We aim to make this paper as self-contained as possible, thus we take the pain of providing as much useful background for readers less familiar with traced symmetric monoidal categories and Hopf monads, as well as introducing notation. In Section 2, we provide a background section on traced symmetric monoidal categories and provide an overview of well-known examples. Similarly, Section 3 is a background section which collects some basics on monads and their algebras, while Section 4 is devoted to bimonads on monoidal categories. In Section 5, we discuss traced monads. In Section 6, we review symmetric Hopf monads. We then introduce trace-coherent Hopf monads in Section 7, and show that it characterises precisely symmetric Hopf monads which lift traced symmetric monoidal structure. Finally, in Section 8, we conclude this paper by presenting examples of traced monads which are not Hopf, and Hopf monads which are not traced.

2 Traced Symmetric Monoidal Categories

In this background section, we review traced *symmetric* monoidal categories. Traced monoidal categories were introduced by Joyal, Street, and Verity in [16], and are balanced monoidal categories equipped with a trace operator. In this paper, we will work with the symmetric version of traced monoidal categories, and so will provide the definition of traced symmetric monoidal categories as found in [14]. Alternate, but equivalent, axiomatizations of traced symmetric monoidal categories can be found in [16, Section 2] and [14, Section 2.1]. We also provide numerous well-known examples of traced symmetric monoidal categories. In particular, we consider three important subclasses of traced symmetric monoidal categories. The first is compact closed categories, whose trace operator captures the trace of matrices from linear algebra. The second is traced Cartesian monoidal categories, whose monoidal product are products and where the trace operator is given by fixed points. The third is traced coCartesian monoidal categories, whose monoidal product are coproducts and where the trace operator is given by iterations.

2.1 Traced Symmetric Monoidal Categories

As the name suggests, the underlying category of a traced symmetric monoidal category is a symmetric monoidal category, which are categories equipped with a tensor product. For an in-depth introduction to (symmetric) monoidal categories, and their axioms written out in commutative diagrams, we refer the reader to [20, Section 3].

Definition 2.1 *[20, Section 3.1] A* ***monoidal category*** *is a sextuple $(\mathbb{X}, \otimes, I, \alpha, \lambda, \rho)$ consisting of a category $\mathbb{X}$, a functor $\otimes : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ called the* ***monoidal product****, an object I called the* ***monoidal unit*** *(or simply unit), a natural isomorphism $\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C$ called the associativity isomorphism, a natural isomorphism $\lambda_A : I \otimes A \xrightarrow{\cong} A$ called the left unit isomorphism, and a natural isomorphism $\rho_A : A \otimes I \xrightarrow{\cong} A$ called the right unit isomorphism, and such that the following equalities hold:*

$$\alpha_{A \otimes B,C,D} \circ \alpha_{A,B,C \otimes D} = (\alpha_{A,B,C} \otimes 1_D) \circ \alpha_{A,B \otimes C,D} \circ (1_A \otimes \alpha_{B,C,D}) \qquad (1_A \otimes \lambda_B) \circ \alpha_{A,I,B} = \rho_A \otimes 1_B \tag{1}$$

where the equality on the left is called the pentagon axiom and the equality on the right is called the triangle axiom.

Definition 2.2 *[20, Section 3.3 & 3.5] A* ***symmetric monoidal category*** *is a septuple $(\mathbb{X}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ consisting of a monoidal category $(\mathbb{X}, \otimes, I, \alpha, \lambda, \rho)$ and a natural isomorphism $\sigma_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A$ called the symmetry isomorphism, such that the following equalities hold:*

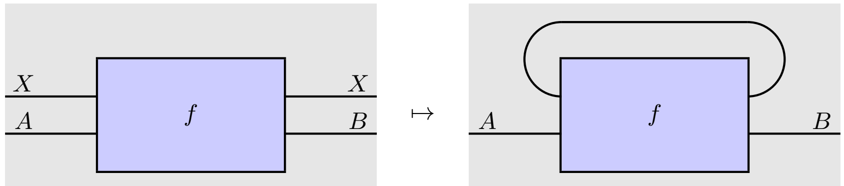
$$\sigma_{B,A} \circ \sigma_{A,B} = 1_{A \otimes B} \qquad (1_B \circ \sigma_{A,C}) \circ \alpha_{B,A,C} \circ (\sigma_{A,B} \otimes 1_C) = \alpha_{B,C,A} \circ \sigma_{A,B \otimes C} \circ \alpha_{A,B,C} \tag{2}$$

where the equality on the left is called the self-inverse axiom and the equality on the right called the hexagon axiom.

Throughout this paper, we will also make use of the graphical calculus for (symmetric) monoidal categories, and we will use more-or-less the same conventions as found in [20]. So in particular, our string diagrams should be read horizontally from left to right. We will not review in full the graphical calculus here, and we refer the reader to [20] for an in-depth introduction.

