Algebraic Exponential Dynamics

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Abstract

Tetration is the process of recurrsive exponentiation of complex numbers, as exponentiation is generated from recurrsive multiplication. ~~The tetration of~~ tetrating a complex number by a second complex numbers is defined. This is for tetrationial analog of the Euler identity for exponentiation. The construction of a Lie algebra for any mapping containing a limit point is given. The map $f(z) = a^z$ demonstrates chaotic properties: containing limit points, limit cycles, and strange attractors; depending on the value of a.

1 Introduction

Tetration [1] is the fourth operation in the series: addition, multiplication, exponentiation, tetration, and pentation. The word tetration is used interchangeable with the exponential map. In the same manner that two complex numbers may be added together; two complex numbers may be "tetrated" together. One of the main currents of mathematical history is the progressive exploration and abstraction of the processes of addition, multiplication, and exponentiation. The addition of tetration as a well defined process may provide fertile ground for abstracting new mathematical concepts; particularly in the area of chaos. As the first chaotic Ackerman General function, tetration may be able to provide unique insights into chaos. An example would be using the tetrationial analog of the Fourier transform for examining the chaotic structure of different processes. Many new questions can be posed. What are the properties of tetrationial spinors? Do tetrationial physical processes exist? See [2] for a complimentary exposition of the Julia sets of exponential dynamics.

The examples of the notational form [1] for tetrationial expressions are in Figure 1. The author independently developed the same notation. Exponential expressions are evaluated from the upper right towards the lower left. This is due to the fact that exponentiation is not commutative. The equation $(a^b)^c$ simplifies to a^{bc}.

$$^1a = a$$

$$^2a = a^a$$

$$^3a = a^{a^a}$$

Figure 1: Tetrationial Notation

It should be understood that all equations in this paper implicitly take into account the orientation on the Riemann surface. An example is the preceding identity $(a^b)^c = a^{bc}$. Without considering the orientation on the Riemann surface, problems can arise as follows:

$$i = (-1)^{1/2} = (-1)^{2(1/4)} = (-1)^{2^{1/4}} = 1^{1/4} = 1.$$

This is obviously not true. Then problem arises because $(-1)^2$ is equivalent to a unary 360 degree rotation, while 1 is not rotated at all. When considering Riemann surfaces, the equation $(-1)^2 = 1$ is incorrect.

The most obvious attribute of tetration is its ability is generate large numbers. An example is the relationship

$$^210 < googol = 10^{100} < \ ^310 < googolplex = 10^{10^{100}} < \ ^410.$$

Examples of limit points, limit cycles, and strange attractors exist in tetration. In other words: the sequence $z_0, f(z_0), f(f(z_0)), f(f(f(z_0))), \ldots$ can generate limit points, limit cycles, or strange attractors: when $f(z) = a^z$, where a is a complex number. Contrasting the size of numbers that tetration can produce, there are many cases where $^\infty a$ is finite. Limit points of the map $f(z) = a^z$ will be denoted by capital letters: A in this example. The generators of the limit points (a in the previous example), are expressed with lower case letters. They relate as follows:

$$a^A = A \tag{1}$$

$$^{\pm\infty}a = A. \tag{2}$$

Each limit point of a exponential map (including Riemann branch) has a unique generator. For example the limit point $e^{\pi i/2}$ of a recurrsive exponential series has a different generator than the limit point $e^{5\pi i/2}$. Since the natural logarithms of both limit points and their generators are used regularly, and they also denote the Riemann branch; the following expressions are utilized.

$$\alpha \equiv Ln(a) \tag{3}$$

$$\lambda \equiv Ln(A) \quad (4)$$

$$\lambda = \alpha A \quad (5)$$

Both α and λ contain information on the Riemann branch of a and A respectively. For example, in Equation (6): it will be implicitly understood that $A^{1/A} = (e^{\alpha})^{1/A} = e^{\alpha/A}$. The Riemann branches used with λ are independent of the Riemann branch of α.

