phi - \tau_C.$$

Pushing forward on tangent spaces sends ν to $\nu \circ \phi$. The bijection between tangent spaces and derivations sends $\nu \circ \phi$ to $\delta_C := \nu \circ \phi - \tau_C$.

Thus, we deduce that the bijection is natural in A.

To complete the proof, let $\psi : B \to D$ be a homomorphism of rings. Then ψ induces a map $\psi_\epsilon : B[\epsilon] \to D[\epsilon]$ by $b_0 + b_1\epsilon \mapsto \psi(b_0) + \psi(b_1)\epsilon$, and therefore a map $T_\xi h^A \to T_{\psi\circ\xi} h^A$. Furthermore, ψ induces a map $\mathrm{Der}_{\mathbf{Z}}(A, B) \to \mathrm{Der}_{\mathbf{Z}}(A, D)$, again by composition.

We verify that for a $\mathbf{Z}$-derivation of A into B, δ, $\psi \circ \delta$ is a derivation of A into D. For any $n \in \mathbf{Z}$, $\psi \circ \delta(n) = 0$ because δ is a derivation. Next, for any $a, b \in A$,

$$\psi \circ \delta(ab) = \psi(a\delta(b) + b\delta(a)) = \psi(a)\psi \circ \delta(b) + \psi(b)\psi \circ \delta(a).$$

Now fix a $\nu \in T_\xi h^A$, and let τ_B denote the trivial lift of ξ. Note that $\tau_C = \psi_\epsilon \circ \tau_B$.[70] The bijection between tangent spaces and derivations sends ν ←70
to $\delta_B := \nu - \tau_B$. The map induced by ψ on derivations then sends δ_B to

$$\psi_\epsilon \circ \delta_B = \psi_\epsilon \circ \nu - \psi_\epsilon \circ \tau_B = \psi_\epsilon \circ \nu - \tau_C.$$

On the other hand, the map between tangent spaces induced by ψ sends ν to $\psi_\epsilon \circ \nu$. The bijection between tangent spaces and derivations then sends $\psi_\epsilon \circ \nu$ to

$$\psi_\epsilon \circ \nu - \tau_C.$$

We conclude that the bijection is natural in B. □

def:tangent-bundle

Definition 12.1.6. Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor. Define a new functor $TX : \mathsf{ComRng} \to \mathsf{Sets}$ by

$$TX(A) = \big\{(\xi, \eta) \,\big|\, \xi \in X(A),\ \eta \in T_\xi X\big\}$$

The functor TX is known as the **tangent bundle** of X.

Exercise 12.1.7. Show that there is a morphism of functors $\pi : TX \to X$ and that the fiber of π over $\xi \in X(A)$ is $T_\xi X$.

Exercise 12.1.8. Show that $T\mathbf{A}^n \simeq \mathbf{A}^n \times \mathbf{A}^n$ and that the projection $T\mathbf{A}^n \to \mathbf{A}^n$ is thus identified with the map

$$\mathbf{A}^n \times \mathbf{A}^n \to \mathbf{A}^n : (x, y) \mapsto x.$$

Exercise 12.1.9. Let k be a field. Compute $T_\xi X$ at all points $\xi \in X(k)$ of $X = V(y^2 - x^3 - x^2) \subset \mathbf{A}^2$.

Exercise 12.1.10. Let k be a field. Compute $T_\xi X$ at all points $\xi \in X(k)$ of $X = V(y^2 - x^3) \subset \mathbf{A}^2$.

Exercise 12.1.11. Let k be a field. Compute $T_\xi X$ at all points $\xi \in X(k)$ of $X = V(x^3 + y^3 + z^3) \subset \mathbf{P}^2$.

[70] τ_C? same typo occurs below at least twice more

12.2 The vector space structure

Exercise 12.2.1. Let k be a field, let $k' = k[\epsilon]/(\epsilon^2)$, and let $k'' = k' \times_k k'$ where the two maps $k' \to k$ are the ones sending ϵ to zero.

(i) Prove that $k'' \cong k[\epsilon_1, \epsilon_2]/(\epsilon_1, \epsilon_2)^2$.

(ii) Construct a map $\delta : k'' \to k' : \epsilon_i \mapsto \epsilon$.

(iii) Show that if X is a scheme then $X(k'') = X(k') \times_{X(k)} X(k')$.

Suppose that $\xi \in X(k)$ and $v, w \in T_\xi X$. Obtain an element $(v, w) \in X(k'')$ and then define $v + w = \delta_*(v, w) \in T_\xi X$.

(iv) Show that this addition law is associative and commutative.

(v) Let λ be any element of k. Construct a k-algebra homomorphism $\mu^\lambda : k' \to k' : \epsilon \mapsto \lambda\epsilon$.

For $v \in T_\xi X$, define $\lambda.v = \mu^\lambda_*(v) \in T_\xi X$.

(vi) Show that this operation, together with the addition defined above, makes $T_\xi X$ into a k-vector space.

In fact, we will see that everything from the last exercise works equally well when k is replaced by an arbitrary commutative ring A. The only step that requires modification is the first one:[71]

Proposition 12.2.2. *Let A be a commutative ring, write $A' = A[\epsilon]/(\epsilon^2)$ and write $A'' = A' \times_A A'$. Let X be a scheme. Then the map*

$$X(A'') \to X(A') \underset{X(A)}{\times} X(A')$$

is a bijection.

Proof. The proposition is obvious when X is an affine scheme. We will reduce to this case.

For each $\xi \in X(A)$, let $X_\xi(A')$ denote the set of all $\xi' \in X(A')$ that induce ξ in $X(A)$. Define $X_\xi(A'')$ similarly. With this notation, note that the proposition is equivalent to the assertion that the maps

$$X_\xi(A'') \to X_\xi(A') \times X_\xi(A')$$

are bijective for all $\xi \in X(A)$. We can therefore assume a single $\xi \in X(A)$ has been fixed.

Choose an open covering sieve $U \subset h^A$ such that for all $\varphi \in U(B)$, the map $\xi \circ \varphi : h^B \to h^A \to X$ factors through some open affine subscheme $V \subset X$. For

[71] Note that this proposition is also an immediate consequence of the universal property of the scheme obtained by gluing $h^{A'}$ to $h^{A'}$ along h^A.

each such B, let $B' = B[\epsilon]/(\epsilon^2)$ and $B'' = B' \times_B B'$. As $X_{\xi\circ\varphi}(B') = V_{\xi\circ\varphi}(B')$ and $X_{\xi\circ\varphi}(B'') = V_{\xi\circ\varphi}(B'')$, it follows that the map

$$X_{\xi\circ\varphi}(B'') \to X_{\xi\circ\varphi}(B') \times X_{\xi\circ\varphi}(B')$$

is a bijection for all $\varphi \in U(B)$.

