Research Outline

Exploring Set Theory through First-Order Logic, and Category Theory through Set Theory

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We will start with a basic signature for a theory, examine first the terms and then the formulae which we can construct, and then explore constructing theorems from these axioms using a Hilbert-style deductive system.

1 First-Order Signature

We begin with a first-order signature containing only the following binary relation symbol:

$$\Sigma = \{\in\}$$

2 Term Algebra

The term algebra over this signature is trivial since we have no constants or functions. Assuming infinite variable symbols, it is equal to the set of variables:

$$T_\Sigma = \{x, y, z, \ldots\}$$

3 Atomic Formulae

Atomic formulae are created by the application of the $\equiv$ symbol and $\in$ relation. These are both binary. It follows that we only need two variables to generate all equivalence classes of atomic formulae under substitution of variables:

Atomic Formulae
$\phi_1 := a \equiv a$
$\phi_2 := a \equiv b$
$\phi_3 := a \in a$
$\phi_4 := a \in b$

4 Generation of Well-Formed Formulae

The formation rules for composite formulae in first-order logic are as follows:

1. **Negation:** If ϕ is a formula, then $\neg\phi$ is a formula.
2. **Conjunction:** If ϕ and ψ are formulae, then $(\phi \wedge \psi)$ is a formula.
3. **Disjunction:** If ϕ and ψ are formulae, then $(\phi \vee \psi)$ is a formula.
4. **Implication:** If ϕ and ψ are formulae, then $(\phi \rightarrow \psi)$ is a formula.
5. **Biconditional:** If ϕ and ψ are formulae, then $(\phi \leftrightarrow \psi)$ is a formula.
6. **Universal Quantification:** If ϕ is a formula and x is a variable, then $\forall x \phi$ is a formula.
7. **Existential Quantification:** If ϕ is a formula and x is a variable, then $\exists x \phi$ is a formula.

This is a table of formulae generated by n rounds of an application of one of the above rules to any combination of the set of formulae available from the previous round. We start with the set of atomic formulae $\Phi = \{\phi_1, \phi_2, \phi_3, \phi_4\}$ and apply the formation rules iteratively. Here's a table showing the generation of new formulae in each round:

Round n	New Formulae
0	$\phi_1 : a \equiv a$ $\phi_2 : a \equiv b$ $\phi_3 : a \in a$ $\phi_4 : a \in b$
1	$\neg\phi_1 : \neg(a \equiv a)$ $\neg\phi_2 : \neg(a \equiv b)$ $\neg\phi_3 : \neg(a \in a)$ $\neg\phi_4 : \neg(a \in b)$ $\phi_1 \wedge \phi_2 : (a \equiv a) \wedge (a \equiv b)$ $\phi_1 \vee \phi_3 : (a \equiv a) \vee (a \in a)$ $\phi_2 \to \phi_4 : (a \equiv b) \to (a \in b)$ $\phi_3 \leftrightarrow \phi_4 : (a \in a) \leftrightarrow (a \in b)$ $\forall a\phi_1 : \forall a(a \equiv a)$ $\exists b\phi_4 : \exists b(a \in b)$
2	$\neg(\phi_1 \wedge \phi_2) : \neg((a \equiv a) \wedge (a \equiv b))$ $(\neg\phi_1) \vee \phi_3 : \neg(a \equiv a) \vee (a \in a)$ $(\phi_1 \wedge \phi_2) \to \phi_4 : ((a \equiv a) \wedge (a \equiv b)) \to (a \in b)$ $\forall a(\phi_1 \vee \phi_3) : \forall a((a \equiv a) \vee (a \in a))$ $\exists b(\phi_2 \wedge \phi_4) : \exists b((a \equiv b) \wedge (a \in b))$ $(\forall a\phi_1) \to (\exists b\phi_4) : (\forall a(a \equiv a)) \to (\exists b(a \in b))$
3	$\neg(\forall a(\phi_1 \vee \phi_3)) : \neg(\forall a((a \equiv a) \vee (a \in a)))$ $((\neg\phi_1) \vee \phi_3) \wedge (\phi_2 \to \phi_4) :$ $\quad (\neg(a \equiv a) \vee (a \in a)) \wedge ((a \equiv b) \to (a \in b))$ $\forall a\exists b(\phi_2 \wedge \phi_4) : \forall a\exists b((a \equiv b) \wedge (a \in b))$ $\exists a\forall b((\phi_1 \wedge \phi_2) \to \phi_4) :$ $\quad \exists a\forall b(((a \equiv a) \wedge (a \equiv b)) \to (a \in b))$

This table shows a selection of example formulae generated in each round. It will be a big task to determine the logical equivalence classes of these formulae. We will use these formulae below when we apply the axiom of specification.

