THE ULTRA-INTUITIONISTIC CRITICISM AND THE ANTITRADITIONAL PROGRAM FOR FOUNDATIONS OF MATHEMATICS *

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1. The aim of this program is to banish faith from the foundations of mathematics, *faith* being defined as any violation of the law of sufficient reason (for sentences). This law is defined as the identification (by definition) of *truth* with the result of a (present or feasible) proof, in spite of the traditional incompleteness theorem, which deals only with a very narrow kinds of proofs (which I call 'formal proofs'). I define *proof* as any fair way of making a sentence incontestable. Of course this explication is related to ethics – the notion *fair* means 'free from any *coercion* and *fraud*' – and to the theory of disputes, indicating the cases in which a sentence is to be considered as incontestable. Of course the methods of traditional mathematical logic are not sufficient for this program: and I have to enlarge the domain of means explicitly studied in logic. I shall work in a domain wherein are to be found only special notions of proof satisfying the mentioned explication. In this domain I shall allow as a means of proof only the strict following of definitions and other rules or principles of using signs. I shall indicate several new logical theories of this kind dealing with modalities, tenses and voices of verbs, with identifications and distinctions, with rules of attention and neglecting, with general principles of semiotics, etc., and with the help of these theories I shall reconstruct arithmetic and prove the consistency of Zermelo-Fraenkel set theory (ZF) with any finite number of inaccessible cardinals. This program has been studied by me in full detail. I don't insist I have eliminated all difficult questions – but I have introduced in my proofs no essentially new hypothesis and even if I *am* using some kind of hypotheses, these are all of a linguistic nature, and are common to all thinkers. I need no hypotheses concerning infinity.

But this program is too large for a short report, and here I shall confine

* Opening address.

myself to the exposition of my ultra-intuitionist criticism and of some main features of the positive program.

2. I begin with a criticism of the following *traditional* assumptions underlying the body of modern mathematics.

T1. The uniqueness (up to isomorphism) of the natural number series;

T2. The existence of the values of primitive *recursive* functions (prf) for every system of arguments for an arbitrary natural number series.

If for each system $x_1, \ldots, x_n$ of arguments in a natural number series N the value of $\varphi(x_1, \ldots, x_n)$ exists in N, I shall call N *closed* with respect to φ.

T3. The principle of mathematical induction from n to n'.

T4. If the axioms of a formal system are true and the rules of inference conserve the truth then each theorem is true.*

T5. The meaningfulness of the relations of identity and distinctness.

I criticize these assumptions above all simply on the ground they are assumptions. Their incontestability has never been established and this is sufficient reason to doubt them. But this radical line of criticism does not help to establish the connections between these assumptions, and I shall examine them more closely as follows.

Above all, proposition T1 uses a quantifier ranging over a domain including natural number series. It is too obscure to place at the beginning of mathematics.

I accept the traditional intuitionistic criticism of Brouwer and go further. I ask: why has such entity as 10^{12} to belong to a natural number series? Nobody has counted up to it (10^{12} seconds constituting more than 20 000 years) and every attempt to construct the 10^{12}-th member of sequence $0, 0', 0'', \ldots$ requires just 10^{12} steps. But the expression 'n steps' presupposes that n is a natural number i.e. a number of a natural number series. So this

* The assumptions T1–T4 have been criticized by many people, e.g. Borel, Frechet, Mannoury, Rieger and van Dantzig doubted T1, or the 'finiteness' of very great numbers like $10^{10^{10}}$ (van Dantzig). Lusin spoke about T1 as a 'poignant problem'. The independence of T2 is a commonplace in the theory of primitive recursive functions; and doubts about the finiteness of $10^{10^{10}}$ lead immediately to a rejection of T2. H. Poincaré in his 'Science et Méthode' (1908) wrote that many people have rejected T3 because they found a vicious circle in its substantiation: he does not name these scientists, but the most plausible conjecture is that they spoke about the circle involving T3 and T4, as discussed in the text below.

But this criticism has never been made in a systematic manner and I never heard of anybody who has criticized T5 or T8.

natural attempt to construct the number 10^{12} in a natural number series involves a vicious circle. This vicious circle is no better than that involved in the impredicative definitions of set theory: and if we have proscribed these definitions we have to proscribe the belief in existence of a natural number 10^{12}, too.

Let us consider the series F of *feasible* numbers, i.e. of those up to which it is possible to count. The number 0 is feasible and if n is feasible then $n < 10^{12}$ and so n' also is feasible. And each feasible number can be obtained from 0 by adding $'$; so F forms a natural number series. But 10^{12} does not belong to F.

