

Assume a symmetric monoidal category s.t.

- 1) Every object carries a comonoid $q \dashv \vdash$
- 2) Every morphism is a homomorphism w.r.t. these comonoids
- 3) The comonoids are unipm:

$$\begin{array}{c} X \quad Y \quad X \quad Y \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ X \quad Y \end{array} = \begin{array}{c} X \quad Y \quad X \quad Y \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ X \quad Y \end{array}$$

Lemma

The monoidal unit is terminal.

Proof

Let $f: X \rightarrow I$ be arbitrary, then $q \circ f \stackrel{(\text{unit})}{=} f \circ q \stackrel{(\text{scalar})}{=} \bullet \stackrel{(2)}{=} \bullet$

Corollary: $\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ X \quad Y \end{array} = \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ X \quad Y \end{array}$ Proof. From uniqueness. \square

Define: $\pi_1 := \begin{array}{c} X \\ | \\ \bullet \\ | \\ X \quad Y \end{array}$, $\pi_2 := \begin{array}{c} Y \\ | \\ \bullet \\ | \\ X \quad Y \end{array}$ and for $f: C \rightarrow X, g: C \rightarrow Y$ $\langle f, g \rangle := f \circ \begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ \bullet \\ | \\ C \end{array} \circ g$

Lemma [$X \otimes Y$ is a product]

$\pi_1 \circ \langle f, g \rangle = f$, $\pi_2 \circ \langle f, g \rangle = g$, and is the unique such morphism.

Proof
 $\pi_1 \circ \langle f, g \rangle = \begin{array}{c} X \\ | \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ C \end{array} \circ g \stackrel{(2)}{=} f \circ \begin{array}{c} X \\ | \\ \bullet \\ | \\ C \end{array} \stackrel{(\text{unit})}{=} \begin{array}{c} X \\ | \\ \bullet \\ | \\ C \end{array} \circ f$, and symmetrically for $\pi_2 \circ \langle f, g \rangle = g$.

Now assume $\begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ \bullet \\ | \\ C \end{array} \circ h$ s.t. $\begin{array}{c} X \\ | \\ \bullet \\ | \\ C \end{array} \circ h = \begin{array}{c} X \\ | \\ \bullet \\ | \\ C \end{array} \circ f$ and $\begin{array}{c} Y \\ | \\ \bullet \\ | \\ C \end{array} \circ h = \begin{array}{c} Y \\ | \\ \bullet \\ | \\ C \end{array} \circ g$. Then:

$$\begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ \bullet \\ | \\ C \end{array} \circ h \stackrel{(\text{unit})}{=} \begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ \bullet \\ | \\ C \end{array} \circ h \stackrel{(3)}{=} \begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ \bullet \\ | \\ C \end{array} \stackrel{(2)}{=} \begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ C \end{array} \stackrel{(\text{ass})}{=} \begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ \bullet \\ | \\ C \end{array} \circ \begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ \bullet \\ | \\ C \end{array} \stackrel{(\text{dot})}{=} \langle f, g \rangle \quad \square$$