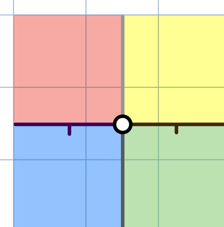
logic in color

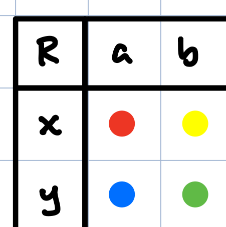
$\tilde{\Sigma}$ The Language $\tilde{\Pi}$

Our story so far

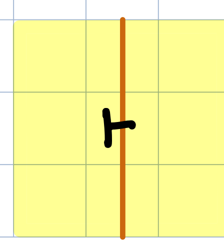
Rel: colors, strings, & beads



[$\mathbb{V}$]: sets & $\mathbb{V}$-matrices

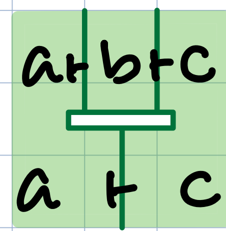


$\underline{\mathfrak{C}}$: monads ~ "cosmic" logic



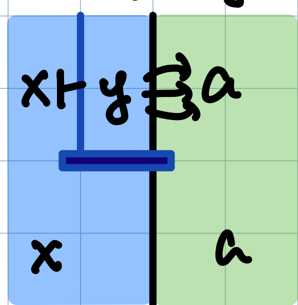
In $\underline{\mathbb{V}} := \mathrm{Mod}(\mathrm{Mat}\,\mathbb{V})$,

types are "logics", ($\mathbb{V}$-categories)

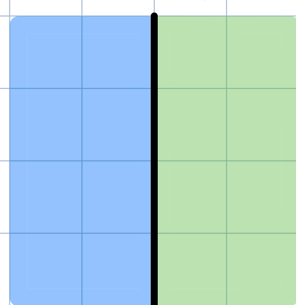


&

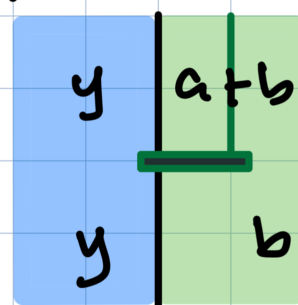
so judgements are "actual"



left action

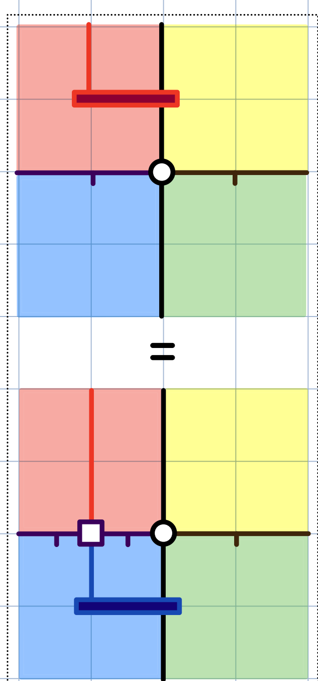


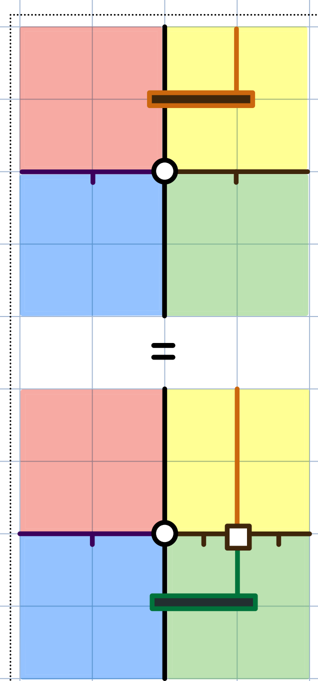
bimodule



right action

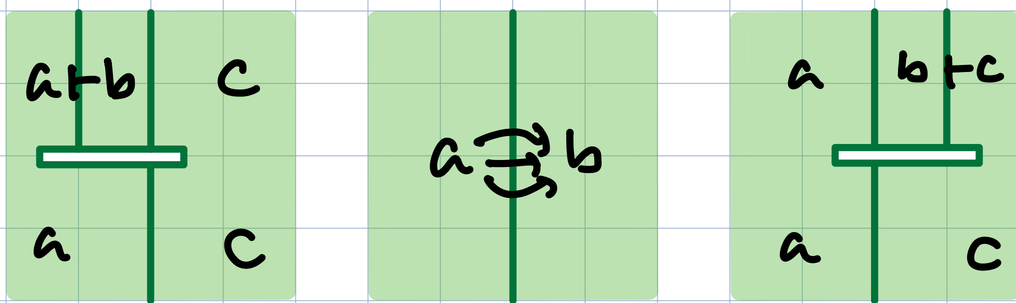
& inferences are "natural".





This is a whole new idea of logic:
everything is flowing, even

identity.