Definition 2.3 *[10, Definition 2.1] A* ***traced symmetric monoidal category*** *is an octuple $(\mathbb{X}, \otimes, I, \alpha, \lambda, \rho, \sigma, \mathsf{Tr})$ consisting of a symmetric monoidal category $(\mathbb{X}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ equipped with a* ***trace operator*** *Tr, which is a family of operators (indexed by triples of objects $X, A, B \in \mathbb{X}$), $\mathsf{Tr}^X_{A,B} : \mathbb{X}(A \otimes X, B \otimes X) \to \mathbb{X}(A, B)$, which is drawn in the graphical calculus as follows:*

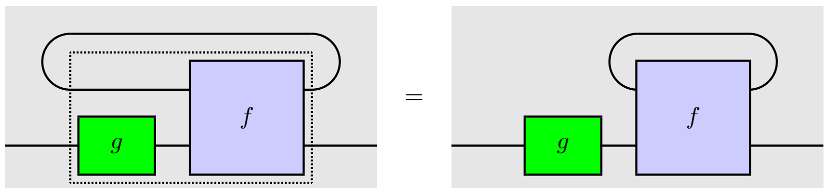
$$f : A \otimes X \to B \otimes X \qquad \mathsf{Tr}^X_{A,B}(f) : A \to B$$



such that the following axioms are satisfied:

[Tightening]: *For every map $f : A \otimes X \to B \otimes X$ and $g : A' \to A$, the following equality holds:*

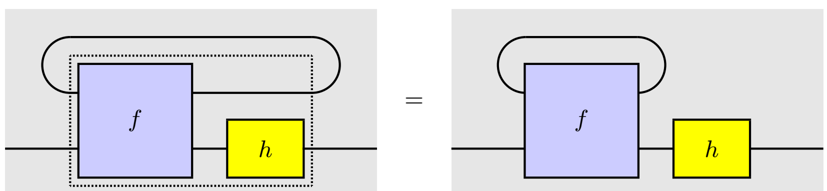
$$\mathsf{Tr}^X_{A',B}\left(f \circ (g \otimes 1_X)\right) = \mathsf{Tr}^X_{A,B}(f) \circ g$$



(3)

and for every map $f : A \otimes X \to B \otimes X$ and $h : B \to B'$, the following equality holds:

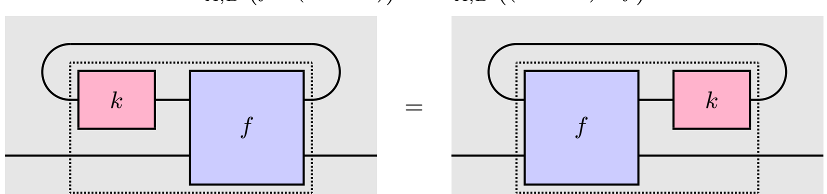
$$\mathsf{Tr}^X_{A,B'}\left((h \otimes 1_X) \circ f\right) = h \circ \mathsf{Tr}^X_{A,B}(f)$$



(4)

[Sliding]: *For every map $f : A \otimes X \to B \otimes X'$ and $k : X' \to X$, the following equality holds:*

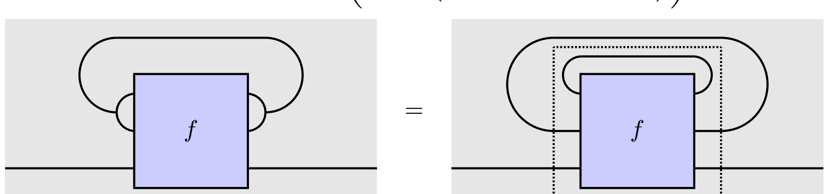
$$\mathsf{Tr}^X_{A,B}\left(f \circ (1_A \otimes k)\right) = \mathsf{Tr}^{X'}_{A,B}\left((1_B \otimes k) \circ f\right)$$



(5)

[Vanishing]: *For every map $f : A \otimes (X \otimes Y) \to B \otimes (X \otimes Y)$, the following equality holds:*

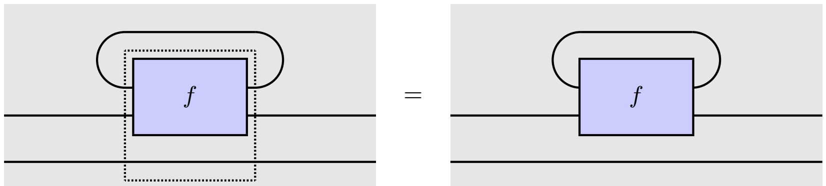
$$\mathsf{Tr}^{X \otimes Y}_{A,B}(f) = \mathsf{Tr}^X_{A,B}\left(\mathsf{Tr}^Y_{A,B}\left(\alpha_{B,X,Y} \circ f \circ \alpha^{-1}_{A,X,Y}\right)\right)$$



(6)

[Superposing]: *For every map $f : A \otimes X \to B \otimes X$ the following equality holds:*

