On model structures from the FEP

Reid Barton

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1 Introduction

There is a method for constructing model category structures based on the so-called *fibration extension property*. The fibration extension property implies in particular that every fibration can be expressed as a pullback of one between fibrant objects. This method has been formulated as a theorem in [CS23] (and perhaps elsewhere?) in the following form:

Theorem 1.1 ([CS23], 3.26). *Let M be a cylindrical premodel category (definition 3.14) such that:*

(D) Every object of M is cofibrant.

(E) Any cofibration with the left lifting property with respect to fibrations between fibrant objects *is an anodyne cofibration. (This condition is implied by the fibration extension property.)*

Then M is a model category.

The theorem is proved in [CS23] by a very explicit method. In this note, we give an alternative proof of this theorem. Our proof exploits the equivalence of cylindrical premodel structures with premodel structures that are *modules* over the monoidal model category $\widehat{\square}$ of *plain cubical sets*. Most of the note is devoted to establishing this equivalence and the associated machinery. Once it is in place, the main proof is short (theorem 4.1).

This proof is not meant to be better (or worse) than other proofs of the theorem, just different. It is designed to make maximum use of the fact that plain cubical sets are already known to be a model category [Cis06]. As such, it may be regarded as non-elementary. I hope that it may help shed more light on elementary proofs of the same theorem.

Notation. We write $j \square_F f$ for the pushout product of two maps j and f with respect to a bifunctor F. Usually we omit F, in which case we usually have in mind a bifunctor named $\otimes$.

This note discusses both plain cubical sets and cartesian cubical sets. We have tried to roughly follow the notation of [Cis06] for plain cubical sets and [Awo23] for cartesian cubical sets. The notation is probably not yet fully consistent.

2 Enriched premodel categories

This section introduces module (or enriched) premodel categories. Some related material can be found in [Bar20], though mostly we just observe that some standard theory for model categories also works for premodel categories.

Let $\mathbb{V}$ be a monoidal premodel category. (Normally, $\mathbb{V}$ will be a model category. We do **not** assume that $\mathbb{V}$ is symmetric monoidal.) Explicitly, this means that the unit object $\mathbb{1}_{\mathbb{V}}$ is cofibrant and that $\otimes : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is a Quillen bifunctor. We want to understand $\mathbb{V}$-module premodel categories (or just $\mathbb{V}$-premodel categories, for short). Specifically, we will be interested in the case where $\mathbb{V}$ is the model category of (plain) cubical sets, and we want an explicit description of the data required to equip a given premodel category M with a $\mathbb{V}$-module structure.

Following Hovey (chapter 4), we define

- a $\mathbb{V}$-*module category* to be a category M equipped with an "action" $\otimes : \mathbb{V} \times M \to M$ together with natural "coherence" isomorphisms $\alpha : (K \otimes L) \otimes A \cong K \otimes (L \otimes A)$, $\lambda : \mathbb{1} \otimes A \cong A$ (where K, $L \in \mathbb{V}$, $A \in M$), satisfying equations involving the associator and unitors of $\mathbb{V}$;
- a *closed* $\mathbb{V}$-*module category* to be a $\mathbb{V}$-module category M in which the action $\otimes : \mathbb{V} \times M \to M$ is part of an adjunction of two variables;
- a $\mathbb{V}$-*module premodel category* to be a $\mathbb{V}$-module category M which is also a premodel category, for which the action $\otimes : \mathbb{V} \times M \to M$ is a Quillen bifunctor.

In particular, a $\mathbb{V}$-module premodel category has an underlying category which is a closed $\mathbb{V}$-module category.

Notation 2.1. For a closed $\mathbb{V}$-module category M, with action or "tensor" $\otimes : \mathbb{V} \times M \to M$, we write $\{-,-\} : \mathbb{V}^{\mathrm{op}} \times M \to M$ for the cotensor and $\mathrm{map}(-,-) : M^{\mathrm{op}} \times M \to \mathbb{V}$ for the enrichment (or $\mathbb{V}$-valued "mapping space"). If the tensor is decorated with a subscript, like $\otimes_{\mathbb{I}}$', then we instead write $\{-,-\}^{\mathbb{I}}$ and $\mathrm{map}^{\mathbb{I}}(-,-)$.

Fix a category M. The data of a $\mathbb{V}$-module category structure on M is the same as the data of a (strong) monoidal functor

$$\Phi : \mathbb{V} \xrightarrow{\otimes} \mathrm{Fun}(M, M), \tag{$*$}$$

as can be seen directly from the definitions. Here, the monoidal structure on $\mathrm{Fun}(M, M)$ is given by composition (in standard order: $F \otimes G = F \circ G$). The coherence isomorphisms α and λ are encoded in the preservation of $\otimes$ and $\mathbb{1}$ respectively.

We are interested in the following two questions.

1. Under reasonable hypotheses on $\mathbb{V}$ and M, what data like ($*$) corresponds to the data of a *closed* $\mathbb{V}$-module category structure on M? (Reasonable hypotheses here are that $\mathbb{V}$ is locally presentable, e.g., a presheaf category, and M is complete and cocomplete.)
2. Suppose furthermore that $\mathbb{V}$ is a monoidal premodel category and M is a premodel category. In terms of the data ($*$), how do we check whether the resulting closed $\mathbb{V}$-module category is a $\mathbb{V}$-module premodel category?

Closed $\mathbb{V}$-module categories

Let M be a category, which we assume is complete and cocomplete. We also assume $\mathbb{V}$ is locally presentable.

Notation 2.2. Write $\mathrm{LAdj}(M, M)$ for the full subcategory of $\mathrm{Fun}(M, M)$ consisting of all functors $F : M \to M$ that are left adjoints (i.e., admit a right adjoint).