This formalizes the idea that
an object is known
through its determinations (connections to everything else).

In CT this is called "Yoneda".

There are many equivalent "Yoneda Lemmas",
but we'll see they all come from
understanding identity as concrete.

This is wild!

Yet if this is <u>really</u> logic,
it needs to be a formal language.

It is — in CT it's called "coend calculus",
but not yet a logical system.

$\tilde{\Pi}$ end : "natural" universal ($\forall$)

$\tilde{\Sigma}$ coend : "bilinear" existential ($\exists$)

In all kinds of generalized logic,
these constructors are <u>so</u> powerful.

' "All concepts are $\tilde{\Sigma}\tilde{\Pi}$" — Mac Lane '
— Michael Scott

$\left.\begin{array}{l} \frac{d}{dx} < \mathrm{Lim} < \tilde{\Pi} \\ \int dx < \mathrm{colim} < \tilde{\Sigma} \end{array}\right\}$ not just "algebra"

First let's remember :
$\forall \sim$ inference
$\exists \sim$ composition
in Rel

Universal ($\forall$) is the key
to higher-order reasoning. ("thinking about thinking")

A relation is
a matrix
of propositions

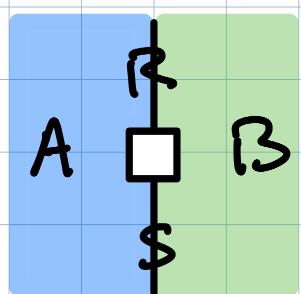
R	b	b'
a	0	1
a'	1	0 ...

+

an inference is
a matrix
of implications,

$R \Rightarrow S$	b	b'
a	$0 \Rightarrow 1$	$1 \Rightarrow 1$
a'	$1 \Rightarrow 1$	$0 \Rightarrow 0$ ...

yet this data is united in one proposition:

$\{$ 

$\} = \forall ab.\ aRb \Rightarrow aSb$

— this is an object in $\mathbb{V}$! *

In $[\mathbb{B}]^{\mathrm{Rel}}$ we have the proposition "$R \Rightarrow S$";
in $[\$]$, we have the <u>set</u> of inferences

$$\left\{\frac{R}{S}\right\} = \prod ab.\ aRb \to aSb$$

This is the inference object $[R \vdash S] : \mathbb{V}$.

Remember,

we compose judgements
with sum & tensor ($-\circ-$ in $\mathbb{V}$)

$$R \circ U = \Sigma b{:}B \; -Rb \otimes bU-$$
$$\exists b{:}B \; aRb \wedge bUc$$

Now, look:

we "reify" inferences
with product & hom $\left(\frac{p \otimes q}{r} \sim \frac{p}{q \multimap r}\right)$

$$R \vdash S = \Pi a{:}A, b{:}B \; aRb \to aSb$$
$$\forall ab \; aRb \Rightarrow aSb$$

(Clearly some duality afoot.)

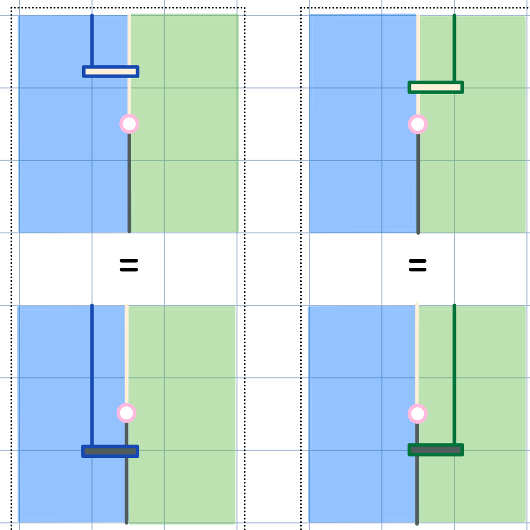
So with $(\mathbb{V}, \otimes, \Sigma)$ we have $[\mathbb{V}]$
& if also $(\text{"}, \to, \Pi)$
then $[\mathbb{V}]$ supports higher-order logic.

— Now to work in "cosmic" logic,
we also need

equalisers $\{a \mid f = g\} \hookrightarrow A$
& quotients $A \twoheadrightarrow A/\langle u \sim v \rangle$.

Why?

