String diagrams for symmetric powers

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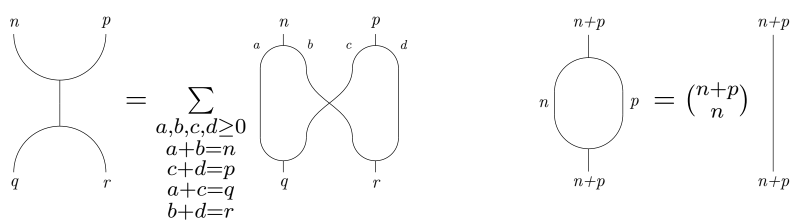
Symmetric powers are an important notion in mathematics, especially in algebra, and computer science, especially for linear logic. In algebra, symmetric powers are used to construct the free symmetric algebra, while in linear logic, they are used to construct free exponential modalities. In physics, symmetric algebras are better known as Fock spaces and have been studied as a model of linear logic [1, 4]. In categorical quantum mechanics, the **Path** calculus [2] gives a string diagrammatic language for Fock spaces. However, nothing ensures that a model is necessarily a symmetric algebra. In the course of our work on graded differential categories [3], we discovered that symmetric powers can be described in a string diagrammatic language that actually characterizes them. We introduce the notion of a graded binomial bialgebra and show that they provide an equivalent characterization of symmetric powers obtained by idempotent splittings.

Definition 1. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal category. A **graded binomial bialgebra** is given by a family*

$$\left(A_n \otimes A_p \underset{\Delta_{n,p}}{\overset{\nabla_{n,p}}{\rightleftarrows}} A_{n+p} \right)_{n,p\geq 0}$$

such that:



Definition 2. *In a symmetric monoidal $\mathbb{Q}^+$-linear category $\mathcal{C}^{\otimes}$, an object A has split symmetric powers if for every n the idempotent*

$$\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sigma : A^{\otimes n} \to A^{\otimes n}$$

split, that is there are maps

$$\left(A^{\otimes n} \underset{s_n}{\overset{r_n}{\rightleftarrows}} A_n \right)_{n\geq 0}$$

such that:

$$r_n; s_n = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sigma \qquad\qquad s_n; r_n = Id$$

Theorem 3. *There is a bijective correspondence between graded binomial bialgebras and objects with split symmetric powers.*

In this talk, we will discuss this theorem, its proof and potential extensions.

References

[1] Blute, R., Panangaden, P. and Seely, R., Fock Space: A Model of Linear Exponential Types (1997).

[2] De Felice, G. and Coecke, B. Quantum Linear Optics via String Diagrams, *Preprint*, *arXiv*, 2204.12985 (2022).

[3] Lemay J.-S. P. and Vienney J.-B. Graded Differential Categories and Graded Differential Linear Logic. *Preprint*, *arXiv*, 2303.10586 (2023).

[4] Melliès, P.-A., Tabareau, N. and Tasson, C. An explicit formula for the free exponential modality of linear logic, *Mathematical Structures in Computer Science* 28 (2017), 1–34.