We note the following.

- $\mathrm{LAdj}(M, M)$ is a monoidal subcategory of $\mathrm{Fun}(M, M)$, i.e., it contains the unit object (the identity functor) and is closed under the monoidal operation (composition).
- $\mathrm{LAdj}(M, M)$ is also closed under colimits in $\mathrm{Fun}(M, M)$, and in particular is cocomplete, with colimits computed pointwise: $(\mathrm{colim}_{i \in I} F_i)(A) = \mathrm{colim}_{i \in I} F_i(A)$. (The right adjoint to this pointwise colimit is given by the pointwise limit of corresponding diagram of right adjoints, which is indexed by I^{op}.)
- The monoidal operation of $\mathrm{LAdj}(M, M)$ preserves colimits in each variable separately. Indeed, $(F, G) \mapsto F \circ G$ preserves colimits in F because colimits in $\mathrm{LAdj}(M, M)$ are computed pointwise, and also in G because, in addition, F preserves colimits (being a left adjoint).

Remark 2.3. In general $\mathrm{LAdj}(M, M)$ might not be locally small even if M is (e.g., if M is small presheaves on a large category). When we talk about the "cocompleteness" of $\mathrm{LAdj}(M, M)$, we mean with respect to the same size of colimits for which the original category M is cocomplete.

If M is locally presentable, then $\mathrm{LAdj}(M, M)$ is also locally presentable, hence in particular locally small and also complete. In general there seems to be no reason to expect $\mathrm{LAdj}(M, M)$ to be complete.

Proposition 2.4. *Closed $\mathbb{V}$-module structures on M correspond to cocontinuous monoidal functors*

$$\Phi : \mathbb{V} \xrightarrow{\otimes,\, \mathrm{colim}} \mathrm{LAdj}(M, M).$$

Proof. Since $\mathrm{LAdj}(M, M) \subseteq \mathrm{Fun}(M, M)$ is a monoidal full subcategory, and we already know that $\mathbb{V}$-module structures on M correspond to monoidal functors $\Phi \to \mathrm{Fun}(M, M)$, it suffices to check that a $\mathbb{V}$-module structure is closed if and only if the corresponding monoidal functor factors through the full subcategory $\mathrm{LAdj}(M, M)$ and preserves colimits.

By definition, a $\mathbb{V}$-module structure $(\otimes : \mathbb{V} \times M \to M, \ldots)$ is closed if and only if the functors $K \otimes - : M \to M$ and $- \otimes A : \mathbb{V} \to M$ admit right adjoints for each $K \in \mathbb{V}$ and $A \in M$ respectively. The condition that each $K \otimes - : M \to M$ admits a right adjoint means exactly that $\Phi : \mathbb{V} \to \mathrm{Fun}(M, M)$ factors through $\mathrm{LAdj}(M, M)$. The condition that $- \otimes A : \mathbb{V} \to M$ admits a right adjoint, by the adjoint functor theorem, is equivalent to the condition that it preserves colimits. Since colimits in $\mathrm{LAdj}(M, M)$ are computed pointwise, imposing this condition for all A is the same as imposing the condition that Φ preserves colimits. □

$\mathbb{V}$-module premodel structures

Now we suppose $\mathbb{V}$ is a monoidal premodel category (still locally presentable) and M is also a premodel category. We want to check whether a given closed $\mathbb{V}$-module structure on M is a $\mathbb{V}$-module premodel structure. Of course, this means by definition that $j \,\square\, f$ is a cofibration of M for any cofibration j of $\mathbb{V}$ and cofibration f of M and is anodyne if either j or f is. Here, $\square$ denotes the pushout product formed using the the action $\otimes : \mathbb{V} \times M \to M$. But we might have some explicitly described class (not necessarily a set) of generating (anodyne) cofibrations for $\mathbb{V}$ and/or M, for which we can check this condition more easily. The following standard fact, which we have specialized to the case of a closed module category structure, gives a few equivalent characterizations of Quillen bifunctors between premodel categories. It is proved in the same way as for model categories.

Proposition 2.5. *For a closed $\mathbb{V}$-module category structure on M, the following are equivalent:*

(i) For any cofibration j in $\mathbb{V}$ and cofibration f in M, the $\otimes$-pushout product $j \,\square\, f$ is again a cofibration in M, which is anodyne if either j or f is. (By definition, this means that $\otimes$ is a Quillen bifunctor, so M is a $\mathbb{V}$-premodel category).

(ii) For any cofibration $j : K \to L$ in $\mathbb{V}$ and fibration $p : X \to Y$ in M, the induced map

$$\{L, X\} \to \{K, X\} \times_{\{K,Y\}} \{L, Y\}$$

is a fibration in M, which is anodyne if either j or p is.

(iii) For any cofibration $f : A \to B$ in M and fibration $p : X \to Y$ in M, the induced map

$$\mathrm{map}(B, X) \to \mathrm{map}(A, X) \times_{\mathrm{map}(A,Y)} \mathrm{map}(B, Y)$$

is a fibration in M, which is anodyne if either f or p is.

Furthermore, given any classes $I_{\mathbb{V}}$, $J_{\mathbb{V}}$ and I_M, J_M of generating (anodyne) cofibrations for $\mathbb{V}$ and M respectively, it suffices to check the conditions involving (anodyne) cofibrations on just the generating ones.

Remark 2.6. *Combinatorial* premodel categories have an internal Hom which represents the multi-category structure given by Quillen bifunctors (and more generally multifunctors), whose underlying category is given by the category of all left adjoints. Hence, if the premodel structure on M is combinatorial, then $\mathrm{LAdj}(M, M)$ is a monoidal premodel category. Informally, the (anodyne) cofibrations of $\mathrm{LAdj}(M, M)$ are defined exactly so as to make evaluation $\mathrm{LAdj}(M, M) \times M \to M$ into a Quillen bifunctor.

