The lattice of signed measures

Notation If ν is a positive measure and f is ~~measurable~~ ν-integrable, denote by ν_f the signed measure satisfying

$$\int g \, d\nu_f = \int g f \, d\nu$$

for all ν-integrable g.

We need ~~an~~ a simple measure-theoretic fact:

Prop. $\nu_f \geq 0$ if and only if $f \geq 0$ ν-a.e.

Pf The reverse direction ($\Leftarrow$) is obvious, so let us prove the forward direction ($\Rightarrow$), or rather its contrapositive. Suppose $f < 0$ on a non-null set. Then

$$0 < \nu\{f<0\} = \nu\Big(\bigcup_{n\in\mathbb{N}} \{f < -\tfrac{1}{n}\}\Big)$$

$$\leq \sum_{n\in\mathbb{N}} \nu\{f < -\tfrac{1}{n}\}$$

Datum Date:

so there exists $\varepsilon := 1/n > 0$ such that
$$\nu\{f < -\varepsilon\} > 0.$$
Conclude that
$$\begin{aligned}\nu_f\{f<-\varepsilon\} &= \int_{f<-\varepsilon} f\, d\nu \\ &\le -\varepsilon \int_{f<-\varepsilon} d\nu \\ &< 0.\end{aligned}$$
□

<u>Cor</u> For $f, g \in L^1(\nu) = L^1(\nu; \mathbb{R})$,
$$\nu_f \le \nu_g \quad \text{iff} \quad f \le g \ \ \nu\text{-a.e.}$$
<u>Pf.</u> Apply the proposition to $\nu_{f-g} = \nu_f - \nu_g$. □

Let $\mathcal{M}(\nu)$ denote the space of signed measures absolutely continuous w.r.t. ν. By the Radon-Nikodym theorem, the map
$$L^1(\nu) \to \mathcal{M}(\nu), \quad f \mapsto \nu_f$$
is a vector space isomorphism. If we recall that the <u>total variation</u> of a signed measure $v \ll \nu$ is defined (independently of ν, it turns out) by $|v| = \nu_{|f|}$ for $f \in L^1(\nu)$ satisfying $v = \nu_f$, then we can define the <u>norm</u> of v to be
$$\|v\| := |v|(\mathcal{X}) = \int |f|\, d\nu = \|f\|_{L^1(\nu)}.$$
Thus the map $f \mapsto \nu_f$ exhibits an isomorphism of $L^1(\nu)$ and $\mathcal{M}(\nu)$ as normed vector spaces and even as Banach spaces:
$$L^1(\nu) \cong \mathcal{M}(\nu) \quad \text{in } \underline{\mathsf{Ban}}.$$
All this is standard textbook material in real analysis.

What is equally obvious, but less commonly remarked on, Datum / Date:

is that (as the corollary shows), the map $f \mapsto \mu_f$ makes $L^1(\mu)$ and $\mathcal{M}(\mu)$ isomorphic as posets:

$$\underset{\substack{\uparrow \\ \text{pointwise order } \mu\text{-a.e.}}}{L^1(\mu)} \cong \underset{\substack{\nwarrow \\ \text{setwise order}}}{\mathcal{M}(\mu)} \quad \text{in } \underline{\mathrm{Pos}}.$$

<u>Cor.</u> $\mathcal{M}(\mu)$ is a lattice, with meets and joins given by

$$\mu_f \wedge \mu_g = \mu_{f \wedge g}$$
$$\mu_f \vee \mu_g = \mu_{f \vee g}$$

<u>Pf.</u> $L^1(\mu)$ has meets and joins, given by

$$f \wedge g = \min(f, g)$$
$$f \vee g = \max(f, g) \quad \text{pointwise}.$$

$\square$

In fact, the lattice structure extends to the space of all signed measures. Let $\mathcal{M}(X)$ denote the space of signed measures on a measurable space X, so that for any positive measure μ on X,

$$\mathcal{M}(\mu) \subseteq \mathcal{M}(X).$$

<u>Thm</u> $\mathcal{M}(X)$ is a lattice, with meets and joins given by

$$\nu \wedge \rho = \mu_{\frac{d\nu}{d\mu} \wedge \frac{d\rho}{d\mu}}, \qquad \nu \vee \rho = \mu_{\frac{d\nu}{d\mu} \vee \frac{d\rho}{d\mu}},$$

where μ is any positive measure dominating ν and ρ.

<u>Pf.</u> First, note that such a measure μ always exists; simply take $\mu = |\nu| + |\rho|$. We must also show that the RHSs are well-defined. Suppose we have two positive measures μ_1, μ_2 with $\nu, \rho \ll \mu_1, \mu_2$. Put $\mu = \mu_1 + \mu_2$, and calculate

$$\left(\frac{d\nu}{d\mu_i} \wedge \frac{d\rho}{d\mu_i}\right) d\mu_i \overset{\text{chain rule}}{=} \left(\frac{d\nu}{d\mu_i} \wedge \frac{d\rho}{d\mu_i}\right) \frac{d\mu_i}{d\mu}\, d\mu$$
$$\overset{\text{positivity}}{=} \left(\frac{d\nu}{d\mu_i}\frac{d\mu_i}{d\mu} \wedge \frac{d\rho}{d\mu_i}\frac{d\mu_i}{d\mu}\right) d\mu$$
$$\overset{\text{chain rule}}{=} \left(\frac{d\nu}{d\mu} \wedge \frac{d\rho}{d\mu}\right) d\mu.$$

Datum Date:

This proves that the quantity defining meets is well-defined, and similarly for joins.
It remains to prove that the quantity actually <u>is</u> a meet.
Let $\sigma \in \mathcal{M}(\mathcal{X})$ be a lower bound for ν, ρ: $\sigma \leq \nu$ and $\sigma \leq \rho$.
Put $\mu = |\nu| + |\rho| + |\sigma|$, so that $\nu, \rho, \sigma \in \mathcal{M}(\mu)$. By what we have proved earlier, $\nu \wedge \rho$ as (well-)defined above is a meet (greatest lower bound) in $\mathcal{M}(\mu)$, hence $\sigma \leq \nu \wedge \rho$.
Since $\sigma \in \mathcal{M}(\mathcal{X})$ is arbitrary, $\nu \wedge \rho$ is also a meet in $\mathcal{M}(\mathcal{X})$.
The argument for joins is dual. $\square$

We will use the lattice structures of signed measures, $\mathcal{M}(\mathcal{X})$, and of positive measures, $\mathcal{M}^+(\mathcal{X})$, to define a composition of subcoupling measures.

<u>Remark</u> One might naively try to define the meet of measures ~~[illegible]~~ ν, ρ by

$$(\nu \wedge \rho)(A) \overset{?}{:=} \nu(A) \wedge \rho(A).$$

However, the resulting "measure" need not even be additive, hence need not be a measure at all.