

# Conversation on Set Theory and Category Theory

JH and Claude 3.5 Sonnet

JH

There are set theories where there is a "class" of all sets. What are the axioms of that theory?

Claude 3.5 Sonnet

The theory you're referring to is likely the von Neumann–Bernays–Gödel (NBG) set theory, which introduces the concept of "classes" alongside sets. NBG is an extension of Zermelo–Fraenkel set theory (ZFC) that allows for the existence of proper classes, including a "class of all sets."

Here's a concise overview of the key axioms of NBG set theory:

- **Extensionality:** Two classes are equal if and only if they have the same elements.
- **Pairing:** For any two sets, there exists a set containing exactly those two sets as elements.
- **Union:** For any set of sets, there exists a set that contains all elements of all those sets.
- **Power Set:** For any set, there exists a set of all its subsets.
- **Infinity:** There exists an infinite set.
- **Replacement:** If a class function is defined on a set, its range on that set is also a set.
- **Separation:** Any subclass of a set is a set.
- **Class Comprehension:** For any formula  $\varphi(x)$ , there exists a class of all sets  $x$  satisfying  $\varphi(x)$ .
- **Choice:** For any set of non-empty sets, there exists a function that chooses one element from each set.
- **Foundation:** Every non-empty class has an  $\in$ -minimal element.

The key difference between NBG and ZFC is the Class Comprehension axiom, which allows for the formation of proper classes like "the class of all sets" or "the class of all ordinals."

Would you like me to elaborate on any specific aspect of NBG set theory or how it compares to other set theories?

JH

In the category of sets, since a subset can be injectively mapped to its parent set, there always exists a monomorphism from a subset of a set to that set. Furthermore, every monomorphism is associated with a subset, because the source object is isomorphic to a subset of the parent set. This means that the collection of all monomorphisms into an object is the set of all subsets.

Imagine that I want to know if there are more important, deeper categorical properties of subsets than just monomorphisms.

What if I select an object and then study all the monomorphisms into that object. This is a slice category (hopefully).

Can properties of that slice category reveal much deeper patterns regarding the relation of a subset to a parent set? (Because the idea is that there is all the information you need about the greater category encoded in the adjacent morphisms of an object).

### Claude 3.5 Sonnet

Your line of thinking is very insightful and touches on some deep connections between set theory and category theory. Let's explore this further:

- **Slice Categories and Subobjects:** You're correct that the category of monomorphisms into an object forms a slice category. Specifically, for an object  $A$  in a category  $C$ , the slice category  $C/A$  (read as "C over A") consists of all morphisms with codomain  $A$ . The subcategory of  $C/A$  consisting only of monomorphisms into  $A$  is indeed closely related to the subobjects of  $A$ .
- **Properties of the Slice Category:** Studying the slice category  $C/A$  can indeed reveal deeper patterns about the relationship between subobjects and their parent object. Here are some properties and insights we can gain:
  - **Partial Order Structure:** The slice category of monomorphisms into  $A$  forms a partial order, which directly corresponds to the subset relation. This partial order structure encodes information about how subsets are nested within each other.
  - **Meets and Joins:** If the category has pullbacks (which **Sets** does), the slice category will have meets, corresponding to intersections of subsets. If it has pushouts, it will have joins, corresponding to unions.
  - **Boolean Algebra:** In the case of **Sets**, this structure forms a Boolean algebra, reflecting the rich logical structure of the powerset.
  - **Subobject Classifier:** In a topos (a category with additional nice properties, including **Sets**), there exists a subobject classifier. This object  $\Omega$  and the characteristic functions to it encode all the information about subobjects in a single object.
- **Deeper Categorical Properties:** Beyond just identifying subsets with monomorphisms, category theory reveals several deeper properties:
  - **Factorization Systems:** In many categories, including **Sets**, every morphism can be factored uniquely into an epimorphism followed by a monomorphism. This property, known as an (Epi, Mono) factorization system, gives insight into how arbitrary morphisms relate to subobjects.
  - **Lattice Structure:** The subobject lattice of an object can reveal important properties of the object and the category. For instance, in the category of groups, the subgroup lattice can tell us a lot about the structure of the group.

