SEMANTICS COLUMN

PRAKASH PANANGADEN,
School of Computer Science, McGill University

It is with great sadness that the SIGLOG community learned of the passing away of Philip J. Scott in December 2023. He was a leader of the community and made major contributions to type theory, polymorphic λ-calculus, bounded linear logic, Geometry of Interaction and many other topics. He and Jim Lambek wrote the classic textbook, "Introduction to Higher-order Categorical Logic" which exposed the computer science community to what were then exotic topics for us: type theory and categories, categorical proof theory and toposes.

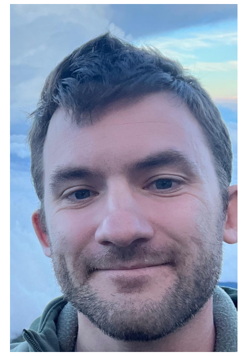
For me he was a personal friend as well as an occasional collaborator. In the 1990's and early 2000's I often visited Ottawa and stayed with him and his wife Marcia. He taught me things, like hyperdoctrines, that I could not find easily in an accessible form. He had been in Montreal as a post-doctoral fellow of the famous McGill category theory group in the 1980's and retained his friendship with many members of that group, particularly Robert Seely. In his last few months he was struggling with health problems, as was I at the time, so we shared medical experiences. I was amazed at how upbeat he was in the face of serious setbacks. As late as November he wrote to me saying, "perhaps we can meet in Edinburgh next year". He and I shared a special fondness for Edinburgh and spent some time there together in 2011. Alas, the Edinburgh reunion that we were hoping for will never happen now.

I have planned two articles on Phil's contributions to the computer science community. Of course, he made many contributions to the logic and category theory community in the time before contacts between these communities became as tight as it is now. It is impossible to cover all his contributions in one article or indeed even a series of articles. I have chosen to have more in-depth coverage of a couple of selected topics rather than attempt a comprehensive coverage.

The following article by Jean-Simon Pascual-Lemay focusses on Geometry of Interaction. After a personal dedication, the article gives a beautiful account of the Geometry of Interaction from a categorical perspective, unlike the operator algebraic formulation originally presented by Girard. It is written with expository skill which should allow anyone in the semantics community to read it easily. A second article is planned for later in the year.

A Brief Survey on the Categorical Semantics for Geometry of Interaction
In Memory of Phil Scott

Jean-Simon Pacaud Lemay, Macquarie University, Australia



Dedication to Phil Scott (1947 – 2023)

Philip Scott, better known to his friends and colleagues as Phil, was a Professor of Mathematics at the University of Ottawa (Canada). Sadly, Phil passed away[1] after a long battle with cancer – which he bravely fought till the very end – on 18 December 2023 at the age of 76. Phil was born on 27 December 1947 in Leeds (UK). When Phil was still a baby, his family moved to North Carolina (USA), where he spent his childhood. He went on to study mathematics at the University of North Carolina at Chapel Hill (USA), before moving to Canada in the 1970s. Phil did his Ph.D. at the University of Waterloo (Canada), supervised by Denis Higgs, and obtained his doctorate in pure mathematics in 1976. Phil joined the Mathematics Department at the University of Ottawa in 1982, eventually becoming Professor Emeritus, and remained there until his passing.

Phil was an amazing mathematician and had an illustrious career. Phil's most celebrated work is most likely his book with Joachim (Jim) Lambek: "Introduction To Higher-Order Categorical Logic" (1986), which is still to this day highly regarded and is one of the most essential books in category theory. Upon the news of Phil's passing, one of the most recurring comments from mathematicians and computer scientists

[1]Obituary on the Canadian Mathematics Society Website:
https://notes.math.ca/en/article/in-memoriam-phil-scott-1947-2023/

from various research fields was how Phil and Jim's book was their introduction to category theory and categorical logic, shaping careers and research interests. Even as he was fighting cancer, Phil continued to be an active member in his mathematical communities.

On top of his research work, Phil was also an outstanding and supportive mentor to young researchers at all levels, as well as a wonderful teacher. He would always find the time for students, both those who were struggling in classes and those who were interested in pursuing a career in mathematics. Phil was also always happy discussing with young researchers about their research and giving many valuable suggestions. Even till the very end, Phil was supporting the young researchers under his supervision at Ottawa, demonstrating his dedication to being an excellent supervisor.

On a personal level, as well as being my friend, Phil was an important figure in the early stages of my academic studies and had quite an impact on the direction of my research interests. I first met Phil in my first year of undergrad, where he was my teacher for an introduction to group theory course[2]. At the time, I was way over my head and completely lost in my pure math courses: I didn't know how to do a proof, let alone what a proof actually was. It wasn't until Phil helped me that I was finally able to understand what was going on. I still have a clear memory of Phil taking the time after class, going through step by step with me on how to do a basic proof. He explained things so clearly. He genuinely seemed to care that I understood and succeeded. I truly believe that this was a pivotal moment in my academic career: without Phil's help, I would have probably kept on struggling and not learned to love mathematics (let alone go on to make it my career).

Throughout the rest of my undergrad, I had Phil again as a teacher for several other courses, such as advanced linear algebra courses and even history of mathematics courses, which he had lots of fun teaching. Sometime near the end of my undergrad, I was trying to figure out what field of research I should go work in. Seeking advice from Phil, I still remember our discussion where I was first introduced to this foreign concept of "category theory". As a motivating example, Phil used the concept that a vector space was not necessarily isomorphic to its double dual. In hindsight, I now understand that Phil was slowly introducing me to star-autonomous categories and Linear Logic. I credit Phil with opening the door to the path that led me to become a category theorist.

After my time at the University of Ottawa, Phil continued to be a mentor: he was always very supportive and happy to discuss my latest research interests or new results. I would meet up with Phil many times at the Foundational Methods in Computer Science (FMCS) workshop – the unofficial yearly Canadian category theory meetup. Anytime I passed through Ottawa, Phil would always bring us to his favourite restaurant, the Green Door – which was a walk or bike ride down the canal from the University of Ottawa. I also got the chance to spend lots of time with Phil when we both happened to be visiting the University of Edinburgh (UK) at the same time.

Phil's passing is a great loss for many communities, including the Canadian mathematics community and the category theory community. Phil was incredibly kind, a fantastic mentor, and a great friend to many of us. Condolences to his loved ones. May he rest in peace.

Acknowledgements

I would first like to thank Prakash Panangaden for inviting me to write this survey paper in memory of Phil. I'd also like to thank Samson Abramsky, Esfandiar Haghverdi,

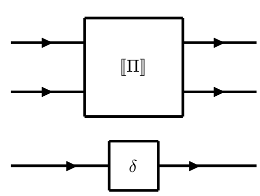
[2]At this point, one should highlight that the University of Ottawa is a bilingual university, teaching courses in both English and French. As such, Phil taught me and many others in French.

and Masahito Hasegawa for looking over the paper and making sure that my history was correct. I'd also like to give a big thank you to Prakash Panangaden, Esfandiar Haghverdi, Rick Blute, Robin Cockett, Chris Heunen, Masahito Hasegawa, and Gordon Plotkin for their encouragement and their insights about Phil and his work.

1. INTRODUCTION

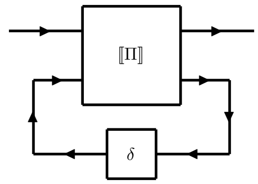
Phil Scott had an illustrious career and made many important contributions[3] in category theory, (linear) logic, and theoretical computer science. Scott is probably best known for his collaborations with J. Lambek on categorical logic and categorical proof theory, and, in particular, their all-important landmark book "Introduction To Higher-Order Categorical Logic" [Lambek and Scott(1988)]. However, another area that Scott was particularly interested in was the theory of *traced monoidal categories* [Joyal et al.(1996)], an area that he and his coauthors significantly contributed to. One aspect of traced monoidal categories in particular that Scott was interested in was using traced monoidal categories to provide a categorical framework for the *Geometry of Interaction*. Scott worked on this mostly with S. Abramsky and E. Haghverdi, the latter of whom worked on this topic for his thesis [Haghverdi(2000)] as a PhD student of Scott at the University of Ottawa. In dedication to Phil Scott, we provide an introductory level brief survey of this story, focusing mainly on the contents of [Abramsky et al.(2002); Haghverdi and Scott(2006)].

Geometry of Interaction was introduced by J.-Y. Girard in [Girard(1989b)], which he followed up on in [Girard(1988); Girard(1989a); Girard(1995)], and is a particular interpretation of Linear Logic [Girard(1987)]. The heart of Geometry of Interaction is to model mathematically the *dynamics* of cut-elimination. Usually, in a categorical model of a logic, formulas are interpreted by objects and proofs by morphisms, where these interpretations are denoted using the brackets $[\![-]\!]$. The soundness of the categorical model means that if a proof Π reduces to another Π' by cut-elimination, then their interpretations must be equal as morphisms in the categorical model, $[\![\Pi]\!] = [\![\Pi']\!]$. Thus, denotational semantics is, in this sense, static, and this kind of interpretation leads to a somewhat bland notion of invariant for cut-elimination. So with the Geometry of Interaction, Girard set out to give a way of modelling the dynamics of cut-elimination with a more meaningful invariant, called the *Execution formula*. Naively, the dynamical interpretation of proofs can be visually represented as follows. For a proof Π of a sequence $\vdash [\Delta], \Gamma$, where Γ is a sequence of formulas and Δ is a sequence of cut formulas that have been applied to the proof of $\vdash \Gamma$, its interpretation $[\![\Pi]\!]$ is graphically represented by an input-output box:



[3]A list of some of Phil Scott's important papers can be found here on his website: https://www.site.uottawa.ca/~phil/papers/

where δ models the cuts in the proof of $\vdash \Gamma$ (i.e. δ models Δ). Then cut-elimination is given by the Execution formula $\mathsf{EX}\left([\![\Pi]\!], \delta\right)$, which involves feedback on δ:



Girard's first model for the Geometry of Interaction was based on operator algebras and Hilbert spaces, and thus was expressed using matrices. Later on, Girard also discussed modelling Geometry of Interaction using von Neumann algebras [Girard(2006); Girard(2011)]. For a more in-depth introduction to the Geometry of Interaction, see [Haghverdi(2000), Chapter 5].

Per [Haghverdi and Scott(2010b)], categorial foundations of the Geometry of Interaction were first considered in lectures by Abramsky and by M. Hyland in the early 1990s. The first formal categorical framework for Geometry of Interaction was given by Abramsky and R. Jagadeesan in their LICS1992 paper [Abramsky and Jagadeesan(1992b)], with the journal version appearing soon after [Abramsky and Jagadeesan(1994b)]. Here, Abramsky and Jagadeesan give a "structurally isomorphic" account of the Geometry of Interaction interpretation based on a different underlying model to Girard's operator algebra model. Abramsky and Jagadeesan's model's setting is instead domain theoretic, so the monoidal structure is given by the product rather than the coproduct, and so feedback is interpreted in terms of least fixed points. Using domain equations, Abramsky and Jagadeesan extend this interpretation to the whole of Linear Logic. In [Abramsky and Jagadeesan(1992a); Abramsky and Jagadeesan(1994a)], Abramsky and Jagadeesan introduced game semantics for Multiplicative Linear Logic, where they also demonstrated how the Geometry of Interaction's Execution formula can be interpreted in the category of sets and partial injective functions, where now the monoidal structure is given by the coproduct and the feedback is interpreted by iteration. This showed that there were two different-looking ways of providing what was essentially structurally the same Geometry of Interaction. One is the "wave style" version, where the monoidal product is a product and the feedback is given by a global flow of information through the system. The other is the "particle style" version, where the monoidal product is a coproduct and feedback is thought of as a token following paths in a graph.

Many of the ideas for a categorical interpretation of some sort of feedback operator appear in [Abramsky and Jagadeesan(1992b); Abramsky and Jagadeesan(1994b); Abramsky and Jagadeesan(1992a); Abramsky and Jagadeesan(1994a)]. The precise axiomatization of these ideas can be given in terms of **traced monoidal categories**, which A. Joyal, R. Street, and D. Verity introduced in [Joyal et al.(1996)]. Motivated by knot theory, traced monoidal categories were defined initially in the *braided* monoidal setting. However, it is instead traced *symmetric* monoidal categories that give the desired setting for the feedback used in Geometry of Interaction. Indeed, Abramsky provides the categorical interpretation of the Geometry of Interaction using traced symmetric monoidal categories in [Abramsky(1996)], as well as providing numerous examples. In particular, Abramsky introduced the GOI-construction in [Abramsky(1996)], which is a construction that yields a *compact closed category* whose composition is given by the Execution formula. This formulation of using traced monoidal categories to provide the categorical semantics of Geometry of Interaction is referred to by Scott and Haghverdi as the "Abramsky program". However, the interpretation of the Geometry of Interaction in [Abramsky(1996)] was only done for the multiplicative fragment of Linear Logic. This left open the problem of interpreting a more expressive fragment, in particular one capable of encoding the λ-calculus.

Following the Abramsky program, Scott, Abramsky, and Haghverdi then introduced the concept of *Geometry of Interaction Situations* in [Abramsky et al.(2002)] which are traced symmetric monoidal categories that have the necessary structure in which to interpret an algebraic model of the multiplicative and *exponential* fragments of Linear Logic, called a *linear combinatory algebra*. Scott, Abramsky, and Haghverdi's Geometry of Interaction Situations gave the first fully developed categorical axiomatization for the Geometry of Interaction, covering all known examples as well. Afterwards, Scott and Haghverdi in [Haghverdi and Scott(2006)] used *unique decomposition category* versions of Geometry of Interaction Situations, which give particle style models, to recapture Girard's original operator algebra model of Geometry of Interaction and Execution formula. In follow-up papers [Haghverdi and Scott(2010a); Haghverdi and Scott(2010b)], Scott and Haghverdi continued applying unique decomposition categories to give a deeper analysis of Girard's construction.