Nevertheless the traditional natural number series containing 10^{12} is generally supposed to exist. And if we accept it there are at least two different natural number series.

This is only the beginning of my criticism. I don't really believe in the *existence* of a series containing 10^{12} – I shall prove the *possibility* of such a series – and I shall not need the notion of feasibility or any other empirical notion for my proof of the possibility of distinct natural number series. But for the present let me continue the consideration of F. Since 10 and 12 are feasible numbers, 10^{12} is not such a number and the function a^b is a prf, T2 is violated for F.

This is not strange. The independence of T2 is many times indicated in the literature. The inductive definition of the natural numbers takes no account of the property T2 of a natural number series.

The mathematicians use the following way of constructing natural number sequences: one chooses a number a_0 and defines a_{n+1} in terms of a_n. Here I am interested only in the case $a_{n+1} > a_n$ for all n. I call a natural number series N *regular* if it is closed with respect to a_n for each such sequence a_n considered as a function of n. Of course, the assumption that every natural number series is regular is even stronger than T2: and so it is wrong.

I doubt the possibility of a regular natural number series. This notion depends on arbitrary sequences of numbers: and so it is connected with problems like that of the continuum. It is natural to consider the notion of T-regularity, T being a class of number sequences, and one may expect to obtain a T-regular series for certain concrete T's.

Of course, with the help of T3 (for arbitrary induction properties) one can prove the regularity of every natural number series but this is an argument against the acceptance of T3, at least in its most general form.

I call T4 *the locality principle* for proofs, – because it asserts that the

property of a tree-figure of being a proof depends only on the local properties of this figure – more strictly, on the properties of its summits and nodes, the only 'integral' property consisting of the fact that the local properties are satisfied by each summit and node. A similar principle may be formulated for deductions from arbitrary premises in terms of 'truth relative to the premisses'.

It is clear that T4 depends on T3. Vice versa, T4 is used in obtaining $P(m)$ for arbitrary m from $P(0)$ and $\forall n(P(n) \supset P(n'))$. This vicious circle forces me to reject both T3 and T4 in the deepest questions of foundations of mathematics. It is essentially on these grounds that I am not searching for any axiomatic theory in my program.

This rejection of the axiomatic method does not mean the expulsion of the axioms and rules of inference as such. But I find that these rules are not sufficient to obtain a proof. We always need some supplementary 'rules of guarantee'. I shall consider the traditional tree-figures of logical proofs and deductions too, but I shall not consider them as such and I shall call them the *bodies* of proof or deduction or more briefly the *demonstroids* and the *deductoids*. The rule of guarantee requires that a demonstroid or deductoid is to be accompanied by an *establishment of its convincingness* (which I call also the *soul* of the proof or deduction). This establishment must constitute a proof of the fact that for the given demonstroid the applications of rules of inference considered together form a way of leading only to truths (and similarly for deductoids and relative truths, i.e. truths relative to the premises of deductoids). This is a new proof, and occasionally it may consist of a new demonstroid with its own establishment of convincingness. But in order that this procedure stops at some time, one has to have some primitive kind of proofs, independent of demonstroids. I call these proofs the *protodemonstrations*. The use of definitions and certain logical rules from domains deeper than the predicate calculus (modality theory, etc.) are the only means available in protodemonstrations.

The idea of establishment of convincingness is not completely strange even for traditional mathematical logic. Everyone understands that each tree-figure of proof requires a reference to T4. This reference (perhaps supplied by another reference to T3) is generally considered as an establishment of convincingness: and it is so uniform that it is always omitted. But I criticize this establishment of convincingness for its vicious circle (with T3 and T4). So I reject this traditional establishment of convincingness. I consider a special *genetic theory* dealing with these establishments.

3. Now I go further with my criticism of traditional assumptions. I shall consider T5. I recall the series F of feasible numbers. N denotes a natural number series with 10^{12}.

It may seem quite natural to say that some numbers of N are feasible and that the number 10^{12} is not. I said this above myself. But this way of considering the numbers of F leads to the situation that the finite set of numbers $\{0, 1, \ldots, 10^{12}\}$ of N contains an infinite part F. Perhaps it is not absurd but it leads to many difficulties. If somebody looks over this set of N-numbers he exhausts this infinite part, but I prefer to say an infinite process cannot be exhausted. This difficulty disappears if we distinguish between the F-numbers and the N-numbers equivalent to them. But sometimes it will be more convenient to identify these numbers belonging to different series. It is a very common thing; sometimes we identify objects that we distinguish in other cases. The rules of doing so must be established in a rigorous way.

