An algebraisation of synthesis problems in switching circuits

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The composition of switching circuits A, B, to new circuits can be reduced to two basic operations:

1. The sequential composition $A \circ B$ of A and B, if the number of outputs of B is equal to the number of inputs of A.
2. The composition of A and B to $A \times B$ analogue to the cartesian product. The number of inputs of $A \times B$ is equal to the sum of the numbers of the inputs of A and B. The same holds for the outputs.

Between different compositions an equivalence is introduced which is compatible with "$\circ$" and "$\times$". When the compositions are carried over to the set of the equivalence classes, one gets an algebraic structure, that forms a category respective to "$\circ$" and a semi-group (monoid) respective to "$\times$". The compositions satisfy the relation

$$(A \times B) \circ (C \times D) = (A \circ C) \times (B \circ D)$$

if $A \circ C$ and $B \circ D$ are defined. A category with this property is called an X-category.

The X-categories $\mathfrak{F}$ studied here can be characterised in the following way:

1. The set of units of $\mathfrak{F}$ form a semi-group generated by one generator relative to "$\times$",
2. $\mathfrak{F}$ has a countable generator set or it holds
3. $\mathfrak{F}$ is a subcategory of a category satisfying 1 and 2.

In a free X-category of this type holds the cancellation law relative to "$\circ$" and "$\times$" under a weak condition.

In the whole theory the X-category of "plane nets", a concept of the combinatorial topology related to the braids of Artin plays an important role. Each of the studied free X-categories $\mathfrak{F}$ may be mapped on to an X-category of plane nets by a functor. Each theorem about $\mathfrak{F}$ is carried over by the functor to a theorem about a category of nets. The theorem proved for the nets can be easily proved in most cases for $\mathfrak{F}$; i.e. the theorems about $\mathfrak{F}$ have an essential combinatorial character. This is the reason that the first chapter only deals about nets.

The connection of the switching circuits represented by the elements of $\mathfrak{F}$ with their function gives a functor from $\mathfrak{F}$ into the X-category $\mathfrak{C}_S$ of the maps of the type $f : S^n \to S^m$, where S is a countable set containing at least two elements. The "$\times$"-product is in $\mathfrak{C}_S$ the cartesian product.

0 Introduction

The motivation for this work is a problem of automata theory: From a given system of building blocks an automaton is to be assembled, whose function is specified. From the various possible solutions the cheapest is sought.

A building block $A \in \mathfrak{A}$ is a physical, generally electrical device with inputs $Q(A)$ and outputs $Z(A)$. For each input a particular set S of input signals is permitted, to which the building block reacts with output signals. We assume, to simplify the technical situation, that the following holds:

1. For each input of elements of $\mathfrak{A}$ the same signal set S is provided, and each element of S^n is allowed as an input signal for A with $n = Q(A)$.

2. The sets of output signals of $A \in \mathfrak{A}$ lie in S^m with $m = Z(A)$.

3. Given the input signal $s \in S^n$ at time t, then the output signal at time t caused by s is unambiguously specified. (Thus we ignore the finite speed of propagation of signals.)

The finite automaton is thus for us completely described by its function $\varphi(A) : S^n \to S^m$. In doing so it is assumed that the inputs and outputs of A are provided with a fixed enumeration with the numbers 1 to $Q(A)$ resp. 1 to $Z(A)$, and the i-th input resp. i-th output is associated to the i-th component of S^n resp. S^m.

An element of $\mathfrak{A}$ is a switching circuit. If A and B are switching circuits with $Q(A)$ resp. $Z(B)$ inputs and $Z(A)$ resp. $Z(B)$ outputs, then we build new switching circuits out of A and B, by combining A and B into an element $A \times B$ with $Q(A) + Q(B)$ inputs and $Z(A) + Z(B)$ outputs, and defining the i-th input of A to be the i-th input of $A \times B$ and the i-th input of B to be the $(Q(A) + i)$-th input of $A \times B$ (fig. 1).

If $Z(A) = Q(B)$, then we obtain from A and B a switching circuit $B \circ A$, by connecting the i-th output of A onto the i-th input of B.

A switching circuit built from elements of $\mathfrak{A}$ is a device that will be described inductively using the previous definitions.

If $\varphi(A)$ resp. $\varphi(B)$ is the function of the switching circuit A resp. B, then $\varphi(A) \times \varphi(B)$ resp. $\varphi(B) \circ \varphi(A)$ for $Q(B) = Z(A)$ is the function of $A \times B$ resp. $B \circ A$.

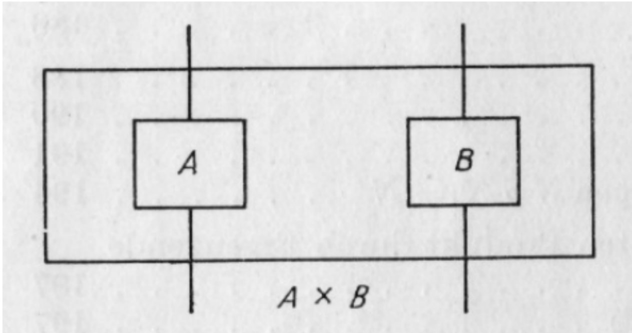

Fig. 1

The costs of the building blocks of $\mathfrak{A}$ are given by the function $L : \mathfrak{A} \to \mathsf{N} \cup \{0\}$; we define:

$$L(A \times B) = L(A) + L(B)$$
$$L(B \circ A) = L(A) + L(B)$$

In this way every switching circuit will be assigned a “price”.

The task now sounds like: One finds for a given $f : S^n \to S^m$ a switching circuit A with $\varphi(A) = f$ and

$$L(A) = \min_{B \in \varphi^{-1}(f)} \{L(B)\}.$$

If f is not defined on the whole of S^n, but only on $R \subset S^n$, then it is the optimum on $\bigcup\limits_{g|R=f} \varphi^{-1}(g)$ that is sought.

If $Q(f) = Z(f) = S^n$