Scott and his coauthors' contributions to the categorical semantics of the Geometry of Interaction are important concepts in the theory of traced monoidal categories, influencing and motivating many research topics in the area. We survey the main concepts of the Abramsky program mentioned above. Of course, for a complete story, we invite the interested reader to see the two main papers surveyed [Abramsky et al.(2002); Haghverdi and Scott(2006)], Haghverdi's thesis [Haghverdi(2000)], an introductory paper on this story by Scott and Haghverdi [Haghverdi and Scott(2010a)], as well as some very nicely written notes by M. Shirahata [Shirahata(2003)].

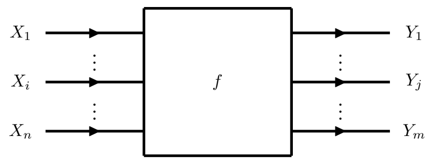
2. TRACED SYMMETRIC MONOIDAL CATEGORIES

Traced monoidal categories were introduced by Joyal, Street, and Verity in [Joyal et al.(1996)], and were initially defined as *balanced* monoidal categories equipped with a *trace operator*. For the story of Geometry of Interaction, one works in the special case of a traced *symmetric* monoidal category, which is instead a symmetric monoidal category equipped with a trace operator. In this section, we review the definition of traced symmetric monoidal categories and their graphical representation.

We assume that the reader is familiar with the basics of monoidal category theory. For an introduction to monoidal category theory, we invite the reader to see [Blute and Scott(2004); Heunen and Vicary(2019); Selinger(2010)]. For a category $\mathbb{C}$, we will denote the class of objects as $Ob(\mathbb{C})$ and denote objects using capital letters $A, B, X, Y, etc. \in Ob(\mathbb{C})$. Homsets will be denoted as $\mathbb{C}(X, Y)$ and maps will be denoted by minuscule letters $f, g, h, etc. \in \mathbb{C}(X, Y)$. Arbitrary maps will be denoted using an arrow $f : X \to Y$, identity maps as $1_X : X \to X$, and for composition we will use *diagrammatic* notation, that is, the composition of $f : X \to Y$ followed by $g : Y \to Z$ is denoted as $f; g : X \to Z$. Following the conventions used in [Abramsky et al.(2002); Haghverdi and Scott(2006)], for simplicity, we will work with *strict* monoidal categories, meaning that the associativity and unit isomorphisms for the monoidal product are equalities. So for an arbitrary symmetric (strict) monoidal category $\mathbb{C}$, we denote its monoidal product as $\otimes$, the monoidal unit as I, and the natural symmetry isomorphism as $\sigma_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$. Strictness allows us to write $A_1 \otimes A_2 \otimes \ldots \otimes A_n$ and $A \otimes I = A = I \otimes A$.

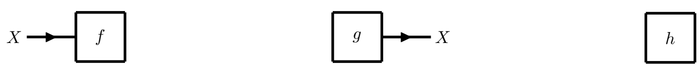
Symmetric monoidal categories enjoy a wonderfully practical graphical calculus, which allows one to write equations of maps in a symmetric monoidal category as string diagrams. This graphical calculus is extremely useful since it provides a more visual representation, which often leads to better intuition for definitions, computations, and proofs. For an in-depth introduction to the graphical calculus of monoidal categories, we invite the reader to see [Heunen and Vicary(2019); Selinger(2010)].

In this paper, following the conventions in [Abramsky et al.(2002); Haghverdi and Scott(2006)], our string diagrams are to be read horizontally from left to right. So for an arbitrary map $f : X_1 \otimes \ldots \otimes X_n \to Y_1 \otimes \ldots \otimes Y_m$ will be denoted as a box with inputs on the left, one wire for each X_i, and outputs on the right, with one wire for each Y_j:



$f : X_1 \otimes \ldots \otimes X_n \to Y_1 \otimes \ldots \otimes Y_m$

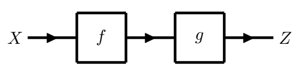
When there is no confusion, we often simply omit the object labels on the wires of our string diagrams. In the special case that one of the inputs or outputs is the monoidal unit I, we do not draw a wire representing it (which lines up with the idea that we are working in the strict case). For example:



$f : X \to I$ $\quad$ $g : I \to X$ $\quad$ $h : I \to I$

Maps of type $I \to I$, so boxes with no input or output wires, are often called "scalars".

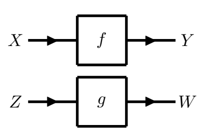
Composition is written sequentially, while identity maps are simply drawn as a wire with no box. For example, in the simplest case:



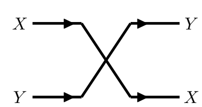
$f; g : X \to Z$

$1_X : X \to X$

Note that the identity for the monoidal unit $1_I : I \to I$ is drawn as nothing. On the other hand, the monoidal product of maps is written by placing the maps in parallel, while the symmetry isomorphism is drawn by crossing the wires:



$f \otimes g : X \otimes Z \to Y \otimes W$

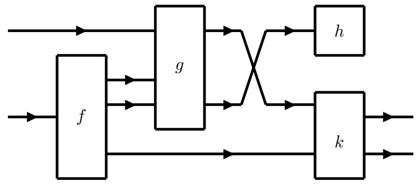


$\sigma_{X,Y} : X \otimes Y \to Y \otimes X$

As such, the usual composition is referred to as "sequential composition", while the monoidal product is referred to as "parallel composition". The advantage of a symmetric monoidal category is that symmetry allows us to cross wires without having to worry about which wire passes on top of the other wire. So we may think of the crossing of wires as the wires passing through each other uninterrupted.

Of course, with all this machinery, we can write out more complex maps in our symmetric monoidal category. For example for maps $f : Y \to A \otimes B \otimes C$, $g : X \otimes A \otimes B \to Z \otimes W$, $h : W \to I$, and $k : Z \otimes C \to E \otimes F$, the composite

$(1_X \otimes f); (g \otimes 1_C); (\sigma_{Z,W} \otimes 1_C); (h \otimes k) : X \otimes Y \to E \otimes F$ is drawn out as follows:



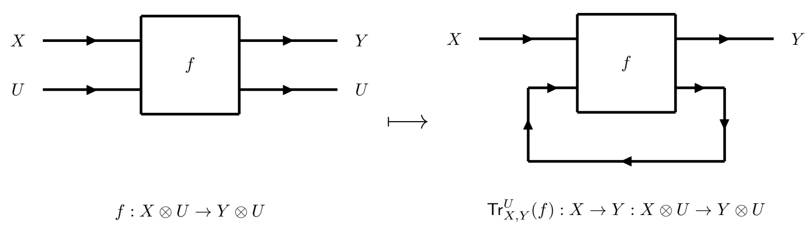
Finally, with all this setup, we can properly add traces to the story. Graphically speaking, traces for symmetric monoidal categories add the possibility of looping back an output wire into an input wire of the same type, which in turn allows loops.

There are many equivalent lists of axioms for a trace operator. Here, we will use the one presented in [Hasegawa and Lemay(2023), Definition 2.2], though we follow the string diagram conventions used in [Abramsky et al.(2002); Haghverdi and Scott(2006)]. For other equivalent axiomatizations of the trace operator, see for example [Haghverdi and Scott(2010a), Definition 4.1] or [Hasegawa(1997), Definition 2.1] or [Hasegawa(2009), Section 3].

Definition 2.1. A **traced symmetric monoidal category** is symmetric monoidal category $\mathbb{C}$ equipped with a **trace operator** Tr, which is a family of operators (indexed by triples of objects $U, X, Y \in \mathbb{C}$):

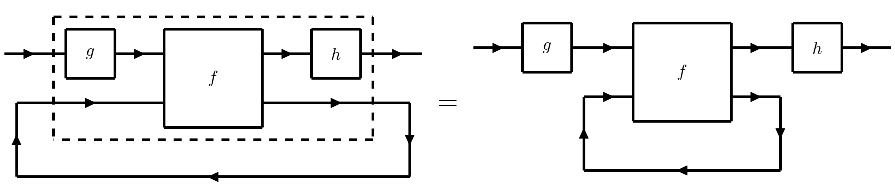
$$\mathsf{Tr}^U_{X,Y} : \mathbb{C}(X \otimes U, Y \otimes U) \to \mathbb{C}(X, Y) \qquad \frac{f : X \otimes U \to Y \otimes U}{\mathsf{Tr}^U_{X,Y}(f) : X \to Y}$$

which is drawn in the graphical calculus as follows:



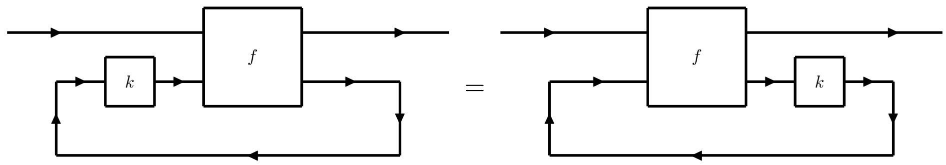
such that the following axioms are satisfied[4] (where the dotted line represents what is being traced when there is possible confusion):

— **[Tightening]**:

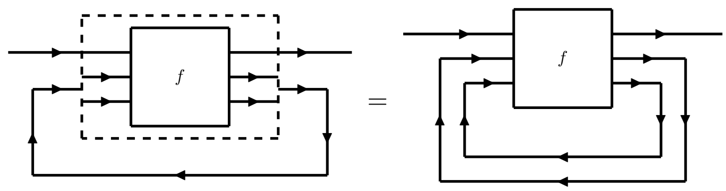


[4]For equational versions of these axioms, see [Hasegawa and Lemay(2023), Definition 2.2]

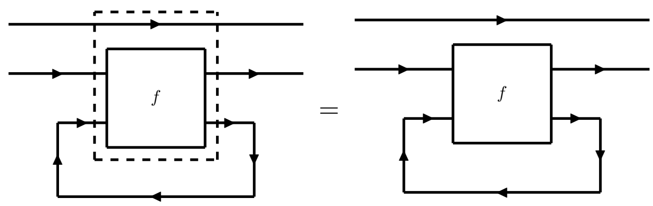
— **[Sliding]**:



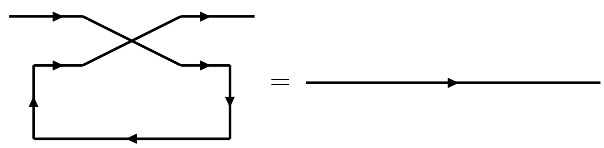
— **[Vanishing]**



— **[Superposing]**



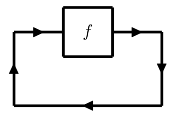
— **[Yanking]**



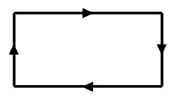
For a map $f : X \otimes U \to Y \otimes U$, the map $\mathsf{Tr}^U_{X,Y}(f) : X \to Y$ is called the **trace** of f.

[Tighetening] says that the trace operator is natural in the X and Y arguments, meaning that we can pull maps in and out of the trace operator if they have no interaction with the argument U that's being traced. In other words, anything that happens on the wires that are not being traced does not affect the trace. **[Sliding]** says that the trace operator is instead *dinatural* in the U argument, and this corresponds to the famous cyclic property of the trace from linear algebra (which we will discuss more about below). **[Vanishing]** says that if we trace out $U_1 \otimes U_2$, its the same thing as tracing out U_2 first then tracing out U_1. Keen-eyed readers will note that in other versions of the axioms, such as in [Abramsky et al.(2002); Haghverdi and Scott(2006)], there is a second part to the vanishing axiom which says that for a map $f : X \to Y$, if seen as a map $f : X \otimes I \to Y \otimes I$, then tracing out the monoidal unit does nothing, $\mathsf{Tr}^I_{X,Y}(f) = f$. It turns out that this is, in fact, provable from the other axioms [Hasegawa(2009), Appendix A], and since graphically there is nothing to draw, we have omitted it here in the above definition. **[Superposing]** says that if there is a part that is completely disconnected from what is being traced, then it can be taken out of the trace. Finally, **[Yanking]** tells us what the trace of the symmetry is, and essentially says that there are no knots in this framework.

Before we consider examples, let's first observe that we can also take traces of endomorphisms to give us actual loops. Indeed, note that an endomorphism $f : U \to U$ can be viewed as a map of type $f : I \otimes U \to I \otimes U$. As such, we may take its trace to obtain the scalar $\mathsf{Tr}^U_{I,I}(f) : I \to I$. Graphically, this is drawn as a loop with a box:

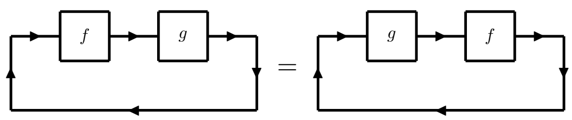


When taking the trace of the identity $1_U : U \to U$, we obtain a scalar which is drawn as an actual loop:



and this is often called the "dimension" of the object U. A simple observation is that we can recapture the famous cyclic property of the trace for matrices in linear algebra. This shows that we can slide endomorphisms around the loop.

LEMMA 2.2. *In a traced symmetric monoidal category, the following equality holds:*



PROOF. This is just a special case of **[Sliding]**. □

Let's now review how the trace operation for matrices gives us an example of a traced symmetric monoidal category.