In this case, one could argue directly (as a general fact about pseudomodules over pseudomonoids) that giving a $\mathbb{V}$-module structure on a combinatorial premodel category M is the same as giving a monoidal left Quillen functor $\Phi : \mathbb{V} \to \mathrm{LAdj}(M, M)$.

Without any accessibility hypothesis on M, probably the best one could hope for is a sort of "left semi-premodel structure" on $\mathrm{LAdj}(M, M)$, which has (anodyne) cofibrations that behave as one would expect without necessarily being part of weak factorization systems. (This is much like how $\mathrm{LAdj}(M, M)$

is cocomplete, as we saw, but seems to have no reason to be complete.) This is enough to say what it means for $\Phi : \mathbb{V} \to \mathrm{LAdj}(M, M)$ to be a left Quillen functor, but it is a bit awkward, and as it turns out not really useful for what we want to do next. We do want to include premodel categories M that are not combinatorial, so we will refrain from regarding $\mathrm{LAdj}(M, M)$ as a premodel category.

Remark 2.7. This section is entitled "Enriched premodel categories". Equipping a premodel category M with a $\mathbb{V}$-module premodel structure is equivalent to giving it an "$\mathbb{V}$-enriched premodel category structure"—the $\mathbb{V}$-enriched mapping objects themselves being given by $\mathrm{map}(-, -) : M^{\mathrm{op}} \times M \to \mathbb{V}$. The $\mathbb{V}$-module perspective turns out to be more convenient here but, for reasons of tradition, we still use the general term "enriched premodel category".

3 Cubical premodel categories

Now we specialize to the case where $\mathbb{V}$ is the model category of *(plain) cubical sets*. We briefly review its definition.

Definition 3.1. The *(plain) cube category* $\square$ has as objects the symbols $\square[n]$ for each $n \geq 0$. The morphisms from $\square[n]$ to $\square[m]$ are given by formal functional expressions

$$(x_1, \ldots, x_n) \mapsto (y_1, \ldots, y_m)$$

where each y_j is either 0, 1, or some x_i, subject to the condition that in $(y_1, \ldots, y_m)$, any terms x_i that appear do so without repetition and in increasing order of i.

In the taxonomy of cube categories, the plain cube category has only face and degeneracy maps (or more precisely their opposites), and no other structure such as diagonals, symmetries, connections, or reversals. There are $2n$ face maps $\delta_n^{i,\varepsilon}$ from $\square[n-1]$ to $\square[n]$, given by

$$\delta_n^{i,\varepsilon}(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1}),$$

and n degeneracies σ_n^i from $\square[n]$ to $\square[n-1]$, given by dropping the ith coordinate.

The category $\square$ has a (strict) monoidal structure. On objects, it is given by $\square[n] \otimes \square[n'] = \square[n + n']$, while on morphisms it is given by "putting functions side-by-side". The unit object is $\mathbb{1} = \square[0]$.

The monoidal structure is important for our purpose because as a monoidal category, $\square$ is generated by a small amount of data. Specifically, we write $\square_{\leq 1}$ for the full subcategory of $\square$ on the objects $\mathbb{1} = \square[0]$ and $\square[1]$. It is generated by the two maps $\delta_1^{1,\varepsilon} : \square[0] \to \square[1]$ and the map $\sigma_1^1 : \square[1] \to \square[0]$, subject to the relations $\sigma_1^1 \circ \delta_1^{1,\varepsilon} = \mathrm{id}_{\square[0]}$ for $\varepsilon = 0, 1$.

Proposition 3.2 ([Cis06], 8.4.6). *Let C be a monoidal category. Then restriction to $\square_{\leq 1}$ induces an equivalence of categories*

$$\mathrm{Fun}^{\otimes}(\square, C) \to \mathrm{Fun}^{\mathbb{1}}(\square_{\leq 1}, C)$$

from the category of monoidal functors from $\square$ to C to the category of functors from $\square_{\leq 1}$ to C sending $\square[0]$ to the unit object of C (up to specified isomorphism).

Explicitly, this means that giving a monoidal functor $\Phi : \square \xrightarrow{\otimes} C$ amounts to specifying an object $I \in C$ (the image of $\square[1]$) together with two maps $d^0, d^1 : \mathbb{1}_C \to I$ and a map $s : I \to \mathbb{1}_C$ such that $s \circ d^\varepsilon = \mathrm{id}_{\mathbb{1}_C}$ for each ε. The corresponding monoidal functor Φ then sends $\square[n] = \square[1]^{\otimes n}$ to $I^{\otimes n} \in C$.

Definition 3.3. The category of *(plain) cubical sets*, $\widehat{\square}$, is the category of presheaves on $\square$. We write $\square_n \in \widehat{\square}$ for the presheaf represented by $\square[n]$.

The category $\widehat{\square}$ has a monoidal structure induced from that of $\square$ (by Day convolution). The unit object is $\mathbb{1} = \square_0$, and the tensor product $\otimes : \widehat{\square} \times \widehat{\square} \to \widehat{\square}$ is characterized as the essentially unique functor preserving colimits in each argument and satisfying $\square_n \otimes \square_{n'} = \square_{n+n'}$.

Notation 3.4. We write $\iota^\varepsilon : \square_0 \to \square_1$ ($\varepsilon = 0, 1$) and $\pi : \square_1 \to \square_0$ for the maps in $\widehat{\square}$ represented by the generators $\delta_1^{1,\varepsilon}$, σ_1^1 of $\square_{\leq 1} \subseteq \square$ respectively. We also write $\iota : \square_0 \amalg \square_0 \to \square_1$ for the map given by ι^0 on the first summand and ι^1 on the second summand.