- **Presheaf Perspective:** We can view the slice category  $C/A$  as a presheaf on  $C^{\text{op}}$  (the opposite category of  $C$ ). This perspective allows us to apply powerful techniques from presheaf theory to study subobjects.
- **Yoneda Embedding:** The Yoneda lemma tells us that studying morphisms into an object  $A$  is equivalent to studying the representable functor  $\text{Hom}(-, A)$ . This gives us a way to study subobjects in terms of functors.
- **Topos Theory:** In a topos, which generalizes many properties of **Sets**, subobjects have a rich logical structure. The subobject classifier allows us to internalize the logic of subobjects within the category itself.
- **Generalizations:** This perspective also allows for powerful generalizations:
  - In categories where not all monomorphisms correspond to "subobjects" in an intuitive sense, we can still study the slice category to understand "generalized subobjects".
  - We can study more general comma categories, not just slice categories, to understand relationships between objects in even more general settings.

In conclusion, your intuition is correct: studying the slice category of monomorphisms into an object reveals deep and fundamental properties about the relationship between subobjects and their parent object. This approach allows us to generalize the notion of "subset" to more abstract categories and uncover structural properties that might not be apparent from a purely set-theoretic perspective.

This categorical viewpoint provides a powerful framework for understanding subobjects, offering insights that go well beyond the simple identification of subsets with monomorphisms.

**JH**

I am trying to link this to the Yoneda construction. A slice category is based on an object, morphisms into that object, and morphisms linking one morphism to the next (what do you call a morphism that "completes" a cospan into a commutative triangle?)

Isn't a Hom functor similar to this? I'm confused.

**Claude 3.5 Sonnet**

Your confusion is understandable, as these concepts are closely related but have some subtle differences. Let's break this down and clarify the connections:

- **Slice Categories:** In a slice category  $C/A$ :
  - Objects are morphisms  $f : X \rightarrow A$  in  $C$
  - Morphisms are commutative triangles: given  $f : X \rightarrow A$  and  $g : Y \rightarrow A$ , a morphism from  $f$  to  $g$  in  $C/A$  is a morphism  $h : X \rightarrow Y$  in  $C$  such that  $g \circ h = f$
- **Morphisms in Commutative Triangles:** The morphism  $h$  that "completes" a cospan into a commutative triangle is sometimes called a "mediating morphism" or a "comparison morphism". In the context of slice categories, these are precisely the morphisms of the slice category.
- **Hom Functors:** The covariant Hom functor  $\text{Hom}(A, -) : C \rightarrow \mathbf{Set}$  sends:

- An object  $X$  to the set  $\text{Hom}(A, X)$  of all morphisms from  $A$  to  $X$
- A morphism  $f : X \rightarrow Y$  to the function  $\text{Hom}(A, f) : \text{Hom}(A, X) \rightarrow \text{Hom}(A, Y)$  that sends each  $g : A \rightarrow X$  to  $f \circ g : A \rightarrow Y$
- **Yoneda Lemma Connection:** The Yoneda lemma states that for any functor  $F : C \rightarrow \mathbf{Set}$ , natural transformations from  $\text{Hom}(A, -)$  to  $F$  are in bijection with elements of  $F(A)$ . In other words,  $\text{Nat}(\text{Hom}(A, -), F) \cong F(A)$ .
- **Linking Slice Categories and Hom Functors:** Here's where the connection becomes clear:
  - The slice category  $C/A$  is closely related to the contravariant Hom functor  $\text{Hom}(-, A) : C^{\text{op}} \rightarrow \mathbf{Set}$ :
    - Objects in  $C/A$  correspond to elements in the sets  $\text{Hom}(X, A)$  for various  $X$
    - Morphisms in  $C/A$  correspond to the action of  $\text{Hom}(-, A)$  on morphisms in  $C$
  - However, there's a key difference:
    - The slice category  $C/A$  keeps track of the "source" objects of the morphisms
    - The Hom functor  $\text{Hom}(-, A)$  only keeps track of the sets of morphisms, not their sources
- **Yoneda Embedding:** The Yoneda embedding  $Y : C \rightarrow [C^{\text{op}}, \mathbf{Set}]$  sends each object  $A$  to the contravariant Hom functor  $\text{Hom}(-, A)$ . This is where the deep connection lies:
  - The Yoneda embedding preserves and reflects all categorical structure
  - Studying  $\text{Hom}(-, A)$  is equivalent to studying  $A$  itself in the category

In essence, while the slice category  $C/A$  gives us a concrete way to study morphisms into  $A$  within  $C$ , the Hom functor  $\text{Hom}(-, A)$  allows us to study  $A$  by looking at how all other objects in  $C$  relate to it. The Yoneda lemma tells us that this functor completely characterizes  $A$ .