It is only due to the existence of limit points in the exponential map that it is possible to give meaningful values to expressions as: ${}^{5}2$ and ${}^{i}2$. When exponential dynamics are viewed from the perspective of a limit point: the algebraic structure undergoes a simplification. In the neighborhood of the limit point the exponential map becomes a logarithmic spiral descending into the limit point. The logarithmic spiral is of the form λ^{x}. A universal property of the exponential map is that the logarithmic spiral is dependent on the position of the limit point (including Riemann branch), regardless of the generator a.

2 Attractors

The exponential map contains examples of limit points, limit cycles, and strange attractors. Repellors found through recurrsive logarithms taken on a the same Riemann branch, exist on the same branch as the generating logarithm.

Theorem 2.1 *If $A = a^{A}$ then*

$$a = A^{1/A}. \quad (6)$$

Proof. $A^{1/A} = (a^{A})^{1/A} = a$. ■

2.1 Limit Points

If $A \neq 0$ then $a^{A+dz} = A + \lambda dz$; thus $A + dz \Rightarrow A + \lambda dz$. If $|\lambda| < 1$ then A is the limit point of an attractor. If $|\lambda| > 1$ then A is the limit point of a repellor. If $|\lambda| = 1$ then A is the center of a limit cycle. By using different Riemann branches, an infinite family of ${}^{-\infty}a$ can be generated for any value of a.

Theorem 2.2 *Given A_1 and A_2 as limit points of the same generator a, then*

$${A_1}^{A_2} = {A_2}^{A_1} \quad (7)$$

Proof. ${A_1}^{1/A_1} = {A_2}^{1/A_2}$. Therefore ${A_1}^{A_2} = {A_2}^{A_1}$. ■

Experimental plots of the logarithmic map in a given base a show the existence of an infinite number of attractors. One for each value of $min\ \alpha =$

$\ln(a) + 2\pi mi$, where m is an integer. Any set R of j Riemann branches, (as $m = [0, 3, -1, 2]$) used cyclicly in the logarithmic mapping; converge to a set of j points $[A_1, A_2, \ldots, A_j]$, which sequentially map into the next point of the set. If only each j^{th} point is plotted in the neighborhood of one of these limit points, the points will lie on the logarithmic spiral $\prod_{i=1}^{j} \lambda_j$ is formed. Computer simulations quickly converge to a set of limit points allowing for the creation of codes. By providing an appropriate single complex number, a fixed number of Riemann branches can be cyclicly traversed; with the Riemann branches indicating the a decoded sequence.

2.2 Limit Cycles

Theorem 2.3 *The locii of the center of limit cycles of the exponential map* $f(z) = a^z$ *is given by*

$$A = e^{e^{2\pi ix}} \tag{8}$$

$0 \geq x \geq 1$.

Proof. For limit cycles $|\lambda| = 1$, $\lambda = e^{2\pi ix}$ where $0 \geq x \geq 1$. Using equation 4 $ln(A) = \lambda = e^{2\pi ix}$; therefore $A = e^{e^{2\pi ix}}$. ■

Theorem 2.4 *The equation for the generators of limit cycles of the exponential map* $f(z) = a^z$ *is*

$$a = e^{e^{2\pi ix - e^{2\pi ix}}}. \tag{9}$$

Proof. Substitute Equation 8 into Equation 6. ■

2.3 Strange Attractors

Strange attractors in the exponential map can be experimentally shown to exist for cases of $|\lambda| > 1$ where a is not a positive real number. Specific strange attractors of the exponential map can be generated with.

$$^{n}a = 1, a \neq 1 \tag{10}$$

where n is a positive integer. An example is $a\ln(a) = 2\pi i$ for $^{2}a = 1$. A solution is $^{2}(2.2136 + 3.1140i) \approx 1$.

3 Tetrationial Algorithms

Two methods are discussed for tetrating numbers.

