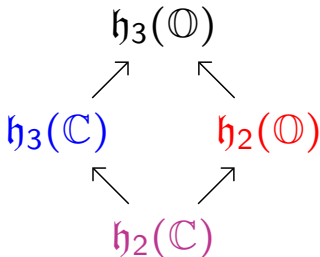


THE STANDARD MODEL GAUGE GROUP

from

THE EXCEPTIONAL JORDAN ALGEBRA



John Baez and Paul Schwahn
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The gauge group of the Standard Model of particle physics is

$$S(U(2) \times U(3)) \quad := \quad \left\{ x \in SU(5) : x = \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix} \right\}$$

$$\cong (U(1) \times SU(2) \times SU(3)) / \mathbb{Z}_6$$

Its representation on one generation of fermions and antifermions comes from restricting the obvious representation of $SU(5)$ on $\Lambda \mathbb{C}^5$ to $S(U(2) \times U(3))$.

Is there something special about this?

Jordan, von Neumann and Wigner proved:

Theorem. Every finite-dimensional Euclidean Jordan algebra is isomorphic to a direct sum of ones on this list:

- ▶ $\mathfrak{h}_n(\mathbb{R})$: $n \times n$ self-adjoint real matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- ▶ $\mathfrak{h}_n(\mathbb{C})$: $n \times n$ self-adjoint complex matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- ▶ $\mathfrak{h}_n(\mathbb{H})$: $n \times n$ self-adjoint quaternionic matrices with $a \circ b = \frac{1}{2}(ab + ba)$.
- ▶ $\mathfrak{h}_n(\mathbb{O})$: $n \times n$ self-adjoint octonionic matrices with $a \circ b = \frac{1}{2}(ab + ba)$, **where $n \leq 3$** .
- ▶ The “spin factor” $\mathbb{R} \oplus \mathbb{R}^n$, with

$$(t, \vec{x}) \circ (t', \vec{x}') = (tt' + \vec{x} \cdot \vec{x}', t\vec{x}' + t'\vec{x}).$$

Building on work of Todorov and Dubois-Violette, Paul Schwahn and I showed:

Theorem 1. Suppose A, B are Jordan subalgebras of $\mathfrak{h}_3(\mathbb{O})$ such that

$$A \cong \mathfrak{h}_2(\mathbb{O}) \quad B \cong \mathfrak{h}_3(\mathbb{C})$$

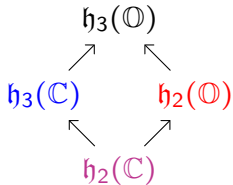
$$A \cap B \cong \mathfrak{h}_2(\mathbb{C})$$

Then

$$\mathrm{Stab}_0(A) \cap \mathrm{Stab}_0(B) = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

where Stab_0 means the *identity component* of the stabilizer.

So, the Standard Model gauge group consists of symmetries of an octonionic qutrit that restrict to act on a complex qutrit and a compatible octonionic qubit.



(More precisely it's the identity component of this symmetry group: there's another component that acts as antiunitary transformations.)

For example, since

$$\mathfrak{h}_3(\mathbb{O}) = \left\{ \begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R}, x, y, z \in \mathbb{O} \right\}$$

we can take

$$A = \left\{ \begin{pmatrix} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R}, z \in \mathbb{O} \right\} \cong \mathfrak{h}_2(\mathbb{O})$$

$$B = \left\{ \begin{pmatrix} \alpha & z & y^* \\ z^* & \beta & x \\ y & x^* & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R}, x, y, z \in \mathbb{C} \right\} \cong \mathfrak{h}_3(\mathbb{C})$$

$$A \cap B = \left\{ \begin{pmatrix} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R}, z \in \mathbb{C} \right\} \cong \mathfrak{h}_2(\mathbb{C})$$

Todorov and Dubois-Violette showed that *in this example*

$$\mathrm{Stab}(A) \cap \mathrm{Stab}_0(B) = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3))$$

So they showed that the theorem holds *in this special case*.

I'll sketch an argument for this.

Schwahn and I showed that this case implies the general case.

We can build the octonions as

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$$

with multiplication defined using only $SU(3)$ -invariant operations: the inner product and a funny cross product on $\mathbb{C}^3$. Thus $SU(3)$ acts as automorphisms of $\mathbb{O}$ preserving $\mathbb{C} \subset \mathbb{O}$.

Conversely, if we start with $\mathbb{O}$ and choose any square root of -1 , $\mathbb{O}$ gets a complex structure, and splits as a copy of $\mathbb{C}$ and a 3d complex vector space as above.

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In physics, this suggests how there are 3 colors of quark ($\mathbb{C}^3$) for each lepton ($\mathbb{C}$), transforming under the fundamental and trivial representations of $SU(3)$, respectively.

Now, $SU(3) \times SU(3)$ acts to give all the Jordan algebra automorphisms of $\mathfrak{h}_3(\mathbb{O})$ preserving $\mathfrak{h}_3(\mathbb{C})$:

- ▶ The left $SU(3)$ acts to *mix up the matrix entries*, conjugating matrices in $\mathfrak{h}_3(\mathbb{O})$ by $SU(3)$ elements.
- ▶ The right $SU(3)$ acts *separately on each matrix entry* as octonion automorphisms that preserve $\mathbb{C} \subset \mathbb{O}$.

Thus, the whole right copy of $SU(3)$ preserves

$$\left\{ \begin{pmatrix} \alpha & z & 0 \\ z^* & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R}, z \in \mathbb{O} \right\} \cong \mathfrak{h}_2(\mathbb{O})$$

This is the strong force gauge group.

Of the left copy, only $S(U(2) \times U(1)) \subset SU(3)$ preserves this block. This is the electroweak force gauge group.

Thus, $S(U(2) \times U(1) \times SU(3))$ acts on $\mathfrak{h}_3(\mathbb{O})$ preserving both $\mathfrak{h}_3(\mathbb{C})$ and $\mathfrak{h}_2(\mathbb{O})$. But there's a $\mathbb{Z}_3$ that acts trivially, and

$$(S(U(2) \times U(1)) \times SU(3))/\mathbb{Z}_3 \cong S(U(2) \times U(3))$$

This is the idea of Theorem 1 in this special case.

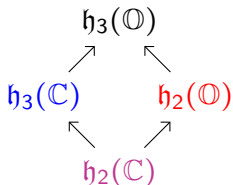
In proving Theorem 1, Paul Schwahn and I noticed:

Theorem 2. Suppose X, B are Jordan subalgebras of $\mathfrak{h}_3(\mathbb{O})$ with

$$X \cong \mathfrak{h}_2(\mathbb{C}) \quad B \cong \mathfrak{h}_3(\mathbb{C}) \quad X \subset B$$

Then

$$\text{Stab}_0(X) \cap \text{Stab}_0(B) = \text{S}(\text{U}(2) \times \text{U}(3))$$



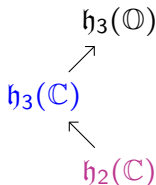
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So, the Standard Model gauge group consists of symmetries of an octonionic qutrit that restrict to act on a complex qutrit contained within it, and complex qubit within that.

$$\begin{array}{c} \mathfrak{h}_3(\mathbb{O}) \\ \nearrow \\ \mathfrak{h}_3(\mathbb{C}) \\ \nwarrow \\ \mathfrak{h}_2(\mathbb{C}) \end{array}$$

(More precisely it's the identity component.)