So, while they're not identical, slice categories and Hom functors are deeply connected ways of studying objects in a category by examining their relationships with other objects. The Yoneda lemma and Yoneda embedding provide the theoretical framework that makes this approach so powerful.

Does this help clarify the connections? Would you like me to elaborate on any specific aspect?

**JH**

Oh ok. So a Hom functor selects an object, maps each other object to its associated Hom set... does that mean there is no functor from a category to its slice category? But only in the reverse direction? Because we can't map an object to each arrow it has with the selected object.

**Claude 3.5 Sonnet**

You've made an excellent observation, and it leads us to some interesting insights about the relationships between categories, slice categories, and functors. Let's break this down:

- **Hom Functor:** You're correct about the Hom functor. For a fixed object  $A$ , the contravariant Hom functor  $\text{Hom}(-, A) : C^{\text{op}} \rightarrow \mathbf{Set}$  maps:

- Each object  $X$  to the set  $\text{Hom}(X, A)$
- Each morphism  $f : X \rightarrow Y$  to the function  $\text{Hom}(f, A) : \text{Hom}(Y, A) \rightarrow \text{Hom}(X, A)$  given by precomposition with  $f$
- **Functor from  $C$  to  $C/A$ :** You're right that there isn't a "natural" functor from  $C$  to  $C/A$ . The reason is exactly as you've intuited: we can't map an object  $X$  in  $C$  to a single "canonical" morphism  $X \rightarrow A$  in  $C/A$ , because there might be many such morphisms (or none at all).
- **Functor from  $C/A$  to  $C$ :** However, there is a natural functor from  $C/A$  to  $C$ , often called the "forgetful" or "domain" functor:
  - It maps each object  $(f : X \rightarrow A)$  in  $C/A$  to its domain  $X$  in  $C$
  - It maps each morphism in  $C/A$  (a commutative triangle) to the underlying morphism in  $C$
- **Adjunctions:** Interestingly, this forgetful functor  $U : C/A \rightarrow C$  has a left adjoint, called the "constant object" functor  $F : C \rightarrow C/A$ :
  - It maps each object  $X$  in  $C$  to the morphism  $(\pi_2 : X \times A \rightarrow A)$  in  $C/A$
  - It maps each morphism  $f : X \rightarrow Y$  in  $C$  to the obvious morphism  $F(f) : F(X) \rightarrow F(Y)$  in  $C/A$
 This adjunction  $F \dashv U$  is important in category theory and has many applications.
- **Connection to Hom Functors:** While we can't directly map  $C$  to  $C/A$  in a functorial way, there is a connection to Hom functors:
  - For each object  $X$  in  $C$ ,  $\text{Hom}(X, A)$  is isomorphic to the set of morphisms from the terminal object  $1$  to  $U^{-1}(X)$  in  $C/A$
  - This gives us a way to "represent" the Hom functor in terms of the slice category
- **Yoneda Perspective:** From the Yoneda perspective, we can view the slice category  $C/A$  as a full subcategory of the functor category  $[C^{\text{op}}, \mathbf{Set}]$ , where it corresponds to the representable functor  $\text{Hom}(-, A)$  and all of its subobjects.

In summary, your intuition is correct. We can't directly map  $C$  to  $C/A$  in a functorial way, but we can go in the reverse direction. This asymmetry reflects the fact that the slice category  $C/A$  contains more specific information (about morphisms into  $A$ ) than the original category  $C$ .

This relationship between categories and their slice categories is a beautiful example of how category theory allows us to study mathematical structures by examining their relationships and mappings, rather than just their internal properties.

**JH**

That's fascinating but I'll need time to think about all these new ideas! It seems like I see my first use of adjoint functors: the domain functor doesn't have an inverse, but it has an adjoint. So adjoints give us an idea of "the reverse direction" in situations where we are more constrained.

