Exploration into Sets

Julius Hamilton

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1 My own philosophical understanding of where set theory comes from

I think that set theory is fundamentally a metaphysical theory. I think it can be seen as coming from mereology, the theory of "what things are made of". Type theory similarly reminds me of philosophical concepts like "monism", "dualism", "holism", etc., regarding "what fundamental kinds of things there are". I think Aristotle and Leibniz are two examples of philosophers who engaged in those questions.

I do not know if set theory came from those kinds of speculations, but I have often thought those kinds of speculations are a good way to motivate set theory on your own. I think that when we axiomatize mathematics, we are trying to reduce the amount of "unexplained" or "unjustified" stuff. I think that means that we want to show how *most* of "mathematics" (including definitions of concepts and proofs of statements) follows necessarily from as few "starting pieces" as possible.

I think that if you spend some time thinking about "what everything in the world is made of", you come across certain concepts which are very abstract and general, but which other people would come across as well. For example, the word *thing* seems very 'general' to me, but if you have ever explored designing an ontology a little bit, it seems like a category that you end up needing.

I think we are quite used to set theory as a developed mathematical theory with its own distinct characteristics, including its notation, terminology, basic operations, and theorems, that you might want to ask, "Why should sets be the fundamental structuring unit of all of mathematics? Why not something else?"

Maybe there are other options, but I think one valid angle on this question is that sets are a common result of that attempt I described above, to keep abstracting and generalizing particular things like colors, names, numbers, information, etc., into some of the most general concepts possible that we can use to build up all the other concepts.

I think that naive set theory, before it became more developed by people like Cantor, Frege, and Russell, might conceive of the "set" as one of those "simplest concepts possible" - for example, maybe "a set of things" just means "some multiplicity of things". Even in as simple a statement as that, I think we can still recognize that there are conceptual building blocks already being implied - it implies there is such a thing as a "thing", and that we can recognize "multiplicity" or that when there is more than one thing, and also that we can group multiple things and call that grouping itself a thing, and so on. I hope the reader can intuit what I would like to get at here - that even though modern set theory has more technical and theoretical details, I think we can at least try to motivate set theory from very elementary first principles, in that we basically just chose some of the simplest ontological categories we could think of, like "there exists something", and show how the rest of the mathematical findings of set theory (i.e., I think interesting early proofs in set theory like Russell's paradox or Cantor's diagonal argument) follow from there.

2 Some more technical details of current set theory

One idea about set theory that has interested me for a long time I think is moderately touched on in this blog post by Mike Shulman and Astra Kolomatskaia, `https://golem.ph.utexas.edu/category/2024/03/semisimplicial_types_part_i_mo.html`:

> There are different ways to describe the relationship between type theory and set theory, but one analogy views set theory as like machine code on a specific CPU architecture, and type theory as like a high level programming language. From this perspective, set theory has its place as a foundation because almost any structure that one thinks about can be encoded through a series of representation choices. However, since the underlying reasoning of set theory is untyped, it can violate the principle of equivalence...
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> Within the programming language analogy, one can fully define a high level programming language and its operational semantics without specifying any particular compiler or any concept of a CPU architecture. Similarly, type theory allows one to reason with concepts defined in a purely operational, as opposed to representational, manner. The goal of type theory is to create expressive and semantically general languages for reasoning about mathematics.

Let me see if I can give a sketch of some of my intuition regarding this. I think one of the 'upsides' of set theory is that everything mathematical that exists is represented as a particular kind of set. Therefore, I think it can be seen as a success, in trying to define "as much as possible in terms of as little as possible".

However, I think the idea that all mathematical concepts are sets can sometimes not accord with one's basic intuition - that the unit circle is a set, a function representing a parabola is a set, a relation is a set, I think the addition operation is a set, and so on. Conceptually, I think that sets are static objects. I think that humans might think of functions, for example, as being an inherently 'dynamic' kind of thing, which can *do* something - take an input, and return an output, or change the input into a new output.

I think the CPU analogy above is a good way to suggest that it can feel like set theory is a low-level code which might structurally map to “a world of things”, but it can feel conceptually disconnected from those things, like it is not actually the most “primary” or “inherent” way of conveying them. I think it feels like the patterns and structures that emerge in set theory are like an “imprint” of some more holistic, inherently existing thing. Excuse me for not being very mathematical here, but I think I can make this more mathematical in time.

Another not-very-mathematical intuition I have had for a long time, relates to how “order”, “structure” or “information” emerges, I think due to axioms that tell us how to construct new sets from sets we already have constructed. Imagine that you have some such axioms. Imagine that there are “rounds” or “generations” that allow you to choose some of the “currently existing or constructed sets”, and apply one of those axioms to construct a new set. I think depending on the axioms, you could go on infinitely constructing new sets, and the sets would becoming very large and complicated.

I think it is interesting to think about at what “generation” certain well-known mathematical concepts first “get constructed”. For example, I think the natural numbers could be defined by a mathematical function or formula, and I think that formula can be encoded in set theory (where I think functions can be represented as particular sets of ordered pairs). I do not know if this idea is valid, but as an example, consider if at some point in constructing new generations of sets, we end up constructing the formula which defines or constructs (the set-theoretic representation of) natural numbers. I think if our axioms allow us to use the sets we construct in further constructions in some way, you can imagine that once the natural numbers get constructed, it enables lots of useful further constructions, which depend on the natural numbers to be defined.

The general intuition I have regarding this is that as you construct more generations of sets, the “information” in the collection of sets increases, because you might construct a set-theoretic representation of a particular definition, like the definition of a group, or a set-theoretic representation of a particular function. I think before the definition of “group” emerged, there was no valid mathematical way within our collection of constructed sets to express that any valid set-theoretic representation of a particular group “is a group”. So, I think when “higher-level” concepts “ get constructed, they relate” to “lower-level” sets that were constructed before them. For example, I think before the natural numbers have been constructed, it is hard to express mathematically what the empty set *is*, other than just “the empty set”. But I think that once we have constructed a set-theoretic representation of the natural numbers, the empty set now comes to *mean* “the number zero”. But I think that this is not the only thing that the empty set can “represent”. I think that there can be some other set-theoretic construction which has a different way of interacting with the empty set. So, I think that as we construct more sets, those sets act like an *interpretation* of other sets. For this reason, I think that set theory might be said to exhibit *polysemy*, in that the more sets we construct, the more ways we have to express a relationship between other sets. I hope this intuitive idea is understandable from this sketch, and I know that it might not be correct.

This is just some philosophical background that I want to explore more; but from here on, I am going to list the axioms of ZFC and define rigorously some set-theoretic constructions that I want to use to explore questions about isomorphic encodings of concepts, and related questions.

3 Building up mathematics in ZFC

3.1 the Kuratowski definition

3.2 Products

3.3 Power sets

3.4 Exponential objects in category theory

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