The functor $\otimes$ is part of an adjunction of two variables, so $\widehat{\square}$ is a closed monoidal category. Note that the monoidal category $\widehat{\square}$ is *not* symmetric.

Example 3.5. The maps $\iota^0 \,\square\, \iota$ and $\iota \,\square\, \iota^0$ are not isomorphic maps of $\widehat{\square}$. They both have codomain $\square_2$, which has no automorphisms (since $\square[2]$ has no automorphisms in $\square$), but they define different subobjects of $\square_2$: two of the four "cubical horns" consisting of three of its edges. (The other two horns are given by $\iota^1 \,\square\, \iota$ and $\iota \,\square\, \iota^1$.) Hence, $\widehat{\square}$ cannot be a symmetric monoidal category.

Proposition 3.6 ([Cis06], 8.4.23). *Let C be a cocomplete monoidal category whose tensor product preserves colimits in each argument separately. Then restriction to $\square_{\leq 1} \subseteq \square \subseteq \widehat{\square}$ induces an equivalence of categories*

$$\mathrm{Fun}^{\otimes,\mathrm{colim}}(\widehat{\square}, C) \to \mathrm{Fun}^{\mathbb{1}}(\square_{\leq 1}, C)$$

from the category of colimit-preserving monoidal functors from $\widehat{\square}$ to C to the category of functors from $\square_{\leq 1}$ to C sending $\square[0]$ to the unit object of C.

We now specialize to the case $C = \mathrm{LAdj}(M, M)$ for a complete and cocomplete category M, and use the explicit description of $\square_{\leq 1}$.

Proposition 3.7. *Closed $\widehat{\square}$-module structures on a complete and cocomplete category M correspond to functors $\Phi_{\leq 1} : \square_{\leq 1} \to \mathrm{LAdj}(M, M)$ sending $\square[0]$ to the identity functor of M.*

In turn, such a $\Phi_{\leq 1}$ amounts to specifying

- *an adjunction $\mathbb{I} : M \rightleftarrows M : \mathbb{P}$, given by $\Phi_{\leq 1}(\square[1])$, and*
- *natural transformations $i^0, i^1 : \mathrm{id} \to \mathbb{I}$ and $p : \mathbb{I} \to \mathrm{id}$, such that $pi^0 = pi^1 = \mathrm{id}$.*

Let us now fix a complete and cocomplete category M as well as data $\mathbb{I} : M \rightleftarrows M : \mathbb{P}$, $i^0, i^1 : \mathrm{id} \to \mathbb{I}$, $p : \mathbb{I} \to \mathrm{id}$ as above. We denote the corresponding action of $\widehat{\square}$ on M by $\otimes_{\mathbb{I}}$. It is part of an adjunction of two variables $(\otimes_{\mathbb{I}}, \{-,-\}^{\mathbb{I}}, \mathrm{map}^{\mathbb{I}})$.

Remark 3.8. We unwind these constructions to give explicit descriptions of these three functors.

- Fix an object $A \in M$. By construction of the action, we have $\square_1 \otimes_{\mathbb{I}} A = \mathbb{I}A$. The n-cube $\square_n \in \widehat{\square}$ is the n-fold monoidal product $\square_1 \otimes \cdots \otimes \square_1$, Hence, we find

$$\square_n \otimes_{\mathbb{I}} A = \square_1 \otimes_{\mathbb{I}} \cdots \otimes_{\mathbb{I}} \square_1 \otimes_{\mathbb{I}} A = \mathbb{I}^n A.$$

  The functoriality of this expression in the cube is provided by the natural transformations i^0, $i^1 : \mathrm{id} \to \mathbb{I}$, $p : \mathbb{I} \to \mathrm{id}$. For example, there are four "edges" $\square_1 \to \square_2$, which act on A by the four morphisms

$$i^0_{\mathbb{I}A} : \mathbb{I}A \to \mathbb{I}^2 A, \quad i^1_{\mathbb{I}A} : \mathbb{I}A \to \mathbb{I}^2 A, \quad \mathbb{I}(i^0_A) : \mathbb{I}A \to \mathbb{I}^2 A, \quad \mathbb{I}(i^1_A) : \mathbb{I}A \to \mathbb{I}^2 A.$$

  In general, a morphism of $\square$ can be expressed uniquely as a monoidal product of copies of $\mathrm{id}_{\square[1]}$, $\delta_1^{1,0}$, $\delta_1^{1,1}$, and σ_1^1, and its action on M is given by the horizontal composition of the corresponding natural transformations.

  For a general cubical set K, we compute $K \otimes_{\mathbb{I}} A$ by writing K as a colimit of cubes $\square_n$ and then forming the corresponding colimit of the objects $\mathbb{I}^n A$ in M.

- For an object $X \in M$, we have

$$\{\square_n, X\}^{\mathbb{I}} = \mathbb{P}^n X$$

  with contravariant functoriality in $\square_n$ determined in a similar way using the natural transformations $e_0 : \mathbb{P} \to \mathrm{id}$, $e_1 : \mathbb{P} \to \mathrm{id}$, $c : \mathrm{id} \to \mathbb{P}$ induced by i^0, i^1, p respectively. For a general $K \in \widehat{\square}$ we can compute $\{K, X\}^{\mathbb{I}}$ by writing K as a colimit of cubes $\square_n$ and then forming the corresponding *limit* of the objects $\mathbb{P}^n X$ in M.