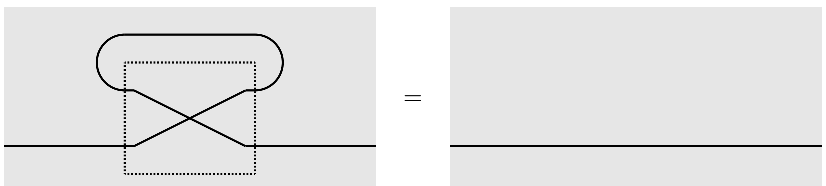
$$\mathsf{Tr}^X_{C\otimes A, C\otimes B}\left(\alpha_{C,A,X} \circ (1_C \otimes f) \circ \alpha^{-1}_{C,A,X}\right) = 1_C \otimes \mathsf{Tr}^X_{A,B}(f)$$



(7)

[Yanking]: *For every object X, the following equality holds:*

$$\mathsf{Tr}^X_{X,X}(\sigma_{X,X}) = 1_X$$



(8)

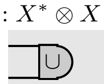
For a map $f : A \otimes X \to B \otimes X$, the map $\mathsf{Tr}^X_{A,B}(f) : A \to B$ is called the ***trace*** *of f.*

2.2 Compact Closed Categories

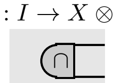
An important class of traced symmetric monoidal categories are *compact closed categories*, which are particularly important in categorical quantum foundations [2]. Compact closed categories are symmetric monoidal category where every object has a dual. Every compact closed category comes equipped with a canonical trace operator that captures the classical notion of (partial) trace for matrices, which is a fundamental operation for both classical quantum theory and categorical quantum foundations [3].

Definition 2.4 *[20, Section 4.8] A* ***compact closed category*** *is a symmetric monoidal category $(\mathbb{X}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ such that for every object X, there is a object X^*, called the dual of X, and maps $\cup_X : X^* \otimes X \to I$, called the cup or evaluation map, and $\cap_X : I \to X \otimes X^*$, called the cap or coevaluation map, which are drawn in the graphical calculus as follows:*

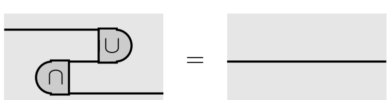
$\cup_X : X^* \otimes X \to I$

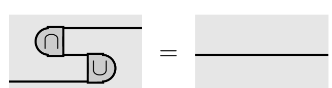


$\cap_X : I \to X \otimes X^*$



such that the following equality holds:





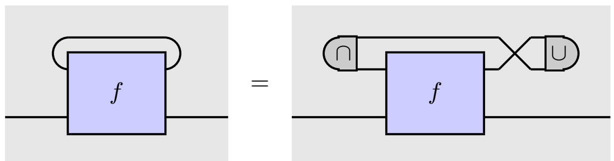
and these equalities are called the snake equations.

Every compact closed category is a traced symmetric monoidal category, where the trace operator is constructed using the cups and caps, and where the trace operator axioms follow from the snake equations. Furthermore, compact closed categories have a *unique* trace operator.

Proposition 2.5 *[16, Proposition 3.1] Let $(\mathbb{X}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ be a compact closed category, with duals $(-)^*$, caps $\cap$, and cups $\cup$. For a map $f : A \otimes X \to B \otimes X$, its trace $\mathsf{Tr}^X_{A,B}(f) : A \to B$ is defined as the following composite: the* *following composite:*

$$\mathsf{Tr}^X_{A,B}(f) := \begin{array}{l} A \xrightarrow{\rho_A^{-1}} A \otimes I \xrightarrow{1_A \otimes \cap_X} A \otimes (X \otimes X^*) \xrightarrow{\alpha_{A,X,X^*}} (A \otimes X) \otimes X^* \xrightarrow{f \otimes 1_{X^*}} \\ (B \otimes X) \otimes X^* \xrightarrow{\alpha^{-1}_{B,X,X^*}} B \otimes (X \otimes X^*) \xrightarrow{1_B \otimes \sigma_{X,X^*}} B \otimes (X^* \otimes X) \xrightarrow{1_B \otimes \cup_X} B \otimes I \xrightarrow{\rho_B} B \end{array}$$

which is drawn the graphical calculus as follows:



Then $(\mathbb{X}, \otimes, I, \alpha, \lambda, \rho, \sigma, \mathsf{Tr})$ *is a traced symmetric monoidal category. Furthermore,* Tr *is the* unique *trace operator on* $(\mathbb{X}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ *[13, Section 3.2].*

Here are now some examples of compact closed categories and their canonical trace operators.