Theorem 3.1 *The Sum of Products gives ${}^n a$, where n is a positive integer, and* $m_0 = 1$.

$$ {}^n a = \sum_{m_1,\dots,m_n}^{\infty} \left[\prod_{i=1}^{n} \frac{1}{m_i!} m_{i-1}^{m_i} \alpha^{m_i} \right] \tag{11} $$

Proof.

$$ a^z = \sum_{m_n=1}^{\infty} \frac{1}{m_n!} \alpha^{m_n} z^{m_n} \tag{12} $$

Given $f(1, z) = a^z$, $f(n, z) = a^{f(n-1,z)}$ and

$$ f(n-1, z) = \sum_{m_1,\dots,m_{n-1}}^{\infty} \left[\prod_{i=1}^{n-1} \frac{1}{m_i!} m_{i-1}^{m_i} \alpha^{m_i} \right] z^{m_{n-1}}; \tag{13} $$

then substituting Equation 12 into z in Equation 13 gives:

$$ \begin{aligned} f(n, z) &= \sum_{m_1,\dots,m_{n-1}}^{\infty} \left[\prod_{i=1}^{n-1} \frac{1}{m_i!} m_{i-1}^{m_i} \alpha^{m_i} \right] \sum_{m_n=1}^{\infty} \frac{1}{m_n!} \alpha^{m_n} (m_{n-1} z)^{m_n} \\ &= \sum_{m_1,\dots,m_n}^{\infty} \left[\prod_{i=1}^{n} \frac{1}{m_i!} m_{i-1}^{m_i} \alpha^{m_i} \right] z^{m_n}. \end{aligned} \tag{14} $$

Set $z = 1$ giving Equation 11. ■

3.1 The Perspective Theorem

For the mapping of an arbitrary function containing at least one limit point, the existence of the regions of order in the neighborhood of limit points enable the existence of continuous functions viewed from the perspective of the limit points.

$$ \begin{aligned} \delta_1 &\equiv f'(\Omega) \equiv \delta \\ \delta_2 &= f''(\Omega) \\ \delta_n &= f^{(n)}(\Omega) \\ \delta_n^m &= (\delta_n)^m \\ \Delta_n &\equiv \frac{1}{\delta^n - \delta} \\ \Delta_n^m &= (\Delta_n)^m \end{aligned} $$

where Δ_0 and Δ_1 are undefined.

Proposition 3.1 *For any recurrsively defined function $f^n(z)$ with at least one limit point Ω, a separate Lie group function can be generated for each limit point. This allows $f^n(z)$ to be defined with n as a complex n*[illegible]

$$
\begin{aligned}
f^n(z) &= \Omega + \delta^n (z - \Omega) \\
&+ \frac{1}{2!}\delta_2 \Delta_2 [\delta \text{[illegible]} z - \Omega \text{[illegible]} \\
&+ \frac{1}{3!}[(\delta_3 \Delta_3 + 3\delta_2 \text{[illegible]} \Delta_3)\delta^{3n} \text{[illegible]} 3\delta \text{[illegible]} \\
&- (\delta_3 \Delta_3 + 3\delta_2^2 \Delta_2 \Delta_3 - 3\delta_2 \text{[illegible]} \delta^n](z - \Omega)^3 \\
&+ \ldots
\end{aligned} \tag{15}
$$

Lemma 3.1

$$Df^n(\Omega) = (f'(\Omega))^n. \tag{16}$$

Proof. Using the Chain Rule:

$$
\begin{aligned}
Df^n(z) &= f'(f^{n-1}(z))Df^{n-1}(z) \\
&= \prod_{i=1}^{n} f'(f^{n-i}(z))
\end{aligned}
$$

At $z = \Omega$, $f^n(\Omega) = \Omega$; therefore $Df^n(\Omega) = (f'(\Omega))^n$. ■

Lemma 3.2

$$D^2 f^n(\Omega) = \frac{f''(\Omega)}{(f'(\Omega))^2 - f'(\Omega)}[(f'(\Omega))^{2n} - (f'(\Omega))^n]. \tag{17}$$

Proof.

$$
\begin{aligned}
D^2 f^n(z) &= D[f'(f^{n-1}(z))Df^{n-1}(z)] \\
&= f''(f^{n-1}(z))[Df^{n-1}(z)]^2 + f'(f^{n-1}(z))D^2 f^{n-1}(z) \\
&= f''(\Omega)[Df^{n-1}(z)]^2 + f'(\Omega)D^2 f^{n-1}(z) \\
&= f''(\Omega)(f'(\Omega))^{2n-2} + f'(\Omega)D^2 f^{n-1}(z)
\end{aligned}
$$

