Lifting Algebraic Reasoning to Generalized Metric Spaces

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Outline

Definitions ($\mathbf{Alg}(\Sigma)$)

A **signature** Σ is a set of operation symbols, each with an arity, we write $\mathrm{op} : n \in \Sigma$ for an operation of arity $n \in \mathbb{N}$ belonging to Σ.

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$$\Sigma : \mathbf{Set} \to \mathbf{Set} = X \mapsto \coprod_{\mathsf{op}:n \in \Sigma} X^n.$$

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A **Σ-algebra** is a function $\Sigma(A) \to A$, i.e. an interpretation $[\![\mathsf{op}]\!]_A : A^n \to A$ for every $\mathsf{op} : n \in \Sigma$. A **homomorphism** is a function $h : A \to B$ such that

$$\begin{array}{ccc} \Sigma(A) & \xrightarrow{\Sigma(h)} & \Sigma(B) \\ {\scriptstyle [\![-]\!]_A}\downarrow & & \downarrow{\scriptstyle [\![-]\!]_B} \\ A & \xrightarrow[h]{} & B \end{array}$$

Equivently, h preserves the operations:

$$h([\![\mathsf{op}]\!]_A(a_1, \ldots, a_n)) = [\![\mathsf{op}]\!]_B(h(a_1), \ldots, h(a_n)).$$

Terms

Definition

The set of **Σ-terms** over a set X is defined inductively:

$$\frac{x \in X}{x \in T_\Sigma X} \qquad \frac{t_1 \in T_\Sigma X \qquad \cdots \qquad t_n \in T_\Sigma X}{\mathsf{op}(t_1, \ldots, t_n) \in T_\Sigma X}$$

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With the evident *syntactical* interpretation of operations, $T_\Sigma X$ is the **free** Σ-algebra on X. Thus, for any assignment of variables $\iota : X \to A$, we get an interpretation of any term in $T_\Sigma X$:

$$\begin{array}{ccc} \Sigma(T_\Sigma X) & \xrightarrow{\Sigma(\llbracket - \rrbracket^\iota_A)} & \Sigma(A) \\ \downarrow & & \downarrow \alpha \\ T_\Sigma X & \xrightarrow[\llbracket - \rrbracket^\iota_A]{} & A \end{array}$$

It does the thing you want it to do:

$$\forall x \in X, \llbracket x \rrbracket^\iota_A = \iota(x)$$

$$\forall t_1, \ldots, t_n \in T_\Sigma X, \mathsf{op} : n \in \Sigma, \llbracket \mathsf{op}(t_1, \ldots, t_n) \rrbracket^\iota_A = \llbracket \mathsf{op} \rrbracket_A(\llbracket t_1 \rrbracket^\iota_A, \ldots, \llbracket t_n \rrbracket^\iota_A).$$

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Example

A semilattice is an algebra for $\Sigma_{\mathcal{P}} = \{\oplus : 2\}$ satisfying the following equations:

$$x \vdash x \oplus x = x \qquad \text{(idempotent)}$$

$$x, y \vdash x \oplus y = y \oplus x \qquad \text{(commutative)}$$

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Definition ($\mathbf{Alg}(\Sigma, E)$)

Given a set E of equations over Σ, $\mathbf{Alg}(\Sigma, E)$ is the full subcategory of Σ-algebras that satisfy all of E. It is the **variety** generated by E.

Free Algebras

Given (Σ, E), the free (Σ, E)-algebra over a set X is given by

$$T_\Sigma X / {\equiv_E},$$

where $\equiv_E$ is the smallest congruence generated by E:

$$\equiv_E \; = \{(s, t) \mid X \vdash s = t \text{ is satisfied by all } \mathbb{A} \in \mathbf{Alg}(\Sigma, E)\}.$$

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Example

The free semilattice (i.e. idempotent, commutative, and associative $\Sigma_\mathcal{P}$-algebra) on a set X is the non-empty finite powerset $\mathcal{P}(X)$ where $\oplus$ is interpreted as the union, and there is a monad isomorphism $T_{\Sigma_\mathcal{P}, E_\mathcal{P}} \cong \mathcal{P}$. We say $(\Sigma_\mathcal{P}, E_\mathcal{P})$ is an **algebraic presentation** of the monad $\mathcal{P}$.