□

Exercise 12.2.3. Let $\xi : A \to B$ be an element of $h^A(B)$.

(i) Show that if $\delta, \delta' : B \to M$ are A-derivations then so is $\delta + \delta'$, where $(\delta + \delta')(x) = \delta(x) + \delta(x')$.

(ii) Show that if $\delta : B \to M$ is an A-derivation and $\lambda \in B$ then $\lambda\delta$ is an A-derivation, where $(\lambda\delta)(x) = \xi(\lambda)\delta(x)$.

(iii) Conclude that $\mathrm{Der}_A(B, M)$ is a B-module.

(iv) Verify that the B-module structure on $\mathrm{Der}_A(B, M)$ constructed above agres with the B-module structure on $T_\xi h^A$ constructed earlier.

12.3 Kähler differentials

ex:differentials

Exercise 12.3.1. Let A be a commutative ring and B a commutative A-algebra. Show that there is a universal A-derivation $B \to \Omega_{B/A}$. You should verify, in other words, that the functor

$$B\text{-}\mathsf{Mod} \to \mathsf{Sets} : M \mapsto \mathrm{Der}_A(B, M)$$

is representable by a B-module $\Omega_{B/A}$. This is known as the **module of relative Kähler differentials** of B over A. When $A = \mathbf{Z}$ we write $\Omega_B = \Omega_{B/\mathbf{Z}}$.

ex:tangent-bundle-scheme

Exercise 12.3.2. (i) Let $X = \operatorname{Spec} A$. Show that TX is isomorphic to $\operatorname{Spec} \operatorname{Sym}_A \Omega_A$.

(ii) Conclude that if X is a scheme then TX is a scheme.

12.4 Relative tangent vectors

Exercise 12.4.1. Let $X, Y : \mathsf{ComRng} \to \mathsf{Sets}$ be functors and let $f : X \to Y$ be a natural transformation.

(i) Show that there is an induced natural map $Tf : TX \to TY$ and that Tf carries $T_\xi X$ into $T_{f(\xi)} Y$ for any $\xi \in X(A)$.

(ii) Show that $T_\xi f : T_\xi X \to T_{f(\xi)} Y$ is a morphism of pointed sets for any $\xi \in X(A)$.

(iii) Assume that X and Y are schemes. With A and ξ as above, show that $T_\xi f$ is an A-module homomorphism.

Definition 12.4.2. Let $X, Y : \mathsf{ComRng} \to \mathsf{Sets}$ be functors and $f : X \to Y$ a morphism. If $\xi \in X(A)$, the **relative tangent space** of X over Y at ξ is[72] ←72

$$T_\xi X/Y = \ker T_\xi f = \left\{ \xi' \in T_\xi(X) \,\middle|\, T_\xi f(\xi') = 0 \right\}$$

where 0 denotes the distinguished point of $T_{f(\xi)}Y$. The **relative tangent bundle** of X over Y is

$$TX/Y(A) = \left\{ \xi' \in TX(A) \,\middle|\, Tf(\xi') = i_* f(p_* \xi') \right\} = \bigcup_{\xi \in TX(A)} T_\xi X/Y(A).$$

Exercise 12.4.3. Let $f : X \to Y$ be a morphism of functors.

(i) Show that TX/Y is a subfunctor of TX.

Assume that X is a scheme.

(ii) Show that $T_\xi X/Y$ is a sub-A-module of $T_\xi X$ for any $\xi \in X(A)$.

def:ramification

Definition 12.4.4. Let k be a field and $f : X \to Y$ a morphism of functors. Suppose that $\xi \in X(k)$. We say that ξ is a **ramification point** of f if $T_\xi X/Y \neq 0$.

Exercise 12.4.5. Let $X = V(x^3+y^3+z^3) \subset \mathbf{P}^2$ and let $Y = \mathbf{P}^1$. Let $f : X \to Y$ be the morphism $f(L, (x, y, z)) = (L, (x, y))$. Determine the ramification points of f.

ex:frobenius

Exercise 12.4.6. Let p be a prime and let k be a field of characteristic p with q elements.

(i) Let A be a commutative k-algebra. Show that the function $F(x) = x^q$ defines a k-algebra homomorphism from A to itself.

Let $\pi : X \to \operatorname{Spec} k$ be a morphism of schemes.

(ii) Show that if $\xi \in X(A)$ then the image of ξ in $h^k(A)$ equips A with the structure of a k-algebra.

(iii) Suppose that $\xi \in X(A)$ and view A and a k-algebra as in the previous part. Define $f(\xi) = F_*(\xi)$ where $F : A \to A$ is the homomorphism constructed in the first part of this exercise. Show that f defines a natural transformation $f : X \to X$.

(iv) Show that if K is a field then the Frobenius morphism gives a *bijection* $X(K) \to X(K)$.

This natural transformation is known as the **Frobenius morphism**.

Exercise 12.4.7. Let k be a finite field of characteristic p and let $f : \mathbf{A}^1_k \to \mathbf{A}^1_k$ be the Frobenius morphism. Determine the ramification points of f.

---

[72] Many different notations are in use for the relative tangent space. The one given here is not standard.

Exercise 12.4.8. Let

$$X(A) = \{(L, p, \varphi) \mid (L, p) \in \mathbf{P}^n(A),\ \varphi \in L^\vee\}.$$

Recall that $L^\vee = \mathrm{Hom}_{A\text{-}\mathsf{Mod}}(L, A)$. Define a morphism $f : X \to \mathbf{A}^{n+1}$ by

$$f(L, (x_0, \dots, x_n), \varphi) = (\varphi(x_0), \varphi(x_1), \dots, \varphi(x_n)).$$

(i) Show that f is a morphism of functors.

(ii) When k is a field and $(x_0, \dots, x_n) \in \mathbf{A}^{n+1}(k)$, determine $f^{-1}(x_0, \dots, x_n)$.

(iii) Find the ramification points of f.

(iv) Show that X is a scheme.

The scheme X is known as the **blow-up** of $\mathbf{A}^{n+1}$ at the origin.

12.5 The tangent bundle of projective space

Our goal in this section will be to describe $T_\xi \mathbf{P}^n$ explicitly where $\xi = (L, p) \in \mathbf{P}^n(A)$ for some commutative ring A.

ex:TPn-field **Exercise 12.5.1.** Show that $T_\xi \mathbf{P}^n$ is an n-dimensional k-vector space when k is a field and $\xi \in \mathbf{P}^n(k)$.

Let $\xi = (L, p)$ be an element of $\mathbf{P}^n(A)$. Recall that $p : A^{n+1} \to L$ is a surjective A-module homomorphism. Let $M = \ker(p)$.

ex:ker-loc-free **Exercise 12.5.2.** Show that M is a locally free A-module of rank n.

