logic in color

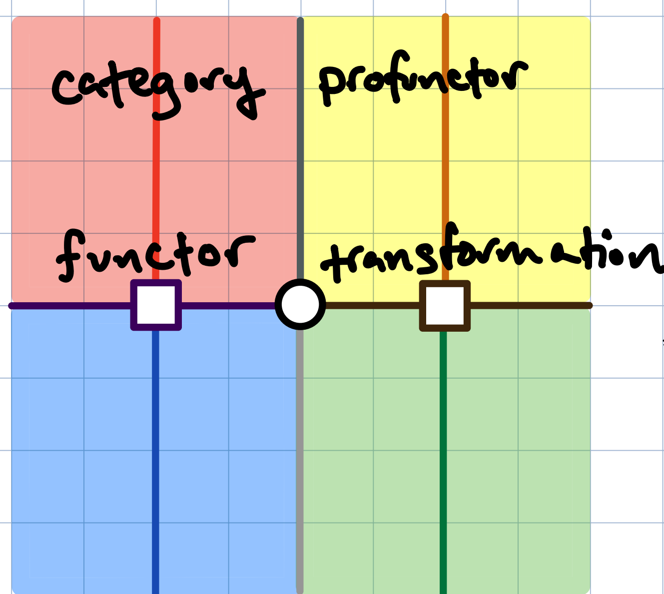
Higher-Order Logic

Recap

$X : x \xrightarrow{R} y : Y$

$A : fx \xrightarrow{S} gy : B$

Cat



$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\psi"] & Y \arrow[d, "g"] \\ A \arrow[r, "\Psi"'] & B \end{tikzcd}$$

From sets + functions,
we built the language of categories.

$$R \vdash T = \tilde{\prod} xy .\ xRy \vdash xTy$$

$$R \circ U = \tilde{\Sigma} x .\ -Rx \circ xU-$$

First, we used the language to verify the basic structure of $\mathbb{C}\mathrm{at}$.

- seq & par comp : double category
- functor → profunctor : fibrant dc (equipment)

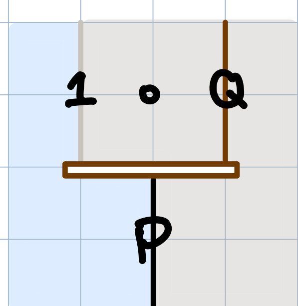
Now, let's actually do logic.

Predicate logic uses $\exists$ + $\forall$.
We used these to form Rel,
but how do we use them in Rel?

Suppose we have a predicate $P : X \mid 1$.
Its universal quantification is $\forall x Px : 1 \mid 1$ (a proposition),
characterized by $\dfrac{Q}{\forall x Px} \sim \forall x \dfrac{Q[x]}{Px}$

In colors, this looks like:

$\forall x \dfrac{Q[x]}{Px}$



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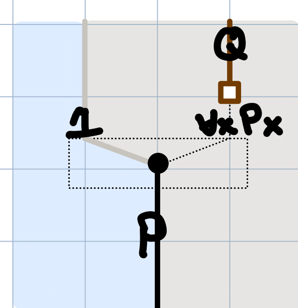


$\dfrac{Q}{\forall x Px}$

$(\forall x Px)$

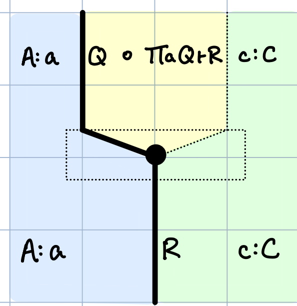
This is the right extension of P along 1.

Let $Q : A|B$ & $R : A|C$.