Inference is conditioned by "naturality":

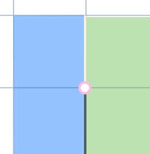


ie for every
$\mathring{a} \to a$
$+\ b \to \dot{b}$

$$
\begin{tikzcd}
\mathring{a}Rb \arrow[r] \arrow[d] \arrow[dr, phantom, "1."] & aRb \arrow[r] \arrow[d] \arrow[dr, phantom, "2."] & aR\dot{b} \arrow[d] \\
\mathring{a}Sb \arrow[r] & aSb \arrow[r] & aS\dot{b}
\end{tikzcd}
$$

"natural transformation"

So, to specify the "subset" of natural inferences:



"equaliser"

$$\{\textstyle\prod ab.\, aRb \to aSb \mid$$
1. $\mathring{a}A \circ Rb \cdot a\gamma b = \mathring{a}\gamma b \cdot \mathring{a}A \circ Sb$
2. $aR \circ B\dot{b} \cdot a\gamma\dot{b} = a\gamma b \cdot aS \circ B\dot{b} \}$

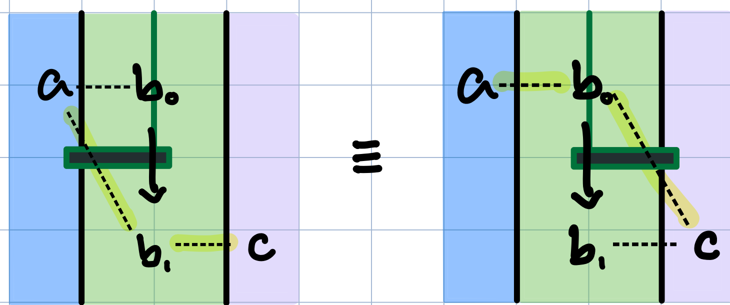
This is the end $\boxed{\widetilde{\textstyle\prod} ab.\, aRb \to aSb}$.

$R \vdash S$ inference object

$\widetilde{\prod}$ is the "natural" universal.

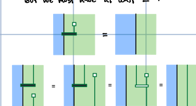
Dually,

composition quotients inner actions:



This is necessary for unitality.

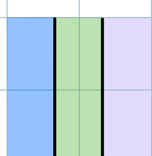
We no longer have ,
but we must have at least $\cong$:



The iso $R \cong A \circ R = \tilde{\Sigma} a{:}A.\ {-}Aa \circ aR{-}$ is known as "coYoneda lemma".

So, composition is defined

$$aR \circ Uc := \Sigma b{:}B.\ aR\,b \circ b\,Uc \;/\; \langle aRB\,b_1 \circ b_1\,Uc \sim aR\,b_0 \circ b_0 B Uc \rangle$$



This is the coend $\boxed{\tilde{\Sigma} b{:}B.\ aRb \circ bUc}$.

$\tilde{\Sigma}$ is the "bilinear" existential.

These work just like $\exists$ + $\forall$.

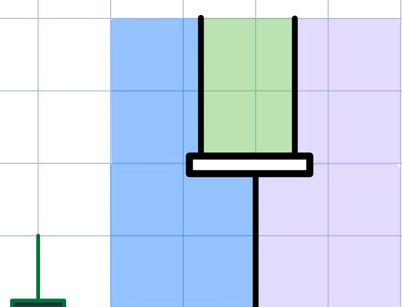
Quantifiers are characterized by a "universal property".

$$\frac{\exists x\, Px}{Q} \sim \forall x\, \frac{Px}{Q}$$

$$\frac{Q}{\forall y\, Ry} \sim \forall y\, \frac{Q}{Ry}$$

(in CT this is called "hom preserves co/limits")

Consider an inference $\dfrac{R \circ U}{Z}$



$$\frac{\tilde{\Sigma} b{:}B.\ aRb \circ bUc}{aZc}$$

An inference from $A \xrightarrow{R \circ U} C$ cannot distinguish the inner actions — so it is "natural" in B.

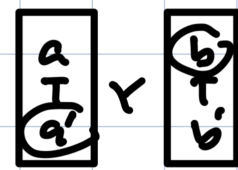
$$\frac{\tilde{\Sigma} b{:}B.\ aRb \circ bUc}{aZc} \sim \tilde{\Pi} b{:}B.\ \frac{aRb \circ bUc}{aZc}$$

+ similarly for $X \vdash \tilde{\Pi}$.

More, $\tilde{\Sigma}$ "bilinearity" provides extensions, adjoints to composition for "internal HOL" — we'll explore next time.

Ab
$A \to [B,C]$
$A \otimes B \to C$

Let's end with a bang.



Let A, B be types. ($\mathbb{V}$-categories)

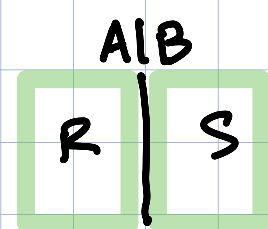
For each $R, S : A \mid B$
there's an object of inferences $[R \vdash S] : \mathbb{V}$.

Does that sound familiar?

$$[A|B](R \vdash S) = \tilde{\prod} a{:}A.\, b{:}B.\; aRb \vdash aSb \; : \mathbb{V}$$

a $\mathbb{V}$-matrix!

and? $[A|B] \circ [A|B] \to [A|B]$



$[A|B]$ is a $\mathbb{V}$-category!

(so, the language is <u>its own</u> metalanguage.)

This language is not yet defined nor used systematically in CT.

What is possible in cosmic logic?

$$[A \mid B] \sim [A^{\circ} \otimes B \vdash \mathbb{Y}]$$

judgements terms