Already Heraclitus pointed out that the notion of identity is not completely clear. But mathematicians prefer to proceed as if Heraclitus had not lived. I cannot continue in this way. The situation when an infinite process can be imbedded in a finite object is an ordinary one in investigations of distinct natural number series, and I shall need an apparatus for the explicit consideration of all identifications used in such cases.

First of all I maintain that the relation of identity or distinctness has no other meaning than that two objects have been identified or distinguished. There are two important kinds of atomic actions: identifications and distinctions. These actions are commonly used according to certain rules and they can be forced by these rules. Actions do not need to possess any meaning but they may have an aim. The rules also can be accepted as a means to achieve some aim. A very common aim of the identification of two objects a and b is to prepare the acceptance of the sentence 'a and b are identical', and similarly with distinctions. I shall consider a general theory of *collations*, i.e. identifications and the distinctions. A more general theory of rules and aims is needed for it.

Generally, one can identify or distinguish any two objects in two arbitrary occurrences of them. But collations play in our arguments a role similar to that of assumptions, and they must be made quite explicit or at least according to explicitly stated rules.

You can identify Fermat's last theorem with the sentence $2 \times 2 = 4$. Then, the last sentence being proved, you can say Fermat's theorem is proved too. This is not awkward if you express it quite explicitly: if Fermat's

last theorem is identified with the proved sentence $2 \times 2 = 4$, it is proved. If the identification were an ordinary one, it could be not mentioned; though in identification theory all rules of the ordinary identifications are to be mentioned.

Although one can identify any two objects in arbitrary occurrences of them, there are cases of awkward identifications. E.g. such is every identification of a present object with an absent one. If you consider identifications used without explicit mention you have to proscribe these dangerous identifications and you have to require that no two objects are identified before they appear.

Let us consider the series F and N once more. Each F-number will be identified with the equivalent N-number – but this presupposes the appearance of the F-number. So the identifications constitute an infinite process P. This process effectuates an imbedding of F in the finite part $\{0, 1, \ldots, 10^{12}\}$ of N: but this imbedding is infinite and you never can say it is over or that F is imbedded in $\{0, 1, \ldots, 10^{12}\}$.

I refer to as a *Zenonian situation* every case in which the events of an infinite process are to be identified with the parts of a finite object; and this finite object I call the *field of a Zenonian situation.* It is useful to establish a connection of this notion with the Zenonian paradox of the runner: A runner cannot get from the point A to the point B; for if he does then the infinite process of runnings of consecutive halves of this distance is over. But the events of this infinite discrete imaginative process are not necessarily the same as the parts of the real continuous process of running. (The continuous character of the process would be more evident if the running is replaced by the flight of a ball.) They are only identified with them: but the series of these identifications is an infinite process, and it can never be over as Zeno supposes in his argument. A similar criticism explains the paradox of Achilles and the tortoise.

The identification of two composite objects $f(a_1, \ldots, a_n)$ and $g(b_1, \ldots, b_n)$ is often produced as the result of a *structural procedure* of identification. The structures of these objects are supposed to be identified (which presupposes the identification of n in both occurrences). One identifies f with g, a_i with b_i for all $i = 1, \ldots, n$, then one makes the *establishment of exhaustion* (of all $f, g, a_1, b_1, \ldots, a_n, b_n$), and only then does one identify the objects $f(a_1, \ldots, a_n)$ and $g(b_1, \ldots, b_n)$. That is a *structural identification.*

The described Zenonian situation with the field $\{0, 1, \ldots, 10^{12}\}$ is not impossible if one considers the series of identifications as an infinite one.

It is only an infinite imbedding of the infinite process F in the finite field. One may say this means the field is finite in some occurrences and infinite in other ones. I call it the Zenonian situation *in the wider sense.*

But generally one considers a finite composite object in many occurrences of it and one identifies the object in these occurrences and one thinks these identifications may be considered as structural ones. Given the Zenonian situation one cannot carry out the structural identification of its field (with an infinite process imbedded in it) on the ground that the establishment of exhaustion becomes impossible. But if the identifications are not stated in an explicit manner it is impossible to notice the fact; and one talks about the field as if *this impossible structural identification were carried out.* That would be the Zenonian situation *in the narrower sense.* This situation is impossible – though without the analysis of identification one can easily overlook it.