Example 2.6 Let $\mathbb{Z}$ be the set of integers. Let $\mathbb{Z}_{\leq}$ be the standard poset category, that is, the category whose objects are integers $n \in \mathbb{Z}$, where there is a map $n \longrightarrow m$ if and only if $n \leq m$. $\mathbb{Z}_{\leq}$ is a compact closed category where the monoidal product is given by addition, so $n \otimes m = n + m$ and $I = 0$, the dual is given by the negative $n^* = -n$, the cap is the equality $0 = n + (-n)$, and the cup is the equality $(-n) + n = 0$. The induced trace operator records the fact that if $n + x \leq m + x$ then $n \leq m$.

Example 2.7 Let R be a commutative ring. Let $\mathsf{MAT}(R)$ be the category of matrices over R, that is, the category whose objects are natural numbers $n \in \mathbb{N}$ and where a map $A : m \longrightarrow n$ is an $m \times n$ matrix A with coefficients in R. $\mathsf{MAT}(R)$ is a compact closed category where the monoidal product is given by multiplication, so $m \otimes n = mn$ and $I = 1$, the dual of n is itself, so $n^* = n$, the cup $\cup_n$ is the $n^n \times 1$ matrix where the n^i-th coefficient is $1 \in R$ and the rest are $0 \in R$, and where the cap $\cap_n$ is the $1 \times n^n$ matrix is the transpose of the cup. The resulting trace operator captures the standard partial trace of matrices from linear algebra. Indeed, recall that for square $n \times n$ matrix A, its trace is equal to the sum of its diagonal coefficients, $\mathsf{Tr}(A) = \sum\limits_{i=1}^{n} A_{i,i}$. Now observe that a map $A : m \otimes k \longrightarrow n \otimes k$, which is an $mk \times nk$ matrix A, can be expressed in terms of square matrices as follows:

$$A = \begin{bmatrix} A(1,1) & A(1,2) & \dots & A(1,n) \\ A(2,1) & A(2,2) & \dots & A(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ A(m,1) & A(m,2) & \dots & A(m,n) \end{bmatrix}$$

where $A(i,j)$ are square $k \times k$ matrices (with $1 \leq i \leq m$ and $1 \leq j \leq n$). Then the trace of A, $\mathsf{Tr}^k_{m,n}(A) : m \longrightarrow n$, is the $m \times n$ matrix $\mathsf{Tr}^k_{m,n}(A)$ whose coefficients are the traces of the square matrices:

$$\mathsf{Tr}^k_{m,n}(A) = \begin{bmatrix} \mathsf{Tr}\left(A(1,1)\right) & \mathsf{Tr}\left(A(1,2)\right) & \dots & \mathsf{Tr}\left(A(1,n)\right) \\ \mathsf{Tr}\left(A(2,1)\right) & \mathsf{Tr}\left(A(2,2)\right) & \dots & \mathsf{Tr}\left(A(2,n)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{Tr}\left(A(m,1)\right) & \mathsf{Tr}\left(A(m,2)\right) & \dots & \mathsf{Tr}\left(A(m,n)\right) \end{bmatrix}$$

Example 2.8 The above example generalises to finite dimensional modules. Let R be a commutative ring, and let $\mathsf{fdMOD}(R)$ be the category of finite dimensional modules over R and R-linear morphisms between them. $\mathsf{fdMOD}(R)$ is a compact closed category, where the monoidal product is the standard algebraic tensor product of modules, so in particular $I = R$, and where the dual of R-module M is given by its algebraic dual, that is, the R-module of R-linear maps from M to R, $X^* = \{\phi : X \longrightarrow R |$ f is R-linear$\}$. The cup $\cup_X : X^* \otimes X \longrightarrow R$ is defined by evaluation, $\cup_X(\phi \otimes x) = \phi(x)$. To define the cap, let $\{e_1, \dots, e_n\}$ be a basis of X, which induces a basis $\{e^*_1, \dots, e^*_n\}$ be a basis of X^*. Then the cap $\cap_X : R \to X \otimes X^*$ is defined as $\cap_X(1) = \sum_{i=1}^{n} e_i \otimes e_i^*$. For an R-linear map $f : M \otimes X \to N \otimes X$, its trace is the R-linear map $\mathsf{Tr}^X_{M,N}(f) : M \to N$ defined as:

$$\mathsf{Tr}^X_{M,N}(f)(m) = \sum_{i=1}^{n}(1_N \otimes e_i^*)\big(f(m \otimes e_i)\big)$$

Note that the definitions of the cap and trace are independent of the choice of basis. In particular, when $R = \mathbb{C}$, the field of complex number, then $\mathsf{fdMOD}(\mathbb{C})$ is equivalent to the category of finite dimensional Hilbert spaces, which is also compact closed and the fundamental category of study in categorical quantum foundations.