We now have an exact sequence

$$0 \to M \to A^{n+1} \to L \to 0.$$

Now consider an extension (L', p') of (L, p) to $A[\epsilon]$.

Exercise 12.5.3. (i) Show that an invertible module B-module is a flat B-module, for any commutative ring B. Deduce that L' is a flat $A[\epsilon]$-module.

(ii) Conclude that there is an exact sequence of $A[\epsilon]$-modules

$$0 \to L \xrightarrow{\epsilon} L' \to L \to 0$$

where ϵ is multiplication by ϵ. (Hint: use the exact sequence $0 \to \epsilon A \to A[\epsilon] \to A \to 0$.)

Continuing to assume that (L', p') is an extension of (L, p) note that we have a map of A-modules

$$A^{n+1} \to A[\epsilon]^{n+1} \xrightarrow{p'} L'.$$

This gives rise to a commutative diagram of A-modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & A^{n+1} & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \epsilon L & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & 0. \end{array} \tag{11}$$

eqn:8

In particular, we obtain an A-module homomorphism $M \to L$.

Proposition 12.5.4. *The construction above identifies* $T_\xi \mathbf{P}^n$ *with* $\mathrm{Hom}_A(M, L)$*.*

Proof. To prove the proposition, we construct an inverse. Namely, given a homomorphism of A-modules, $\varphi : M \to L$, we construct an element of $T_\xi \mathbf{P}^n$. Let L' be the pushout of the diagram

$$\begin{array}{ccc} M & \longrightarrow & A^{n+1} \\ {\scriptstyle \varphi}\downarrow & & \downarrow \\ L & \longrightarrow & L'. \end{array}$$

This induces a commutative diagram (11) with exact rows. In particular, we have a homomorphism of A-modules $A^{n+1} \to L'$. This induces, by the universal property of tensor product, a homomorphism $p' : A[\epsilon]^{n+1} \to L'$. We must verify the following things:

(i) L' is an invertible $A[\epsilon]$-module,

(ii) p' is surjective, and

(iii) (L', p') extends (L, p).

To do the second of these, consider the diagram of $A[\epsilon]$-modules induced from (10):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M[\epsilon] & \longrightarrow & A[\epsilon]^{n+1} & \longrightarrow & L[\epsilon] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow {\scriptstyle p'} & & \downarrow & & \\ 0 & \longrightarrow & \epsilon L & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & 0. \end{array}$$

We know that $L[\epsilon] \to L$ is surjective. It follows therefore that if $x \in L'$ there is some $y \in A[\epsilon]^{n+1}$ such that $x - p'(y)$ lies in ϵL. But then choose some $z \in A$ such that $\epsilon p(z) = x - p'(y)$ (using the surjectivity of p) and we have $x = p'(y) + \epsilon p(z)$.

To do the first, we can assume A is a local ring and therefore that $L \cong A$. Choose a generator $x \in L$ and let $x' \in L'$ be a lift of x to L'. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \epsilon A & \longrightarrow & A[\epsilon] & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle x} & & \downarrow {\scriptstyle x'} & & \downarrow {\scriptstyle x} & & \\ 0 & \longrightarrow & \epsilon L & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & 0. \end{array}$$

Then the outer vertical arrows are isomorphisms so the middle arrow is as well, by the 5-lemma.

Finally, to check that (L', p') extends (L, p), simply note that one has a commutative diagram of $A[\epsilon]$-modules:

$$\begin{array}{ccc} A[\epsilon]^{n+1} & \longrightarrow & A^{n+1} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ L' & \longrightarrow & L. \end{array}$$

Exercise 12.5.5. Verify that the maps between $T_\xi \mathbf{P}^n$ and $\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(M, L)$ are inverse homomorphisms.

□

ex:euler-seq

Exercise 12.5.6 (Euler sequence)**.** Let (L, p) be an element of $\mathbf{P}^n(A)$ where $p = (x_0, \ldots, x_n)$. Show that there is an exact sequence

$$0 \to A \xrightarrow{p^T} L^{n+1} \to T_{(L,p)}\mathbf{P}^n \to 0$$

and that this sequence is natural in A. (Hint: Consider the exact sequence

$$0 \to M \to A^{n+1} \to L \to 0$$

and apply the functor $\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(-, L)$. Use the fact that an invertible A-module is projective.)

Exercise 12.5.7. We will show that $T_{(L,p)}\mathbf{P}^1 \simeq L^{\otimes 2}$.

(i) Consider an exact sequence of locally free modules over a commutative ring

$$0 \to A \to B \to C \to 0$$

where A, B, and C have ranks a, b, and c, respectively. Show that $\bigwedge^b B \simeq \bigwedge^a A \otimes \bigwedge^c C$.

(ii) Now let (L, p) be an element of $\mathbf{P}^1(A)$. Let $M = \ker(p)$. Apply the result from above to demonstrate that $M \otimes L \simeq \bigwedge^2 A^2 \simeq A$.

(iii) Conclude that $T_{(L,p)}\mathbf{P}^1 \simeq \mathrm{Hom}_{A\text{-}\mathsf{Mod}}(M, L) \simeq L^{\otimes 2}$.

12.6 The Grassmannian

Let k and n be non-negative integers. Define $\mathrm{Grass}(k, n)$ to be the functor $X : \mathsf{ComRng} \to \mathsf{Sets}$ where $X(A)$ is the set of all pairs (V, p) where

GRASS1 V is a locally free A-module of rank k, and

GRASS2 $p : A^n \to V$ is a surjective homomorphism of A-modules.

GRASS3 We consider (V, p) to be equivalent to (V', p') if there is a commutative diagram of A-modules

$$\begin{array}{ccc} & & V \\ & \overset{p}{\nearrow} & \\ A^n & & \downarrow f \\ & \underset{p'}{\searrow} & \\ & & V' \end{array}$$

where f is an isomorphism.

Exercise 12.6.1. Let $\xi = (V, p)$ be an element of $\operatorname{Grass}(k, n)(A)$. Following the computation of the tangent space of projective space, construct a bijection between $T_\xi \operatorname{Grass}(k, n)$ and $\operatorname{Hom}_{A\text{-}\mathsf{Mod}}(\ker(p), V)$. Verify that this bijection is natural in A.

12.7 Jets

Definition 12.7.1. Let $X : \mathsf{ComRng} \to \mathsf{Sets}$ be a functor and let $\xi \in X(A)$ for some commutative ring A. For a non-negative integer n, an ***n*-jet** of X at ξ is an extension of ξ to an element of $X(A[\epsilon]/(\epsilon^{n+1})$.

Denote by $J^n_\xi(X)$ the set of n-jets of X at ξ. Let $J^n X(A) = X(A[\epsilon]/(\epsilon^{n+1})$. Write $J^\infty X$ for $X(A[[\epsilon]])$ [73] and $J^\infty_\xi X$ for the set of extensions of $\xi \in X(A)$ to $X(A[[\epsilon]])$.