Example 2.3. Let R be a commutative ring. Let $\mathsf{MAT}(R)$ be the category of matrices over R, that is, the category whose objects are natural numbers $n \in \mathbb{N}$ and where a map $A : m \to n$ is an $m \times n$ matrix A with coefficients in R, with composition given by matrix multiplication and identity maps given by identity matrices. $\mathsf{MAT}(R)$ is a trace symmetric monoidal category whose trace operator is the standard partial trace of matrices from linear algebra. The monoidal product for $\mathsf{MAT}(R)$ is defined on objects by multiplication $m \otimes n = mn$, and on maps as the usual tensor product of matrices. For the trace operator, first recall that for square $n \times n$ matrix A, its trace is equal to the sum of its diagonal coefficients, $\mathsf{Tr}(A) = \sum\limits_{i=1}^{n} A_{i,i}$. Now observe that a map $A : m \otimes k \to n \otimes k$, which is an $mk \times nk$ matrix A, can be expressed in terms of square matrices as follows:

$$A = \begin{bmatrix} A(1,1) & A(1,2) & \dots & A(1,n) \\ A(2,1) & A(2,2) & \dots & A(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ A(m,1) & A(m,2) & \dots & A(m,n) \end{bmatrix}$$

where $A(i,j)$ are square $k \times k$ matrices (with $1 \leq i \leq m$ and $1 \leq j \leq n$). Then the trace $\mathsf{Tr}^k_{m,n}(A) : m \to n$ is the $m \times n$ matrix $\mathsf{Tr}^k_{m,n}(A)$ whose coefficients are the traces of the square matrices:

$$\mathsf{Tr}^k_{m,n}(A) = \begin{bmatrix} \mathsf{Tr}\left(A(1,1)\right) & \mathsf{Tr}\left(A(1,2)\right) & \dots & \mathsf{Tr}\left(A(1,n)\right) \\ \mathsf{Tr}\left(A(2,1)\right) & \mathsf{Tr}\left(A(2,2)\right) & \dots & \mathsf{Tr}\left(A(2,n)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{Tr}\left(A(m,1)\right) & \mathsf{Tr}\left(A(m,2)\right) & \dots & \mathsf{Tr}\left(A(m,n)\right) \end{bmatrix}$$

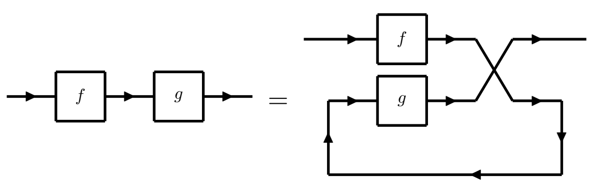
Now note that an endomorphism $A : n \to n$ is precisely a square $n \times n$ matrix A. Therefore, in this case, since the monoidal unit is $I = 1$, we get that the trace operator for endomorphisms gives us back precisely the usual trace for square matrices, that is, $\mathsf{Tr}^n_{1,1}(A) = \mathsf{Tr}(A)$. So Lemma 2.2 corresponds precisely to the all-famous identity that $\mathsf{Tr}(AB) = \mathsf{Tr}(BA)$.

There are many other interesting examples of traced symmetric monoidal categories, see [Haghverdi and Scott(2010a), Examples 4.5 & 4.6] for lists of more examples. In particular, below we will review two important classes of traced symmetric monoidal categories: compact closed categories and traced unique decomposition categories. Other important examples worth mentioning are when the monoidal product is a product, coproduct, or even a biproduct. For products, trace operators capture the notion of feedback via fixpoints, and in fact, to give a trace operator for products is equivalent to providing a special kind of fixpoint operator (called a *Conway operator*) [Hasegawa(1997), Theorem 3.1]. On the other hand, for coproducts, trace operators capture the notion of feedback via iteration, and similarly, to give a trace operator for coproducts is equivalent to giving a special kind of iteration operator [Selinger(2010), Proposition 6.8]. For biproducts, to give a trace operator is equivalent to giving a repetition operator on endomorphisms [Selinger(2010), Proposition 6.11]. Scott and his coauthors often liked to refer to traces for compact closed categories and products as "product or wave style traces" [Haghverdi and Scott(2010a), Examples 4.5], and traces for unique decomposition categories and coproducts as "sum or particle style traces" [Haghverdi and Scott(2010a), Examples 4.5]. Briefly, the intuition is that a product style trace can be understood as passing information in a global information wave, while a sum style trace is interpreted by streams of particles or tokens flowing around a network – see [Abramsky(1996); Abramsky et al.(2002)] for more explanation on these ideas.

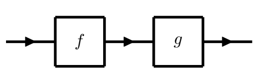
An important identity in a traced symmetric monoidal category is that sequential composition can be reformulated in terms of parallel composition using the trace.

THEOREM 2.4. *[Abramsky et al.(2002), Proposition 2.4] In a traced symmetric monoidal category, the following equality holds:*

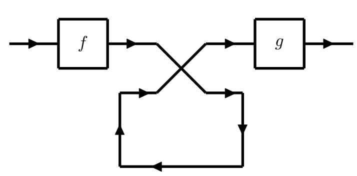
— ***[Generalized Yanking]***

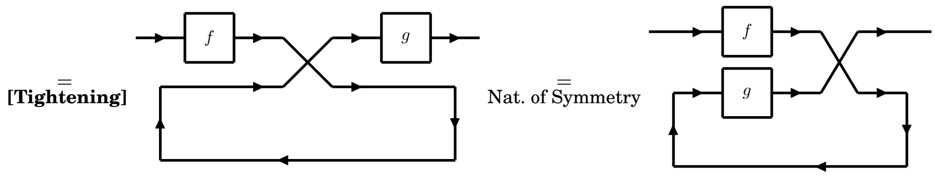


PROOF. To demonstrate the usefulness of working with string diagrams, let's prove this using the graphical calculus. So we compute:



$\underset{\textbf{[Yanking]}}{=}$





We invite the reader to compare these computations with the equational version of this computation found in the proof of [Abramsky et al.(2002), Proposition 2.4]. □

It is also possible to axiomatize trace operators with **[Generalized Yanking]** as an axiom instead, see [Haghverdi(2000), Proposition 2.1.21]. Moreover, using **[Generalized Yanking]**, we get a normal form theorem for maps in traced symmetric monoidal categories.

THEOREM 2.5. *[Abramsky et al.(2002), Theorem 2.5] Let T be a set of maps in a traced symmetric monoidal category $\mathbb{C}$. Then any expression E built from maps in T built from using the monoidal product, composition, and trace can be expressed as $\mathsf{Tr}(\pi; F; \tau)$ where F consists of a monoidal product of maps in T, and π and τ are permutations (i.e. maps constructed from symmetry and identity maps using monoidal product and composition).*

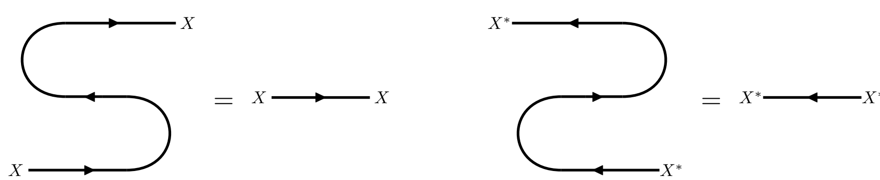
Essentially, from the point of view of the graphical calculus, this normal form theorem says we can always re-draw a map in such a way that we don't have any sequential composition of boxes, and only have parallel composition of boxes.

3. COMPACT CLOSED CATEGORIES

An important class of traced symmetric monoidal categories are *compact closed categories*, which are particularly important in categorical quantum foundations [Abramsky and Coecke(2004); Heunen and Vicary(2019)]. Compact closed categories are symmetric monoidal categories where every object has a dual. Every compact closed category comes equipped with a canonical trace operator that captures the classical notion of (partial) trace for matrices, which is a fundamental operation for both classical quantum theory and categorical quantum foundations [Abramsky and Coecke(2005)]. For a more in-depth introduction to compact closed categories, we invite the reader to see [Selinger(2010), Section 4.8] or [Heunen and Vicary(2019), Chapter 3].

Definition 3.1. A **compact closed category** is a symmetric monoidal category such that for every object X, there is an object X^*, called the **dual** of X, and maps $\cup_X : X^* \otimes X \to I$, called the **cup**, and $\cap_X : I \to X \otimes X^*$, called the **cap**, which are drawn in the graphical calculus as follows:

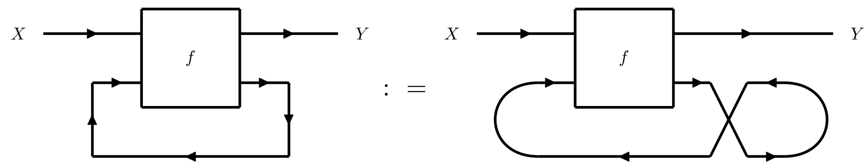
such that the following equality holds:



and these equalities are called the snake equations.

Every compact closed category is a traced symmetric monoidal category, where the trace operator is constructed using the cups and caps, and where the trace operator axioms follow from the snake equations. Furthermore, compact closed categories have a *unique* trace operator.

PROPOSITION 3.2. *[Joyal et al.(1996), Proposition 3.1] A compact closed category is a traced symmetric monoidal category where the trace operator is defined as follows:*



Furthermore, this is the unique *trace operator on a compact closed category [Hasegawa(2009), Section 3.2].*

It is also worth mentioning that, as their name suggests, compact closed categories are also symmetric monoidal *closed* categories. In fact, compact closed categories are also *star-autonomous* categories. It is also interesting to note that the converse is also true, that is, *traced* star-autonomous categories are compact closed [Hajgató and Hasegawa(2013), Theorem 3.6].

There are many interesting examples of compact closed categories. In fact, we have already encountered one, as the category of matrices from Example 2.3 is a compact closed category. However, for a more intuitive example, let's review how finite dimensional vector spaces form a compact closed category.

Example 3.3. Let k be a field, and let FVEC_k be the category of finite dimensional k-vector spaces and k-linear transformations between them. FVEC_k is a compact closed category, where the monoidal product is the standard algebraic tensor product of vector spaces, so in particular $I = k$, and where the dual of a k-vector space U is given by its algebraic dual, that is, $U^* = \{\phi : U \to k |\ \phi \text{ is } k\text{-linear}\}$. The cup $\cup_U : U^* \otimes U \to k$ is defined by evaluation:

$$\cup_U(\phi \otimes x) = \phi(x)$$

To define the cap, let $\{e_1, \ldots, e_n\}$ be a basis of U, which induces a basis $\{e_1^*, \ldots, e_n^*\}$ for U^*. Then the cap $\cap_U : k \to U \otimes U^*$ is defined as:

$$\cap_U(1) = \sum_{i=1}^{n} e_i \otimes e_i^*$$

For a k-linear map $f : V \otimes U \to W \otimes U$, its trace is the k-linear map $\mathsf{Tr}^U_{V,W}(f) : V \to W$ defined as:

$$\mathsf{Tr}^U_{V,W}(f)(m) = \sum_{i=1}^{n} (1_W \otimes e_i^*)\,(f(m \otimes e_i))$$

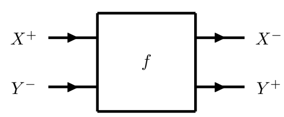
Note that the definitions of the cap and trace are independent of the choice of basis. In particular, when $k = \mathbb{C}$, the field of complex numbers, then $\mathsf{FVEC}_{\mathbb{C}}$ is equivalent to the category of finite dimensional Hilbert spaces, which is also compact closed and the fundamental category of study in categorical quantum foundations.

On the other hand, not every traced symmetric monoidal category is a compact closed category. That said, every traced symmetric monoidal category embeds fully and faithfully into a compact closed category via the INT-construction [Joyal et al.(1996), Section 4]. The name INT-construction comes from the fact that when you apply this construction to the natural numbers $\mathbb{N}$ you get the integers $\mathbb{Z}$ [Haghverdi and Scott(2010a), Example 4.13]. Abramsky introduced another way of constructing a compact closed category from a traced symmetric monoidal category via the so-called "Geometry of Interaction construction" [Abramsky(1996), Section 3], or simply GOI-construction for short. It turns out that the resulting compact closed categories from the INT-construction and the GOI-construction are isomorphic [Haghverdi(2000), Proposition 2.3.6]. However, it is the GOI-construction that better isolates the fundamental properties of Girard's original ideas for Geometry of Interaction, where, in particular, the composition in this compact closed category corresponds to Girard's Execution formula. As such, it is the GOI-construction that plays a more central role in the categorical approaches to Geometry of Interaction in [Abramsky(1996); Abramsky et al.(2002); Haghverdi(2000); Haghverdi and Scott(2010a)].

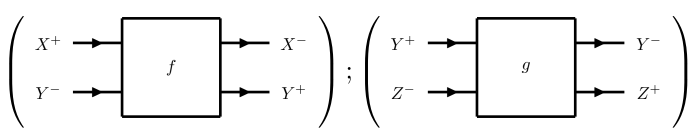
Definition 3.4. For a traced symmetric monoidal category $\mathbb{C}$, define the compact closed category $\mathcal{G}(\mathbb{C})$ as follows:

(i) The objects are pairs of objects of $\mathbb{C}$, so $\mathcal{O}(\mathcal{G}(\mathbb{C})) = \mathcal{O}(\mathbb{C}) \times \mathcal{O}(\mathbb{C})$. We will suggestively denote objects of $\mathcal{G}(\mathbb{C})$ as pairs (X^+, X^-), though this is simply notation and there is no relationship between X^+ and X^-.

