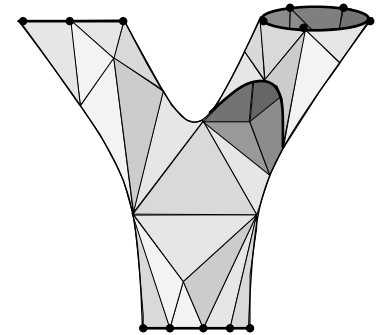
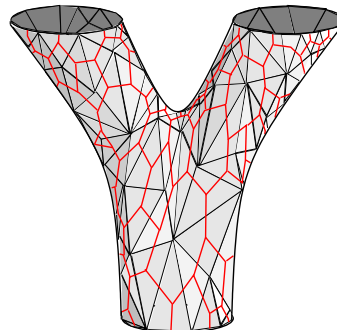
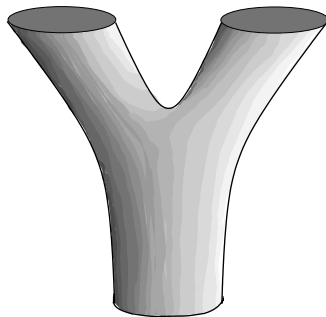


# Spans in Quantum Theory

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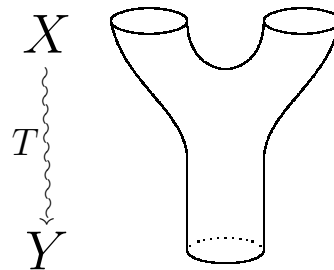
for more, see:

<http://math.ucr.edu/home/baez/span/>

Quantizing gravity is tough. But:

*Quantum theory will make more sense when seen as part of a theory of spacetime.*

Why? Quantum theory seems ‘weird’ because — seen from the eyes of category theory — *Hilbert spaces* and *operators* behave less like *sets* and *functions* than *manifolds* describing choices of ‘space’ and *cobordisms* describing choices of ‘spacetime’:



	<b>object</b> ●	<b>morphism</b> ● → ●
<b>SET THEORY</b> Set	<b>set</b>	<b>function between sets</b>
<b>QUANTUM THEORY</b> Hilb	<b>Hilbert space</b> (state)	<b>operator between Hilbert spaces</b> (process)
<b>GENERAL RELATIVITY</b> $n\text{Cob}$	<b>manifold</b> (space)	<b>cobordism between manifolds</b> (spacetime)

The category  $\text{Set}$  is ‘cartesian’, while  $\text{Hilb}$  and  $n\text{Cob}$  are ‘dagger compact’.

This means that  $\text{Set}$  is a mathematical universe in which classical logic applies, while a very different sort of logic reigns in  $n\text{Cob}$  and  $\text{Hilb}$ . In a cartesian category:

- We can freely duplicate and delete information.
- We cannot reverse most processes.
- We do not have generalized Bell states.

In a dagger compact category:

- We cannot freely duplicate and delete information.
- Every process  $T$  has a ‘reverse version’,  $T^\dagger$ .
- We have generalized Bell states.

How can we bridge the chasm between the classical world of cartesian categories and the quantum world of dagger compact categories?

We can bridge it with the concept of a ‘span’:

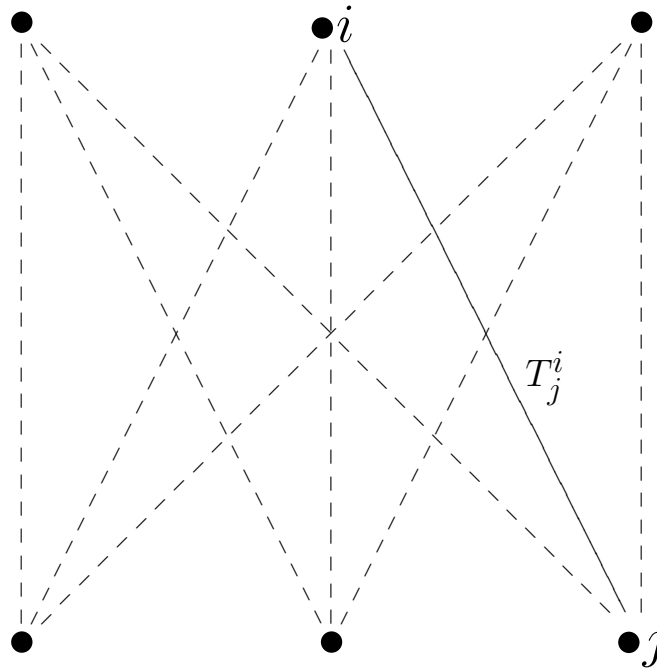
$$\begin{array}{ccc} & T & \\ p \swarrow & & \searrow p' \\ X & & Y \end{array}$$

We’ll see that any cartesian category  $C$  with pullbacks gives a dagger compact category  $\text{Span}(C)$ .

