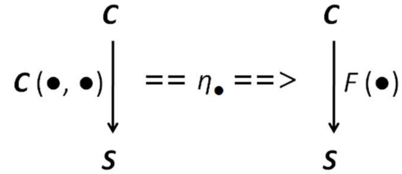
Yoneda Lemma

We begin with an intuitive introduction to the mathematical content of Yoneda lemma (Lawvere and Rosebrugh, 2003, pp. 175–176, 249). Every object (of a category) is completely determined by its relationships with all objects of the category. Let us now look at a simple illustration of the Yoneda lemma. For ease of exposition, let us consider a single-morphism category (one object along with its identity morphism). Consider a one-morphism category C, with $1_\bullet: \bullet \to \bullet$ as its morphism. Next, consider a set-valued functor F: C → S, with F (●) = A, a set, and $F(1_\bullet) = 1_A$. Yoneda lemma says: A ≈ N, the set of natural transformations

$$\eta: C(\bullet, -) \to F$$

where C (●, -): C → S is a functor assigning to each object in the category C, the set of ●-shaped figures in the object (see Lawvere and Rosebrugh, 2003, p. 241 for the definition of natural transformation). Since there is only one object ●, along with its identity morphism $1_\bullet: \bullet \to \bullet$ in C, we calculate the natural transformations

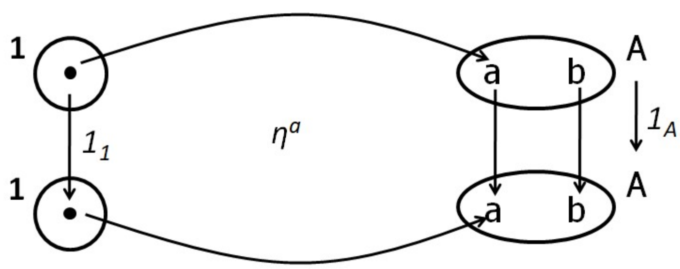
$$\eta_\bullet: C(\bullet, \bullet) \to F(\bullet).$$

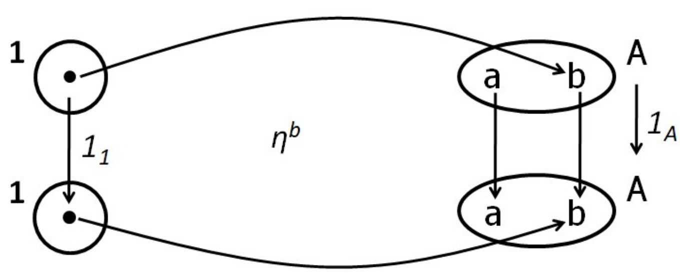


Since C (•, •) = 1 and with F (•) = A, a set, we obtain natural transformations η_1

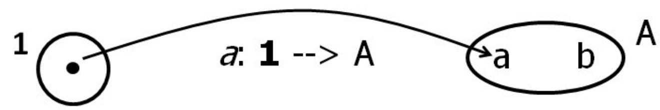
$$\begin{array}{ccc} \mathbf{C}(\bullet, \bullet) = \mathbf{1} & \text{---}\ \eta_1\ \text{--->} & F(\bullet) = A \\ 1_1 \big\downarrow & & \big\downarrow 1_A \\ \mathbf{C}(\bullet, \bullet) = \mathbf{1} & \text{---}\ \eta_1\ \text{--->} & F(\bullet) = A \end{array}$$

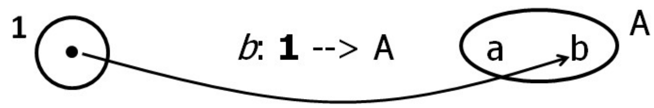
With the set A = {a, b}, we find the two natural transformations corresponding to the two elements of the set A = {a, b}.





The two natural transformations correspond to the two points (functions from the terminal set 1) of the set A.





Thus, with F (•) = A and F ($1_\bullet$) = 1_A we found that A ≈ N, the set of natural transformations η: C (•, -) --> F, where A = {a, b} and N = {η^a, η^b}, in accord with the Yoneda lemma.