Exercise 12.7.2. Construct maps $J^n X \to J^m X$ for every $m, n \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$ with $m < n$.

Exercise 12.7.3. Assume X is a scheme.

(i) Prove that $J^n_\xi X \times_{J^{n-1}_\xi X} J^n_\xi X \simeq J^n_\xi X \times T_\xi X$ for all $\xi \in X(A)$ and $n \in \mathbf{Z}_{\geq 0}$. (Hint: Glue $h^{A[\epsilon]/(\epsilon^{n+1})}$ to itself along $h^{A[\epsilon]/(\epsilon^n)}$.)

(ii) Use the construction from the last part to describe an action of TX on $J^n X$ for all $n \in \mathbf{Z}_{\geq 0}$.

(iii) Conclude that $J^n X \to J^{n-1} X$ is a pseudo-torsor under the action of the family of groups TX over X.

Exercise 12.7.4. Show that if A is a local ring then $J^\infty X(A) = \varprojlim_n J^n X(A)$.

12.8 Infinitesimal extensions

def:inf-ext **Definition 12.8.1.** An **infinitesimal extension** or **nilpotent thickening** of a commutative ring A is a surjection $A' \to A$ whose kernel is a nilpotent ideal. We also describe the map $h^A \to h^{A'}$ as an infinitesimal extension of affine schemes or a nilpotent thickening of affine schemes.[74]

A nilpotent thickening $A' \to A$ with ideal I is called a **square-zero extension** if $I^2 = 0$.

[73] Here, $A[[\epsilon]]$ is the formal power series ring.

[74] Later we will define infinitesimal extensions of non-affine schemes.

Exercise 12.8.2. Show that if $S \to S'$ is an infinitesimal extension of affine schemes then

(i) $S(k) \to S'(k)$ is a bijection for all field k, and

(ii) the map $U \mapsto U \cap S$ from the set of open subsets of S' to the set of open subsets of S is a bijection.

formal-smooth-etale-unramf **Definition 12.8.3.** A morphism of functors $f : X \to Y$ is said to be (i) formally smooth, (ii) formally étale, or (iii) formally unramified if, given any diagram of solid lines

$$\begin{array}{ccc} S & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ S' & \longrightarrow & Y, \end{array} \tag{12}$$

eqn:10

in which $S \to S'$ is an infinitesimal extension of affine schemes, there is (i) at least one, (ii) exactly one, or (iii) at most one dashed arrow making the whole diagram commute. We say that X is formally smooth, étale, or unramified if the map $X \to h^{\mathbf{Z}}$ has the corresponding property.

Note that being formally étale is the conjunction of being formally smooth and formally unramified.

Exercise 12.8.4. Show that $\mathbf{A}^n$ is formally smooth but not formally unramified unless $n = 0$.

Exercise 12.8.5. Show that any injection $X \to Y$ is formally unramified.

Exercise 12.8.6. Show that if $f : X \to Y$ is formally unramified then $T(X/Y) = 0$.

Exercise 12.8.7. Show that the inclusion of an open subset $U \to X$ is formally étale.

ex:square-zero-suffice **Exercise 12.8.8.** Show that to prove a morphism $f : X \to Y$ is formally smooth, formally étale, or formally unramified it is sufficient to demonstrate that Diagram (12) has the appropriate lifting property only for square-zero extensions.

Proposition 12.8.9. $\mathbf{P}^n$ *is formally smooth.*

In order to prove the proposition, we must show that for any $(L, p) \in \mathbf{P}^n(A)$ and any square-zero extension $A' \to A$, there is some $(L', p') \in \mathbf{P}^n(A')$ lifting (L, p).

The proof may be broken up into two lemmas:

lem:Pn-form-smooth-1 **Lemma 12.8.10.** *Let A' be a square-zero extension of A with ideal J and let K' be a flat A'-module. Define $K = K' \otimes_{A'} A$. Let L be an A-module and*

$p : K \to L$ a surjective A-module homomorphism. Let $M = \ker(p)$. Then there is an exact sequence of A-modules

$$0 \to J \underset{A}{\otimes} L \to E \xrightarrow{r} M \to 0 \tag{13}$$ eqn:11

such that splittings [75] *of* (13) *are in bijection with extensions* (L', p') *of* (L, p) *to an* A'*-module* L' *and a surjection* $p' : K' \to L'$. [76]

Proof. Since K' is flat, taking the tensor product of K' with the exact sequence

$$0 \to J \to A' \to A \to 0,$$

and observing that $J \otimes_{A'} K' = J \otimes_A A \otimes_{A'} K' = J \otimes_A K$ because $J^2 = 0$, yields an exact sequence

$$0 \to J \underset{A}{\otimes} K \to K' \to K \to 0.$$

Pushing out via the homomorphism $\mathrm{id}_J \otimes p : J \otimes_A K \to J \otimes_A L$ gives us a commutative diagram with exact rows:

$$\begin{CD}
0 @>>> J \otimes_A K @>>> K' @>>> K @>>> 0 \\
@. @V{\mathrm{id}_J \otimes p}VV @VVV @| @. \\
0 @>>> J \otimes_A L @>>> F @>>> K @>>> 0
\end{CD}$$

Now, let $i : M \to K$ be the inclusion and pull back the sequence above to obtain the following commutative diagram with exact rows:

$$\begin{CD}
0 @>>> J \otimes_A L @>>> E @>{r}>> M @>>> 0 \\
@. @| @VVV @VV{i}V @. \\
0 @>>> J \otimes_A L @>>> F @>>> K @>>> 0
\end{CD}$$

The sequence in the first row of the equation displayed above is the one that will serve to satisfy the assertion of the lemma.

Exercise 12.8.11. By construction E is automatically an A'-module. Verify that E is in fact an A-module.

[75] Recall that a splitting of the sequence is a homomorphism $s : M \to E$ such that $r \circ s = \mathrm{id}_M$.

[76] The precise meaning of an extension here is a commutative diagram of A'-modules with exact rows having the following form:

$$\begin{CD}
0 @>>> J \otimes_A K' @>>> K' @>>> K @>>> 0 \\
@. @V{\mathrm{id}_J \otimes p}VV @VV{p'}V @VV{p}V @. \\
0 @>>> J \otimes_A L @>>> L' @>>> L @>>> 0
\end{CD}$$

The remainder of the proof is contained in the following exercise, in which we construct a bijection between splittings of (13) and extensions of (L, p) to $p' : K' \to L'$.

Exercise 12.8.12. (i) Show that splittings of (13) are in bijection with choices of A'-module homomorphisms $\varphi : M \to F$ making the diagram

$$\begin{array}{ccc} & & M \\ & \overset{\varphi}{\swarrow} & \downarrow{\scriptstyle i} \\ F & \longrightarrow & K \end{array}$$

commutative.