This construction is a modern version of *matrix mechanics!*

But first, recall the original idea of matrix mechanics...

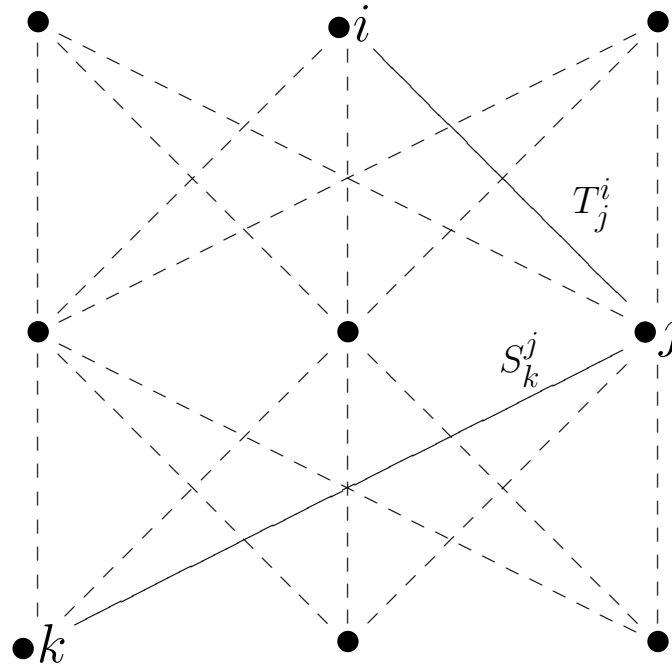
Heisenberg used matrices with complex entries to describe processes in quantum mechanics:



For each input state  $i$  and output state  $j$ , the process  $T$  gives a complex number, the *amplitude* to go from  $i$  to  $j$ :

$$T_j^i \in \mathbb{C}$$

To compose processes, we sum over paths:



$$(ST)_k^i = \sum_j S_k^j \times T_j^i$$

In the continuum limit, such sums become *path integrals*.

Matrix mechanics also works with other rigs (= rings without negatives) replacing the complex numbers:

- $[0, \infty)$  with its usual  $+$  and  $\times$

Now  $T_j^i$  gives the *probability* to go from  $i$  to  $j$ .

- $\{T, F\}$  with OR as  $+$  and AND as  $\times$

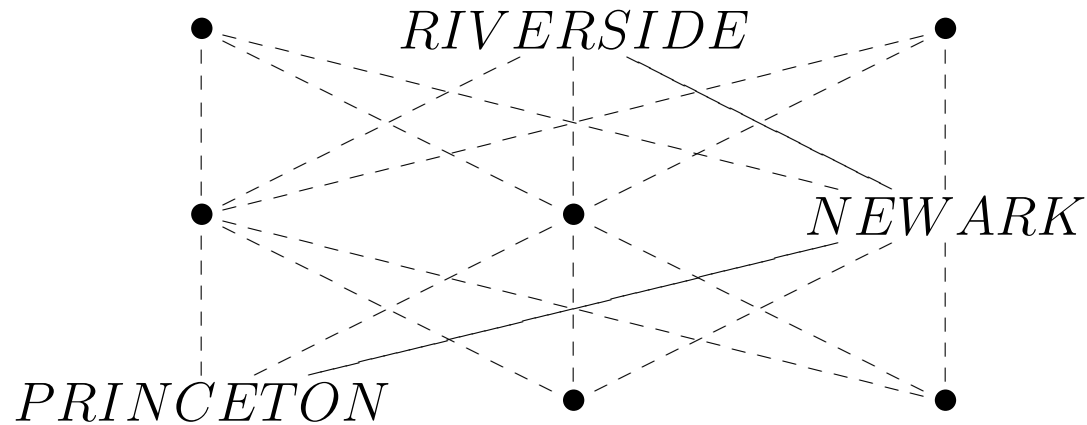
Now  $T_j^i$  gives the *possibility* to go from  $i$  to  $j$ .

- $\mathbb{R}^{\text{MIN}} = \mathbb{R} \cup \{+\infty\}$  with MIN as  $+$  and  $+$  as  $\times$

Now  $T_j^i$  gives the *cost, or action*, to go from  $i$  to  $j$ .