Let $f(n) \equiv x^{bn+c} + xf(n-1)$, $f(0) = 0$, $f(1) \neq 0$; then

$$f(n) = \frac{x^{b+c-1}}{x^{b-1} - 1}[x^{bn} - x^n]. \tag{18}$$

Lemma 3.3 *The higher derivatives of $f^n(z)$ can be decomposed into lower derivatives of $f^{n-1}(z)$ which are known and a single derivative of $f^{n-1}(z)$ of equal degree to the original derivative which is solved as a geometrical progression.*

From inspection of Lemma 3.2 it can be seen that after substituting Ω for $f^{n-1}(z)$, the Chain rule expansion of $D^2\ f^n(z)$ contains no derivatives greater than the second derivative; and only one derivative of the second degree. Given

$$D^m\ f^n(z) = g(z) + h(z)D^m\ f^{n-1}(z);$$

then

$$D^{m+1}\ f^n(z) = g'(z) + h'(z)D^m\ f^{n-1}(z) + h(z)D^{m+1}\ f^{n-1}(z).$$

Take $m = 2$; then after substituting Ω for $f^{n-1}(z)$ it is seen that $g(z)$ and $h(z)$ only contain derivatives of less than degree m or 2. Therefore in $D^{m+1}\ f^n(z)$; $g'(z)$ and $h'(z)$ only contain derivatives of less than degree $m+1$ or 3. Using the principle of induction, Lemma 3.3 is proved. ■

Since x^{bn} has the same coefficient as $-x^n$, all calculations of the second derivative and greater are composed of these terms. This is due to $f^0(z) = z$ being linear.

Lemma 3.4 *For any recurrsively defined function $f^n(z)$ with at least one limit point Ω; a solution of the Perspective equation, (eq. 15), can be generated for each limit point; defined with n as any whole number.*

From the definition of the Taylor series the constant must equal Ω. Lemma 3.1 gives the linear term, Lemma 3.2 the quadratic term, and Lemma 3.3 show how to develop all higher terms. ■

Lemma 3.5 *The Perspective equation, (eq. 15), can be generated for each limit point; defined with n as any integer.*

Since a specific limit point defines the Riemann branch used, the Perspective equation is as valid for $g^n(z)$, where $g(z) = f^{-1}(z)$, as for $f^n(z)$. Therefore the Perspective equation is valid for $f^{-n}(z)$. ■

Theorem 3.2 *A solution to the Perspective equation can be generated for each limit point, defined with n as a complex number.*

The evaluation of $f^n(z)$ where n is a non-integer number is made by determining a j, where j is an integer; such that $f^j(z)$ is in the neighborhood of the limit point. This gives $f^j(z) = \Omega + (f'(\Omega))^j(z - \Omega)$. In the neighborhood of the limit point, $f^s(z)$ may be defined where s is a complex number, such that s is a close to j and $s - n$ is an integer. This is due to $f^j(z)$ sweeping out a logarithmic spiral. Finally $f^n(z)$ can be determined by a projection of $f^s(z)$. Since the Perspective equation handles integer values of n everywhere, complex values of n in the neighborhood of the limit point; it is capable of directly computing $f^n(z)$ where n is complex. ■

Table 1 lists dynamical derivative operators. The notation provided expresses

$$\begin{aligned} D^3 f^n(z) &= f'''(\Omega)[D\ f^{n-1}(z)]^3 \\ &\quad + f''(\Omega)3[D\ f^{n-1}(z)][D^2\ f^{n-1}(z)] \\ &\quad + f'(\Omega)[D^3\ f^{n-1}(z)] \end{aligned}$$

as D^3 =[3]+3[1.1]+[0.0.1]. A second example is:

$$[2,4,3,1] = f^{(10)}(\Omega)[D\ f^{n-1}(z)]^2\ [D^2\ f^{n-1}(z)]^4\ [D^3\ f^{n-1}(z)]^3\ [D^4\ f^{n-1}(z)].$$

The coefficient $f^{(10)}(\Omega)$ is the tenth derivative because $10 = 2 + 4 + 3 + 1$.