(ii) Show that maps φ as above are in bijection with commutative diagrams of A'-modules with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & M & = & M & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J \otimes_A L & \longrightarrow & F & \longrightarrow & K & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J \otimes_A L & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{14}$$

eqn:12

(iii) Show that commutative diagrams (14) with exact rows are in bijection with commutative diagrams with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J \otimes_A K & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_J \otimes p} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & J \otimes_A L & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & 0. \end{array}$$

□

lem:Pn-form-smooth-2

Lemma 12.8.13. *Let (L, p) be an element of $\mathbf{P}^n(A)$ and let $M = \ker(p)$. Then M is a projective A-module.* [77] *In particular, every exact sequence*

$$0 \to N' \to N \to M \to 0$$

splits.

[77] Recall that an A-module M is called projective if $\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(M, N) \to \mathrm{Hom}_{A\text{-}\mathsf{Mod}}(M, N')$ is surjective whenever $N \to N'$ is surjective.

Exercise 12.8.14. Prove Lemma 12.8.13:

(i) Show that M is projective if and only if L is projective. It therefore suffices to show L is projective.

(ii) Demonstrate that an A-module that is locally finitely generated [78] is finitely generated. (Hint: Write $Q = \bigcup Q_i$ where the Q_i are finitely generated submodules of Q. Show that there is a finite collection $f_1, \ldots, f_n \in A$ and indices i_j for each $j = 1, \ldots, n$ such that $(f_1, \ldots, f_n)A = A$ and such that the maps $Q_{i_j} \otimes A[f_j^{-1}] \to Q \otimes A[f_j^{-1}]$ are all surjective. Let Q_k be a finitely generated submodule of Q containing all of the Q_{i_j}. Deduce that $Q_k = Q$.)

(iii) Conclude from the last part that an A-module that is locally finitely presented is finitely presented.

(iv) Show that a finitely presented A-module Q is projective if and only if $Q \otimes_A A_\mathfrak{p}$ is projective for every prime ideal $\mathfrak{p}$ of A. (Hint: Present Q as $A^n \to A^m \to Q \to 0$. Then consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(Q, N)_\mathfrak{p} & \longrightarrow & \mathrm{Hom}(A^m, N)_\mathfrak{p} & \longrightarrow & \mathrm{Hom}(A^n, N)_\mathfrak{p} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}(Q_\mathfrak{p}, N_\mathfrak{p}) & \longrightarrow & \mathrm{Hom}(A_\mathfrak{p}^m, N_\mathfrak{p}) & \longrightarrow & \mathrm{Hom}(A_\mathfrak{p}^n, N_\mathfrak{p}) \end{array}$$

associated to an A-module surjection $N \to N'$ and use the 5-lemma.)

(v) Conclude that L is projective and therefore that M is.

13 Finite type and finite presentation

We generally say an object of a category is of *finite presentation* if it can be specified with a finite amount of data. For example, an A-module M is said to be of finite presentation if there is an exact sequence

$$A^n \to A^m \to M \to 0$$

where n and m are positive integers. Thus M can be recovered as the cokernel of a $m \times n$ matrix with entries in A.

We say that an object is of *finite type* if it is *determined* by a finite amount of information, even though a finite amount of information may not necessarily suffice to specify the object. An A-module M is said to be of finite type if it is finitely generated, i.e., if there is a sequence

$$A^m \to M \to 0$$

[78] An A-module Q is said to be locally finitely generated if there is a collection of $f \in A$ that generate the unit ideal such that $Q \otimes_A A[f^{-1}]$ is finitely generated.

where m is finite. Thus a map from M into some A-module N is determined by the images of the m generators of M, though not every choice of images for those generators is guaranteed to give a map $M \to N$.

Similarly, an A-algebra B is said to be of finite type if it can be generated, as an A-algebra, by finitely many elements. In other words, it is of finite type if there is a surjection of A-algebras $A[t_1, \dots, t_m] \to B$. In other words, there should be an exact sequence

$$A[t_1, \dots, t_m] \to B.$$

We say that B is of finite presentation if B is of finite type and the relations among the finitely many generators of B constitute a finitely generated ideal. One should thus have an exact sequence

$$(f_1, \dots, f_n)A[t_1, \dots, t_m] \to A[t_1, \dots, t_m] \to B \to 0$$

where m and n are finite.

One of the goals of this section will be to describe what it means for an object to be of finite presentation in terms of the functor it represents.

13.1 References

There is a thorough discussion of local finite presentation in [GD, IV.8].

13.2 Filtered diagrams

Definition 13.2.1. A diagram $\mathscr{D}$ is said to be filtered if it satisfies the following two conditions:

fil:1 **FIL1** if A and B are any two objects in $\mathscr{D}$ then there is a third object C in $\mathscr{D}$ along with maps $A \to C$ and $B \to C$ in $\mathscr{D}$, and

fil:2 **FIL2** if $u, v : A \to B$ are two morphisms in $\mathscr{D}$ then there is a third morphism $w : B \to C$ in $\mathscr{D}$ with $w \circ u = w \circ v$.

Example 13.2.2. The simplest, and perhaps most important, example of a filtered diagram is a sequence of objects A_i, $i \in \mathbf{Z}$ and morphisms $u_{ij} : A_i \to A_j$ for $i \le j$ with $u_{jk} \circ u_{ij} = u_{ik}$. Such a diagram is uniquely specified by the sequence of maps $u_{i,i+1}$:

$$\cdots \xrightarrow{u_{-2,-1}} A_{-1} \xrightarrow{u_{-1,0}} A_0 \xrightarrow{u_{0,1}} A_1 \xrightarrow{u_{1,2}} \cdots$$

Such a filtered diagram is called a **directed system**.

An important special kind of filtered diagram is one where all of the morphisms are monomorphisms. [79] We will call such a diagram a **filtered system**. [80]

[79] In general, a morphism $X \to Y$ is called a monomorphism if $\mathrm{Hom}(W, X) \to \mathrm{Hom}(W, Y)$ is injective for all W. In familiar settings, like A-modules or A-algebras, monomorphisms are the same as injections.

[80] This terminology is non-standard.

Exercise 13.2.3. Suppose that $\mathscr{D}$ is a diagram in which all morphisms are monomorphisms and Axiom **FIL1** is satisfied. Show that Axiom **FIL2** is equivalent to the hypothesis that there is at most one map between any two objects in $\mathscr{D}$.

ex:localizations-filtered **Exercise 13.2.4.** Let $\mathfrak{p}$ be a prime ideal in a commutative ring A. Denote by $\mathscr{D}$ the collection of all A-algebras $A[f^{-1}]$ with $f \notin \mathfrak{p}$. Show that $\mathscr{D}$ is a filtered diagram of commutative rings and $\varinjlim_{B\in\mathscr{D}} B \simeq A_{\mathfrak{p}}$.