Example 2.9 Let REL be the category of sets and relations, that is, the category whose objects are sets and where a map $R : X \to Y$ is a subset of the Cartesian product of X and Y, $R \subseteq X \times Y$. REL is a compact closed category where the monoidal product is given by the Cartesian product of sets, so $X \otimes Y = X \times Y$ and $I = \{*\}$, the dual of set X is itself, so $X^* = X$, and where the cups and caps are the relations which relate the singleton's element to diagonal pairs:

$$\cap_X = \{\big(*, (x,x)\big) \mid \forall x \in X\} \subset \{*\} \times (X \times X) \qquad \cup_X = \{\big((x,x), *\big) \mid \forall x \in X\} \subset (X \times X) \times \{*\}$$

For a map $R : A \times X \to B \times X$, which is a subset $R \subseteq (A \times X) \times (B \times X)$, its trace $\mathsf{Tr}^X_{A,B}(R) : A \to B$, is the subset $\mathsf{Tr}^X_{A,B}(R) \subseteq A \times B$ defined as follows:

$$\mathsf{Tr}^X_{A,B}(R) = \Big\{(a,b) \mid \exists x \in X \text{ s.t. } \big((a,x),(b,x)\big) \in R\Big\}$$

It is important to note that the Cartesian product $\times$ is not the categorical product and the singleton $\{*\}$ is not the terminal object in REL.

Every traced symmetric monoidal category embeds fully and faithfully into a compact closed category via the INT-construction [16, Section 4]. That said, not every traced symmetric monoidal category is compact closed. Indeed, there are many interesting examples of non-compact closed traced symmetric monoidal categories that are especially important in computer science, some of which we review below. Here is a simple example of a traced symmetric monoidal category that is not compact closed:

Example 2.10 Let $\mathbb{N}$ be the set of natural numbers. Let $\mathbb{N}_\leq$ be the standard poset category, that is, the category whose objects are natural numbers $n \in \mathbb{N}$, where there is a map $n \to m$ if and only if $n \leq m$. $\mathbb{N}_\leq$ is a symmetric monoidal category where the monoidal product is given by addition $n \otimes m = n + m$ and the unit is zero $I = 0$. Furthermore, $\mathbb{N}_\leq$ is also a traced symmetric monoidal category where the trace operator records the fact that if $n + x \leq m + x$ then $n \leq m$. However, $\mathbb{N}_\leq$ is not compact closed, since this would mean that for every n there is a n^* such that $n^* + n \leq 0$ and $0 \leq n + n^*$, which implies that $n + n^* = 0$, but this is not case for all $n \neq 0$. Applying the INT-construction on $\mathbb{N}_\leq$ results in the compact closed category $\mathbb{Z}_\leq$ from Example 2.6.

2.3 Traced Cartesian Monoidal Categories and Fixed Point Operators

Any category with finite products is a symmetric monoidal category, called a Cartesian monoidal category. In a traced Cartesian monoidal category, the trace operator captures the notion of feedback via fixed points. In fact, for Cartesian monoidal categories, trace operators are in bijective correspondence with Conway operators, which are special kinds of fixed point operators (which is a result proved by Hyland and the first author independently [10]).

There are multiple equivalent ways to define a category with finite products. Since we are interested in their induced symmetric monoidal structure, we will define a category with finite products from this perspective, that is, as a symmetric monoidal whose monoidal product is a product and whose monoidal unit is a terminal object. Of course, any category with finite products is a Cartesian monoidal category, where the symmetric monoidal structure is derived using the universal property of the product, and conversely, every Cartesian monoidal category is a category with finite products. Alternatively, a Cartesian monoidal category can be defined as a symmetric monoidal category with natural copy and delete maps satisfying the axioms found in [20, Table 7].

Definition 2.11 *[20, Section 6.1 & Section 6.4] A **Cartesian monoidal category** is a symmetric monoidal category $(\mathbb{X}, \times, \top, \alpha, \lambda, \rho, \sigma)$ whose monoidal structure is a given by **finite products**, that is:*

(i) *The monoidal unit $\top$ is a* ***terminal object****, that is, for every object A there exists a* unique map $t_A : A \to \top$*;*

(ii) *For every pair of objects A and B, $A \times B$ a* ***product*** *of A and B with* ***projection*** *maps $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ defined as following composites:*

$$\pi_0 := \; A \times B \xrightarrow{1_A \times t_B} A \times \top \xrightarrow{\rho_A} A \qquad\qquad \pi_1 := \; A \times B \xrightarrow{t_A \times 1_B} \top \times B \xrightarrow{\lambda_B} B$$

that is, for every pair of maps $f_0 : C \to A$ and $f_1 : C \to B$, there is a unique *map $\langle f_0, f_1 \rangle : C \to A \times B$, called the* ***pairing*** *of f_0 and f_1, such that $\pi_0 \circ \langle f_0, f_1 \rangle = f_0$ and $\pi_1 \circ \langle f_0, f_1 \rangle = f_1$.*

A ***traced Cartesian monoidal category*** *is a traced symmetric monoidal category $(\mathbb{X}, \times, \top, \alpha, \lambda, \rho, \sigma, \mathsf{Tr})$ whose underlying symmetric monoidal category $(\mathbb{X}, \times, \top, \alpha, \lambda, \rho, \sigma)$ is a Cartesian monoidal category.*

Traced Cartesian monoidal categories can equivalently be defined as a Cartesian monoidal category equipped with a Conway operator. Here we provide the Conway operator axiomatization found in [10], but equivalent alternative axiomatizations can be found in [12, 22].