- For $A, X \in M$, we have

$$\mathrm{map}^{\mathbb{I}}(A, X)_n = \mathrm{Hom}(\square_n, \mathrm{map}^{\mathbb{I}}(A, X)) = \mathrm{Hom}(\square_n \otimes_{\mathbb{I}} A, X) = \mathrm{Hom}(\mathbb{I}^n A, X).$$

  Thus, the vertices of $\mathrm{map}^{\mathbb{I}}(A, X)$ are the maps in M from A to X, while the 1-cubes are *homotopies* defined using $\mathbb{I}$ as cylinder functor, the 2-cubes are two-dimensional homotopies, and so on. The cubical structure of $\mathrm{map}^{\mathbb{I}}(A, X)$ is again derived from i^0, i^1, p. Of course, we can equivalently compute $\mathrm{map}^{\mathbb{I}}(A, X)_n$ as $\mathrm{Hom}(A, \mathbb{P}^n X)$.

We will actually make use of only one of these computations: $\mathrm{map}^{\mathbb{I}}(A, X)_0 = \mathrm{Hom}(A, X)$.

Our remaining tasks in this section are to introduce the model category structure on $\widehat{\square}$, and determine the conditions under which a closed $\widehat{\square}$-module structure on a *premodel* category M gives M the structure of a $\widehat{\square}$-module premodel category. This will lead us inevitably to the notion of a "cylindrical premodel category".

Notation 3.9 ([Cis06], 8.4.20, 8.4.34). We write $\partial\square_n \in \widehat{\square}$ for the "boundary" of $\square_n$, the union of its proper faces. We write $\sqcap_n^{k,\varepsilon}$ for the union of the proper faces of $\square_n$ other than the one that is the image of $\delta_n^{k,\varepsilon} : \square_{n-1} \to \square_n$. These are subobjects of $\square_n$, and we denote their inclusions by

$$i_n : \partial\square_n \to \square_n, \qquad u_n^{k,\varepsilon} : \sqcap_n^{k,\varepsilon} \to \square_n.$$

Theorem 3.10 ([Cis06], 8.4.38). *The category $\widehat{\square}$ admits a monoidal model category structure with generating (acyclic) cofibrations given by*

$$I = \{\, i_n \mid n \geq 0 \,\}, \qquad J = \{\, u_n^{k,\varepsilon} \mid n \geq 1,\ 1 \leq k \leq n,\ \varepsilon \in \{0,1\} \,\}.$$

Proposition 3.11. *The morphisms i_n, $u_n^{k,\varepsilon}$ satisfy the relations (up to isomorphisms)*

$$\begin{aligned} i_n &= i_1 \,\square \cdots \square\, i_1 \qquad (n \text{ copies of } i_1), \\ u_n^{k,\varepsilon} &= i_k \,\square\, u_1^{1,\varepsilon} \,\square\, i_{n-k-i}. \end{aligned}$$

These are not formulated explicitly in [Cis06], but are implicit in the proof of 8.4.27 and 8.4.36.

Lemma 3.12. *Suppose M is a closed $\widehat{\square}$-module category with a premodel structure. In order for M to be a $\widehat{\square}$-module premodel category, it is necessary and sufficient that all of the following conditions hold:*

- *For $f : A \to B$ a cofibration of M, $i_1 \,\square\, f$ is also a cofibration, which is anodyne if f is.*
- *For $f : A \to B$ a cofibration of M, $u_1^{1,0} \,\square\, f$ and $u_1^{1,1} \,\square\, f$ are anodyne cofibrations.*

Proof. The necessity is obvious as i_1 is a cofibration and $u_1^{1,0}$, $u_1^{1,1}$ are anodyne cofibrations of $\widehat{\square}$. Conversely, suppose the listed conditions hold; we must check that M is a $\widehat{\square}$-module premodel category. By proposition 2.5, it is enough to consider the generating (anodyne) cofibrations for $\widehat{\square}$. Specifically, if $f : A \to B$ is any cofibration of M, we must check that

- for each n, $i_n \,\square\, f$ is a cofibration which is anodyne if f is,
- for each n, k, ε, $u_n^{k,\varepsilon} \,\square\, f$ is an anodyne cofibration.

But in view of the identifications

$$\begin{aligned} i_n \,\square\, f &= i_1 \,\square \cdots \square\, i_1 \,\square\, f, \\ u_n^{k,\varepsilon} \,\square\, f &= i_k \,\square\, u_1^{1,\varepsilon} \,\square\, i_{n-k-i} \,\square\, f, \end{aligned}$$

these statements follow from the cases where $n = 1$. □

We combine this with proposition 3.7 to obtain a description of $\widehat{\square}$-module premodel structures.

Proposition 3.13. *Let M be a premodel category. Then giving a $\widehat{\square}$-model premodel structure on M is equivalent to giving*

(a) *an adjunction* $\mathbb{I} : M \rightleftarrows M : \mathbb{P}$,

(b) *natural transformations* $i^0, i^1 : \mathrm{id} \to \mathbb{I}$ *and* $p : \mathbb{I} \to \mathrm{id}$, *such that* $pi^0 = pi^1 = \mathrm{id}$,

(c) *and such that for any cofibration* $f : A \to B$ *of* M, *the induced map*

$$B \amalg_A \mathbb{I}A \amalg_A B \to \mathbb{I}B$$

is a cofibration which is anodyne if f *is, while the induced maps*

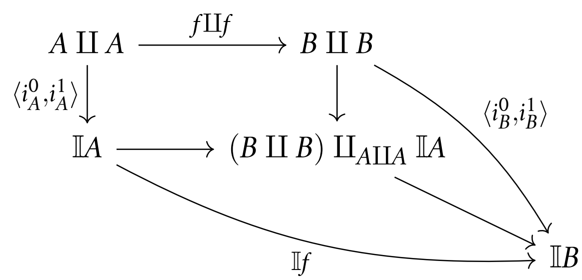
$$B \amalg_A \mathbb{I}A \to \mathbb{I}B, \qquad \mathbb{I}A \amalg_A B \to \mathbb{I}B$$

are anodyne cofibrations.