The <u>right extension</u> of R along Q is:

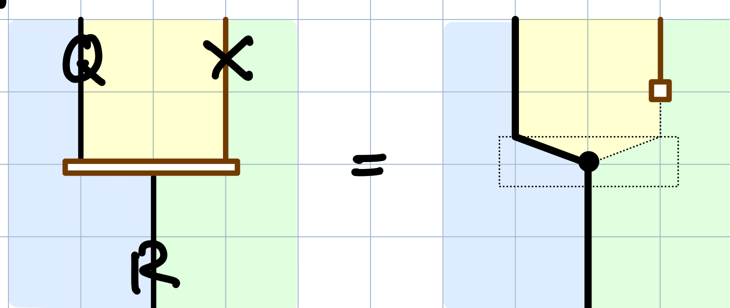
$\Pi a. Q \vdash R : B|C$

$$\frac{A{:}a \quad Q \circ (\Pi a. Q \vdash R) \quad c{:}C}{A{:}a \quad R \quad c{:}C}$$



"eval"

so that for every inference $\gamma : Q \circ X \vdash R$ there is a unique $\bar{\gamma} : X \vdash \Pi a. Q \vdash R$ so that



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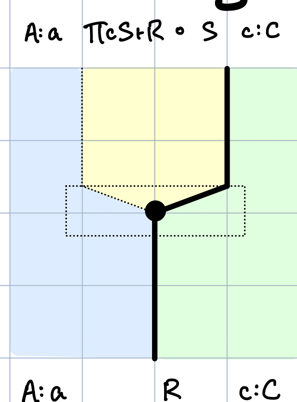
The <u>right lifting</u> of R along S is:

$$\frac{A{:}a \quad (\Pi c. S \vdash R) \circ S \quad c{:}C}{A{:}a \quad R \quad c{:}C}$$



such that (same).

How is this useful?

Higher-order logic is based on right adjoints:

$$
\begin{array}{ccc}
\text{props} & \qquad & \text{sets} \\
\dfrac{X \wedge Y \vdash Z}{X \vdash Y \Rightarrow Z} & & \dfrac{A \times B \vdash C}{A \vdash B \to C}
\end{array}
$$

Right extension along Q is right adjoint to precomposition by Q:

$$
\begin{array}{lrl}
 & Q \circ X & \vdash R \\
= \Pi ac.\ (\Sigma b\ aQb \circ bXc) & & \vdash aRc \\
\sim \Pi ac.\ \Pi b\ \ aQb \circ bXc & & \vdash aRc \\
\sim \Pi abc. & bXc & \vdash (aQb \vdash aRc) \\
\sim \Pi bc. & bXc & \vdash \Pi a.\, aQb \vdash aRc \\
 & X & \vdash \text{extension}
\end{array}
$$

$$
\begin{array}{lrl}
 & Q \circ X & \vdash R \\
= \Pi ac\ (\Sigma b\ aQb \circ bXc) & & \vdash aRc \\
\sim \Pi ac\ \ \Pi b\ aQb \circ bXc & & \vdash aRc \\
\sim \Pi abc & bXc & \vdash (aQb \vdash aRc) \\
\sim \Pi bc & bXc & \vdash \Pi a.aQb \vdash aRc \\
= & X & \vdash \underline{\Pi a.Q \vdash R}
\end{array}
$$

(we can prove $[B|C] \underset{\Pi Q}{\overset{Q \circ -}{\rightleftarrows}} [A|C]$.)