Definition 2.12 *[10, Theorem 3.1] A* ***Conway operator*** *on a Cartesian monoidal category $(\mathbb{X}, \times, \top, \alpha, \lambda, \rho, \sigma)$ is a family of operators $\mathsf{Fix}^X_A : \mathbb{X}(A \times X, X) \to \mathbb{X}(A, X)$ such that:*

[Parametrized Fixed Point] *For every map $f : A \times X \to X$ the following equality holds:*

$$\mathsf{Fix}^X_A(f) = f \circ \left\langle 1_A, \mathsf{Fix}^X_A(f) \right\rangle \tag{9}$$

[Naturality] *For every map $f : A \times X \to X$ and every map $g : A' \to A$ the following equality holds:*

$$\mathsf{Fix}^X_A \left(f \circ (g \times 1_X) \right) = \mathsf{Fix}^X_{A'}(f) \circ g \tag{10}$$

and for every map $f : A \times X \to X'$ and every map $k : X' \to X$ the following equality holds:

$$\mathsf{Fix}^X_A(k \circ f) = k \circ \mathsf{Fix}^{X'}_A \left(f \circ (1_A \times k) \right) \tag{11}$$

[Bekič Lemma] *For every map $f : A \times (X \times Y) \to X$ and $g : A \times (X \times Y) \to Y$ the following equality holds:*

$$\mathsf{Fix}^{X \times Y}_A(\langle f, g \rangle) = \langle \pi_1, \mathsf{Fix}^Y_{A \times X}(g \circ \alpha^{-1}_{A,X,Y}) \rangle \circ \left\langle 1_A, \mathsf{Fix}^X_A \left(f \circ \alpha^{-1}_{A,X,Y} \circ \langle 1_{A \times X}, \mathsf{Fix}^Y_{A \times X}(g \circ \alpha^{-1}_{A,X,Y}) \rangle \right) \right\rangle \tag{12}$$

For a map $f : A \times X \to X$, $\mathsf{Fix}^X_A(f) : A \to X$ is called the ***parametrized fixed point*** *of f.*

Proposition 2.13 *[10, Theorem 3.1] Let $(\mathbb{X}, \times, \top, \alpha, \lambda, \rho, \sigma)$ be a Cartesian monoidal category:*

(i) *Let Fix be a Conway operator on $(\mathbb{X}, \times, \top, \alpha, \lambda, \rho, \sigma)$. Then for a map $f : A \times X \to B \times X$, its trace $\mathsf{Tr}^X_{A,B}(f) : A \to B$ is defined as follows:*

$$\mathsf{Tr}^X_{A,B}(f) := \; A \xrightarrow{\langle 1_A, \mathsf{Fix}^X_A(\pi_1 \circ f) \rangle} A \times X \xrightarrow{\quad f \quad} B \times X \xrightarrow{\quad \pi_0 \quad} B$$

Then $(\mathbb{X}, \times, \top, \alpha, \lambda, \rho, \sigma, \mathsf{Tr})$ is a traced Cartesian monoidal category.

(ii) *Let $(\mathbb{X}, \times, \top, \alpha, \lambda, \rho, \sigma, \mathsf{Tr})$ be a Cartesian monoidal category. For a map $f : A \times X \to X$, its parametrized fixed point $\mathsf{Fix}^X_A(f) : A \to X$ is defined as follows:*

$$\mathsf{Fix}^X_A(f) := \mathsf{Tr}^X_{A,X}(\langle f, f \rangle)$$

Then Fix is a Conway operator on $(\mathbb{X}, \times, \top, \alpha, \lambda, \rho, \sigma)$.

Furthermore, these constructions are inverses of each other.

Here is now an example of a traced Cartesian monoidal category, which is particularly important in domain theory:

Example 2.14 Let ω-CPPO be the category whose objects are ω-complete partial orders with bottom element (ω-cppo) and whose maps are (Scott) continous functions between them. ω-CPPO is a traced Cartesian (closed) monoidal category where the monoidal product is given by the Cartesian product, $X \otimes Y = X \times Y$ and $I = \{\perp\}$, and where the Conway operator and trace operator are induced by the parametrized Tarski least fixed-point operator. Indeed, for any continous function $f : X \to X$, there exists a least fixed point for f, that is, there exists an $\mathsf{fix}(f) \in X$ such that $f\left(\mathsf{fix}(f)\right) = \mathsf{fix}(f)$ and for every $x \in X$ such that $f(x) = x$, we have that $\mathsf{fix}(f) \leq x$. So for a continous function $g : A \times X \to X$, its parametrized fixed point $\mathsf{Fix}^X_A(g) : A \to X$ is defined as the least fixed point of the curry of g, that is, $\mathsf{Fix}^X_A(g)(a) = \mathsf{fix}\left(g(a, -)\right)$. Then for a continous function $h : A \times X \to B \times X$, where $h(a,x) = \left(h_0(a,x)_B, h_1(a,x)\right)$ for continous functions $h_0 : A \times X \to B$ and $h_1 : A \times X \to X$, its trace $\mathsf{Tr}^X_{A,B}(h) : A \to B$ is defined as follows:

$$\mathsf{Tr}^X_{A,B}(h)(a) = h_0\left(a, \mathsf{fix}\left(h_1(a,-)\right)\right)$$