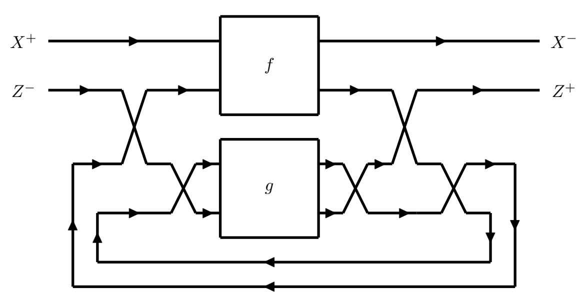
(ii) The homsets are $\mathcal{G}(\mathbb{C})\,((X^+, X^-), (Y^+, Y^-)) = \mathbb{C}(X^+ \otimes Y^-, X^- \otimes Y^+)$. So a map $f : (X^+, X^-) \to (Y^+, Y^-)$ in $\mathcal{G}(\mathbb{C})$ is a map of type $f : X^+ \otimes Y^- \to X^- \otimes Y^+$ in $\mathbb{C}$.



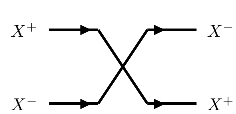
(iii) Composition is defined as follows:



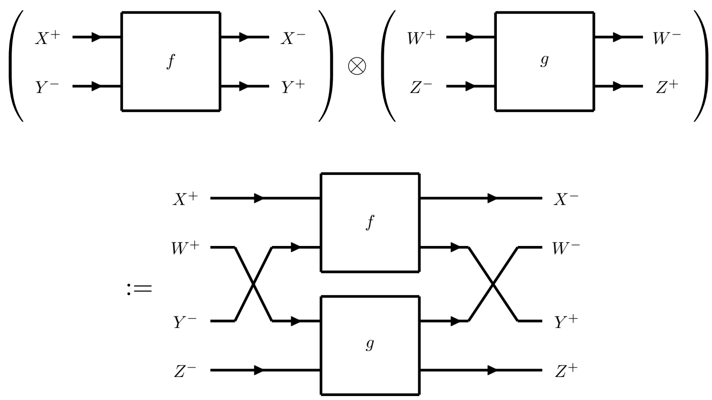
:=



(iv) The identity map $1_{(X^+,X^-)} : (X^+, X^-) \to (X^+, X^-)$ is the symmetry map in $\mathbb{C}$, that is, $1_{(X^+,X^-)} = \sigma_{X^+,X^-}$:

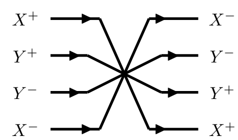


(v) On objects, the monoidal product is defined as the pointwise monoidal product of pairs $(X^+, X^-) \otimes (Y^+, Y^-) = (X^+ \otimes Y^+, X^- \otimes Y^-)$, while on maps, the monoidal product is defined as follows:

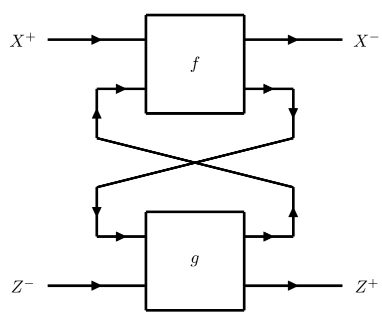


(vi) The monoidal unit is the pair (I, I);

(vii) The symmetry $\sigma_{(X^+,X^-),(Y^+,Y^-)} : (X^+, X^-) \otimes (Y^+, Y^-) \to (Y^+, Y^-) \otimes (X^+, X^-)$ is the permutation $X^+ \otimes Y^+ \otimes Y^- \otimes X^- \to X^- \otimes Y^- \otimes Y^+ \otimes X^+$:



A more intuitive way of understanding the composition in $\mathcal{G}(\mathbb{C})$ is by slightly abusing the flexibility of the wires in the graphical calculus and drawing composition as follows:

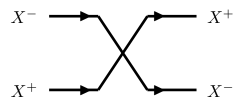


Drawn like this, we see that composition in $\mathcal{G}(\mathbb{C})$ is given by symmetric feedback. Moreover, observe that the composition in $\mathcal{G}(\mathbb{C})$ only depends on the parallel composition of boxes, similar to that of **[Generalized Yanking]**. As mentioned above, $\mathcal{G}(\mathbb{C})$ is a compact closed category, where it turns out that the cups and caps are simply given by the symmetry maps. Furthermore, we can also embed $\mathbb{C}$ into $\mathcal{G}(\mathbb{C})$, and this embedding preserves the traced symmetric monoidal structure strictly.

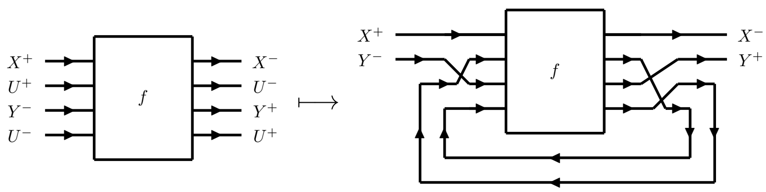
PROPOSITION 3.5. *[Abramsky et al.(2002), Proposition 2.8] For a traced symmetric monoidal category $\mathbb{C}$, $\mathcal{G}(\mathbb{C})$ is compact closed where:*

(i) *The dual of* (X^+, X^-) *is* $(X^+, X^-)^* = (X^-, X^+)$

(ii) *The cup* $\cup_{(X^+,X^-)} : (X^-, X^+) \otimes (X^+, X^-) \to (I, I)$ *and the cap* $\cap_{(X^+,X^-)} : (I, I) \to (X^+, X^-) \otimes (X^-, X^+)$*, which are both maps of type* $X^- \otimes X^+ \to X^+ \otimes X^-$ *in* $\mathbb{C}$*,* *are given by the symmetry map in* $\mathbb{C}$*, that is,* $\cup_{(X^+,X^-)} = \cap_{(X^+,X^-)} = \sigma_{X^-,X^+}$*:*



(iii) *The induced traced of a map* $f : (X^+, X^-) \otimes (U^+, U^-) \to (Y^+, Y^-) \otimes (U^+, U^-)$*, which is a map of type* $f : X^+ \otimes U^+ \otimes Y^- \otimes U^- \to X^- \otimes U^- \otimes Y^+ \otimes U^+$ *in* $\mathbb{C}$*, is the double trace of* f *in* $\mathbb{C}$*, that is,* $\mathsf{Tr}^{(U^+,U^-)}_{(X^+,X^-),(Y^+,Y^-)}(f) : (X^+, X^-) \to (Y^+, Y^-)$*, which is drawn on the right as follows:*



Moreover, there is a full and faithful embedding functor $\mathcal{N} : \mathbb{C} \to \mathcal{G}(\mathbb{C})$ *which is defined on objects as* $\mathcal{N}(X) = (X, I)$ *and on maps as* $\mathcal{N}(f) = f$*. Furthermore,* $\mathcal{N}$ *preserves the traced symmetric monoidal structure strictly, so in particular we have that* $\mathcal{N}(\mathsf{Tr}^U_{X,Y}(f)) = \mathsf{Tr}^{\mathcal{N}(U)}_{\mathcal{N}(X),\mathcal{N}(Y)}(\mathcal{N}(f))$*.*

Some explicit descriptions of the GOI-construction applied to certain traced symmetric monoidal categories are described in [Abramsky(1996), Section 4]. In particular, when applying the GOI-construction to the category of partial injections or a certain subcategory of Hilbert spaces, it recaptures Girard's original model of Geometry of Interaction. Also, as mentioned above, we have an isomorphism of compact closed categories $\mathcal{G}(\mathbb{C}) \simeq \mathcal{INT}(\mathbb{C})$, where the latter is the INT-construction over $\mathbb{C}$. As such, the GOI-construction is also the free compact closed category over $\mathbb{C}$ in an appropriate bicategorical sense [Haghverdi and Scott(2010a), Proposition 4.14]. For more details on the INT-construction, and its comparision to the GOI-construction, see [Haghverdi(2000), Section 2.2 & 2.3].

4. UNIQUE DECOMPOSITION CATEGORIES

Another important sort of traced symmetric monoidal category that Scott and Haghverdi studied for the categorical interpretations of Geometry of Interaction were *traced unique decomposition categories* [Haghverdi and Scott(2006); Haghverdi and Scott(2010a)]. In particular, in [Haghverdi and Scott(2006)], Scott and Haghverdi explain how traced unique decomposition categories provide the correct categorical framework which properly captures Girard's original Hilbert space model of Geometry of Interaction.

Unique decomposition categories (UDC) were originally introduced by Haghverdi in their PhD thesis [Haghverdi(2000), Chapter 4], and are generalizations of Manes and Arbib's partially additive categories [Manes and Arbib(2012), Section 3.2]. Briefly, a UDC is a symmetric monoidal category with an appropriate notion of countable sums and a matrix representation for maps. This countable sum property is captured by the fact that the homsets of a UDC are **Σ-monoids**, which are generalizations of partially additive monoids [Manes and Arbib(2012), Section 3.1] by dropping the so-called limit axiom. So a **Σ-monoid** is a non-empty set M equipped with a partial operation Σ called the **partial infinitary sum** on countable families of elements of M satisfying a partition-associativity axiom and a unary sum axiom. For a countable family $\{x_i\}_{i \in I}$

we denote its image as $\Sigma_{i\in I} x_i$ and say that $\{x_i\}_{i\in I}$ is **summable** if $\Sigma_{i\in I} x_i$ is defined. Then, the two necessary axioms of a Σ-monoid are:

(i) **Partitition-Associativity Axiom**: If $\{x_i\}_{i\in I}$ is a countable family and if $\{I_j\}_{j\in J}$ is a countable partition of I, then $\{x_i\}_{i\in I}$ is summable if and only if $\{x_i\}_{i\in I_j}$ is summable for every $j \in J$ and $\{\sum_{i\in I_J} x_i\}_{j\in J}$ is summable. Moreover, in this case $\sum_{i\in I} x_i = \sum_{j\in J}\sum_{i\in I_j} x_i$.

(ii) **Unary Sum Axiom:** Every singleton $\{x\}$ is summable and $\sum_{i\in I} x_i = x$ where $\{x\}$ is seen as an $I = \{i\}$ indexed set.

In any Σ-monoid, the empty family is summable, which we denote as 0, and it is an additive unit. A Σ-monoid morphism is a function $f : M \to M'$ between the underlying sets which preserves summability and the partial infinitary sums, that is, if $\{x_i\}_{i\in I}$ is summable in M then $\{f(x_i)\}_{i\in I}$ is summable in M' and $f\left(\sum_{i\in I} x_i\right) = \sum_{i\in I} f(x_i)$.

Now Σ-monoids and Σ-monoid morphisms form a symmetric monoidal closed category ΣMON. Therefore, we may consider categories enriched in ΣMON. Explicitly, a ΣMON-category is a category $\mathbb{C}$ such that each homset is a Σ-monoid where composition preserves the partial infinitary sums, that is, if $\{f_i\}_{i\in I}$ is a summable family of maps in $\mathbb{C}(X, Y)$, then for any suitable maps we also have that $\{g; f_i; h\}_{i\in I}$ is also summable and:

$$g;\left(\sum_{i\in I} f_i\right); h = \sum_{i\in I} g; f_i; h$$

Note that a ΣMON-category has non-empty homsets since the sum of the empty set gives us the map $0_{X,Y} : X \to Y$, which is in fact an actual zero morphism in the categorical sense. Now a symmetric monoidal ΣMON-category is a symmetric monoidal category $\mathbb{C}$ which is also a ΣMON-category such that the monoidal product preserves the partial infinitary sums, that is, if $\{f_i\}_{i\in I}$ and $\{g_i\}_{j\in J}$ are summable family of maps, then $\{f_i \otimes g_j\}_{(i,j)\in I\times J}$ is also a summable family of maps and:

$$\left(\sum_{i\in I} f_i\right) \otimes \left(\sum_{j\in J} g_j\right) = \sum_{i\in I}\sum_{j\in J} f_i \otimes g_j$$

Then a UDC is a symmetric monoidal ΣMON-category which comes equipped with quasi-injections and quasi-projections that satisfy analogues of the axioms for biproducts.

Definition 4.1. A **unique decomposition category (UDC)** is a symmetric monoidal ΣMON-category such that for all finite family of objects $\{X_1, \ldots, X_n\}$ and every $1 \leq j \leq n$ there are maps $\iota_j : X_j \to X_1 \otimes \ldots \otimes X_n$, called the **quasi-injections**, and maps $\rho_j : X_1 \otimes \ldots \otimes X_n \to X_j$, called the **quasi-projections**, such that:

(i) $\iota_j; \rho_j = 1_{X_j}$ and $\iota_j l \rho_i = 0_{X_i,X_j}$ if $i \neq j$

(ii) $\sum_{j=1}^{n} \iota_j; \rho_j = 1_{X_1\otimes\ldots\otimes X_n}$

It is important to note that these quasi-injections and quasi-projections do not make the monoidal product a biproduct. Nevertheless, with these quasi-injections and quasi-projections, we can give a matrix representation for maps in a UDC [Haghverdi(2000), Proposition 4.0.6] similar to the matrix representation we can give for maps in a category with finite biproducts [Heunen and Vicary(2019), Section 2.2.4]. Indeed, for a $f : X_1 \otimes \ldots \otimes X_n \to Y_1 \otimes \ldots \otimes Y_m$, there exists a unique summable finite family of maps

$\{f_{i,j} : X_i \to Y_j\}_{i\in I, j\in J}$ such that:

$$f = \sum_{i\in I, j\in J} \rho_i; f_{i,j}; \iota_j$$

Explicitly, the maps $f_{i,j}$ are given by pre-compositing and post-composing f by the appropriate quasi-injection and quasi-projection respectively, that is, $f_{i,j} := \iota_i; f; \rho_j$. As such, we may represent f as an $n \times m$ matrix with (i,j)-component the map $f_{i,j} : X_i \to Y_j$:

$$f = \begin{bmatrix} f_{1,1} & \dots & f_{1,m} \\ \vdots & \ddots & \vdots \\ f_{n_1} & \dots & f_{n,m} \end{bmatrix}$$

This matrix representation is particularly useful since composition in a UDC then corresponds to matrix multiplication.