Proof. By proposition 3.7, giving M a closed $\widehat{\square}$-module structure amounts to specifying the data in (a) and (b). To apply the lemma, we should compute $i_1 \,\square\, f$ and $u_1^{1,\varepsilon} \,\square\, f$ for a morphism $f : A \to B$ of M. The map

$$i_1 : \square_0 \amalg \square_0 \cong \partial\square_1 \to \square_1$$

is the one induced by ι_0 and ι_1. Hence $\partial\square_1 \otimes_{\mathbb{I}} A = A \amalg A$, and $i_1 \otimes_{\mathbb{I}} A : A \amalg A \to \mathbb{I}A$ is the map induced by i^0_A and i^1_A, and likewise for B. We conclude that $i_1 \,\square\, f$ is the "corner map" in



which, by rearranging pushouts, we may also describe as the induced map $B \amalg_A \mathbb{I}A \amalg_A B \to \mathbb{I}B$. The computation of $u_1^{1,\varepsilon} \,\square\, f$ is similar except that the domain of $u_1^{1,\varepsilon}$ is only one copy of $\square_0$. $\square$

Definition 3.14 ([CS23], 3.13). A *cylindrical premodel structure* on a category M consists of a premodel category structure on M together with the data specified in (a)–(c) of the previous proposition.

The conclusion of everything so far is

a cylindrical premodel category is the same as a $\widehat{\square}$-module premodel category

and, from now on, the only fact we really need to know about $\widehat{\square}$ is that it is a model category that makes the statement above true.

Remark 3.15. Suppose we add more "structure" to our plain cube category, obtaining a new category $\square'$. Then closed $\widehat{\square'}$-module categories will be described by a correspondingly-extended version of (b) of the proposition. For example, if we add reversals to the cube category, then we should also ask for a natural transformation $r : \mathbb{I} \to \mathbb{I}$ satisfying $r^2 = \mathrm{id}$, $ri_0 = i_1$, $ri_1 = i_0$, $pr = p$. Depending on the kind of structure we add, however, there may or may not be a model category structure on $\widehat{\square'}$ whose generating (acyclic) cofibrations can be described in a convenient way, to give a corresponding version of condition (c).

If our objective is just to get an enrichment in any model category, then the plain cube category is the most convenient choice, since it involves the least amount of structure on the cylinder functor.

We end this section with some ways to construct cylindrical premodel structures, including the case of cartesian cubical sets.

Construction 3.16. Suppose M is a *monoidal* premodel category. In this case, we could ask that the cylinder functor $\mathbb{I} : M \to M$ be of the form $\mathbb{I}A = I \otimes A$ for some "interval object" $I \in M$, and that the natural transformations i^0, i^1, p are also induced by morphisms between $\mathbb{1}$ and I. In this case, the required data amounts to an anodyne cylinder object on $\mathbb{1} \in M$ (defined below).

The corresponding $\widehat{\square}$-module structure of M actually arises from a monoidal Quillen functor from $\widehat{\square}$ to M itself, carrying $\square_1$ to I. Hence, we could have constructed it by considering M directly instead of working with $\mathrm{LAdj}(M, M)$. This situation has also been analyzed in [Gri21].

Definition 3.17 ([Bar20], 3.1.2)**.** For M a premodel category and $A \in M$ a cofibrant object, an *anodyne cylinder object* on A consists of an object $C \in M$, anodyne cofibrations $i^0, i^1 : A \to C$, and a map $p : C \to A$ with $pi^0 = pi^1 = \mathrm{id}_A$, such that the map $\langle i^0, i^1 \rangle : A \amalg A \to C$ is a cofibration.

If M is a *model* category, then any cofibrant object A of M must admit an anodyne cylinder object. Indeed, as usual, we factor the fold map $A \amalg A \to A$ into a cofibration $\langle i^0, i^1 \rangle : A \amalg A \to C$ followed by a weak equivalence $p : C \to A$. The maps i^0 and i^1 are individually cofibrations (because A is cofibrant) as well as weak equivalences by two-out-of-three, hence anodyne cofibrations.

Therefore, if we are trying to prove that a given premodel category M is a model category, we had better be able to at least construct an anodyne cylinder object on any given object. The previous construction says that when M is monoidal, an anodyne cylinder object on the unit object $\mathbb{1} \in M$ gives rise to a cylindrical premodel structure and hence a $\widehat{\square}$-module premodel structure on M.

Construction 3.18. Suppose M is instead a premodel category with a $\mathbb{V}$-module structure for some arbitrary monoidal model category $\mathbb{V}$. As described above, we can choose an anodyne cylinder object $\mathbb{1} \amalg \mathbb{1} \to C \to \mathbb{1}$ on the unit object $\mathbb{1} \in \mathbb{V}$. We then construct a cylindrical premodel structure on M by setting $\mathbb{I}A = C \otimes A$, and so on.

These constructions indicate that $\widehat{\square}$ is a kind of "universal monoidal model category", in the sense that a premodel category that is a module over any model category $\mathbb{V}$ can also be made into a $\widehat{\square}$-module. This is however not a true universal property since there are generally many nonisomorphic choices of anodyne cylinder object in $\mathbb{V}$, which give rise to many nonisomorphic $\widehat{\square}$-module structures on M.

Construction 3.19 (Cartesian cubical sets). We show below (proposition 3.20) that the premodel category of *cartesian cubical sets* [Awo23] is a monoidal premodel category (for the cartesian monoidal structure). This is more than we strictly need to know in order to give it a cylindrical premodel structure, but it seems better to get it out of the way at some point.

