Lie Algebras from Commutative Differential Algebras

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Abstract

We present a construction of a Lie algebra from any commutative differential algebra, generalizing the Witt algebra in positive characteristic. To the best of our knowledge, this result has not been previously documented in the literature.

1 Introduction

The Witt algebra $W(1;\underline{1})$ over a field of characteristic $p > 0$ is defined as $k[X]/\langle X^p \rangle$ with bracket

$$[f + \langle X^p \rangle, g + \langle X^p \rangle] = f\frac{d}{dX}(g) - g\frac{d}{dX}(f) + \langle X^p \rangle,$$

where $f, g \in k[X]$ and $\frac{d}{dX}$ denotes differentiation. This paper generalizes the Witt algebra by constructing Lie algebras from commutative differential algebras over any field, regardless of characteristic.

2 Main Result

Proposition 2.1. *Let (R, D) be a commutative differential algebra over a field k, where $D : R \to R$ is a k-linear derivation. Define a bracket $[x, y] = xD(y) - yD(x)$ for $x, y \in R$. Then $(R, [-, -])$ is a Lie algebra over k.*

Proof. Bilinearity, antisymmetry and the Jacobi identity can be verified by direct computation. □

3 Examples

Example 3.1. *Let k be a field. The usual derivative $\frac{d}{dX}$ on $k[X]$ defines a Lie algebra with bracket*

$$[f, g] = f\frac{d}{dX}(g) - g\frac{d}{dX}(f).$$

Example 3.2. *If k is a field of characteristic $p > 0$, then $\frac{d}{dX}$ on $k[X]/\langle X^p \rangle$ induces a Lie algebra with bracket*

$$[f + \langle X^p \rangle, g + \langle X^p \rangle] = f\frac{d}{dX}(g) - g\frac{d}{dX}(f) + \langle X^p \rangle,$$

recovering the Witt algebra.

Note that the well-definedness of $\frac{d}{dX}$ on $k[X]/\langle X^p \rangle$ is a consequence of $\operatorname{char} k = p$.

References

[1] H. Strade, *Simple Lie Algebras over Fields of Positive Characteristic. I. Structure Theory*, de Gruyter, 2004.