**Claude 3.5 Sonnet**

You've grasped a profound insight about adjoint functors! Your observation is spot-on and touches on one of the fundamental ideas in category theory. Let's expand on this:

- **Adjoint Functors as Generalized Inverses:** You're absolutely right that adjoint functors can be thought of as a way to capture the idea of "going in the reverse direction" when a true inverse doesn't exist. This is one of the key intuitions behind adjunctions.
- **Constraints and Flexibility:** As you noted, adjoints give us a way to work with situations where we're more constrained. In many mathematical contexts, strict inverses are too much to ask for, but adjoints provide a more flexible notion of "reversing" that still captures important structural relationships.
- **Examples of Adjunctions:**
  - Free/Forgetful adjunctions (like the one between groups and sets)
  - Product/Diagonal adjunctions
  - Exponential/Product adjunctions in cartesian closed categories
  - Tensor/Hom adjunctions in monoidal categories

In each case, one functor (usually the right adjoint) performs some kind of "forgetful" operation, while its left adjoint provides a way to "freely generate" structure.

- **Universal Properties:** Adjunctions are closely related to universal properties. In fact, many universal constructions in mathematics (like products, coproducts, limits, colimits) can be characterized in terms of adjoint functors.
- **Balancing Generality and Specificity:** Adjoint functors often balance between generality and specificity. The forgetful functor (right adjoint) moves towards more general structures, while the free functor (left adjoint) moves towards more specific structures.
- **Preservation of Information:** While an adjunction doesn't give an isomorphism between categories, it does preserve certain kinds of information. This is formalized in concepts like adjunctions preserving limits and colimits.
- **Equivalence of Categories:** In some special cases, when the unit and counit of the adjunction are isomorphisms, we get an equivalence of categories, which is as close as we can get to an isomorphism between categories.

Your intuition about adjoints providing a notion of "reverse direction" in constrained situations is a valuable way to think about these concepts. As you delve deeper into category theory, you'll find that this intuition applies in many contexts and helps unify seemingly disparate mathematical ideas.

Would you like to explore any specific examples of adjunctions or dive deeper into how they relate to the slice category construction we discussed earlier?

JH

So what is the connection between adjoint functors and universal properties? Like the universal property of the product. What adjoints is it associated with?

Claude 3.5 Sonnet

Excellent question! The connection between adjoint functors and universal properties is deep and fundamental in category theory. Let's explore this connection, using the product as a specific example.

- **General Connection:** Universal properties and adjoint functors are closely related because:
  - Universal properties often define objects uniquely up to isomorphism.
  - Adjoint functors can be characterized by universal properties.
  - Many universal constructions arise as adjoint functors.
- **Universal Property of Products:** The universal property of the product states that for objects  $A$  and  $B$  in a category  $C$ , their product  $P = A \times B$  (if it exists) is characterized by:
  - Projection morphisms  $\pi_1 : P \rightarrow A$  and  $\pi_2 : P \rightarrow B$
  - For any object  $X$  with morphisms  $f : X \rightarrow A$  and  $g : X \rightarrow B$ , there exists a unique morphism  $h : X \rightarrow P$  such that  $\pi_1 \circ h = f$  and  $\pi_2 \circ h = g$
- **Adjunction Related to Products:** The product is associated with an adjunction between two functors:
  - Diagonal functor  $\Delta : C \rightarrow C \times C$ , which maps an object  $X$  to  $(X, X)$
  - Product functor  $\times : C \times C \rightarrow C$ , which maps a pair  $(A, B)$  to their product  $A \times B$

The adjunction is:  $\times \dashv \Delta$  (Product is left adjoint to Diagonal)
- **How the Adjunction Works:**
  - For objects  $A, B, X$  in  $C$ , we have a natural isomorphism:  $\text{Hom}(A \times B, X) \cong \text{Hom}((A, B), \Delta(X))$
  - This isomorphism says that morphisms from a product  $A \times B$  to  $X$  correspond to pairs of morphisms from  $A$  to  $X$  and  $B$  to  $X$ .
- **Universal Property from Adjunction:** The universal property of the product arises from this adjunction:
  - The unit of the adjunction at  $(A, B)$  gives the projection morphisms  $\pi_1$  and  $\pi_2$
  - The universal property's "unique morphism" corresponds to the bijection in the adjunction
- **Generalizing to Other Universal Properties:** This pattern repeats for many universal properties:
  - Coproducts are left adjoint to the diagonal functor in the opposite direction
  - Exponentials (in cartesian closed categories) are right adjoint to the product functor
  - Limits are right adjoint to the constant diagram functor
  - Colimits are left adjoint to the constant diagram functor
- **Significance:** Understanding universal properties in terms of adjunctions allows us to:
  - Unify many constructions under a common framework
  - Prove general theorems about universal constructions

- Understand how different universal properties relate to each other

In essence, adjoint functors provide a general framework for understanding and working with universal properties. The universal property of products is just one example of how a universal construction can be understood in terms of an adjunction.