I did matrix multiplication in  $\mathbb{R}^{\text{MIN}}$  to find the cheapest flight here:



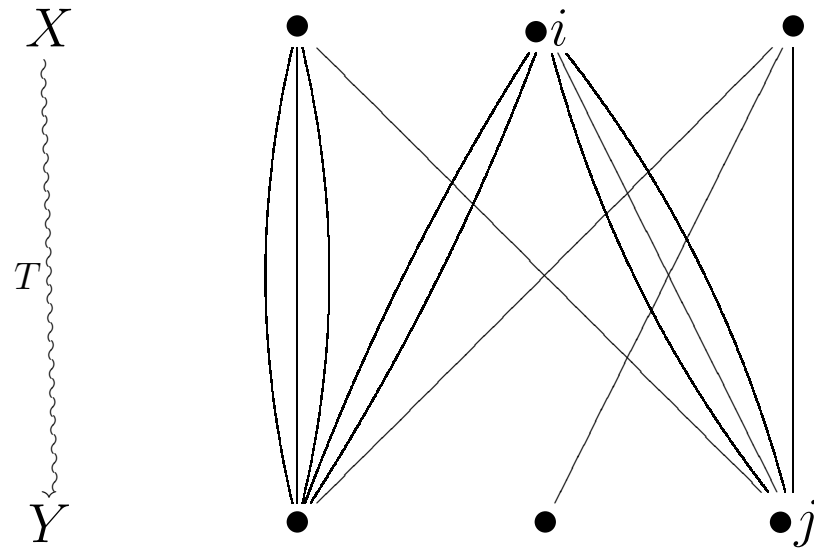
In nature, the continuum limit of this sort of calculation gives *the principle of least action*.

For more on the analogy

$\mathbb{R}^{\text{MIN}} : \mathbb{C} :: \text{classical mechanics} : \text{quantum mechanics}$

read about ‘idempotent analysis’.

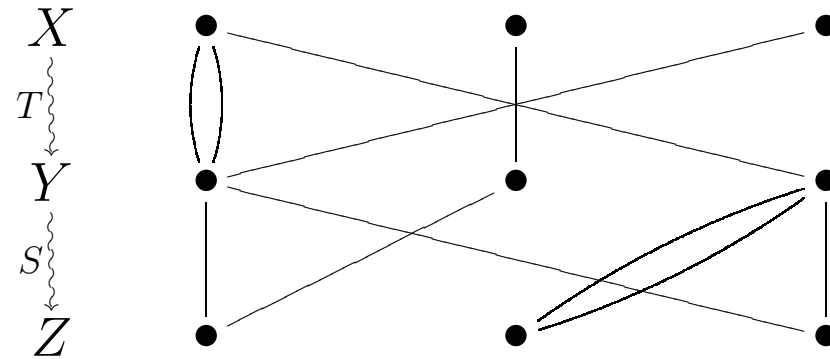
Perhaps the most fundamental example is when  $T_j^i$  is the *set of ways* to go from  $i \in X$  to  $j \in Y$ :



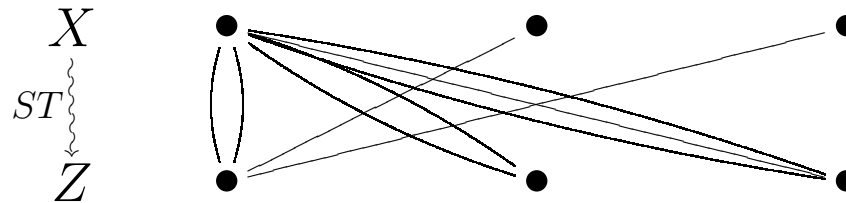
However, now the matrix  $T$  has entries taking values not in a rig but in a category: the category  $\text{Set}$ .

So, in this example we have ‘categorified’ matrix mechanics!