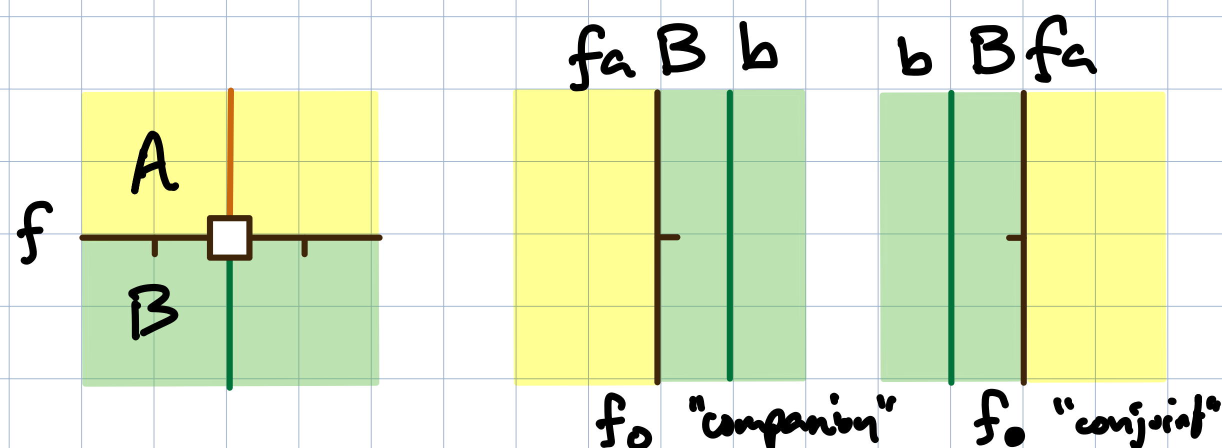
This generalizes $\forall$ to categories.

- exercise: do this for postcomposition.

So, for any pair $Q : A|B$ + $R : A|C$
there's a "universal judgement" $\Pi a. Q \vdash R : B|C$.

This is great — yet it's a judgement.
Logic is really "about" *terms*.

Recall that every functor determines a dual pair of "profunctors" (relations)



A *representation* of a profunctor $R : A|B$ is a functor $f : A \to B$ or $g : B \to A$ & an invertible transform
$$R \sim f_\circ \quad \text{or} \quad R \sim g_\bullet .$$

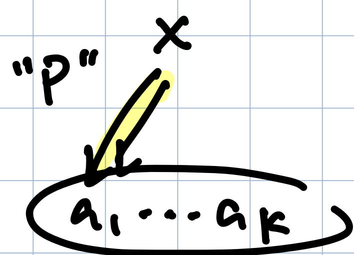
Representations of universal judgements are extremely useful.

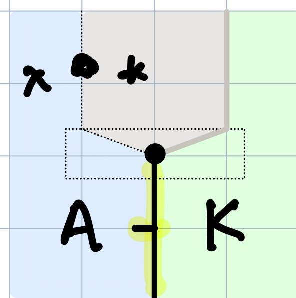
Let $a_{(-)} : \overset{\text{set}}{K} \to A$ be a function. (picks out $\{a_k\} : A$)

This defines a profunctor

"∀" $a_\bullet = A(-, a_{(-)}) : A \mid K$ ("conjoint")

The right lifting along $* : 1 \mid K$ defines $\Pi k. * \vdash a_\bullet : A \mid 1$.

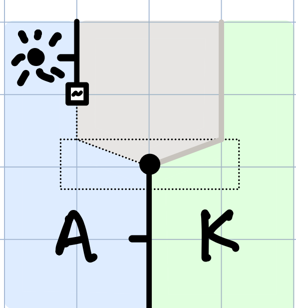


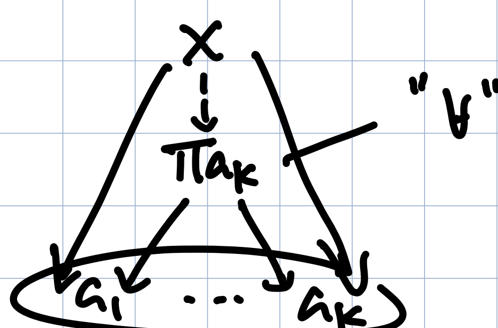


$\Pi k. * \vdash A(-, a_k)$

cone

A representation of $\Pi k. A(-, a_k)$ is a <u>product</u> $\Pi a_k : A$.





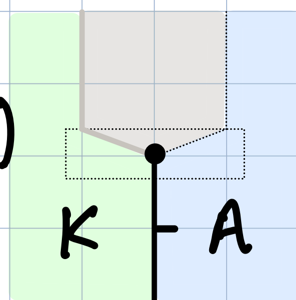
In general, a <u>limit</u> of a functor $f : I \to A$ is a representation of $\Pi i. A(-, fi)$.