Finitary Monads

The free (Σ, E)-algebra monad is always **finitary**, i.e. it preserves filtered colimits, and (Σ, E)-algebras correspond to Eilenberg–Moore algebras for $T_{\Sigma,E}$:

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Moreover, for any finitary monad $M : \mathbf{Set} \to \mathbf{Set}$, you can:

- define the signature $\Sigma_M = \bigcup_{n \in \mathbb{N}} M(n)$,
- embed $\mathbf{EM}(M)$ as a full subcategory of $\mathbf{Alg}(\Sigma_M)$ (using finitary assumption),
- show it is closed under homomorphic images, subalgebras, and products, and
- conclude, using Birkhoff's variety theorem, that $\mathbf{EM}(M) \cong \mathbf{Alg}(\Sigma_M, E_M)$ for some set of equations E_M.[1]

[1] You can also construct E_M explicitly, see [Man76, Theorem 5.40].

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After a bit more work, you obtain a dual equivalence between the category of finitary monads and the category of finitary varieties.

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Universal Algebra

Quantitative Algebras

Lifting Presentations

Future Work

Universal Quantitative Algebra

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$$d_A(\llbracket \mathsf{op} \rrbracket_A(a_1, \ldots, a_n), \llbracket \mathsf{op} \rrbracket_A(b_1, \ldots, b_n)) \leq \sup_i d_A(a_i, b_i).$$

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Definition

A **quantitative Σ-algebra** is a metric space (A,d) and a Σ-algebra on the same carrier, i.e. interpretations $\llbracket \mathsf{op} \rrbracket_A : A^n \to A$ for every $\mathsf{op} : n \in \Sigma$.

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$$\begin{array}{ccc} \mathbf{QAlg}(\Sigma) & \longrightarrow & \mathbf{Alg}(\Sigma) \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbf{Met} & \longrightarrow & \mathbf{Set} \end{array}$$

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$[0, \infty]$-spaces

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Definition ($[0,\infty]$**Spa**)

An $[0,\infty]$**-space** is a set A equipped with a distance function $d_A : A \times A \to [0,\infty]$.
Morphisms are nonexpansive maps: $f : A \to B$ such that $d_B(f(a), f(a')) \leq d_A(a, a')$.

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Definition ($[0,\infty]$**Str**)

An $[0,\infty]$**-structure** is a set A equipped with a family of binary predicates $=_\varepsilon \subseteq A \times A$ indexed by $[0,\infty]$ satisfying

$$\varepsilon \leq \varepsilon' \implies =_\varepsilon \subseteq =_{\varepsilon'} \quad \text{and} \quad =_{\inf S} = (\cap_{\varepsilon \in S} =_\varepsilon).$$

Morphisms are functions preserving the predicates: $a =_\varepsilon a' \implies f(a) =_\varepsilon f(a')$.

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Proposition

$[0,\infty]$**Spa** $\cong [0,\infty]$**Str** *by understanding* $a =_\varepsilon a'$ *as* $d_A(a, a') \leq \varepsilon$.

Given a complete lattice L (e.g. $[0,\infty]$ or $[0,1]$ or $\{0,1\}$)

Definition (**LSpa**)

An **L-space** is a set A equipped with a distance function $d_A : A \times A \to \mathsf{L}$.
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An **L-structure** is a set A equipped with a family of binary predicates $=_\varepsilon \subseteq A \times A$ indexed by L satisfying

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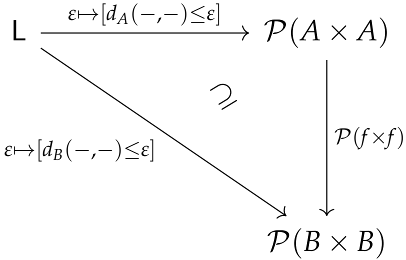
Proposition

LSpa $\cong$ **LStr**

Bonus

L**Spa** is a lax comma category of continuous functors $\mathsf{L} \to (\mathcal{P}(A \times A), \subseteq)$:



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We say a quantitative algebra $(A, d, [\![-]\!])$ **satisfies** $\mathbf{X} \vdash s =_\varepsilon t$ if for all nonexpansive assignments $\iota : \mathbf{X} \to (A, d)$, $d([\![s]\!]^\iota, [\![t]\!]^\iota) \leq \varepsilon$.