Given this, we take the anodyne interval object

$$1 \amalg 1 \xrightarrow{\langle \delta_0, \delta_1 \rangle} \mathrm{I} \to 1$$

where $I \in$ cSet is the cartesian cubical interval and δ_0, δ_1 are the endpoint inclusions. The maps δ_0 and δ_1 are anodyne cofibrations by Remark 31 and the map $\langle \delta_0, \delta_1 \rangle$ is a cofibration by Remark 20. Hence, there is a corresponding cylindrical premodel structure on cSet whose interval functor is given by $\mathbb{I}A := I \times A$.

The corresponding plain cubical enrichment of cSet can be described as follows:

- $\square_n \otimes A = I^n \times A$. More generally, $K \otimes A = j_! K \times A$, where j is the functor between cube categories sending the plain cube $\square[n]$ to the cartesian cube $[n]$, and $j_!$ is left Kan extension along this functor.
- Likewise, $\{\square_n, X\} = X^{\mathrm{I}^n}$ and more generally $\{K, X\} = X^{j_! K}$.
- For A, $X \in$ cSet, the plain cubical mapping space is given by $\mathrm{map}(A, X) = j^*(X^A)$, since its n-cubes are given by

$$\mathrm{map}(A, X)_n = \mathrm{Hom}(\mathrm{I}^n \times A, X) = \mathrm{Hom}(\mathrm{I}^n, X^A) = \mathrm{Hom}(j_!(\square_n), X^A).$$

These are easy to write down from first principles; the purpose of all the machinery is to verify that the functors $(\otimes, \{-,-\}, \mathrm{map})$ satisfy the conclusions of proposition 2.5.

Proposition 3.20. *The premodel category of cartesian cubical sets is cartesian monoidal as a premodel category.*

Proof. We need to check that:

1. The pushout product of two cofibrations is a cofibration.
2. The pushout product of a cofibration and an anodyne cofibration is an anodyne cofibration.

The first statement follows quickly from the axioms on cofibrations, which include that cofibrations are closed under pullback, that every cofibration is a monomorphism, and that if the inclusions of two subobjects of an object are cofibrations, then so is the inclusion of their union.

To check the second statement, it suffices to consider the generating (anodyne) cofibrations. A typical generating anodyne cofibration is of the form $I_!(c \,\square_i\, \delta)$ where c is a morphism of $\mathrm{cSet}_{/I}$ whose underlying morphism $I_! c$ is a cofibration. Suppose b is a cofibration of cSet. We want to show that $b \,\square\, I_!(c \,\square_i\, \delta)$ can be expressed in the same form. We will compute it by considering the morphisms (squares) from this morphism to an arbitrary morphism f of cSet. Below, Hom and $\mathrm{Hom}_{/I}$ denotes the Hom-set in *the arrow category* of either cSet or $\mathrm{cSet}_{/I}$, respectively.

$$
\begin{aligned}
\mathrm{Hom}(b \,\square\, I_!(c \,\square\, \delta), f) &= \mathrm{Hom}(I_!(c \,\square\, \delta), b \Rightarrow f) \\
&= \mathrm{Hom}_{/I}(c \,\square\, \delta, I^*(b \Rightarrow f)) \\
&= \mathrm{Hom}_{/I}(c \,\square\, \delta, I^*b \Rightarrow I^*f) \\
&= \mathrm{Hom}_{/I}(I^*b \,\square\, c \,\square\, \delta, I^*f) \\
&= \mathrm{Hom}(I_!((I^*b \,\square\, c) \,\square\, \delta), f)
\end{aligned}
$$

A direct calculation shows that $I_!(I^*b \,\square\, c) = b \,\square\, I_!c$, which is a cofibration. □

4 The weak equivalences

We now set out to prove the following theorem.

Theorem 4.1 ([CS23], 3.26). *Let M be a premodel category such that:*

1. *Every object of M is cofibrant.*
2. *M admits a cylindrical structure.*
3. *Any cofibration with the left lifting property with respect to fibrations* between fibrant objects *is an anodyne cofibration.*

Then M is a model category.

Remark 4.2. If M satisfies the "fibration extension property", then the third condition on M is satisfied, because any fibration of M is a pullback of a fibration between fibrant objects (obtained by extending the fibration to a fibrant replacement of its codomain).

We comment first on the necessity of the conditions. The first condition is obviously not necessary for M to be a model category, but it will quickly become clear that it is needed for our proof strategy to succeed. The second condition is probably not strictly necessary, but in practice model categories tend to satisfy it. For example, we saw it is automatic in a monoidal or enriched model category. The third condition is necessary, as it is a true fact in all model categories: If $f : A \to B$ is a cofibration with the left lifting property with respect to fibrations between fibrant objects, then by lifting f against a fibrant replacement of f we can deduce that f is a weak equivalence.

Proof. By proposition 3.13, M admits a $\widehat{\square}$-module structure $(\otimes, \{-,-\}, \mathrm{map})$. We define a map $f : A \to B$ of M to be a *weak equivalence* if for every fibrant object $X \in M$, the induced map

$$
\mathrm{map}(f, X) : \mathrm{map}(B, X) \to \mathrm{map}(A, X)
$$

is a weak equivalence of $\widehat{\square}$.

This is a sensible definition because we expect $\mathrm{map}(A, X) \in \widehat{\square}$ to represent the "correct" space of maps from A to X provided A is cofibrant and X is fibrant; and we assumed that every object of M is cofibrant. Furthermore, the weak equivalences of M are supposed to be the maps which are inverted in the ∞-categorical localization $\mathrm{Ho}^{\infty} M$ of M. By the Yoneda lemma we can test this by looking at the space of maps into all objects of $\mathrm{Ho}^{\infty} M$, and these will all be represented by fibrant objects of M.