This connection between universal properties and adjunctions is a powerful tool in category theory, allowing us to see deep structural similarities across different mathematical contexts.

Would you like to explore another specific example of how a universal property relates to an adjunction, or dive deeper into any aspect of this connection?

**JH**

I think I get the idea.

So coming all the way back to subsets: I wanted to know if they have an associated universal property. Do they?

**Claude 3.5 Sonnet**

Excellent question! Coming back to subsets, we can indeed discuss them in terms of universal properties and related categorical concepts. Let's break this down:

- **Subobjects and Monomorphisms:** In category theory, subobjects are often represented by monomorphisms. In the category of sets, these correspond precisely to injective functions, which we use to represent subsets.
- **Universal Property of Subobjects:** Subobjects do have an associated universal property, but it's slightly different from the universal properties we usually discuss for constructions like products or limits. The universal property of a subobject is often described in terms of a factorization property:
  - Given a monomorphism  $m : S \rightarrow A$  representing a subobject of  $A$ , for any morphism  $f : X \rightarrow A$  that "factors through"  $S$ , there exists a unique morphism  $g : X \rightarrow S$  such that  $m \circ g = f$ .

In set-theoretic terms, this means that if the image of  $f$  is contained in the subset  $S$  of  $A$ , then there's a unique way to define  $f$  as a function into  $S$ .

- **Subobject Classifier:** In certain categories (specifically, elementary toposes, which include the category of sets), there's a more powerful way to characterize subobjects using a universal property. This involves the subobject classifier:
  - There exists an object  $\Omega$  (in **Sets**, this is the two-element set  $\{\text{true}, \text{false}\}$ )
  - There's a distinguished morphism  $\text{true} : 1 \rightarrow \Omega$  (in **Sets**, this maps the single element to true)
  - For any monomorphism  $m : S \rightarrow A$ , there exists a unique morphism  $\chi_m : A \rightarrow \Omega$  (called the characteristic function) such that the following square is a pullback:

$$\begin{array}{ccc} S & \xrightarrow{m} & A \\ \downarrow & & \downarrow \chi_m \\ 1 & \xrightarrow{\text{true}} & \Omega \end{array}$$



This universal property essentially says that subobjects are in one-to-one correspondence with morphisms to  $\Omega$ .

- **Adjunction Perspective:** While not immediately apparent, this universal property is related to an adjunction. The functor  $\text{Sub}(-)$  that assigns to each object its poset of subobjects is right adjoint to the functor that assigns to each poset its set of global elements.
- **Slice Category Connection:** Remember our earlier discussion about slice categories? The subobjects of an object  $A$  can be identified with the isomorphism classes of objects in the slice category  $C/A$ . This provides another way to study subobjects categorically.
- **Generalization to Other Categories:** This approach to subobjects generalizes nicely to other categories. For instance, in the category of groups, subgroups can be characterized by a similar universal property involving monomorphisms.

In conclusion, while subsets (or more generally, subobjects) don't have a universal property in quite the same way as products or limits, they do have a characterization in terms of a universal property involving factorization. In sufficiently nice categories, this can be strengthened to the universal property involving the subobject classifier, which provides a powerful tool for working with and reasoning about subobjects.

This categorical view of subobjects provides a rich framework for understanding inclusion and subset relationships across different mathematical contexts.

JH

Ok, damn. Wow.

So imagine that I am making a diagram where I want to express the relationship between a function  $f : X \rightarrow Y$  versus the product  $X \times Y$ . Is there a categorical construction for that?

... I guess it's the exponential object  $Y^X$ ... but what is the relationship between an exponential and a product? I think that's in the definition of the universal property of exponents...