Set is a categorified rig, with disjoint union as  $+$  and cartesian product as  $\times$ . We use this to multiply matrices of sets. If we multiply these:



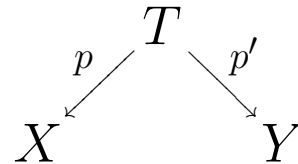
we get this:



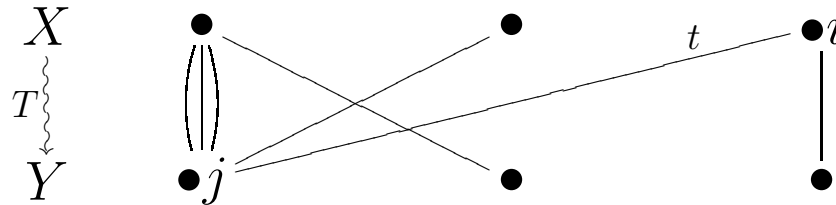
where

$$(ST)_k^i = \bigsqcup_j S_k^j \times T_j^i$$

A Set-valued matrix is the same as a ‘span’ of sets:



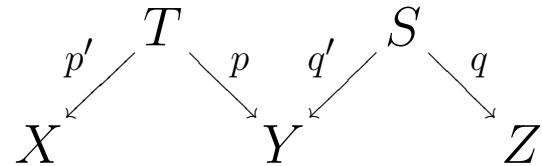
Think of  $T$  as a set of ‘paths’ from points of  $X$  to points of  $Y$ . Each path  $t \in T$  goes from some point  $p(t) = i$  to some point  $p'(t) = j$ :



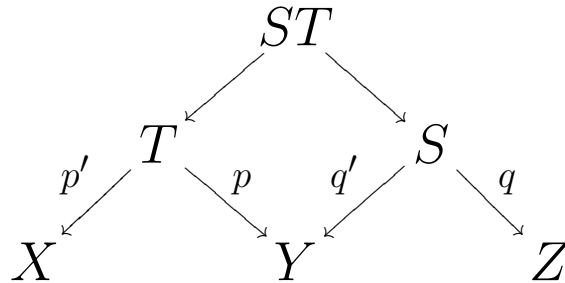
To get a matrix of sets from a span of sets, define:

$$T_j^i = \{t \in T : p(t) = i, p'(t) = j\}$$

Multiplying matrices of sets is the same as ‘composing spans’ in  $\text{Set}$ . Given spans like this:



we compose them as follows:

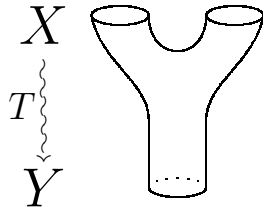


where  $ST$  is defined using a ‘pullback’:

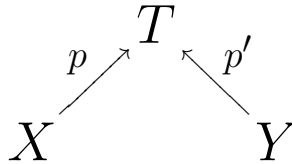
$$ST = \{(s, t) \in S \times T : q(s) = p'(t)\}$$

In words: a path in  $ST$  is a path  $s \in S$  and a path  $t \in T$ , such that  $s$  starts where  $t$  ends.

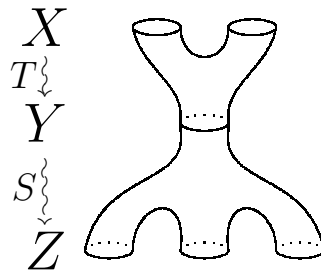
Cobordism are examples of spans in  $\text{Diff}^{op}$ , the opposite of the category of manifolds with boundary, since a cobordism:



is really a diagram like this in  $\text{Diff}$ :



$\text{Diff}^{op}$  doesn't have *all* pullbacks, but it has enough to compose cobordisms:



So, we've seen:

- Spans are a categorified version of matrix mechanics.
- A cobordism is a specially nice span in  $\text{Diff}^{\text{op}}$ .

This suggests that the common features of  $n\text{Cob}$  and  $\text{Hilb}$  are *features of categories of spans*.

This is true!

To make this precise, we'll use the concept of a 'dagger compact category'.

Suppose  $C$  is any category with pullbacks. Then we can form a category  $\text{Span}(C)$ , in which:

- an object is an object of  $C$ ;
- a morphism  $T: X \rightsquigarrow Y$  is an isomorphism class of spans

$$\begin{array}{ccc} & T & \\ p \swarrow & & \searrow p' \\ X & & Y \end{array}$$

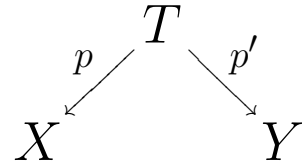
where two spans  $T_1, T_2$  are isomorphic if there's a commutative diagram:

$$\begin{array}{ccccc} & & T_1 & \xrightarrow{\sim} & T_2 & & \\ & p_1 \swarrow & & & & \searrow p'_2 & \\ & X & & & & Y & \\ & & p_2 \swarrow & & p'_1 \swarrow & & \end{array}$$

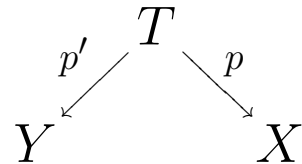
We compose morphisms in  $\text{Span}(C)$  via pullback, as sketched already.