Now, since maps in a UDC have a matrix representation like in a category with biproducts, it is natural to ask if an analogue of the well-known trace formula for countable biproducts gives a trace for UDC. If the necessary sum is always defined, then the answer is yes! As such, we say that a UDC is **traced** if for every map of type $f : X \otimes U \to Y \otimes U$, which in matrix form is:

$$f := \begin{bmatrix} f_{1,1} : X \to Y & f_{1,2} : X \to U \\ f_{2,1} : U \to Y & f_{2,2} : U \to U \end{bmatrix}$$

the sum $f_{1,1} + \sum_{n\in\mathbb{N}} f_{1,2}; f_{2,2}^n; f_{2,1}$ exists, which is a map of type $X \to Y$. This then gives us the *standard trace formula* for a trace operator, justifying the name traced UDC.

THEOREM 4.2. *[Haghverdi(2000), Proposition 4.0.11] A traced UDC is a traced symmetric monoidal category with trace operator defined on a map $f : X \otimes U \to Y \otimes U$ as follows:*

$$\mathsf{Tr}^U_{X,Y}(f) = f_{1,1} + \sum_{n\in\mathbb{N}} f_{1,2}; f_{2,2}^n; f_{2,1}$$

We note that the matrix representation greatly simplifies the proof of showing that the standard trace formula does indeed give a trace operator. It is also worth mentioning that a (traced) UDC could have another trace operator which is not given by the standard trace formula. However, when talking about a traced UDC, we are referring to the trace operator given above by the standard trace formula.

The two main examples of traced UDC are the category of partial injections and a certain subcategory of Hilbert spaces. These two examples, especially the latter, capture Girard's operator algebra model of Geometry of Interaction.

Example 4.3. Let PINJ be the category of sets and partial injections between them. Then PINJ is a traced UDC [Haghverdi and Scott(2006), Example 8] where:

- *(i)* The monoidal product is given by the disjoint union of sets, $X \otimes Y = X \sqcup Y := (\{1\} \times X) \cup (\{2\} \times Y)$, while the monoidal unit is the empty set, $I = \emptyset$;
- *(ii)* $\mathsf{PINJ}(X, Y)$ is a Σ-monoid where a countable family of partial injections $\{f_i : X \to Y\}_{i\in I}$ is summable if they have pairwise disjoint domains and codomains, and in this case, their sum is given as follows:

$$\left(\sum_{i\in I} f_i\right)(x) = \begin{cases} f_j(x) & \text{if } x \in \mathsf{dom}(f_j) \text{ for some } j \in I \\ \text{undefined} & \text{o.w.} \end{cases}$$

(iii) The quasi-injection $\iota_j : X_j \to X_1 \sqcup \ldots \sqcup X_n$ inserts X_j into the disjoint union, while the quasi-projection $\rho_j : X_1 \sqcup \ldots \sqcup X_n \to X_j$ is only defined on the X_j part:

$$\iota_j(x) = (j, x) \qquad\qquad \rho_j(i, x) = \begin{cases} x & \text{if } i = j \text{, so } x \in X_j \\ \text{undefined} & \text{if } i \neq j \end{cases}$$

(iv) For a partial injection $f : X \sqcup U \to Y \sqcup Y$, its induced trace is worked out to be the partial injection $\mathsf{Tr}^U_{X,Y}(f) : X \to Y$ defined as follows:

$$\mathsf{Tr}^U_{X,Y}(f)(x) = \begin{cases} y & \text{if } f(1,x) = (1,y) \\ & \\ & \text{if } \exists\, n \in \mathbb{N}\ \exists\, u_0, u_1, \ldots, u_n \in U \text{ s.t.} \\ y_{n+1} & f(1,x) = (2,u_0) \text{ and } f(2,u_0) = (2,u_1) \text{ and} \\ & \ldots \text{ and } f(2,u_n) = (1, y_{n+1}) \\ & \\ \text{undefined} & \text{o.w.} \end{cases}$$

More intuitively, the trace starts by taking an $x \in X$, applying f and if $f(1,x)$ is defined, checking if the second component of $f(1,x)$ is in Y or in U. If it's in Y, then we are done and output the Y component. If it's in U, then we apply f again, and if $f(2, f(1,x))$ is defined, then we check again if its second component is in Y or U. If it's in Y, we are done and output the Y component. Otherwise, if it's in U, we repeat the process again. We do this until if after a finite number of iterations, we land in Y and then output the result. Otherwise, the trace is undefined.

Example 4.4. Let HILB be the category of Hilbert spaces and linear contractions (with norm ≤ 1). Now while HILB is a symmetric monoidal category, where the monoidal product is the usual tensor product of Hilbert spaces, it is not a traced UDC. To obtain a UDC, we must instead consider a subcategory of HILB. So for a set X, let $\ell_2(X)$ be the set of complex-valued functions $a : X \to \mathbb{C}$ such that the sum $\sum_{x \in X} |a(x)|^2$ is finite (or equivalently the set of square summable X-indexed families). Then $\ell_2(X)$ is a Hilbert space with inner product $\langle a, b \rangle = \sum_{x \in X} a(x)\overline{b(x)}$ and induced norm $\|a\| = \left(\sum_{x \in X} |a(x)|^2\right)^{\frac{1}{2}}$. This induces a contravariant faithful functor $\ell_2 : \mathsf{PINJ} \to \mathsf{HILB}$ [Barr(1992), Section 3] which maps a set X to $\ell_2(X)$, and maps a partial injection $f : X \to Y$ to the linear contraction $\ell_2(f) : \ell_2(Y) \to \ell_2(X)$ defined as follows:

$$\ell_2(f)(a)(y) = \begin{cases} a(f(x)) & \text{if } x \in \mathsf{dom}(f) \\ 0 & \text{if } x \notin \mathsf{dom}(f) \end{cases}$$

Note that $\ell_2(f)$ is in fact a partial isometry. In fact, ℓ_2 sends certain special maps in PINJ to important kinds of maps in HILB, such as the fact total bijections get mapped to unitaries in HILB – see [Haghverdi and Scott(2006)] for more details. Heunen also studies the functor ℓ_2 in more detail in [Heunen(2013)]. Now consider the image of ℓ_2, that is, define $\mathsf{HILB}_2 := \ell_2(\mathsf{PINJ})$ to be the category whose objects are Hilbert spaces of the form $\ell_2(X)$ for some set X and whose maps are of the form $\ell_2(f)$ for some partial injection f. Then HILB_2 is naturally equivalent to PINJ, and therefore HILB_2 is a traced UDC [Haghverdi and Scott(2006), Example 9] whose structure is induced from the one of PINJ. Explicitly, the traced UDC structure of HILB_2 is given as follows:

(i) The monoidal product is given by the biproduct $\oplus$ of Hilbert spaces since we have that:

$$\ell_2(X \sqcup Y) \cong \ell_2(X) \oplus \ell_2(Y)$$

However, it is important to note that this is not a biproduct in HILB_2. The monoidal unit is the zero Hilbert space, $\ell_2(\emptyset) = \{0\}$;

(ii) $\mathsf{HILB}_2(\ell_2(Y), \ell_2(X))$ is a Σ-monoid where a countable family $\{\ell_2(f)_i : \ell_2(Y) \to \ell_2(X)\}_{i \in I}$ is summable if and only if $\{f_i : X \to Y\}_{i \in I}$ is summable in PINJ. In this case, their sum is:

$$\ell_2\left(\sum_{i \in I} f_i\right) := \ell_2\left(\sum_{i \in I} f_i\right)$$

(iii) The quasi-injection/projections are the quasi-projections/injections from PINJ, so we have that $\ell_2(\rho_j) : \ell_2(X_j) \to \ell_2(X_1 \sqcup \ldots \sqcup X_n)$ are the quasi-injections and similarly that $\ell_2(\iota_j) : \ell_2(X_1 \sqcup \ldots \sqcup X_n) \to \ell_2(X_j)$ are the quasi-projections. From another perspective, the quasi-injections are the usual injections for the biproduct $\ell_2(X_j) \to \ell_2(X_1) \oplus \ldots \oplus \ell_2(X_n)$, while the quasi-projections are the usual projections for the biproduct $\ell_2(X_1) \oplus \ldots \oplus \ell_2(X_n) \to \ell_2(X_j)$.

(iv) For a map $\ell_2(f) : \ell_2(Y \sqcup U) \to \ell_2(X \sqcup U)$, its trace is the image of the trace of its associated partial injection $f : X \sqcup U \to Y \sqcup U$ in PINJ:

$$\mathsf{Tr}^{\ell_2(U)}_{\ell_2(Y), \ell_2(X)}(\ell_2(f)) := \ell_2\left(\mathsf{Tr}^{U}_{X,Y}(f)\right)$$

Other examples of traced UDC can be found in [Haghverdi(2000), Example 4.0.15] and [Haghverdi and Scott(2010a), Example 4.11], which include partially additive categories and categories with countable biproducts.

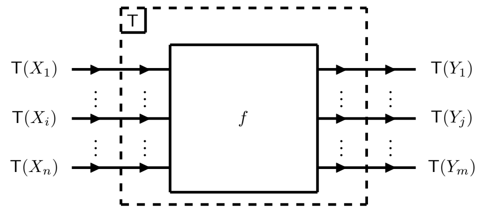
5. GEOMETRY OF INTERACTION SITUATIONS

Traced symmetric monoidal categories and the GOI-construction only account for the Geometry of Interaction interpretation of the *multiplicative* fragment of Linear Logic. One of the main objectives of [Abramsky et al.(2002)] was to provide the categorical semantics for the Geometry of Interaction interpretation of the *exponential* fragment of Linear Logic as well. As such, towards this goal, Scott, Abramsky, and Haghverdi introduced *Geometry of Interaction Situations* [Abramsky et al.(2002), Definition 4.1], which are in particular traced symmetric monoidal categories equipped with a *traced symmetric monoidal functor* and various structural maps, which is used to interpret the exponential modality ! and its rules in the GOI-construction.

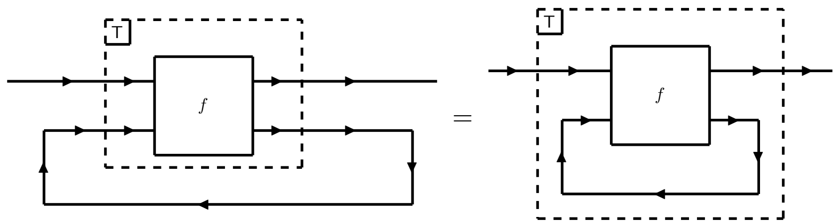
A **traced symmetric monoidal endofunctor** [Abramsky et al.(2002), Section 2] is an endofunctor on a traced symmetric monoidal category which preserves the traced symmetric monoidal structure up to isomorphism. So for a symmetric monoidal category $\mathbb{C}$, recall that an endofunctor $\mathsf{T} : \mathbb{C} \to \mathbb{C}$ is said to be *strong symmetric monoidal* if it has natural isomorphisms $\mathsf{T}(X \otimes Y) \cong \mathsf{T}(X) \otimes \mathsf{T}(Y)$ and $\mathsf{T}(I) \cong I$ which are compatible with the symmetric monoidal structure. In the graphical calculus, we will draw the application of the strong symmetric monoidal functor using a dotted functor box. That T preserves the monoidal structure up to isomorphism allows us to draw wires coming in and out of the box without any issues. So for a map $f : X_1 \otimes \ldots \otimes X_n \to Y_1 \otimes \ldots \otimes Y_m$, the composite:

$$\mathsf{T}(X_1) \otimes \ldots \otimes \mathsf{T}(X_n) \xrightarrow{\cong} \mathsf{T}(X_1 \otimes \ldots X_n) \xrightarrow{\mathsf{T}(f)} \mathsf{T}(Y_1 \otimes \ldots Y_m) \xrightarrow{\cong} \mathsf{T}(Y_1) \otimes \ldots \otimes \mathsf{T}(Y_m)$$

is drawn as follows:



Then if $\mathbb{C}$ is a traced symmetric monoidal category, a *traced symmetric monoidal endofunctor* is a strong symmetric monoidal endofunctor $\mathsf{T} : \mathbb{C} \to \mathbb{C}$ which also preserves the trace in the sense that the following equality holds:



Another key part of the definition of a Geometry of Interaction Situation are *retraction* pairs. Using the same notation as in [Abramsky et al.(2002)], a retraction pair will be denoted by $f : X \triangleleft Y : g$ to mean that X is a retract of Y via f and g, that is, $f; g = 1_X$.