Claude 3.5 Sonnet

Your intuition is spot on, and you're moving towards some deep and fundamental ideas in category theory. Let's unpack this step by step:

- **Functions and Products:** You're right to think about the relationship between a function  $f : X \rightarrow Y$  and the product  $X \times Y$ . This relationship is indeed captured by the exponential object  $Y^X$  in a cartesian closed category (which includes the category of sets).
- **Exponential Objects:** In category theory,  $Y^X$  represents the object of all morphisms from  $X$  to  $Y$ . In **Set**, this is literally the set of all functions from  $X$  to  $Y$ .
- **Universal Property of Exponentials:** The universal property of the exponential  $Y^X$  is defined in terms of its relationship with the product. Specifically:
  - For any object  $Z$ , there is a natural bijection:

$$\text{Hom}(Z \times X, Y) \cong \text{Hom}(Z, Y^X)$$

- This bijection is natural in  $Z$  and  $Y$ .
- **Curry-Howard Correspondence:** This universal property is a categorical expression of currying in lambda calculus and functional programming. It says that a function of two arguments ( $Z \times X \rightarrow Y$ ) corresponds to a function that takes one argument and returns a function ( $Z \rightarrow (X \rightarrow Y)$ ).
- **Adjunction between Product and Exponential:** This universal property actually defines an adjunction:

$$(-) \times X \dashv (-)^X$$

This means "the functor that takes products with  $X$  is left adjoint to the functor that takes exponentials to the  $X$  power."

- **Relationship to Your Original Question:** Now, to directly address your question about the relationship between  $f : X \rightarrow Y$  and  $X \times Y$ :
  - $f : X \rightarrow Y$  corresponds to a single element of  $Y^X$
  - The evaluation map  $ev : Y^X \times X \rightarrow Y$ , defined by  $ev(f, x) = f(x)$ , captures how individual functions relate to the product
  - The universal property tells us that any map  $g : Z \times X \rightarrow Y$  factors uniquely through  $ev$  via a map  $h : Z \rightarrow Y^X$
- **Categorical Constructions:** There are several categorical constructions that relate to this:
  - The graph of  $f : X \rightarrow Y$  is a subobject of  $X \times Y$
  - The transpose of  $f : X \rightarrow Y$  is the corresponding morphism  $1 \rightarrow Y^X$
  - The evaluation map  $ev : Y^X \times X \rightarrow Y$  is a universal element for the functor  $\text{Hom}(-, Y)$

In essence, the exponential object  $Y^X$ , along with its universal property, provides a rich framework for understanding the relationship between functions and products in a categorical setting. This framework generalizes the familiar notions from set theory to a wide range of mathematical contexts, revealing deep connections between seemingly disparate concepts.

This connection between products and exponentials is a cornerstone of categorical logic and the foundation for many advanced concepts in category theory and theoretical computer science.

**JH**

The evaluation map seems very relevant to what I want to do.

I lost my train of thought. I guess all I wanted was a canonically categorical way to express the set of all functions from  $X$  to  $Y$ . This is a Hom set, or it's an exponential object. Is there a diagram that shows that these objects are isomorphic?

**Claude 3.5 Sonnet**

You're right on track, and your question touches on a fundamental aspect of category theory. Let's clarify this and construct a diagram to illustrate the isomorphism you're asking about.

- **Hom-sets and Exponential Objects:** In a cartesian closed category (like **Set**), the Hom-set  $\text{Hom}(X, Y)$  is indeed isomorphic to the exponential object  $Y^X$ . This isomorphism is not just a set-theoretic bijection, but a natural isomorphism in the categorical sense.
- **Diagram Showing the Isomorphism:** We can represent this isomorphism with a diagram, but first, let's set up the context:
  - Let  $C$  be a cartesian closed category.
  - Let  $1$  be the terminal object in  $C$ .
  - We know that  $\text{Hom}(1, Y^X) \cong \text{Hom}(X, Y)$  for any  $X$  and  $Y$ .

