Referee's report on

The Existence and Uniqueness of the Taylor Series of Iterated Functions

by

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The aim of the present paper is an interesting endeavor. Hierarchically one can view arithmetic operations on integers in four steps

(0) Addition by 1. $a' = a + 1$.

(1) This leads immediately to addition $a+b$, where a and b are integers.

(2) b iterations of: a, $a + a \cdots + a$ leads to multiplication $a \cdot b$.

(3) The next step is exponentiation $a \cdot \cdots \cdot a$ (b times) which is denoted by a^b.

All three operation (1)–(3) have natural extensions to complex numbers even if (3) requires the logarithm function which is naturally multivalued.

Continuing the above construction one is lead to consider

(4) a, a^a, a^{a^a}, etc. until

$$a^{\cdot^{\cdot^{\cdot^{a}}}} \qquad (b \quad \text{times}).$$

Formally one can define recursively

$$x_{n+1} = a^{x_n},$$

with $x_1 = a$, $n = 1, \ldots, b-1$ and define

$${}^b a = x_b.$$

The present author call this operation *teration*. It has been treated historically by several authors, but also recently in forums on the internet.

The main proble which is addressed in the present paper is how (or in which sense) the operation of teration can be extended to complex numbers and in terms of analytic functions.

One can view this as the problem of solutions of the functional equation

$$F(z+1) = a^{F(z)}, \tag{1}$$

where F is analytic in some domain. This solution cannot be unique.

This can be compared to the factorial function $n!$, which has the extension to complex numbers by the gamma funtion, through the functional equation

$$\Gamma(z+1) = z\Gamma(z).$$

By the Bohr-Mollerup theorem $\Gamma(x)$ is the only function defined for $x > 0$ such that

- $f(1) = 1$, and
- $f(x+1) = xf(x)$, for $x > 0$, and
- f is logarithmically convex,

and one could perhaps hope that a similar result hold for $z \mapsto {}^{z}a$ at least for $z = x > 0$ and a real and positive. In particular is there a natural definition of $z \mapsto {}^{z}a$, which makes it analytic?

The present paper tries to address this question, which is somewhat off the common track, but a reasonable answer would be interesting.

Part of the the problem to define complex teration is to extend iteration from integers to real numbers

This is done by embedding the iteration into a flow and this is well-known. For an analytic map to a periodic point with multiplier λ, $|\lambda| \neq 1$, this is done by solving Schröder's functional equation and for $\lambda = 1$ there is the theory of parabolic conjugation (Abel's equation). For $|\lambda| = 1$, with $\lambda = e^{2\pi\theta}$, with θ Diophantine Siegel proved the existence of analytic conjugation and this is mentioned in the present paper (without reference to Siegel).

However the present author does not mention the work of Bruno and Yoccoz, which give necessary and sufficient conditions for linearization and then also gives possibilities to extend of maps to flows.

In sections 2.1–2.4, the author writes a recursion formula for the n:th derivative of the the t:th iterate f^t, where t may be a non-integer. This results in the construction of the Taylor series of f^t, where t is an integer, formula (7) in the paper.

In Section 2.6.1 the author classifies fixed points, but omits the results of Bruno and Yoccoz mentioned before.

In Section 2.6.2 (Theorem 4) the author gives *without proof* the first terms of a power series expansion of the solution of the Schröder equation. By Siegel's theorem this power series converges also in the Diophantine case and the first coefficients are easy to compute by computing coefficients for ϕ in the functional equation.

$$f \circ \varphi(z) = \varphi(\lambda z).$$

In the parabolic case $\lambda = 1$, Geisler gives the first terms of the formal solutions which are not expected to converge. Analytic solutions require Fatou coordinate and gives approximations by asymptotic expansions. For a recent discussion of this see eg the thesis of Dudko

`http://www.math.toronto.edu/graduate/Dudko-thesis.pdf`.

Écalle and Voronin described a set of invariants which completely determine a conjugacy class. The approach of Écalle gives rice to the so called *Resurgence theory*. Basically one starts with the analytic germ

$$f(z) = z + az^2 + \mathcal{O}(z^3).$$

If $a \neq 0$ one can by a change of coordinates make $a = 1$.

By another change of coordinates $w = -1/z$ and considering the germ at infinity

$$f(z) = -/F(-1/z)$$

one gets

$$f(z) = z + 1 + \mathcal{O}(1/z)$$

It is known that there are constants $c > 0$, $\pi > \alpha > \pi/2$ so that the functional equation

$$\phi(f(z)) = \phi(z) + 1 \tag{2}$$

has analytic solutions in the sectors

$$\{|\arg(z - c)| < \alpha \qquad \text{and} \qquad \{|\arg(z + c) - \pi| < \alpha\}.$$

with asymptotics

$$\phi(z) = \text{const} + z + A \log z + \mathcal{O}(z^{-1}).$$

These solutions are so called Fatou coordinates. The resurgence theory of Écalle is based on the observation that the equation (2) has a formal solution

$$\tilde{\phi}(z) = \text{const} + z + A \log z + \sum_{j=1}^{\infty} b_j z^{-j}, \tag{3}$$

where $\sum_{j=1}^{\infty} b_j z^{-j}$ is a divergent power series.

The Fatou coordinates can be obtained from (3) by Borel summation of the divergent series.

In Section 2.6.3, Geisler gives without any proof or motivation a formal power series for the iterated function $\sin^t(z)$. There is no discussion of the convergence. In view of Ecalle's results one would expect that the power series is divergent but Borel summable.

The following facts are fairly well known:

The functional equation $(f \circ f)(x) = e^x$ has a real analytic solution (Kneser, 1950). The solution given by Kneser is to reduce the problem to the solution of the Abel equation as follows:

Suppose we want to solve

$$\psi \circ \varphi(x) = F(x).$$

Then consider the Abel functional equation

$$\psi(F(x)) = \pi(x) + \beta.$$

This solution is not single-valued (Baker, 1955) and there is no uniqueness. Baker proved that

$$e^x - 1 = x + \frac{x^2}{2} + \dots$$

cannot be embedded in an analytic flow.

He proved that the formal power series obtained, $\varphi_t(z)$, where $\varphi_1(z) = e^z - 1$ only has a positive radius of convergence when $t = n$, an integer.

In Section 3.1 (Extending teration) the author discusses fixed points of teration and this does in the view of the referee not make sense since the teration itself is not well defined. The referee believes that he tries to find the boundary of the set of z:s such that z_n remains bounded as $n \to \infty$, where $z_0 = z$

$$z_{n+1} = z^{z_n},$$

and the exponentiation is defined by the principle branch of the logaritm.

This boundary has previously been studied by Baker and Rippon "Convergence of infinite exponentials." (Ann. Acad. Sci. Fenn. Ser. A I Math. 8 (1983), no. 1, 179–186), and "A note on complex iteration." Amer. Math. Monthly 92 (1985), no. 7, 501–504.

The main result is in Section 3.2, Existence and Uniqueness of Hyperoperators is in my view totally incomplete. The author start by defining the first hyperoperator teration and this requires a fixed point which is not well defined. Also the linearizing power series is not well defined and there are definite problems with the convergence of the power series, compare the discussion of formal power series ofthe flow $\varphi_t(z)$ in the fixed point $z = 0$ of $z \mapsto e^z - 1$.

The results, formula number (11), of Theorem 9 in the paper are not verified at all.

In summary, the problem of the existence of solutions to the functional equation

$$F(z+1) = e^{F(z)},$$

and whether there are "natural" definition of (analytic) solutions is still an interesting open problem but the present paper does not solve it. The paper is not recommended for publication in Annals of Mathematics.