Definition 5.1. A **Geometry of Interaction Situation (GoI Situation)** is a triple $(\mathbb{C}, \mathsf{T}, U)$ where $\mathbb{C}$ is a traced symmetric monoidal category, $\mathsf{T} : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal endofunctor, and $U \in \mathcal{O}(\mathbb{C})$ is an object, called the **reflexive object**, such that T comes equipped with the following retraction pairs of *monoidal* natural transformations:

— **[Digging]**: $\mathsf{e}_X : \mathsf{TT}(X) \triangleleft \mathsf{T}(X) : \mathsf{e}'_X$
— **[Dereliction]**: $\mathsf{d}_X : X \triangleleft \mathsf{T}(X) : \mathsf{d}'_X$
— **[Contraction]**: $\mathsf{c}_X : \mathsf{T}(X) \otimes \mathsf{T}(X) \triangleleft \mathsf{T}(X) : \mathsf{c}'_X$
— **[Weakening]**: $\mathsf{w}_X : I \triangleleft \mathsf{T}(X) : \mathsf{w}'_X$

and also that U comes equipped with the following retraction pairs:

$$\mathsf{j} : U \otimes U \triangleleft U : \mathsf{k} \qquad \mathsf{a} : I \triangleleft U : \mathsf{b} \qquad \mathsf{u} : \mathsf{T}(U) \triangleleft U : \mathsf{v}$$

As mentioned above, the trace symmetric monoidal functor of a GoI Situation will be used to interpret the exponential modality. So, as their names suggest, the four retraction pairs of monoidal natural transformations correspond to the four main rules of the exponential from Linear Logic. The reflexive object will be used to build an algebraic model of Linear Logic.

The two main examples of a GoI Situation are PINJ and HILB_2, and these can both be seen as capturing the main essence of Girard's original model of Geometry of Interaction.

Example 5.2. PINJ is a GoI Situation where the endofunctor is $\mathsf{T}(-) = \mathbb{N} \times -$ and the reflexive object is $\mathbb{N}$ [Abramsky et al.(2002), Proposition 5.2]. The necessary retraction pairs are given as follows:

(i) For $m, n \in \mathbb{N}$ let $\langle m, n \rangle = \frac{(m+n+1)(m+n)}{2} + n$ be Cantor's pairing function (which recall is a bijection). Then $\mathsf{e}_X : \mathbb{N} \times \mathbb{N} \times X \to \mathbb{N} \times X$ is defined as $\mathsf{e}_X(m, n, x) =$

$(\langle m,n\rangle, x)$, and $\mathsf{e}'_X : \mathbb{N}\times X \to \mathbb{N}\times\mathbb{N}\times X$ is defined as $\mathsf{e}'_X(n,x) = (n_1,n_2,x)$ where $\langle n_1,n_2\rangle = n$ (which is well-defined since n_1 and n_2 exists and are unique);

(ii) For some fixed $n_0 \in \mathbb{N}$, $\mathsf{d}_X : X \to \mathbb{N}\times X$ and $\mathsf{d}'_X : X \to \mathbb{N}\times X$ are defined as follows:

$$\mathsf{d}_X(x) = (n_0, x) \qquad\qquad \mathsf{d}_X(n,x) = \begin{cases} x & \text{if } n = n_0 \\ \text{undefined} & \text{o.w.} \end{cases}$$

(iii) $\mathsf{c}_X : (\mathbb{N}\times X)\sqcup(\mathbb{N}\times X) \to \mathbb{N}\times X$ and $\mathsf{c}'_X : \mathbb{N}\times X \to (\mathbb{N}\times X)\sqcup(\mathbb{N}\times X)$ are defined respectively as follows:

$$\mathsf{c}_X(i,n,x) = \begin{cases} (2n,x) & \text{if } i = 1 \\ (2n+1,x) & \text{if } i = 2 \end{cases} \qquad \mathsf{c}'_X(n,x) = \begin{cases} (1,\frac{n}{2},x) & \text{if } n \text{ is even} \\ (2,\frac{n-1}{2},x) & \text{if } n \text{ is odd} \end{cases}$$

(iv) $\mathsf{w}_X : \emptyset \to \mathbb{N}\times X$ and $\mathsf{w}'_X : \mathbb{N}\times X \to \emptyset$ are the empty partial functions which are nowhere defined.

(v) $\mathsf{j} : \mathbb{N}\sqcup\mathbb{N} \to \mathbb{N}$ and $\mathsf{k} : \mathbb{N} \to \mathbb{N}\sqcup\mathbb{N}$ are defined respectively as follows:

$$\mathsf{j}(i,n) = \begin{cases} 2n & \text{if } i = 1 \\ 2n+1 & \text{if } i = 2 \end{cases} \qquad \mathsf{k}(n) = \begin{cases} (1,\frac{n}{2}) & \text{if } n \text{ is even} \\ (2,\frac{n-1}{2}) & \text{if } n \text{ is odd} \end{cases}$$

Note that j and k are, up to isomorphism, $\mathsf{c}_{\{*\}}$ and $\mathsf{c}'_{\{*\}}$ for a singleton $\{*\}$.

(vi) $\mathsf{a} : \emptyset \to \mathbb{N}$ and $\mathsf{b} : \mathbb{N} \to \emptyset$ are the empty partial functions which are nowhere defined. Again, note that a and b are, up to isomorphism, $\mathsf{w}_{\{*\}}$ and $\mathsf{w}'_{\{*\}}$ for a singleton $\{*\}$.

(vii) $\mathsf{u} : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ is defined as $\mathsf{u}(m,n) = \langle m,n\rangle$ and $\mathsf{v} : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ is its inverse $\mathsf{v} = \mathsf{u}^{-1}$, so $\mathsf{v}(n) = (n_1,n_2)$ where $\langle n_1,n_2\rangle = n$. Similarly, note again that u and v are, up to isomorphism, $\mathsf{e}_{\{*\}}$ and $\mathsf{e}'_{\{*\}}$ for a singleton $\{*\}$.

Example 5.3. HILB_2 is a GoI Situation where the endofunctor is defined as $\mathsf{T}(\ell_2(-)) = \ell_2(\mathbb{N}\times -)$ and the reflexive object is $\ell_2(\mathbb{N})$ [Haghverdi and Scott(2006), Proposition 22]. Note that we can also express the endofunctor in terms of the tensor product of Hilbert spaces since $\ell_2(\mathbb{N}\times X) \cong \ell_2(\mathbb{N})\otimes\ell_2(X)$. The necessary retraction pairs are given by applying ℓ_2 to the GoI Situation structure from PINJ. So, up to natural isomorphism, we may write the retraction pairs as follows:

$$\ell_2(\mathsf{e}_X) : \ell_2(\mathbb{N})\otimes\ell_2(\mathbb{N})\otimes\ell_2(X) \lhd \ell_2(\mathbb{N})\otimes\ell_2(X) : \ell_2(\mathsf{e}'_X)$$
$$\ell_2(\mathsf{d}_X) : \ell_2(X) \lhd \ell_2(\mathbb{N})\otimes\ell_2(X) : \ell_2(\mathsf{d}'_X)$$
$$\ell_2(\mathsf{c}_X) : (\ell_2(\mathbb{N})\otimes\ell_2(X))\oplus(\ell_2(\mathbb{N})\otimes\ell_2(X)) \lhd \ell_2(\mathbb{N})\otimes\ell_2(X) : \ell_2(\mathsf{c}'_X)$$
$$\ell_2(\mathsf{w}_X) : \ell_2(\emptyset) \lhd \ell_2(\mathbb{N})\otimes\ell_2(X) : \ell_2(\mathsf{w}'_X)$$
$$\ell_2(\mathsf{j}) : \ell_2(\mathbb{N})\oplus\ell_2(\mathbb{N}) \lhd \ell_2(\mathbb{N}) : \ell_2(\mathsf{k})$$
$$\ell_2(\mathsf{a}) : \ell_2(\emptyset) \lhd \ell_2(\mathbb{N}) : \ell_2(\mathsf{b})$$
$$\ell_2(\mathsf{u}) : \ell_2(\mathbb{N})\otimes\ell_2(\mathbb{N}) \lhd \ell_2(\mathbb{N}) : \ell_2(\mathsf{v})$$

In particular, in [Haghverdi and Scott(2006), section 6], Scott and Haghverdi explain how Girard's C^*-algebra model of Geometry of Interaction is perfectly captured using this GoI Situation HILB_2.

Multiple other examples of GoI Situations can be found in [Abramsky et al.(2002), Section 5].

Usually, a categorical model of the multiplicative and exponential fragments of Linear Logic (MELL), sometimes called a *linear category*, is a symmetric monoidal closed category equipped with a symmetric monoidal comonad, which interprets the exponential modality, that comes equipped with extra natural transformations, which interpret the rules for the exponential. For full details of the categorical semantics of Linear Logic, see [Blute and Scott(2004)]. However, the GOI-construction on a GoI Situation will not necessarily result in a linear category. Instead, Scott, Abramsky, and Haghverdi introduced the notion of a **weak linear category** [Abramsky et al.(2002), Definition 3.1], which is similar to a linear category but with weaker axioms that are nevertheless still sufficient to obtain a Geometry of Interaction interpretation of MELL. Briefly, a weak linear category is a symmetric monoidal closed category $\mathbb{C}$, equipped with a symmetric monoidal functor $! : \mathbb{C} \to \mathbb{C}$, and the following monoidal *pointwise* natural transformations:

$$\mathsf{dig}_X : !(X) \to !!(X) \quad \mathsf{der}_X : X \to !(X) \quad \mathsf{con}_X : !(X) \to !(X) \otimes !(X) \quad \mathsf{weak}_X : !(X) \to I$$

By pointwise natural, we mean natural only with respect to points, i.e., maps of type $I \to X$. See [Abramsky et al.(2002), Appendix II] and [Shirahata(2003), Section 5] for discussions on why pointwise naturality is sufficient in this context. Applying the GOI-construction to a GoI Situation results in a weak linear category.

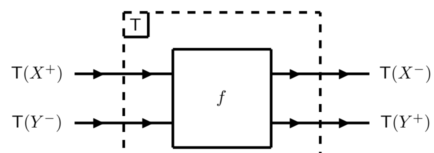
PROPOSITION 5.4. *[Abramsky et al.(2002), Proposition 4.2.(i)] Let $(\mathbb{C}, \mathsf{T}, U)$ be a GoI Situation. Then $\mathcal{G}(\mathbb{C})$ is a weak linear category where:*

(i) *The endofunctor $! : \mathcal{G}(\mathbb{C}) \to \mathcal{G}(\mathbb{C})$ is defined on objects as:*

$$!(X^+, X^-) := (\mathsf{T}(X^+), \mathsf{T}(X^-))$$

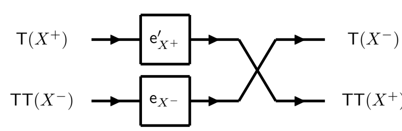
and for a map $f : (X^+, X^-) \to (Y^+, Y^-)$, $!(f) : !(X^+, X^-) \to !(Y^+, Y^-)$, which is a map of type $\mathsf{T}(X^+) \otimes \mathsf{T}(Y^-) \to \mathsf{T}(X^-) \otimes \mathsf{T}(Y^+)$ in $\mathbb{C}$, is defined as follows:

$!(f) :=$



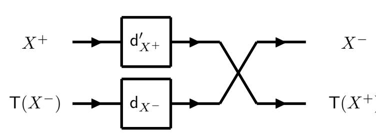
(ii) *The digging $\mathsf{dig}_{(X^+,X^-)} : !(X^+, X^-) \to !!(X^+, X^-)$, which is a map of type $\mathsf{T}(X^+) \otimes \mathsf{TT}(X^-) \to \mathsf{T}(X^-) \otimes \mathsf{TT}(X^+)$ in $\mathbb{C}$, is defined as:*

$\mathsf{dig}_{(X^+,X^-)} :=$



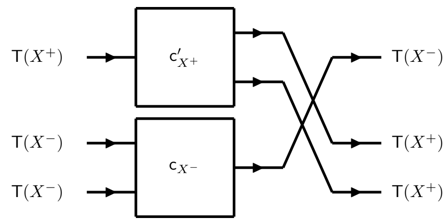
(iii) *The dereliction $\mathsf{der}_{(X^+,X^-)} : (X^+, X^-) \to !(X^+, X^-)$, which is a map of type $X^+ \otimes \mathsf{T}(X^-) \to X^- \otimes \mathsf{T}(X^+)$ in $\mathbb{C}$, is defined as:*

$\mathsf{der}_{(X^+,X^-)} :=$



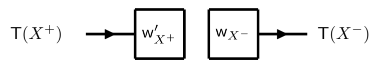
(iv) *The contraction* $\mathsf{con}_{(X^+,X^-)} : !(X^+, X^-) \to !(X^+, X^-) \otimes !(X^+, X^-)$*, which is a map of type* $\mathsf{T}(X^+) \otimes \mathsf{T}(X^-) \otimes \mathsf{T}(X^-) \to \mathsf{T}(X^-) \otimes \mathsf{T}(X^+) \otimes \mathsf{T}(X^+)$ *in* $\mathbb{C}$*, is defined as:*

$\mathsf{con}_{(X^+,X^-)} :=$



(v) *The weakening* $\mathsf{weak}_{(X^+,X^-)} : !(X^+, X^-) \to (I, I)$*, which is a map of type* $\mathsf{T}(X^+) \to \mathsf{T}(X^-)$ *in* $\mathbb{C}$*, is defined as:*

$\mathsf{weak}_{(X^+,X^-)} :=$

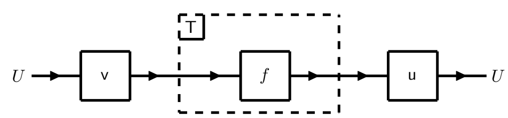


Scott, Abramsky, and Haghverdi also introduced the notion of a **linear combinatory algebra** [Abramsky et al.(2002), Definition 3.4] which briefly is a set A equipped with applicative structure $\cdot$, a unary operator $! : A \to A$, and eight distinguished elements of A which satisfy various axioms. The idea is that a linear combinatory algebra is an algebraic model of MELL. Moreover, in a weak linear category, if V is a special kind of *reflexive object* in the sense of [Abramsky et al.(2002), Definition 3.2], then the homset $\mathbb{C}(I, V)$ can be made into a linear combinatory algebra [Abramsky et al.(2002), Theorem 3.5]. We can apply this to GoI Situations. So for a GoI Situation $(\mathbb{C}, \mathsf{T}, U)$, (U, U) will be a reflexive object with the necessary properties in the weak linear category $\mathcal{G}(\mathbb{C})$. As such, we get that $\mathcal{G}(\mathbb{C})\,((I, I), (U, U))$ is a linear combinatory algebra. However, note that a map of type $(I, I) \to (U, U)$ in $\mathcal{G}(\mathbb{C})$ corresponds to a map of type $U \to U$ in $\mathbb{C}$. Therefore, we have that $\mathbb{C}(U, U)$ is in fact a linear combinatory algebra.