In this situation  $\text{Span}(C)$  is always a ‘dagger category’.  
 In other words, each span  $T: X \rightsquigarrow Y$ :



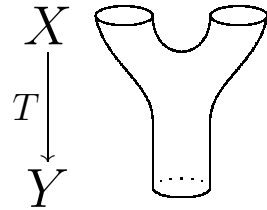
has an ‘adjoint’  $T^\dagger: Y \rightsquigarrow X$  given by switching input and output:



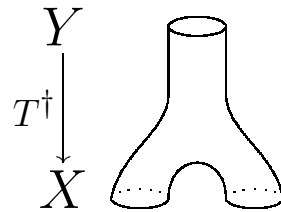
The axioms of a dagger category hold:

$$1_X^\dagger = 1_X \quad (ST)^\dagger = T^\dagger S^\dagger \quad T^{\dagger\dagger} = T$$

The dagger operation on spans makes  $n\text{Cob}$  into a dagger category where the adjoint of



is



When  $C = \text{Set}$ , spans are Set-valued matrices and  $T^\dagger: X \rightsquigarrow Y$  is just the *transpose* of  $T: Y \rightsquigarrow X$ .

Next suppose  $C$  is also ‘cartesian’, i.e. has cartesian products and a terminal object  $1$ . Then  $\text{Span}(C)$  is a symmetric monoidal category, where the tensor product of two spans:

$$\begin{array}{ccc} & T_1 & \\ p_1 \swarrow & & \searrow p'_1 \\ X_1 & & Y_1 \end{array}$$

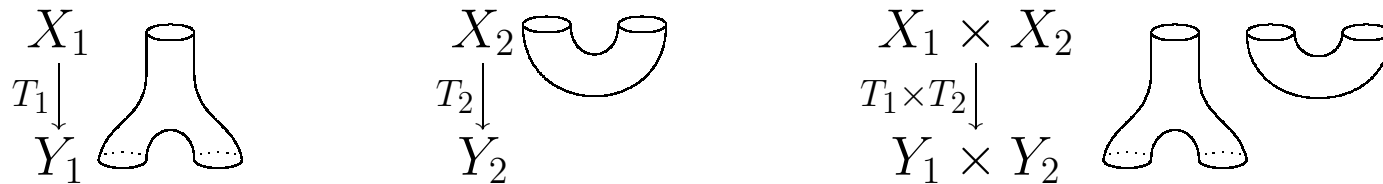
and

$$\begin{array}{ccc} & T_2 & \\ p_2 \swarrow & & \searrow p'_2 \\ X_2 & & Y_2 \end{array}$$

is given by ‘doing them in parallel’:

$$\begin{array}{ccc} & T_1 \times T_2 & \\ p_1 \times p_2 \swarrow & & \searrow p'_1 \times p'_2 \\ X_1 \times X_2 & & Y_1 \times Y_2 \end{array}$$

When  $C = \text{Diff}^{\text{op}}$ , this tensor product on  $\text{Span}(C)$  includes the usual tensor product of cobordisms as a special case:



Note that  $\times$  in  $\text{Diff}^{\text{op}}$  is disjoint union!

When  $C = \text{Set}$ , the tensor product on  $\text{Span}(C)$  is the more or less obvious ‘tensor product’ of set-valued matrices.



In Hilb,  $X^*$  is the dual Hilbert space. We have:

$$\begin{array}{ll} e_X: X^* \otimes X \rightarrow \mathbb{C} & i_X: \mathbb{C} \rightarrow X \otimes X^* \\ \ell \otimes \psi \mapsto \ell(\psi) & c \mapsto c 1_X \end{array}$$

and the zig-zag identities say familiar things about linear algebra.

In quantum theory, the unit  $i_X$  describes a ‘generalized Bell state’. The zig-zag identities are fundamental in Abramsky and Coecke’s treatment of quantum teleportation.

A ‘dagger compact category’ is a dagger category that is also compact, with some extra equations relating the two structures — see the papers by Abramsky & Coecke and Selinger for the precise definition. One can show:

**THEOREM.** If  $C$  is a cartesian category with pullbacks,  $\text{Span}(C)$  is a dagger compact category.

This result illustrates how the span construction, as a generalized version of matrix mechanics, relates ‘classical’ categories like  $\text{Set}$  to ‘quantum’ categories like  $n\text{Cob}$  and  $\text{Hilb}$ .

There’s much more to say, but perhaps not now!