Let X be an object of a category $\mathscr{C}$ and let $\mathscr{D}$ be a filtered diagram in $\mathscr{C}$. Assume that $\varinjlim_{Y\in\mathscr{D}} Y$ exists in $\mathscr{C}$. Then there is a canonical map

$$\varinjlim_{Y\in\mathscr{D}} \mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{C}}(X, \varinjlim_{Y\in\mathscr{D}} Y). \tag{15}$$ eqn:13

Exercise 13.2.5. Show by example that the map (15) is not always an isomorphism.

def:ft-fp **Definition 13.2.6.** Let X be an object of a category $\mathscr{C}$ that contains all filtered colimits. We say that X is **of finite type** if the map (15) is a bijection for any filtered *system* $\mathscr{D}$ in $\mathscr{C}$. We say that X is **of finite presentation** if the map (15) is a bijection for any filtered *diagram* $\mathscr{D}$ in $\mathscr{C}$.

Observe that the definition above can be extended to any functor $X : \mathscr{C} \to \mathsf{Sets}$:

def:lft-lfp **Definition 13.2.7.** Let $F : \mathscr{C} \to \mathsf{Sets}$ be a functor and assume that $\mathscr{C}$ contains all filtered colimits. We say that F is **locally of finite type** or **locally of finite presentation** if the function

$$\varinjlim_{Y\in\mathscr{D}} F(Y) \to F(\varinjlim_{Y\in\mathscr{D}} Y)$$

is, respectively, a bijection for all filtered systems $\mathscr{D}$ or filtered diagrams $\mathscr{D}$.

13.3 Examples

ex:fpftmod **Exercise 13.3.1.** (i) Show that an A-module M is of finite type in the sense of Definition 13.2.6 if and only if is finitely generated as an A-module.

(ii) Show that an A-module M is of finite presentation in the sense of Definition 13.2.6 if and only if there is an exact sequence

$$A^m \to A^n \to M \to 0.$$

ex:fpftmod **Exercise 13.3.2.** (i) Show that an A-algebra B is of finite type in the sense of Definition 13.2.6 if and only if is finitely generated as an A-algebra.

(ii) Show that an A-module B is of finite presentation in the sense of Definition 13.2.6 if and only if

$$B \simeq A[t_1, \ldots, t_n]/(f_1, \ldots, f_m)$$

for some non-negative integers n and m.

Exercise 13.3.3. Suppose that B is an A-algebra of finite presentation and A' is any A-algebra. Show that $B \otimes_A A'$ is of finite presentation as an A'-algebra.

ex:loc-not-lfp **Exercise 13.3.4.** Show that the local ring $\mathbf{Z}_{(p)}$ is not of finite presentation as a $\mathbf{Z}$-algebra.

Exercise 13.3.5. (i) Show that a topological space is compact if and only if

$$\mathrm{Hom}(X, \bigcup_{U \in \mathscr{D}} U) \leftarrow \bigcup_{U \in \mathscr{D}} \mathrm{Hom}(X, U)$$

is a bijection whenever $\mathscr{D}$ is a filtered system of open inclusions. [81]

(ii) Show that a topological space is compact Hausdorff if and only if

$$\mathrm{Hom}(X, \varinjlim_{U \in \mathscr{D}} U) \leftarrow \varinjlim_{U \in \mathscr{D}} \mathrm{Hom}(X, U)$$

is a bijection whenever $\mathscr{D}$ is a filtered diagram of local homeomorphisms.

13.4 Morphisms locally of finite presentation

Definition 13.4.1. Let $\mathscr{C}$ be a category that possesses all filtered colimits. Consider functors $X, Y : \mathscr{C} \to \mathsf{Sets}$ and a morphism $f : X \to Y$. Say that f is **locally of finite presentation** if, for any filtered diagram $\mathscr{D}$ of objects of Y with colimit A, [82] we have the following two properties:

lfp:1 **LFP1** If $\xi \in X(A)$ lifts η, in the sense that $f(\xi) = \eta$, then there is some $B \in \mathscr{D}$ and some $\xi_B \in X(B)$ lifting η_B such that $\xi = \xi_B\big|_A$.

lfp:2 **LFP2** If $\xi_B \in X(B)$ and $\xi_{B'} \in X(B')$ are lifts of $\eta_B \in Y(B)$ and $\eta_{B'} \in Y(B')$ such that $\xi_B\big|_A = \xi_{B'}\big|_A$ then there are maps $u : B \to C$ and $u' : B' \to C$ in $\mathscr{D}$ such that $u_* \xi_B$ and $u'_* \xi_{B'}$ define the same lift of $u_* \eta_B = u'_* \eta_{B'}$.

Exercise 13.4.2. Consider a sequence of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$.

(i) Assume that g is locally of finite presentation. Show that f is locally of finite presentation if and only if gf is.

(ii) Assume that g is locally of finite type. Show that f is locally of finite type if and only if gf is.

[81] Note that a filtered system of open inclusions is the same thing as a filtered system of local homeomorphisms.

[82] This means that $\mathscr{D}$ is a diagram in $\mathscr{C}$ and for each $B \in \mathscr{D}$ we have an object $\eta_B \in Y(B)$. The images of the η_B under the various tautological maps $B \to A$, for each $B \in \mathscr{D}$, will not depend on the choice of B. We denote this object by η.

13.5 Schemes locally of finite presentation

Exercise 13.5.1. Let X be a functor and $U \subset X$ an open subfunctor.

(i) Show that the inclusion $U \to X$ is locally of finite presentation.

(ii) Conclude that an open subfunctor of a functor that is locally of finite presentation is itself locally of finite presentation.

(iii) Conclude likewise that an open subfunctor of a functor that is locally of finite type is itself locally of finite type.

thm:scheme-lfp

Theorem 13.5.2. *A scheme $X : \mathsf{ComRng} \to \mathsf{Sets}$ is (i) locally of finite presentation (ii) locally of finite type if and only if X is covered by open affine schemes h^A where A is a commutative ring of (i) finite presentation (ii) finite type.*

Proof. Step 1: Suppose first that X is locally of finite presentation. Let $U \subset X$ be an open affine subscheme of X. Then U is also locally of finite presentation. By definition, if $U \simeq h^A$, this means that A is a commutative ring of finite presentation.

Step 2: For the converse, we suppose that X is covered by affine schemes of finite presentation[83] and begin with the proof of **LFP2**. Suppose that $B = \varinjlim_{C \in \mathscr{D}} C$ and that $\xi \in X(C)$ and $\xi' \in X(C')$ for some $C, C' \in \mathscr{D}$. Assume furthermore that $\xi\big|_B = \xi'\big|_B$. We will show that there is some $C'' \in \mathscr{D}$, along with maps $C \to C''$ and $C' \to C''$ such that $\xi\big|_{C''} = \xi'\big|_{C''}$. This will demonstrate that the map

$$\varinjlim_{C \in \mathscr{D}} X(C) \to X(B)$$

is injective.