The symbolic derivation of $D^3\ f^n(z)$ follows:

$$\begin{aligned} D^3 &= [3] + 3[1.\ 1] + [0.\ 0.\ 1] \\ &= \delta_3\delta^{3(n-1)} + \delta_2[\delta^(n - 1)] \\ &\quad [\delta_2\Delta_2(\delta^{2(n-1)} - \delta^{n-1}] + \delta D^3\ f^{n-1}(z) \\ &= (\delta_3 + 3\delta_2^2\Delta_2)\delta^{3n-3} \\ &\quad - 3\delta_2^2\Delta_2\delta^{2n-2} + D^3\ f^{n-1}(z) \\ &= (\delta_3\Delta_3 + 3\delta_2^2\Delta_2\Delta_3)\delta^{3n} - 3\delta_2^2\Delta_2^2\delta^{2n} \\ &\quad - (\delta_3\Delta_3 + 3\delta_2^2\Delta_2\Delta_3 - 3\delta_2^2\Delta_2^2)\delta^n \end{aligned}$$

Table 1: Dynamical Derivatives

D =[1]
D^2 =[2]+[0,1]
D^3 =[3]+3[1.1]+[0,0,1]
D^4 =[4]+6[2.1]+4[1,0,1]+3[0.2]+[0.0.0.1]
D^5 =[5]+10[3,1]+15[1,2]+10[2,0.1]+10[0.1.1]+5[1.0.0.1]+[0.0.0.0 ,1]
D^6 =[6]+45[2.2]+20[3,0.1]+15[4.1]+15[2.0.0.1]+60[1.1.1]
+15[0,3]+6[1,0,0.0,1]+15[0,1.0.1]+10[0.0.2]+[0.0.0.0.0,1]
D^7 =[7]+21[5,1]+35[4,0,1]+105[3.2]+35[3.0.0.1]+210[2.1.1]+105[1 .3]
+21[2,0,0.0.1]+105[1.1.0,1]+70[1.0.2]+105[0.2.1]+7[1,0,0.0.0.1]
+21[0,1,0,0,1]+35[0,0.1,1]+[0.0.0,0.0,0,1]

3.2 The Transexponential Function

$T_\lambda^n(z)$ denotes the transexponential function with the following identities:

$$\begin{aligned} T_\lambda^a \circ T_\lambda^b(z) &= T_\lambda^{a+b}(z) \\ T_\lambda^0(z) &= z \\ T_1^1(z) &= \exp(z) \\ T_1^{-1}(z) &= \ln(z) \\ T_\lambda^1(z) &= a^z && \text{using eq.(4)} \\ T_\lambda^{-1}(z) &= \log_a(z) && \text{on the appropriate Riemann branch} \\ T_\lambda^b(1) &= {}^b a \end{aligned}$$

For convenience a new expression is defined:

$$L_m^n \equiv \left(\frac{1}{1-\lambda^m}\right)^n. \tag{19}$$

Theorem 3.3

$$\begin{aligned} T_\lambda^n(z) &= A + \lambda^n (z-A) + \frac{\alpha}{2!}[L_1\lambda^{2n} - L_1\lambda^n](z-A)^2 \\ &+ \frac{\alpha^2}{3!}[(L_2 + 3L_1L_2)\lambda^3 - 3L_1^2\lambda^{2n} && (20) \\ &+ (3L_1^2 - L_2 - 3L_1L_2)\lambda^n](z-A)^3 \\ &+ \ldots && (21) \end{aligned}$$

Proof. The derivation is the same as that of the Perspective Theorem. The specific calculations are carried out for the sake of clarity.

The transexponential version of Lemma 3.1 is: $T_\lambda^n(z)$ can be approximated by the Taylor series at $z = A$.

$$DT_\lambda^n(z) = \alpha T_\lambda^n(z)\; DT_\lambda^{n-1}(z) = \alpha^n \prod_{j=1}^{n} T_\lambda^j(z)$$

If $z = A$, then $T_\lambda^n(A) = A$; thus $DT_\lambda^n(A) = \alpha^n A^n = \lambda^n$.