A Cartesian monoidal category is compact closed if and only if it degenerate, that is, every object is isomorphic to the terminal object. Therefore, non-degenerate traced Cartesian monoidal categories (such as the above example) are examples of traced symmetric monoidal categories that are not compact closed.

2.4 Traced CoCartesian Monoidal Categories and Iterations

Any category with finite coproducts is a symmetric monoidal category, called a coCartesian monoidal category. Traced coCartesian monoidal categories are the dual notion of traced Cartesian monoidal categories, in the sense that the opposite category of one is a form of the other. In a traced coCartesian monoidal category, the trace operator captures the notion of feedback via iteration. In fact, for coCartesian monoidal categories, trace operators are in bijective correspondence with iteration operators, which are the dual notion of Conway operators.

As for the product case, we are interested in the symmetric monoidal structure induced by finite coproducts. So we will define coCartesian monoidal categories as symmetric monoidal categories whose monoidal product is a coproduct and whose monoidal unit is an initial object. Once again, any category with finite coproducts is a coCartesian monoidal category where the symmetric monoidal structure can be fully derived from the couniversal property of the coproduct, and conversely, every coCartesian monoidal category is a category with finite coproducts.

Definition 2.15 *[20, Section 6.2 & Section 6.4] A* ***coCartesian monoidal category*** *is a symmetric monoidal category* $(\mathbb{X}, \oplus, \perp, \alpha, \lambda, \rho, \sigma)$ *whose monoidal structure is a given by* ***finite coproducts****, that is:*

(i) *The monoidal unit* $\perp$ *is an* ***initial object****, that is, for every object* A *there exists a unique map* $i_A : \perp \to A$.

(ii) *For every pair of objects* A *and* B, $A \oplus B$ *a* ***coproduct*** *of* A *and* B *with* ***injection*** *maps* $\iota_0 : A \to A \oplus B$ *and* $\iota_1 : B \to A \oplus B$ *defined as following composites:*

$$\iota_0 := \; A \xrightarrow{\rho_A^{-1}} A \oplus \perp \xrightarrow{1_A \times i_B} A \oplus B \qquad\qquad \iota_1 := \; B \xrightarrow{\lambda_B^{-1}} \perp \oplus A \xrightarrow{i_A \times 1_B} A \oplus B$$

that is, for every pair of maps $f_0 : A \to C$ *and* $f_1 : B \to C$*, there is a* unique *map* $[f_0, f_1] : A \oplus B \to C$*, called the* ***copairing*** *of* f_0 *and* f_1*, such that* $[f_0, f_1] \circ \iota_0 = f_0$ *and* $[f_0, f_1] \circ \iota_1 = f_1$.

A ***traced coCartesian monoidal category*** *is a traced symmetric monoidal category* $(\mathbb{X}, \oplus, \perp, \alpha, \lambda, \rho, \sigma, \mathsf{Tr})$ *whose underlying symmetric monoidal category* $(\mathbb{X}, \oplus, \perp, \alpha, \lambda, \rho, \sigma)$ *is a coCartesian monoidal category.*

As mentioned above, a traced coCartesian monoidal category can equivalently be characterised as a coCartesian monoidal category $\mathbb{X}$ equipped with a iteration operator, which is a family of operators $\mathsf{Iter}^X_A : \mathbb{X}(X, A \oplus X) \to \mathbb{X}(X, A)$, such that the dual axioms of a Conway operator hold. Since iteration operators do not play a crucial role in this paper, we will not review the full definition here and invite curious readers to see [20, Section 6.4]. That said, here are now some examples of coCartesian traced monoidal categories which show how the trace is given by iteration.