"$\exists$"

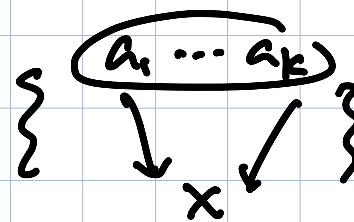
Dually, $a_{(-)} : K \to A$ defines
$a_\circ = A(a_{(-)}, -) : K \mid A$. ("companion")

The right extension along $* : K \mid 1$
defines $\Pi k.\, * \vdash a_\circ$

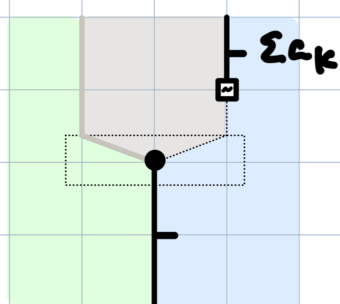
$\Pi k.\, * \vdash A(a_k, -)$

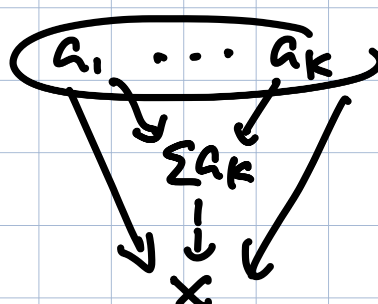


$x \mapsto$



A representation of $\Pi k.\, A(a_k, -)$
is a <u>coproduct</u> $\Sigma a_k : A$. "$\exists$"





In general, a <u>colimit</u> of $f : I \to A$
is a representation of $\Pi i.\, A(fi, -)$.

Puzzle: let $I = \boxed{0 \rightrightarrows 1}$ & $f : I \to A$.
What is the limit & colimit of f?

(equations...)

Note: We've only used $* : I \to 1$.
This gives "cones & cocones",
which consist of individual morphisms.

Let $f : I \to A$ & $W : C | I$.
Then the W-weighted limit of f,
if it exists, is a representation

$$(\Pi W.f)_\bullet \sim \Pi i. W \vdash f_\bullet \quad .$$

If $W : I | C$, then the W-weighted colimit
is a representation

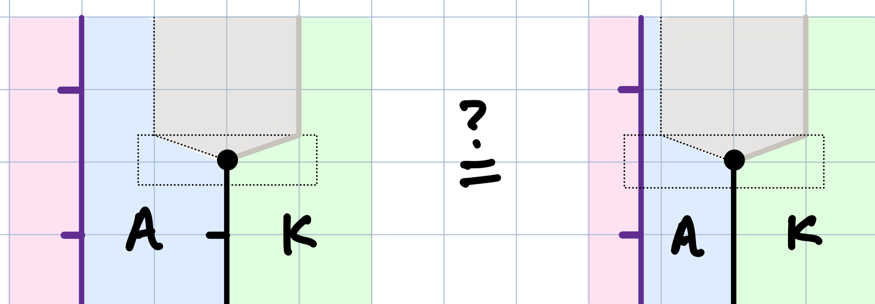
$$(\Sigma W.f)_\circ \sim \Pi i. W \vdash f_\circ \quad .$$

(puzzle: what are these?)

Preservation

Limits & colimits are precious jewels, and we care whether composition *preserves* them.

For example, given $\prod a_k : A$ & $f : A \to B$, there is a canonical $f(\prod a_k) \to \prod f(a_k)$. If it is invertible, the product is preserved.



(formal)

Puzzle: right adjoints preserve weighted limits (right lifts)

& left adjoints preserve weighted colimits (right exts)

Dually, there are left exts & lifts — but they don't always exist.

Try it: what is needed?

Questions / Thoughts?