As per Lemma 3.2:

$$D^2T_\lambda^n(z) = D[\alpha T_\lambda^n(z)\; DT_\lambda^{n-1}(z)] = \alpha^2 T_\lambda^n(z)[DT_\lambda^{n-1}(z)]^2 + \alpha T_\lambda^n(z)\; D^2\; T_\lambda^{n-1}(z)$$

therefore

$$D^2\; T_\lambda^n(A) = \alpha\lambda^{2n-1} + \lambda D^2\; T_\lambda^{n-1}(A).$$

For Lemma 3.3: Let $f(n) \equiv \lambda^{bn+c} + \lambda f(n-1), f(0) = 0, f(1) \neq 0$; then $f(n) = \frac{\lambda^{b+c-1}}{\lambda^{b-1}-1}[\lambda^{bn} - \lambda^n]$.

$$D^2\; T_\lambda^n(A) = \frac{\alpha}{\lambda - 1}[\lambda^{2n} - \lambda^n].$$

In practice $f(n)$ will always equal $\lambda^{bn-b+1} + \lambda f(n-1)$. Therefore $f(n) = L_{b-1}[\lambda^{bn} - \lambda^{n}]$. Together with the following notation, the calculation of the symbolic terms of the transexponential function can be expedited.

Theorem 3.4 *The transexponential function can be used to tetrate one complex number by a second complex number:*

$$T_{\lambda}^{b}(1) = {}^{b}a \tag{22}$$

Substitute ${}^{0}a$ for z; since ${}^{0}a = 1$, all $z^{n} = 1$.■

The transexponential function indicates that where b is not an integer, the tetrated number is actually $\aleph_0$ multi-valued. Each of the values of a number corresponds to one of the $\aleph_0$ limit points.

4 Experimental Results

Computers are not only invaluable, but necessary for a thorough study of tetration. Equations 6, 8, and 9 are verified in examples:

$$A = e^{e^{2\pi i/3}} \approx .3929 + .4620i. \tag{23}$$

with

$$a = e^{e^{\frac{2\pi i}{3} - e^{\frac{2\pi i}{3}}}} \approx .03095 + 1.73922i \tag{24}$$

and

$$A = e^{e^{2\pi i/5}} \approx .7911 + 1.1087i \tag{25}$$

with

$$a = e^{e^{\frac{2\pi i}{5} - e^{\frac{2\pi i}{5}}}} \approx 1.96514 + .44124i. \tag{26}$$

The saddle node bifurcation [2] on the real line sets $x = 0$ in Equations 8, and 9; giving

$$A = e^{e^{0}} = e, \tag{27}$$

with

$$a = e^{e^{-e^{0}}} = e^{1/e}. \tag{28}$$

The transexponential function, Equation 20 has been primarily researched for the case $T_1^{.5}(z - A)$; using the Mc Clurant series at $z = A \approx .31813150520 + 1.33723570142i$.

The function $T_1^{.5}$ is the hemi-exponential function: hmexp(z); such that hmexp(hmexp(z)) = exp(z). The coefficients for the hemi-exponential function are listed in Figure 4.

5 Future Directions

At this time, the author is investigating four line of research.

1. The development of computer algorithms for computing and confirming the existence of the transexponential function.
2. The inverse of the tetrationial function.
3. The generalization of the Perspective Theorem for periodic limit points.
4. The Tetrationial Hypothesis: the idea that tetration is the basis of quantum field theory. This is predicated on the existence of the Perspective Theorem for periodic limit points, and replacing the Feynman Path Integral with the transexponential function.

References

[1] R.v.B. Rucker *Infinity and the mind* pp. 68-69. Boston; Basel; Stuttgart; Birkhäuser 1982

[2] R.L. Devaney *An introduction to chaotic dynamical systems* pp. 306-315 The Benjamin/Cummings Publishing Co., Inc.