Example 2.16 Let PAR be the category of sets and partial functions. PAR is a traced coCartesian monoidal category where the monoidal product is given by the disjoint union of sets, so $A \oplus B = X \sqcup Y$ and $\perp = \emptyset$, and where the trace operator is induced from the natural feedback operator. Indeed, given a partial function $f : A \sqcup X \to B \sqcup X$, its trace $\mathsf{Tr}^X_{A,B}(f) : A \to B$ is the partial function defined as follows:

$$\mathsf{Tr}^X_{A,B}(f)(a) = \begin{cases} f(a) & \text{if } f(a) \in B \\ b & \text{if } \exists\, n \in \mathbb{N}\ \exists\, x_0, x_1, \ldots, x_n \in X \text{ s.t. } f(a) = x_0 \text{ and } f(x_0) = x_1 \text{ and } \ldots \text{ and } f(x_n) = b \\ \text{undefined} & \text{o.w.} \end{cases}$$

Example 2.17 Let REL be the category of sets and relations as defined in Example 2.9. REL is a traced coCartesian monoidal category where this time (unlike Example 2.9) the monoidal product is given by the disjoint union of sets, so $X \otimes Y = X \sqcup Y$ and $\perp = \emptyset$, and where for a map $R : A \sqcup X \to B \sqcup X$, which is a subset $R \subseteq (A \sqcup X) \times (B \sqcup X)$, its trace $\mathsf{Tr}^X_{A,B}(R) : A \to B$, is the subset $\mathsf{Tr}^X_{A,B}(R) \subseteq A \times B$ defined as follows:

$$\mathsf{Tr}^X_{A,B}(R) = \big\{(a,b)|\ (a,b) \in R \text{ or } \exists\ n \in \mathbb{N}\ \exists\ x_0, x_1, \ldots, x_n \in X \text{ s.t. } (a, x_0), (x_0, x_1), \ldots, (x_n, b) \in R\big\} \subseteq A \times B$$

In this example, note that the monoidal product $\sqcup$ is also a (bi)product, so REL is also a traced Cartesian monoidal category in this way.

Example 2.18 The above two examples both fall in the same class of traced coCartesian monoidal categories called partially additive categories [9, Definition 2.16]. Briefly, a partially additive category is a category with countable coproducts (and therefore a symmetric monoidal category) with suitable partially defined countable (possibly infinite) sums. Every partially additive category is a traced symmetric monoidal category where the trace operator is defined via a well-defined infinite sum formula [9, Proposition 2.20]. In a partially additive category, a map $f : A \otimes X \to B \otimes X$ is uniquely decomposed into four map $f_1 : A \to B$, $f_2 : A \to X$, $f_3 : X \to B$, and $f_4 : X \to X$, and so its trace $\mathsf{Tr}^X_{A,B}(f) : A \to B$ is defined as follows:

$$\mathsf{Tr}^X_{A,B}(f) = f_1 + \sum_{n=0}^{\infty} f_3 \circ \underbrace{f_4 \circ \ldots \circ f_4}_{n\text{-times}} \circ f_2$$

Both PAR and REL are partially additive categories [9, Example 2.3], and their induced traces are precisely those described in the examples above. Slightly more generally, partially additive categories are examples of unique decomposition categories [9, Definition 2.4], which under mild assumptions are also traced symmetric monoidal categories [9, Proposition 2.21].

A coCartesian monoidal category is compact closed if and only if it degenerate, that is, every object is isomorphic to the initial object. Therefore, non-degenerate traced Cartesian monoidal categories (such as the above examples) are examples of traced symmetric monoidal categories that are not compact closed.

3 Monads

In this section we review monads and their algebras, which form a category called the Eilenberg-Moore category of the monad. As discussed, in this paper we are interested in lifting structure from the base category up to the Eilenberg-Moore category, specifically lifting traced symmetric monoidal structure, which we discuss in the sections below. We also discuss the main examples of monads we will consider thoroughout this paper. These include monads induced by monoids, whose algebras are the modules over the monoids, and idempotent monads, whose algebras form a subcategory of the base category. We refer the reader to [6, Chapter 4] for a detailed introduction to monads.

3.1 Monads and their Algebras

Monads are endofunctors with some extra structure that satisfy associativity and unit axioms.

Definition 3.1 *[6, Definition 4.1.1] A **monad** on a category $\mathbb{X}$ is a triple (T, μ, η) consisting of a functor $T : \mathbb{X} \to \mathbb{X}$, a natural transformation $\mu_A : TT(A) \to T(A)$, called the monad multiplication, and a natural transformation $\eta_A : A \to T(A)$, called the unit multiplication, such that the following equalities hold:*

$$\mu_A \circ T(\eta_A) = 1_{T(A)} = \mu_A \circ \eta_{T(A)} \qquad\qquad \mu_A \circ T(\mu_A) = \mu_A \circ \mu_{T(A)} \tag{13}$$

We now review algebras of a monad, which were called modules of a monad in [7, 8].

Definition 3.2 *[6, Definition 4.1.2] Let (T, μ, η) be a monad on a category $\mathbb{X}$. A T**-algebra** is a pair (A, a) consisting of an object A and a map $a : T(A) \to A$ of $\mathbb{X}$, called the T-algebra structure, such that the following equalities hold:*

$$a \circ \eta_A = 1_A \qquad\qquad a \circ \mu_A = a \circ T(a) \tag{14}$$

Among the algebras of the monad are the free algebras, where the algebra structure is the monad multiplication.