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We can replace **Met** with L**Spa**, but $=$ and $=_0$ are separate. Also, **Met** is a quantitative variety $\mathbf{QAlg}(\emptyset, E)$, and we call any such category **GMet**, e.g. **Poset**, **UMet**, **Graph**, etc.

Given a signature Σ and quantitative equations E, the free quantitative (Σ, E)-algebra over a generalized metric space $\mathbf{X}$ is given by

$$\widehat{T}_{\Sigma,E}\mathbf{X} = (T_\Sigma X/{\equiv_E}, d_E),$$

where $\equiv_E$ and d_E are a congruence and metric generated by E with quantitative equational logic:

$$\equiv_E \;=\; \{(s,t) \mid \mathbf{X} \vdash s = t \text{ is satisfied by all } \widehat{\mathbb{A}} \in \mathbf{QAlg}(\Sigma, E)\}$$
$$d_E([s],[t]) = \inf\left\{\varepsilon \in \mathsf{L} \mid \mathbf{X} \vdash s =_\varepsilon t \text{ is satisfied by all } \widehat{\mathbb{A}} \in \mathbf{QAlg}(\Sigma, E)\right\}.$$

This yields a monad $\widehat{T}_{\Sigma,E} : \mathbf{GMet} \to \mathbf{GMet}$.

Axiomatization of Hausdorff Distance

The Hausdorff lifting takes a metric on X to a metric on $\mathcal{P}X$:

$$(X, d) \mapsto (\mathcal{P}X, d_{\mathsf{H}}) \text{ where } d_{\mathsf{H}}(S, T) = \max\left\{\max_{x \in S} \min_{y \in T} d(x, y), \max_{y \in T} \min_{x \in S} d(x, y)\right\}.$$

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The quantitative $\Sigma_{\mathcal{P}}$-algebra over $(\mathcal{P}X, d_{\mathsf{H}})$ ($\oplus$ is union again) is the free algebra over (X,d) in the following theory:

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Therefore, the monad $\mathcal{P}_{\mathsf{H}} = (X,d) \mapsto (\mathcal{P}X, d_{\mathsf{H}})$ is presented by that theory.

Axiomatization of Not Hausdorff Distance

After removing that last quantitative equation, the free algebras are given by

$$(X, d) \mapsto (\mathcal{P}X, \widehat{d}) \text{ where } \widehat{d}(S, T) = \begin{cases} 0 & S = T \\ d(x, y) & S = \{x\} \text{ and } T = \{y\} \\ 1 & \text{otherwise} \end{cases}.$$

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For the Hausdorff distance, $\oplus$ is a nonexpansive operation $\mathcal{P}_\mathsf{H}\mathbf{X} \times \mathcal{P}_\mathsf{H}\mathbf{X} \to \mathcal{P}_\mathsf{H}\mathbf{X}$.
Not the case for the "not Hausdorff" distance.

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After removing that last quantitative equation, the free algebras are given by

$$(X,d) \mapsto (\mathcal{P}X, \widehat{d}) \text{ where } \widehat{d}(S,T) = \begin{cases} 0 & S = T \\ d(x,y) & S = \{x\} \text{ and } T = \{y\} \\ 1 & \text{otherwise} \end{cases} .$$

For the Hausdorff distance, $\oplus$ is a nonexpansive operation $\mathcal{P}_{\mathsf{H}}\mathbf{X} \times \mathcal{P}_{\mathsf{H}}\mathbf{X} \to \mathcal{P}_{\mathsf{H}}\mathbf{X}$. Not the case for the "not Hausdorff" distance. The former defines an enriched monad, the latter does not.

Outline

Universal Algebra

Quantitative Algebras

Lifting Presentations

Future Work

Lifting Presentations

Let (M, η, μ) be a monad on **Set**, and (Σ, E) be an algebraic presentation for it via $\rho : T_{\Sigma,E} \cong M$.