Making use of the fact that $\mathscr{D}$ is filtered, we may first of all assume without loss of generality, that $C = C'$. By assumption, we may cover X by open affine subschemes h^{A_i} with each A_i of finite type. We may now select $f_1, \dots, f_n \in B$ such that $\xi_j = \xi\big|_{B[f_j^{-1}]} \in h^{A_i}(B[f_j^{-1}])$ for some i. There is also nothing stopping us from numbering the A_i in unison with the f_j and therefore obtaining $\xi_j \in h^{A_j}(B[f_j^{-1}])$ for all $j = 1, \dots, n$.

Now, replacing C with another object of $\mathscr{D}$ if necessary, we can assume (because $\varinjlim_{C \in \mathscr{D}} C = B$) that all of the f_j lie in the image of the map $C \to B$. Let us choose lifts $g_j \in C$ of the f_j. Now consider the diagram $\mathscr{D}'$ consisting of all pairs (C', v) where $v : C \to C'$ is a map in $\mathscr{D}$. For each j, note that $B[f_j^{-1}] = \varinjlim_{(C',v) \in \mathscr{D}} C'[v(g_j)^{-1}]$. This is a filtered colimit and we have $\xi_j\big|_{B[f_j^{-1}]} = \xi'_j\big|_{B[f_j^{-1}]}$ as elements of $h^{A_j}(B[f_j^{-1}])$. As A_j is of finite type we may deduce that there is some $(C', v) \in \mathscr{D}'$ such that $\xi_j\big|_{C'} = \xi'_j\big|_{C'}$.

Applying the above for all j and using the fact that $\mathscr{D}'$ is filtered, we find a single $(C'', v) \in \mathscr{D}'$ such that $\xi_j\big|_{C''} = \xi'_j\big|_{C''}$ for all $j = 1, \dots, n$. Thus the two

[83] In fact, it suffices to assume that X is covered by affine schemes of finite type.

objects $\xi\big|_{C''}, \xi'\big|_{C''} \in X(C'')$ agree on an open cover of $h^{C''}$. We conclude that $\xi = \xi'$, and deduce the sought-after injectivity.

Step 3: Now we prove **LFP1**, namely the surjectivity of the map

$$\varinjlim_{C \in \mathscr{D}} X(C) \to X(B)$$

whenever $\mathscr{D}$ is a filtered diagram (resp. system) and X has an open cover by affine schemes of finite presentation (resp. finite type). Suppose that $\xi \in X(B)$. As in Step 2, choose an open cover of X by affines h^{A_j} and elements $f_1, \ldots, f_n \in B$ such that $\xi\big|_{B[f_j^{-1}]} \in h^{A_j}(B[f_j^{-1}])$. Let $\xi_j = \xi\big|_{B[f_j^{-1}]}$.

As before, we may select a $C \in \mathscr{D}$ such that all of the f_j lie in the image of $C \to B$. We may further select lifts g_j of the f_j and consider the diagram $\mathscr{D}'$ of all (C', v) where $v : C \to C'$ is a morphism in $\mathscr{D}$.

We remark again that $B[f_j^{-1}] = \varinjlim_{(C',v)\in\mathscr{D}'} C'[v(g_j)^{-1}]$ and that furthermore, the diagram of $C'[v(g_j)^{-1}]$ is a directed *system* if the diagram $\mathscr{D}$ is.[84] Applying, as the case warrants, the assumption that A_j is of finite presentation or finite type, we deduce that there is some $(C', v) \in \mathscr{D}'$ and some $\eta_j \in h^{A_j}(C') \subset X(C')$ with $\eta_j\big|_B = \xi_j$.

Using the fact that $\mathscr{D}'$ is a filtered diagram, we may assume that there is a single $(C', v) \in \mathscr{D}'$ such that η_j is defined in $h^{A_j}(C'[v(g_j)^{-1}])$ for all $j = 1, \ldots, n$. We would like to glue these together to obtain an element of $X(C')$ inducing $\xi \in X(B)$. Unfortunately, there is no guarantee at this point that a gluing exists. However, we will be able to adjust C' to ensure that gluing works.

Consider the elements $\eta_j\big|_{C'[v(g_j)^{-1},v(g_k)^{-1}]}$ and $\eta_k\big|_{C'[v(g_j)^{-1},v(g_k)^{-1}]}$. These both lie in $X(C'[v(g_j)^{-1}, v(g_k)^{-1}])$ and they have the same image $\xi\big|_{B[f_j^{-1},f_k^{-1}]}$ in $X(B[f_j^{-1}, f_k^{-1}])$. Therefore, since $B[f_j^{-1}, f_k^{-1}]$ is the colimit of the filtered diagram (resp. filtered system) of $C''[w(g_j)^{-1}, w(g_k)^{-1}]$ for $(C'', w) \in \mathscr{D}'$, it follows from the local finite presentation of X that there is some $(C'', w) \in \mathscr{D}'$ for which $\eta_j\big|_{C''[w(g_j)^{-1},w(g_k)^{-1}]} = \eta_k\big|_{C''[w(g_j)^{-1},w(g_k)^{-1}]}$.

Once again making use of the fact that $\mathscr{D}'$ is filtered, we can acquire a single $(C'', w) \in \mathscr{D}'$ such that the above property holds for all pairs $j, k \in \{1, \ldots, n\}$. Now the $\eta_j\big|_{C''[w(g_j)^{-1}]}$ are compatible and, since X is a scheme, they may be glued to give a single element $\eta \in X(C'')$ that induces $\xi \in X(B)$. □

Exercise 13.5.3. Adapt the proof of Theorem 13.5.2 to show that a morphism of schemes $f : X \to Y$ is (i) locally of finite presentation, or (ii) locally of finite type if and only if X and Y can be covered by open affine schemes U and V such that $f\big|_U : U \to Y$ factors through V and the induced map $U \to V$ is (i) of finite presentation, or (ii) of finite type.

[84] In other words, $C_1'[v_1(g_j)^{-1}] \to C_2'[v_2(g_j)^{-1}]$ is injective if $C_1' \to C_2'$ is.