I am pretty sure this is called a free cylindrical algebra. The below is just a description of that I obtained, which I haven't studied yet:

Axioms of a Free Cylindrical Algebra on 4 Generators

Let $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, d_{ij} \rangle_{i,j<\omega}$ be a cylindrical algebra, where A is the underlying set with generators $\{g_1, g_2, g_3, g_4\}$. The axioms are as follows:

1. **Boolean Algebra Axioms:** For all $x, y, z \in A$:

$$
\begin{aligned}
x + (y + z) &= (x + y) + z \\
x + y &= y + x \\
x + (x \cdot y) &= x \\
x + (-x) &= 1 \\
x \cdot (y \cdot z) &= (x \cdot y) \cdot z \\
x \cdot y &= y \cdot x \\
x \cdot (x + y) &= x \\
x \cdot (-x) &= 0
\end{aligned}
$$

2. **Cylindrification Axioms:** For all $x, y \in A$ and $i, j < \omega$:

$$
\begin{aligned}
c_i 0 &= 0 \\
x &\leq c_i x \\
c_i(x \cdot c_i y) &= c_i x \cdot c_i y \\
c_i c_j x &= c_j c_i x
\end{aligned}
$$

3. **Diagonal Element Axioms:** For all $i, j, k < \omega$:

$$
\begin{aligned}
d_{ii} &= 1 \\
d_{ij} &= d_{ji} \\
d_{ik} &\leq d_{ij} \cdot d_{jk} \\
c_k(d_{ik} \cdot d_{jk}) &= d_{ij} \text{ if } i \neq j, k
\end{aligned}
$$

4. **Commutativity of Cylindrification with Diagonal Elements:** For all $i, j < \omega$:

$$c_i d_{jk} = d_{jk} \text{ if } i \neq j, k$$

5. **Freeness Conditions:** For the generators $\{g_1, g_2, g_3, g_4\}$:

    No non-trivial equation holds among g_1, g_2, g_3, g_4 except those derivable from the above axioms.

In this algebra, c_i represents cylindrification (existential quantification) with respect to the i-th variable, and d_{ij} represents the diagonal element (equality) between the i-th and j-th variables.

5 Axioms of ZFC

The following are the axioms of ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice) expressed in first-order logic, that I got from somewhere:

Axioms of ZFC

1. **Axiom of Extensionality:**
$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \to x = y)$$

2. **Axiom of Empty Set:**
$$\exists x \forall y (y \notin x)$$

3. **Axiom of Pairing:**
$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow w = x \lor w = y)$$

4. **Axiom of Union:**
$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (z \in w \land w \in x))$$

5. **Axiom of Power Set:**
$$\forall x \exists y \forall z (z \in y \leftrightarrow \forall w (w \in z \to w \in x))$$

6. **Axiom Schema of Separation:** For any formula $\phi(x)$ with free variable x:
$$\forall z \exists y \forall x (x \in y \leftrightarrow x \in z \land \phi(x))$$

7. **Axiom Schema of Replacement:** For any formula $\phi(x, y)$ that is functional in x:
$$\forall A (\forall x \in A \exists! y \phi(x, y) \to \exists B \forall y (y \in B \leftrightarrow \exists x \in A \phi(x, y)))$$

8. **Axiom of Infinity:**
$$\exists x (\emptyset \in x \land \forall y (y \in x \to y \cup \{y\} \in x))$$

9. **Axiom of Foundation:**
$$\forall x (x \neq \emptyset \to \exists y \in x (y \cap x = \emptyset))$$

10. **Axiom of Choice:**
$$\forall x (\forall y \in x (y \neq \emptyset) \to \exists f : x \to \bigcup x \forall y \in x (f(y) \in y))$$

Notes:

- The Axiom Schema of Separation and the Axiom Schema of Replacement are actually infinite collections of axioms, one for each formula ϕ in the language of set theory.
- The Axiom of Choice is stated here in terms of a choice function. There are many equivalent formulations of this axiom.
- The symbol $\exists!$ in the Axiom Schema of Replacement means "there exists a unique".

6 Deductive System

We will use a Hilbert-style deductive system with only modus ponens and universal instantiation as inference rules.

7 Set Generation

Let's examine the sets we can generate in each "round" of deduction:

7.1 Round 0

- $\emptyset$ (Empty Set)
- ω (Set of natural numbers)

7.2 Round 1

- $\{\emptyset\}$ (Singleton of empty set)
- $\{\emptyset, \{\emptyset\}\}$ (Von Neumann ordinal 2)
- $\mathcal{P}(\emptyset) = \{\emptyset\}$ (Power set of empty set)

For specification, we can create sets based on an ordering of the well-formed formulae we generated earlier. For example:

- $\{x \in \omega : x = x\}$ (All natural numbers)
- $\{x \in \omega : x \in x\}$ (Empty set)
- $\{x \in \omega : \exists y(y \in x)\}$ (Non-empty natural numbers)
- $\{x \in \omega : \forall y(y \in x \rightarrow y \in y)\}$ (Empty set)
- $\{x \in \omega : x \notin x\}$ (All natural numbers)

We haven't defined any functions yet. In ZFC, functions are typically defined as special kinds of relations (sets of ordered pairs). We would need to construct ordered pairs first before defining functions.

8 Research Questions

1. How does the complexity of definable sets grow with each round of deduction?
2. Can we characterize all sets definable within the first n rounds for some small n?
3. At what point do we gain the ability to define non-trivial functions?
4. How does this step-by-step construction relate to the cumulative hierarchy of sets?
5. In which round of application do we see the first category come into existence?

9 Next Steps

1. Formalize the process of set generation in each round.
2. Investigate the role of the Axiom of Infinity in this construction.
3. Explore how the introduction of function symbols would change our term algebra and subsequent constructions.
4. Consider alternative deductive systems and their impact on set generation, especially a generalized theory of deductive systems that encompasses Hilbert-style systems, natural deduction and sequent calculus as special cases of it.

References

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