Definitions

A **monad lifting** of M to **Met** is a monad $\widehat{M} : \mathbf{Met} \to \mathbf{Met}$ whose functor, unit and multiplication coincide with those of M after applying $U : \mathbf{Met} \to \mathbf{Set}$.

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A **quantitative extension** of E is a collection of quantitative equations $\widehat{E}$ on the same signature Σ satisfying for all $\mathbf{X} \in \mathbf{Met}$ and $s, t \in T_\Sigma X$,

$$X \vdash s = t \text{ satisfied in } \mathbf{Alg}(\Sigma, E) \iff \mathbf{X} \vdash s = t \text{ satisfied in } \mathbf{QAlg}(\Sigma, \widehat{E}).$$

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Theorem

There is a "correspondence" between monad liftings of M and quantitative extensions of E. More categorically, there is a dual equivalence between the category of monad liftings of M and the category of quantitative varieties whose forgetful functor factors through $\mathbf{Alg}(\Sigma)$*, with appropriate restrictions of morphisms.*

Extension to Lifting (Easy)

- The equivalence

$$X \vdash s = t \in \mathfrak{Th}(E) \iff \mathbf{X} \vdash s = t \in \mathfrak{QTh}(\widehat{E})$$

really says that $\equiv_E = \equiv_{\widehat{E}}$, so the functors $T_{\Sigma,E}$ and $\widehat{T}_{\Sigma,\widehat{E}}$ agree on sets.

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  really says that $\equiv_E = \equiv_{\widehat{E}}$, so the functors $T_{\Sigma,E}$ and $\widehat{T}_{\Sigma,\widehat{E}}$ agree on sets.
- It follows from the syntactic definitions that the units and multiplications also coincide, hence $\widehat{T}_{\Sigma,\widehat{E}}$ is a monad lifting of $T_{\Sigma,E}$.
- Via the isomorphism $\rho : T_{\Sigma,E} \cong M$, we can construct the monad lifting by

$$\widehat{M}(X,d) = (MX, \widehat{d}), \text{ where } \widehat{d}(m,m') = d_{\widehat{E}}(\rho^{-1}m, \rho^{-1}m').$$

Lifting to Extension

- Put some equations in $\widehat{E}$:

$$\text{For all } X \vdash s = t \in E, \text{ add } \mathbf{X}_{\perp} \vdash s = t \text{ to } \widehat{E}.$$

Lifting to Extension

- Put some equations in $\widehat{E}$:

$$\text{For all } X \vdash s = t \in E, \text{ add } \mathbf{X}_\bot \vdash s = t \text{ to } \widehat{E}.$$

- Put some quantitative equations in $\widehat{E}$:

$$\text{For all } (X, d) \in \mathbf{Met} \text{ and } s, t \in T_\Sigma X, \text{ add } (X, d) \vdash s =_{\widehat{d}(\rho[s], \rho[t])} t \text{ to } \widehat{E}.$$

Lifting to Extension

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$$\text{For all } (X, d) \in \mathbf{Met} \text{ and } s, t \in T_\Sigma X, \text{ add } (X, d) \vdash s =_{\widehat{d}(\rho[s], \rho[t])} t \text{ to } \widehat{E}.$$

- Show that nothing else is entailed by exhibiting $\widehat{M}(\mathbf{X})$ as the free Σ-algebra satisfying $\widehat{E}$ generated by $\mathbf{X}$.

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Functorial Semantics

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- the arity of operations are discrete, yet quantitative equations would be enforced with non-discrete arities.

Functorial Semantics

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- the arity of operations are discrete, yet quantitative equations would be enforced with non-discrete arities.
- even if we allowed non-discrete operations (see [FMS21]), we do not use the product/exponential in **Met**.

Other Stuff

- Can the monad lifting-theory extension correspondence be made fibered in some sense?
- How to compose two liftings of monads when their underlying **Set** monads compose via composite theories?
- What about infinitary theories?
- Further simplify the entry point to quantitative algebraic reasoning (find lots of examples).

Merci !

References I

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