Here's a diagram that captures this isomorphism:

$$\begin{array}{ccc}
 Y^X \times X & \xrightarrow{\text{ev}} & Y \\
 \uparrow & & \uparrow \\
 \phi & & f \\
 \uparrow & & \uparrow \\
 1 & \xrightarrow{\psi(f)} & Y^X
 \end{array}$$

Where:

- $f : X \rightarrow Y$  is an arbitrary morphism
- $\text{ev} : Y^X \times X \rightarrow Y$  is the evaluation map
- $\phi : 1 \times X \rightarrow X$  is the natural isomorphism (left unitor)
- $\psi(f) : 1 \rightarrow Y^X$  is the transpose of  $f$
- **Explanation of the Diagram:**
  - The outer square commutes, meaning  $\text{ev} \circ (\psi(f) \times \text{id}_X) = f \circ \phi$
  - $\psi(f)$  is the unique morphism that makes this square commute
  - This uniqueness is precisely what establishes the isomorphism between  $\text{Hom}(X, Y)$  and  $\text{Hom}(1, Y^X)$
- **The Evaluation Map:** The evaluation map  $\text{ev} : Y^X \times X \rightarrow Y$  plays a crucial role here. It's the canonical morphism that "applies" a function to an argument. In **Set**, it would be defined as  $\text{ev}(f, x) = f(x)$ .
- **Natural Isomorphism:** The isomorphism  $\text{Hom}(X, Y) \cong \text{Hom}(1, Y^X)$  is natural in both  $X$  and  $Y$ . This means it respects composition with morphisms in either variable.
- **Connection to Your Original Question:** This diagram and the associated isomorphism provide the canonical categorical way to express the set of all functions from  $X$  to  $Y$  that you were looking for. It shows how:
  - Elements of  $Y^X$  (represented by morphisms  $1 \rightarrow Y^X$ )
  - Correspond exactly to morphisms  $X \rightarrow Y$

In the category of sets, this precisely captures the idea that  $Y^X$  is the set of all functions from  $X$  to  $Y$ .

This construction is powerful because it generalizes beyond just sets to any cartesian closed category, providing a uniform way to talk about "function objects" in a wide variety of mathematical contexts.

Does this diagram and explanation help clarify the relationship between Hom-sets and exponential objects? Would you like to explore any specific aspect of this construction further?

**JH**

That seems like what I wanted. Thank you.

The thing is that in my diagram, I have an element which is a function from a set of function symbols  $\text{Func}$  to the set of natural numbers. Any such function assigns each function symbol an arity. Each function doing so can be thought of as a signature for a language in logic. But it depends on what function symbols you're using. Maybe it doesn't matter - we never specify what the elements of  $\text{Func}$  are, because this should work with generality, for any set of function symbols.

The thing is that now, the function mapping function symbols to arities is a morphism in my diagram. How do I "pull out that morphism" from a Hom set to depict it as an object? Perhaps using a terminal object like 1 to "select" it?

**Claude 3.5 Sonnet**

Your intuition about using the terminal object to "select" a specific morphism is exactly right. This is a common technique in category theory to represent elements or specific morphisms as objects in a category. Let's break this down:

- **Your Setup:**
  - You have a set  $\text{Func}$  of function symbols
  - You have a set  $\mathbb{N}$  of natural numbers (arities)
  - A signature is a function  $\sigma : \text{Func} \rightarrow \mathbb{N}$  assigning arities to function symbols
- **Categorical Representation:** In the category of sets,  $\sigma$  is indeed a morphism in  $\text{Hom}(\text{Func}, \mathbb{N})$ .
- **Representing  $\sigma$  as an Object:** To "pull out"  $\sigma$  and represent it as an object, we can use the terminal object 1 (a singleton set in **Set**). Here's how:
  - The morphism  $\sigma : \text{Func} \rightarrow \mathbb{N}$  corresponds uniquely to a morphism  $\sigma' : 1 \rightarrow \mathbb{N}^{\text{Func}}$
  - This  $\sigma'$  is an element of the exponential object  $\mathbb{N}^{\text{Func}}$ , which represents all possible signatures
- **Diagram:** Here's a diagram representing this situation:

$$\begin{array}{ccc} 1 & \xrightarrow{\sigma'} & \mathbb{N}^{\text{Func}} \\ & \searrow & \downarrow \\ & & \mathbb{N} \end{array}$$

Where:

- 1 is the terminal object
- $\sigma' : 1 \rightarrow \mathbb{N}^{\text{Func}}$  is the "element" representing your specific signature
- $\sigma : \text{Func} \rightarrow \mathbb{N}$  is your original signature function
- **Interpretation:**
  - The morphism  $\sigma' : 1 \rightarrow \mathbb{N}^{\text{Func}}$  "selects" your specific signature from the object of all possible signatures ( $\mathbb{N}^{\text{Func}}$ )
  - The commutativity of the diagram ensures that  $\sigma'$  and  $\sigma$  represent the same signature
- **Generalization:** This approach works not just for your specific case, but for any morphism you want to "pull out" as an object. It's a general technique in category theory for turning morphisms into objects.
- **Universal Property:** This construction relates to the universal property of exponentials we discussed earlier. The bijection  $\text{Hom}(1, \mathbb{N}^{\text{Func}}) \cong \text{Hom}(\text{Func}, \mathbb{N})$  is precisely what allows us to move between these two representations of a signature.

By representing your signature  $\sigma$  as  $\sigma' : 1 \rightarrow \mathbb{N}^{\text{Func}}$ , you've effectively turned a morphism ( $\sigma$ ) into an object (the image of  $\sigma'$  in  $\mathbb{N}^{\text{Func}}$ ). This allows you to manipulate and reason about specific signatures as objects in your category, which can be very useful for further constructions or proofs.

This approach gives you the flexibility to work with both the function-as-morphism view ( $\sigma : \text{Func} \rightarrow \mathbb{N}$ ) and the function-as-object view ( $\sigma' : 1 \rightarrow \mathbb{N}^{\text{Func}}$ ), choosing whichever is more convenient for a given situation in your work with logical signatures.

JH

That's amazing but that still does not depict the specific function as an object - it depicts it as a morphism into the set of all functions on those sets. How do I make it into an object?

Claude 3.5 Sonnet

You're absolutely right, and I apologize for not addressing your question more directly. You're looking to represent the specific function as an object itself, not just as a morphism into a set of functions. Let's correct this and provide a more appropriate solution.

To represent a specific function as an object in the category, we can use the graph of the function. This approach turns the function into a subobject of the product, which is indeed an object in the category. Here's how we can do this:

- **Graph of a Function:** For your function  $\sigma : \text{Func} \rightarrow \mathbb{N}$ , its graph is the set:

$$G(\sigma) = \{(f, \sigma(f)) \mid f \in \text{Func}\}$$

- **Categorical Construction:** In category-theoretic terms,  $G(\sigma)$  is a subobject of  $\text{Func} \times \mathbb{N}$ .

- **Diagram:** Here's a diagram representing this construction:

$$\begin{array}{ccc}
 G(\sigma) & \xrightarrow{i} & \mathbf{Func} \times \mathbb{N} \\
 \pi_1 \downarrow & & \downarrow (\pi_1, \sigma \circ \pi_1) \\
 \mathbf{Func} & \xrightarrow{\sigma} & \mathbb{N}
 \end{array}$$

Where:

- $G(\sigma)$  is the graph of  $\sigma$ , now an object in the category
- $i : G(\sigma) \rightarrow \mathbf{Func} \times \mathbb{N}$  is the inclusion morphism
- $\pi_1$  and  $\pi_2$  are the projection morphisms from the product
- $(\pi_1, \sigma \circ \pi_1)$  is the morphism that makes  $G(\sigma)$  the pullback of  $\sigma$  along  $\pi_2$

- **Properties:**

- $G(\sigma)$  is isomorphic to  $\mathbf{Func}$  (via  $\pi_1 \circ i$ )
- The diagram commutes:  $\sigma \circ \pi_1 \circ i = \pi_2 \circ i$

- **Interpretation:**

- $G(\sigma)$  is now an object in your category, representing the specific function  $\sigma$
- It contains all the information of  $\sigma$ , but as an object rather than a morphism
- You can recover  $\sigma$  from  $G(\sigma)$  by composing the inclusion with the second projection:  $\sigma = \pi_2 \circ i$

This construction turns your specific signature function  $\sigma$  into an object  $G(\sigma)$  in the category. It's no longer just a morphism or an element of a function space, but a full-fledged object that you can manipulate, relate to other objects, or use in further constructions.

This approach gives you what you asked for: a way to depict your specific function (signature) as an object in your categorical diagram, while still maintaining its functional nature through its relationship with  $\mathbf{Func}$  and  $\mathbb{N}$ .

Is this more in line with what you were looking for? This representation allows you to treat your signature as a first-class object in your categorical framework.