13.6 Pro-affine schemes

If $\mathscr{D}$ is a filtered diagram of commutative rings and $X : \mathsf{ComRng} \to \mathsf{Sets}$ is a functor, we will write

$$\mathrm{Hom}(\text{“}\varprojlim_{B\in\mathscr{D}}\text{”}\, h^B, X) = \varinjlim_{B\in\mathscr{D}} X(B).$$

If A is the colimit of $\mathscr{D}$ then there are compatible maps $B \to A$ for every $B \in \mathscr{D}$. Therefore one gets a map

$$\mathrm{Hom}(\text{“}\varprojlim_{B\in\mathscr{D}}\text{”}\, h^B, X) = \varinjlim_{B\in\mathscr{D}} X(B) \to X(A) = \mathrm{Hom}(h^A, X)$$

One may therefore imagine that we have a map $h^A \to \text{“}\varprojlim_{B\in\mathscr{D}}\text{”}\, h^B$. One should regard this only as a convenient abuse of notation and should not get the impression that $\text{“}\varprojlim_{B\in\mathscr{D}}\text{”}\, h^B$ actually is an object of the category $\mathrm{Hom}(\mathsf{ComRng}, \mathsf{Sets})$.
[85]

Exercise 13.6.1. Let $X, Y : \mathsf{ComRng} \to \mathsf{Sets}$ be a pair of functors and let $f : X \to Y$ be a morphism of functors. Show that f is **locally of finite presentation** if, for any filtered diagram of commutative rings $\mathscr{D}$ with colimit A, the diagram

$$\begin{array}{ccc} h^A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \text{“}\varprojlim_{B\in\mathscr{D}}\text{”}\, h^B & \longrightarrow & Y \end{array} \qquad (16)$$

eqn:9

admits a *unique* lift.

14 Morphisms of schemes

14.1 Smooth, étale, and unramified morphisms

sec:smooth

def:smooth-etale-unramified

Definition 14.1.1. A morphism of *schemes* is said to be (i) **smooth**, (ii) **étale**, or (iii) **unramified** if it is locally of finite presentation and, respectively, (i) formally smooth, (ii) formally étale, or (iii) formally unramified.

Warning 14.1.2. Although it makes sense to say that a functor $f : X \to Y$ is formally smooth (or formally étale or formally unramified) and locally of finite presentation, this does not give a reasonable definition of smoothness for an arbitrary morphism of functors!

[85]In fact, $\text{“}\varprojlim_{B\in\mathscr{D}}\text{”}\, h^B$ is known as a **pro-object** of the category $\mathrm{Hom}(\mathsf{ComRng}, \mathsf{Sets})$.

14.2 Separated morphisms

sec:separated

Let $u, v : Z \to X$ be two morphisms of functors. Define the *equalizer* of u and v to be the subfunctor $W = \mathrm{eq}(u, v)$ of Z with

$$W(A) = \{\xi \in Z(A) \mid u(\xi) = v(\xi)\}.$$

ex:separated

Exercise 14.2.1. Let $f : X \to Y$ be a morphism of functors. Show that the following properties are equivalent:

(i) The diagonal map $X \to X \times_Y X$ is a closed embedding.

(ii) If $u, v : Z \to X$ are two morphisms such that $fu = fv$ then $\mathrm{eq}(u, v)$ is a closed subfunctor of Z.

(iii) If $u, v \in X(A)$ then there is an ideal $I \subset A$ such that, for any homomorphism $\varphi : A \to B$ we have $\varphi_*(u) = \varphi_*(v)$ if and only if $\varphi(I) = 0$.

def:separated

Definition 14.2.2. A morphism $f : X \to Y$ is said to be **separated** if it satisfies the equivalent conditions of Exercise 14.2.1.

14.3 Proper morphisms

14.4 Embeddings

Exercise 14.4.1. Give an example of an embedding that is not locally closed.

15 Line bundles, projective space, and linear series

sec:line-bundles

Let X be a scheme. By a line bundle over X we will mean a scheme L, equipped with a projection $p : L \to X$, such that

LB1 there is an cover of X by open subschemes U_i and isomorphisms $\varphi_i : p^{-1}(U_i) \simeq U_i \times \mathbf{A}^1$ over U_i, such that

LB2 over $U_i \cap U_j$, the composition $\varphi_i\big|_{U_i \cap U_j} \circ \varphi_j\big|_{U_i \cap U_j}^{-1}$ is of the form $(u, x) \mapsto (u, \lambda x)$ where λ is a map $U \to \mathbf{G}_m$.

References

sga4 [AGV] M. Artin, A. Grothendieck, and J. Verdier. Sga4, 1963/64. *Springer Lecture Notes*, 269, 270, 305, 1972–3.

AM [AM] M.F. Atiyah and I.G. Macdonald. *Introduction to commutative algebra*, volume 19. Addison-Wesley Reading, MA, 1969.

Bredon [Bre] G.E. Bredon. *Sheaf theory*, volume 170. Springer Verlag, 1997.

FAG [FGI$^+$] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained.

ega [GD] A. Grothendieck and J. Dieudonné. *Eléments de géométrie algébrique*.

GM [GM] S. I. Gelfand and Y. I. Manin. *Methods of homological algebra*.

Godement [God] Roger Godement. *Topologie algébrique et théorie des faisceaux*. Hermann, Paris, 1973. Troisième édition revue et corrigée, Publications de l'Institut de Mathématique de l'Université de Strasbourg, XIII, Actualités Scientifiques et Industrielles, No. 1252.

Tohoku [Gro1] A. Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Mathematical Journal*, 9(3):119–221, 1957.

SGA1 [Gro2] A. Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)].

TDTE-I [Gro3] Alexander Grothendieck. Technique de descente et théorèmes d'existence en géometrie algébrique. I. Généralités. Descente par morphismes fidèlement plats. In *Séminaire Bourbaki, Vol. 5*, pages Exp. No. 190, 299–327. Soc. Math. France, Paris, 1995.

Hartshorne [Har] R. Hartshorne. *Algebraic geometry*, volume 52. Springer, 1977.

IR [IR] K. Ireland and M. Rosen. *A classical introduction to modern number theory*, volume 84. Springer, 1990.

Iversen [Ive] B. Iversen. *Cohomology of sheaves*. Springer-Verlag Berlin, Heidelberg, New York, 1986.

Matsumura [Mat] H. Matsumura. *Commutative algebra*, volume 120. WA Benjamin New York, 1970.

red-book [Mum] D. Mumford. *The red book of varieties and schemes: includes the Michigan Lectures (1974) on curves and their Jacobians*, volume 1358. Springer, 1999.

FAC [Ser] Jean-Pierre Serre. Faisceaux algébriques cohérents.(french). *Ann. of Math.(2)*, 61:197–278, 1955.

stacks [Sta] The Stacks Project Authors. *Stacks Project*. http://stacks.math.columbia.edu.

Vakil [Vak] R. Vakil. Foundations of algebraic geometry, 2012. math.stanford.edu/∼vakil/216blog.

Weil [Wei] A. Weil. Numbers of solutions of equations in finite fields. *Bull. Amer. Math. Soc*, 55(5):497–508, 1949.