Write W for the weak equivalences and C, AC, F, AF for the cofibrations, anodyne cofibrations, fibrations and anodyne fibrations of M respectively. The weak equivalences of M satisfy the two-out-of-six property: if f, g, h are composable maps such that fg and gh are weak equivalences, then f, g, h are all weak equivalences. This is because two-out-of-six holds in any model category, such as $\widehat{\square}$.

Thus, it remains to verify that $\mathsf{AC} = \mathsf{C} \cap \mathsf{W}$ and $\mathsf{AF} = \mathsf{F} \cap \mathsf{W}$. By a well-known argument, it is actually enough to prove $\mathsf{AC} = \mathsf{C} \cap \mathsf{W}$ and $\mathsf{AF} \subseteq \mathsf{W}$, as we know that $\mathsf{AF} \subseteq \mathsf{F}$ and then a retract argument proves that $\mathsf{F} \cap \mathsf{W} \subseteq \mathsf{AF}$. We will separately prove $\mathsf{AC} \subseteq \mathsf{C} \cap \mathsf{W}$, $\mathsf{C} \cap \mathsf{W} \subseteq \mathsf{AC}$, and $\mathsf{AF} \subseteq \mathsf{W}$.

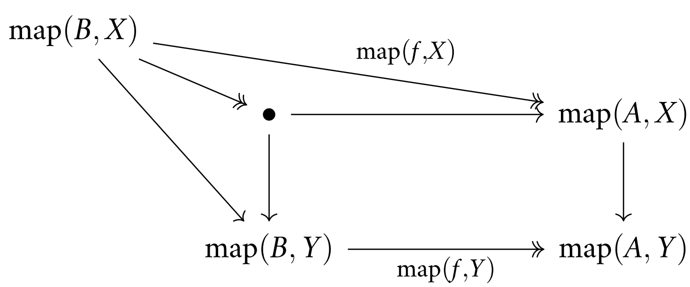
$\mathsf{AC} \subseteq \mathsf{C} \cap \mathsf{W}$: Suppose $f : A \to B$ is an anodyne cofibration of M and X is a fibrant object, so $X \to 1$ is a fibration. By proposition 2.5, the induced map

$$\mathrm{map}(f, X) : \mathrm{map}(B, X) \to \mathrm{map}(A, X)$$

is then an acyclic fibration of $\widehat{\square}$, hence in particular a weak equivalence.

$\mathsf{C} \cap \mathsf{W} \subseteq \mathsf{AC}$: Suppose $f : A \to B$ is a cofibration and a weak equivalence. We need to show that f is an anodyne cofibration. By assumption, it suffices to prove that f has the left lifting property with respect to any fibration $p : X \to Y$ between fibrant objects of M.

Applying proposition 2.5, we see that in the diagram



the double-headed arrows are fibrations, since X and Y are fibrant (and A and B are cofibrant). Furthermore, the two arrows of the form $\mathrm{map}(f, -)$ are weak equivalences, because f is one. Using the pullback closure of acyclic fibrations and two-out-of-three, we deduce that the corner map

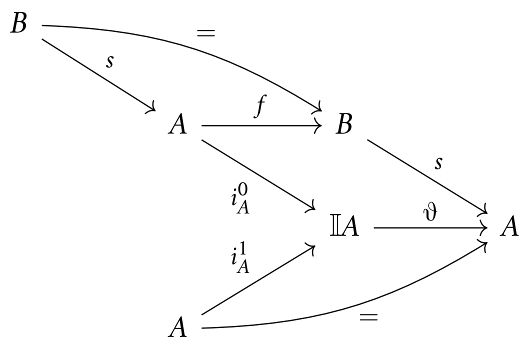
$$e : \mathrm{map}(B, X) \to \mathrm{map}(A, X) \times_{\mathrm{map}(A,Y)} \mathrm{map}(B, Y)$$

is also a weak equivalence, hence an acyclic fibration. In particular e has the right lifting property with respect to $\varnothing \to \mathbb{1}$. Because $\mathrm{Hom}(\mathbb{1}, \mathrm{map}(B, X)) = \mathrm{Hom}(\mathbb{1} \otimes B, X) = \mathrm{Hom}(B, X)$ and so on, this exactly means that f has the left lifting property with respect to p.

AF $\subseteq$ W**:** Suppose $f : A \to B$ is an anodyne fibration. Using its lifting property against $\varnothing \to B$ and then against $A \amalg A \to \mathbb{I}A$, we deduce the existence of:

- first, a map $s : B \to A$ with $fs = \mathrm{id}_B$;
- second, a "homotopy" $\vartheta : \mathbb{I}A \to A$ with $\vartheta \circ i_A^0 = sf$, $\vartheta \circ i_A^1 = \mathrm{id}$.

We can assemble this data into the following diagram:



We also know that the maps $i_A^0, i_A^1 : A \to \mathbb{I}A$ are anodyne cofibrations, hence weak equivalences (as shown earlier). Using two-out-of-three, we deduce that ϑ is a weak equivalence and so both sf and fs are weak equivalences. Hence, by two-out-of-six, f is also a weak equivalence (as is s). □

Remark 4.3. The proof used one specific definition of the weak equivalences of M. But once we know that M is a model category, we also know *a fortiori* other descriptions of the weak equivalences that must agree. For example, the weak equivalences are exactly the maps which can be expressed as a composition of an acyclic cofibration and an acyclic fibration, since this is the case in any model category. We also see in this way that the resulting model structure does not depend on the choice of cylindrical structure (or, equivalently, the cubical enrichment).

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