PROPOSITION 5.5. *[Abramsky et al.(2002), Proposition 4.2.(ii)] Let* $(\mathbb{C}, \mathsf{T}, U)$ *be a GoI Situation. Then* $\mathbb{C}(U, U)$ *is a linear combinatory algebra where:*

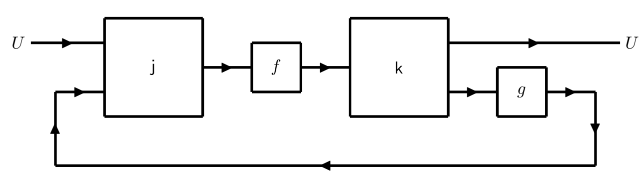
(i) *For a map* $f : U \to U$*,* $!f : U \to U$ *is defined as follows:*

$!f :=$



(ii) *For maps* $f : U \to U$ *and* $g : U \to U$*,* $f \cdot g : U \to U$ *is defined as follows:*

$f \cdot g :=$



and the eight distinguished elements are defined as in [Abramsky et al.(2002), Section 4].

Moreover, by standardization [Abramsky et al.(2002), Theorem 3.7], every linear combinatory algebra gives rise to a combinatory algebra in the usual sense [Abramsky et al.(2002), Definition 3.6]. So for a GoI Situation $(\mathbb{C}, \mathsf{T}, U)$, we also get that $\mathbb{C}(U, U)$ is a combinatory algebra.

6. GEOMETRY OF INTERACTION INTERPRETATIONS

In [Haghverdi and Scott(2006)], Scott and Haghverdi pushed this idea of using GoI Situations for the categorical semantics of the Geometry of Interaction interpretation of MELL further by considering *UDC-GoI Situations*. In particular, one of the main objectives of [Haghverdi and Scott(2006)] was to recapture precisely Girard's original model and matrix intuition for Geometry of Interaction, as well as the Execution formula, in the framework of a certain UDC-GoI Situation. For a review on Linear Logic, we refer the reader to see [Haghverdi and Scott(2010a), Section 3] and [Haghverdi(2000), Chapter 2].

By a **UDC-GoI Situation**, we mean a GoI Situation $(\mathbb{C}, \mathsf{T}, U)$ where $\mathbb{C}$ is a traced UDC and T is also an *additive* functor in the sense of preserving the partial infinitary sums. Then one can provide a Geometry of Interaction interpretation for MELL without units in a UDC-GoI Situation $(\mathbb{C}, \mathsf{T}, U)$. Formulas are interpreted as certain subsets of $\mathbb{C}(U, U)$, proofs are interpreted in $\mathbb{C}(U, U)$, and cut-elimination is interpreted by the Execution formula, which uses the trace operator. For the remainder of this section, we assume that we are working in a UDC-GoI Situation $(\mathbb{C}, \mathsf{T}, U)$.

Starting with the interpretation of formulas, we first need to define an orthogonality relation. Now a map $f : U \to U$ is *nilpotent* if there is some $k \in \mathbb{N}$ such that $f^k = 0_{U,U}$. Then we say that maps $f : U \to U$ and $g : U \to U$ are *orthogonal* if their composite $g; f$ (or equivalently $f; g$) is nilpotent. We write $f \perp g$ to say that f and g are orthogonal. Now given a subset $X \subseteq \mathbb{C}(U, U)$, define $X^\perp \subseteq \mathbb{C}(U, U)$ as the subset of all maps $U \to U$ which are orthogonal to those in X, that is:

$$X^\perp = \{f : U \to U \mid \forall g \in X.\ f \perp g\}$$

Then a **type** [Haghverdi and Scott(2006), Definition 14] is a subset $X \subseteq \mathbb{C}(U, U)$ such that $X^{\perp\perp} = X$. Note that types are always inhabited since the zero map is orthogonal to every map, so if X is a type then $0_{U,U} \in X$.

We will then use types to interpret MELL formulas in our UDC-GoI Situation. To do so, we will make use of our GoI Situation retraction pairs. So recall that we have maps $\mathsf{u} : \mathsf{T}(U) \to U$ and $\mathsf{v} : U \to \mathsf{T}(U)$, as well as $\mathsf{j} : U \otimes U \to U$ and $\mathsf{k} : U \to U \otimes U$, and also $\mathsf{u} : \mathsf{T}(U) \to U$ and $\mathsf{v} : U \to \mathsf{T}(U)$. Moreover, since we are also in a UDC, j and k also have matrix representations:

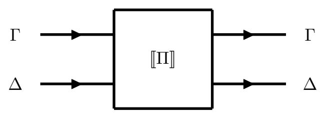
$$\mathsf{j} = \begin{bmatrix} \mathsf{j}_1 : U \to U \\ \mathsf{j}_2 : U \to U \end{bmatrix} \qquad \qquad \mathsf{k} = \begin{bmatrix} \mathsf{k}_1 : U \to U & \mathsf{k}_2 : U \to U \end{bmatrix}$$

Now let A be a MELL formula. Then the **Geometry of Interaction interpretation** [Haghverdi and Scott(2006), Definition 15] of A, denoted as ΘA, is defined inductively as follows:

(i) If $A \equiv \alpha$ an atom, then $\Theta A = X$ where X is a type;
(ii) If $A \equiv \alpha^\perp$, where α is an atom, then $\Theta A = X^\perp$ where $\Theta\alpha = X$;
(iii) If $A \equiv B \otimes C$, then $\Theta A = Y^{\perp\perp}$ where $Y = \{\mathsf{k}_1; b; \mathsf{j}_1 + \mathsf{k}_2; c; \mathsf{j}_2 \mid b \in \Theta B \text{ and } c \in \Theta C\}$;
(iv) If $A \equiv B \parr C$, then $\Theta A = Y^\perp$ where $Y = \{\mathsf{k}_1; b; \mathsf{j}_1 + \mathsf{k}_2; c; \mathsf{j}_2 \mid b \in (\Theta B)^\perp \text{ and } c \in (\Theta C)^\perp\}$;
(v) If $A \equiv\ !B$, then $\Theta A = Y^{\perp\perp}$ where $Y = \{\mathsf{v}; \mathsf{T}(b); \mathsf{u} \mid b \in \Theta B \text{ and } c \in \Theta C\}$;
(vi) If $A \equiv\ ?B$, then $\Theta A = Y^\perp$ where $Y = \{\mathsf{v}; \mathsf{T}(b); \mathsf{u} \mid b \in (\Theta B)^\perp \text{ and } c \in \Theta C\}$.

It follows from the definition that $(\Theta A)^\perp = \Theta A^\perp$.

Now we turn towards understanding interpretations of proofs in our UDC-GoI Situation. For simplicity, let us denote the tensor product of n-copies of U as $U^n = U \otimes \ldots \otimes U$, where by convention $U^0 = I$. Now every MELL (without units) formula will be of the form $\vdash [\Delta], \Gamma$ where Γ is a sequence of formulas and Δ is a sequence of cut formulas that have been applied to the proof of $\vdash \Gamma$. Suppose that Γ consists of n formulas and Δ consists of $2m$ formulas. Therefore, a proof Π of $\vdash [\Delta], \Gamma$ will be interpreted by a map of type $[\![\Pi]\!] : U^{n+2m} \to U^{n+2m}$. Since we have that $U \otimes U \lhd U$, a map of type $U^{n+2m} \to U^{n+2m}$ corresponds to a map of type $U \to U$. As such, the interpretation of our proof is in $\mathbb{C}(U, U)$. However, following [Haghverdi and Scott(2006)], it will be much more convenient to work with interpreting proofs in $\mathbb{C}(U^{n+2m}, U^{n+2m})$. Also, we slightly abusing notation and denote Γ for U^n and Δ for U^{2m}. As such, we get that the interpretation of our proof is a map of type $[\![\Pi]\!] : \Gamma \otimes \Delta \to \Gamma \otimes \Delta$, which is drawn as on the left below, and since we are in a UDC, also has a matrix representation given by a 2×2-matrix as on the right below:



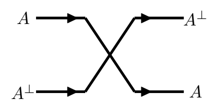
$$[\![\Pi]\!] = \begin{bmatrix} \Pi_{1,1} : \Gamma \to \Gamma & \Pi_{1,2} : \Gamma \to \Delta \\ \Pi_{2,1} : \Delta \to \Gamma & \Pi_{2,2} : \Delta \to \Delta \end{bmatrix}$$

So let Π be a proof of $\vdash [\Delta], \Gamma$. Then the **Geometry of Interaction interpretation** [Haghverdi and Scott(2006), Section 4] of Π, denoted as $[\![\Pi]\!] : \Gamma \otimes \Delta \to \Gamma \otimes \Delta$, is defined inductively on the length of Π as follows:

(i) If Π is an axiom:

$$\overline{\vdash A, A^\perp}$$

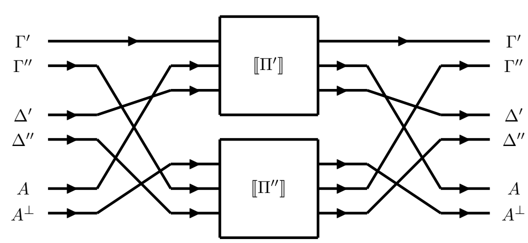
then its interpretation $[\![\Pi]\!]$ is the symmetry map:



(ii) If Π is obtained using the cut rule:

$$\dfrac{\begin{matrix}\Pi' \\ \vdots \\ \vdash [\Delta'], \Gamma', A\end{matrix} \qquad \begin{matrix}\Pi'' \\ \vdots \\ \vdash [\Delta''], A^\perp, \Gamma''\end{matrix}}{\vdash \left[\Delta, \Delta', A, A^\perp\right], \Gamma, \Gamma'} \text{ (CUT)}$$

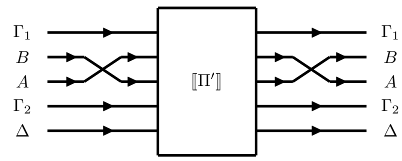
then its interpretation $[\![\Pi]\!]$ is drawn as follows:



(iii) If Π is obtained using the exchange rule:

$$\dfrac{\begin{matrix}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma_1, A, B, \Gamma_2\end{matrix}}{\vdash [\Delta], \Gamma_1, B, A, \Gamma_2} \text{ (EXCH.)}$$

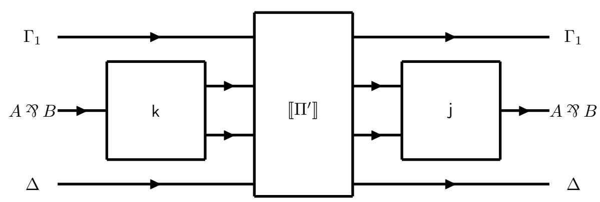
then its interpretation $[\![\Pi]\!]$ is drawn as follows:



(iv) If Π is obtained using the $\parr$ rule:

$$\dfrac{\begin{matrix}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma, A, B\end{matrix}}{\vdash [\Delta], \Gamma, A \parr B}\ (\parr)$$

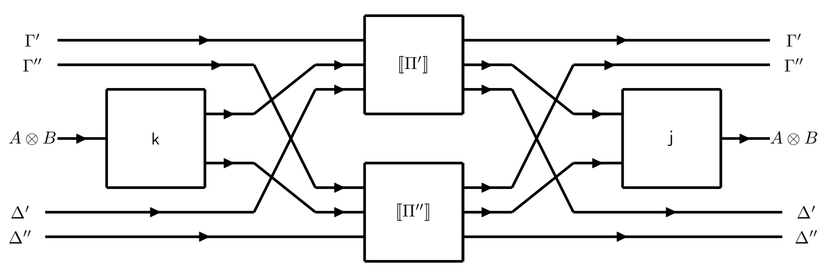
then its interpretation $[\![\Pi]\!]$ is drawn as follows:



(v) If Π is obtained using the $\otimes$ rule:

$$\dfrac{\begin{matrix}\Pi' \\ \vdots \\ \vdash [\Delta'], \Gamma', A\end{matrix} \qquad \begin{matrix}\Pi'' \\ \vdots \\ \vdash [\Delta''], \Gamma'', B\end{matrix}}{\vdash [\Delta', \Delta''], \Gamma', \Gamma'', A \otimes B}\ (\otimes)$$

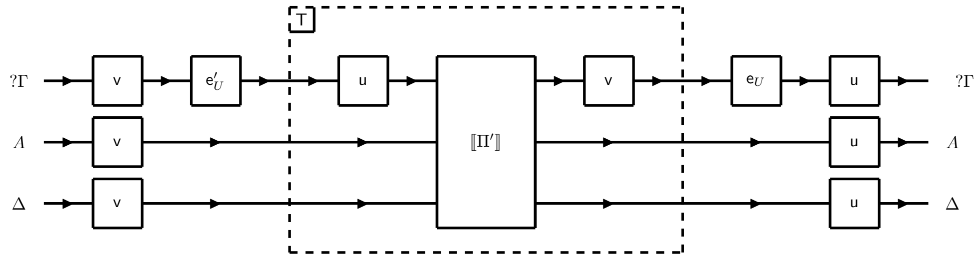
then its interpretation $[\![\Pi]\!]$ is drawn as follows:



(vi) If Π is obtained using the ! rule:

$$\dfrac{\begin{matrix}\Pi' \\ \vdots \\ \vdash [\Delta], ?\Gamma, A\end{matrix}}{\vdash [\Delta], ?\Gamma, !A}\ (!)$$

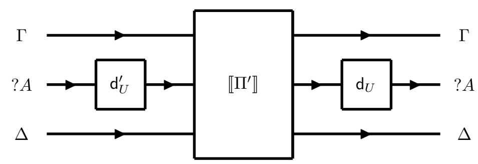
then its interpretation $[\![\Pi]\!]$ is drawn as follows:



(vii) If Π is obtained using the dereliction rule:

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma, A\end{array}}{\vdash [\Delta], \Gamma, ?A} \text{ (DER.)}$$

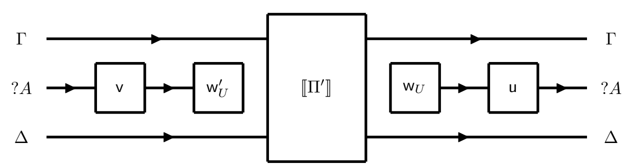
then its interpretation $[\![\Pi]\!]$ is drawn as follows:



(viii) If Π is obtained using the weakening rule:

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma\end{array}}{\vdash [\Delta], \Gamma, ?A} \text{ (WEAK.)}$$

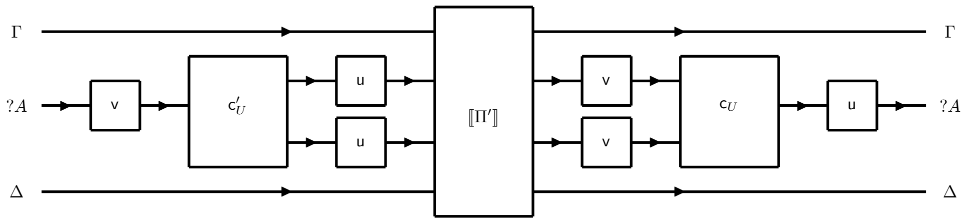
then its interpretation $[\![\Pi]\!]$ is drawn as follows:



(ix) If Π is obtained using the contraction rule:

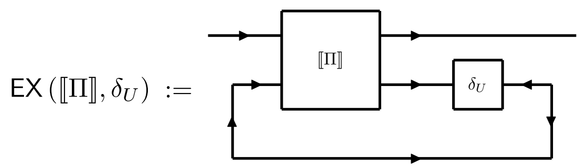
$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma, ?A, ?A\end{array}}{\vdash [\Delta], \Gamma, ?A} \text{ (CON.)}$$

then its interpretation $[\![\Pi]\!]$ is drawn as follows:



Finally, the last thing that remains to be interpreted is cut-elimination. The Geometry of Interaction interpretation of cut-elimination is given by the *Execution formula*, which, as mentioned above, is defined using the trace operator. So consider a proof Π of $\vdash [\Delta], \Gamma$, so with interpretation $[\![\Pi]\!] : \Gamma \otimes \Delta \to \Gamma \otimes \Delta$. While in theory we can already take the trace of $[\![\Pi]\!]$ and trace out Δ, in order to get the correct formula, we will need some extra permutations. So let $\delta_U : U^{2m} \to U^{2m}$ be the tensor product of m-copies of the symmetry map, so $\delta_U = \sigma_{U,U} \otimes \ldots \otimes \sigma_{U,U}$. We may write this permutation as a map of type $\delta_U : \Delta \to \Delta$. Then δ_U represents the cuts in the proof $\vdash \Gamma$, or in other words, δ_U models Δ. This permutation is necessary since it rearranges the cut-formulas so that when we close loops (i.e. take the trace), every cut-formula gets connected to its dual formula.

Then for a proof Π of $\vdash [\Delta], \Gamma$, the **Execution formula** [Haghverdi and Scott(2006), Section 4.1] is defined as follows:



Since we are in UDC, we can also use the standard trace formula to express the Execution formula as follows:

$$\mathsf{EX}\,([\![\Pi]\!], \delta_U) = \Pi_{1,1} + \sum_{n \in \mathbb{N}} \Pi_{1,2}; \delta; (\Pi_{2,2}; \delta)^n; \Pi_{2,1}$$

This standard trace formula version of the Execution formula allowed Scott and Haghverdi to show that the Execution formula in HILB_2 is the same as Girard's original execution formula for the operator algebra model [Haghverdi and Scott(2006), Proposition 23].

The main result of [Haghverdi and Scott(2006)] is, of course, the soundness of this Geometry of Interaction interpretation of MELL in a UDC-GoI Situation. In this case, soundness can be described as saying that if a proof is reduced via cut-elimination to its cut-free form, then applying the Execution formula to the starting proof gives a finite sum and results in the interpretation of the cut-free form. In other words, if we run the result of the Execution formula, then it terminates after finitely many steps (so a finite sum) and yields a datum (a cut-free proof).

THEOREM 6.1. *[Haghverdi and Scott(2006), Theorem 21] Let Π be a proof of $\vdash [\Delta], \Gamma$. Then:*

(i) $\mathsf{EX}\,([\![\Pi]\!], \delta_U)$ *is a finite sum.*
(ii) *If Π reduces to Π' by any sequence of cut-elimination steps and Γ does not contain any fomrulas of the form $?A$, then $\mathsf{EX}\,([\![\Pi]\!], \delta_U) = \mathsf{EX}\,([\![\Pi']\!], \delta'_U)$. So $\mathsf{EX}\,([\![\Pi]\!], \delta_U)$ is an invariant of reduction.*

(iii) *In particular if* Π' *is any cut-free proof obtained from* Π *by cut-elimination, then we have that* $\mathsf{EX}\,([\![\Pi]\!], \delta_U) = [\![\Pi']\!]$.

In [Haghverdi and Scott(2005)], Scott and Haghverdi also explain how to construct a denotational model for MELL (without units), and thus provide a more direct relation between the Geometry of Interaction semantics to the denotational semantics in the case of MELL. Moreover, they also construct star-autonomous categories from a UDC-GoI Situation via an orthogonality construction [Haghverdi and Scott(2005), Section 5] or via a double glueing construction [Haghverdi and Scott(2005), Section 6.1].

7. FURTHER READING

Scott and his coauthors continued to work on traced monoidal categories and develop other important concepts in the area. In [Haghverdi and Scott(2010b)], Scott and Haghverdi developed a typed version of Geometry of Interaction and its categorical semantics. To do so, they introduced the concept of *partially traced categories*, which are a generalization of traced monoidal categories where the trace operator is now only partially defined. The theory of partially traced categories was developed further by Scott, Malherbe, and Selinger in [Malherbe et al.(2012)]. In [Hamano and Scott(2018)], Scott and Hamano also developed a Geometry of Interaction for *Polarized* Linear Logic, as well as a polarized version of GoI Situations.

REFERENCES

S. Abramsky. 1996. Retracing some paths in process algebra. In *International Conference on Concurrency Theory*. Springer, Berlin, Heidelberg, 1–17.

S. Abramsky and B. Coecke. 2004. A categorical semantics of quantum protocols. In *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004.* IEEE, Turku, Finland, 415–425. DOI:https://doi.org/10.1109/LICS.2004.1319636

S. Abramsky and B. Coecke. 2005. Abstract physical traces. *Theory and Applications of Categories* 14, 6 (2005), 111–124.

S. Abramsky, E. Haghverdi, and P. Scott. 2002. Geometry of Interaction and Linear Combinatory Algebras. *Mathematical Structures in Computer Science* 12, 5 (2002), 625–665. DOI:https://doi.org/10.1017/S0960129502003730

S. Abramsky and R. Jagadeesan. 1992a. Games and full completeness for multiplicative linear logic. In *International Conference on Foundations of Software Technology and Theoretical Computer Science*. Springer, Berlin, Heidelberg, 291–301.

S. Abramsky and R. Jagadeesan. 1992b. New foundations for the geometry of interaction. In *[1992] Proceedings of the Seventh Annual IEEE Symposium on Logic in Computer Science*. IEEE, Turku, Finland, 211–222. DOI:https://doi.org/10.1109/LICS.1992.185534

S. Abramsky and R. Jagadeesan. 1994a. Games and full completeness for multiplicative linear logic. *The Journal of Symbolic Logic* 59, 2 (1994), 543–574.

S. Abramsky and R. Jagadeesan. 1994b. New foundations for the geometry of interaction. *Inf. Comput.* 111, 1 (may 1994), 53–119. DOI:https://doi.org/10.1006/inco.1994.1041

M. Barr. 1992. Algebraically compact functors. *Journal of Pure and Applied Algebra* 82, 3 (1992), 211–231.

R. Blute and P. Scott. 2004. Category theory for linear logicians. *Linear logic in computer science* 316 (2004), 3–64.

J.-Y. Girard. 1987. Linear logic. *Theoretical computer science* 50, 1 (1987), 1–101.

J.-Y. Girard. 1988. Geometry of interaction 2: Deadlock-free algorithms. In *International Conference on Computer Logic*. Springer, Berlin, Heidelberg, 76–93.

J.-Y. Girard. 1989a. Geometry of interaction 1: Interpretation of System F. *Studies in Logic and the Foundations of Mathematics* 127 (1989), 221–260.

J.-Y. Girard. 1989b. Towards a geometry of interaction. *Contemp. Math.* 92, 69-108 (1989), 6.

J.-Y. Girard. 1995. Geometry of interaction III: accommodating the additives. *London Mathematical Society Lecture Note Series* , (1995), 329–389.

J.-Y. Girard. 2006. Le point aveugle. *Cours de logique* 1 (2006).

J.-Y. Girard. 2011. Geometry of interaction V: logic in the hyperfinite factor. *Theoretical Computer Science* 412, 20 (2011), 1860–1883.

E. Haghverdi. 2000. *A categorical approach to linear logic, geometry of proofs and full completeness.* Ph. D. Dissertation. University of Ottawa (Canada).

E. Haghverdi and P. Scott. 2005. From geometry of interaction to denotational semantics. *Electronic Notes in Theoretical Computer Science* 122 (2005), 67–87.

E. Haghverdi and P. Scott. 2006. A Categorical Model for the Geometry of Interaction. *Theoretical Computer Science* 350, 2-3 (2006), 252–274.

E. Haghverdi and P. Scott. 2010a. Geometry of Interaction and the Dynamics of Proof Reduction: a tutorial. *New Structures for Physics* 813 (2010), 357–417.

E. Haghverdi and P. Scott. 2010b. Towards a typed geometry of interaction. *Mathematical Structures in Computer Science* 20, 3 (2010), 473–521.

T. Hajgató and M. Hasegawa. 2013. Traced star-autonomous categories are compact closed. *Theory and Applications of Categories* 28, 7 (2013), 206–212.

M. Hamano and P. Scott. 2018. On geometry of interaction for polarized linear logic. *Mathematical Structures in Computer Science* 28, 10 (2018), 1639–1694.

M. Hasegawa. 1997. Recursion from cyclic sharing: traced monoidal categories and models of cyclic lambda calculi. In *Typed Lambda Calculi and Applications*. Springer, Berlin, Heidelberg, 196–213. DOI:https://doi.org/10.1007/3-540-62688-3_37

M. Hasegawa. 2009. On traced monoidal closed categories. *Mathematical Structures in Computer Science* 19, 2 (2009), 217–244. DOI:https://doi.org/10.1017/S0960129508007184

M. Hasegawa and J.-S. P. Lemay. 2023. Traced Monads and Hopf Monads. *Compositionality* 5 (Oct. 2023), 1–34. Issue 10. DOI:https://doi.org/10.32408/compositionality-5-10

C. Heunen. 2013. On the functor ℓ^2. In *Computation, Logic, Games, and Quantum Foundations. The Many Facets of Samson Abramsky: Essays Dedicated to Samson Abramsky on the Occasion of His 60th Birthday*. Springer, Berlin, Heidelberg, 107–121.

C. Heunen and J. Vicary. 2019. *Categories for Quantum Theory: an introduction*. Oxford University Press, Oxford, UK. DOI:https://doi.org/10.1093/oso/9780198739623.001.0001

A. Joyal, R. Street, and D. Verity. 1996. Traced Monoidal Categories. *Mathematical Proceedings of the Cambridge Philosophical Society* 119, 3 (1996), 447–468. DOI:https://doi.org/10.1017/S0305004100074338

J. Lambek and P. Scott. 1988. *Introduction to higher-order categorical logic*. Vol. 7. Cambridge University Press, Cambridge, UK.

O. Malherbe, P. Scott, and P. Selinger. 2012. Partially traced categories. *Journal of Pure and Applied Algebra* 216, 12 (2012), 2563–2585.

E. G. Manes and M. A. Arbib. 2012. *Algebraic approaches to program semantics*. Springer Science & Business Media, Berlin, Heidelberg.

P. Selinger. 2010. A survey of graphical languages for monoidal categories. In *New Structures for Physics*. Springer, Berlin, Heidelberg, 289–355. DOI:https://doi.org/10.1007/978-3-642-12821-9_4

Ma. Shirahata. 2003. Geometry of Interaction explained. *Surikaisekikenkyusho Kokyuroku* 1318 (2003), 160–187. https://repository.kulib.kyoto-u.ac.jp/dspace/handle/2433/43041 Algebra, Logic and Geometry in Informatics (Kyoto, 2002).