4. Before going further I introduce now some terminological conventions.

The events of a natural number series N I call the N-numbers. If $N_1(0_1, 0_1', \ldots)$ and $N_2(0_2, 0_2', \ldots)$ are two such series, then I call 0_1 and 0_2 *equivalent* and I denote it by $0_1 \simeq 0_2$; if $m \simeq n$ (where $m \in N_1$ and $n \in N_2$) then I permit it to be said* that $m' \simeq n'$ also, and the relation $m \simeq n$ is always to be established in this way. If for each N_1-number m there is an equivalent N_2-number n, I say that N_1 *is not longer than* N_2 and I denote it by $N_1 \leqslant N_2$. If $N_1 \leqslant N_2$ and $N_2 \leqslant N_1$, then I call the series N_1 and N_2 *isomorphic.* If $N_1 \leqslant N_2$ and N_1 is not isomorphic to N_2, I say that N_1 is *shorter* than N_2 – but intuitionistically this does not mean the existence of an N_2-number equivalent to no N_1-number. If $N_1 \leqslant N_2$ and there is an N_2-number q equivalent to no N_1-number, I say that N_1 is *shorter* than q and I denote it by $N_1 \prec q$; in this case I say also that N_1 is *explicitly shorter* than N_2 (or that N_2 is *explicitly longer* than N_1) and I denote it by $N_1 \prec N_2$.

If one tries to establish an isomorphism between two series N_1 and N_2 by considering the pairs $\langle 0_1, 0_2 \rangle$, $\langle 0_1', 0_2' \rangle$, . . ., since the consideration of each pair $\langle m, n \rangle$ presupposes the appearance of m and n one obtains only

* Usually this is expressed by 'if $m \simeq n$ then $m' \simeq n'$'. But such expressions overlook the fact that $m' \simeq n'$ can appear only after $m \simeq n$ has appeared. This negligence may lead to the platonistic representation of a realm of all equivalences of the kind $m \simeq n$. Of course I have to reject this, and I prefer to speak here about the rules governing the appearance of the equivalences rather than speaking of the equivalences themselves without mentioning the rules.

the consideration of the third series of these pairs which is not longer than each of N_1 and N_2 and nothing more.

I find it useful to consider here the analysis of an ordinary supposed justification of T3. Given $P(0)$ and $\forall n(P(n) \supset P(n'))$ one indicates the possibility of obtaining the deductions for $P(0) \supset P(0')$ and $P(0')$, then for $P(0') \supset P(0'')$ and for $P(0'')$ and so on. This justification of T3 uses:

(a). An application of Carnap's Rule

$$\text{(Ca)} \qquad \frac{\ldots P(m) \ldots}{\forall m\, P(m)}$$

where m denotes an arbitrary value of the variable m.

(b). The *hypothesis of potential feasibility* according to which, given a method of construction requiring an arbitrary finite number m of steps, each step being feasible on the supposition that all preceding steps are fulfilled, one considers as feasible the result of this construction.

(c). The localization principle T4 for deductions.

(d). A *principle of parallelism* according to which at the m-th step of the described deduction one obtains just the sentence $P(m)$.

(e). The features of the interpretation of signs $\forall$ and $\supset$ used in the justification of the logical steps leading from $P(0)$ and $\forall n(P(n) \supset P(n'))$ to each $P(m)$.

(f). The consideration of the truth of $\forall m P(m)$ (relative to $P(0)$ and $\forall n(P(n) \supset P(n'))$) as denoting the deductibility of each $P(m)$ (from the premisses just mentioned); this consideration is used in the justification of (a).

(g). The indirect clause of the inductive definition of a natural number (leading to the conclusion that each m can be obtained from 0 by adding ').

I should mention also the fact that the result of the construction used for each m is a deduction figure for $P(m)$ (to which T4 is to be applied). I think one can obtain this fact from the definition of a deduction figure, the application of (b) and a kind of argument similar to (d).

5. Now I describe briefly the structure of this antitraditional program:

It contains a *central nucleus* consisting of two theories: the *ontological theory* and the *genetic theory* and a consistency proof for ZF (and some of its extensions) based on these theories. The *ontological theory* deals with the substantiation of the possibility of natural number series and other discrete processes required for this consistency-proof; and the *genetic theory* deals

with the notion of proof for sentences related to the events of distinct processes. The genetic theory deals especially with the elaboration of the notion of the establishment of convincingness for given processes called the *studied processes*. Also some fundamental theories called *prototheories* are needed in the central nucleus – these are e.g. collation theory, the modality and tense theories sketched below, and some others. Generally, only proto-demonstrations are available as means of proving in the prototheories and even in the ontological and in the genetic theories. Outside the central nucleus I shall have several *extreme directions* of the program concerned with the substantiations of some hypotheses occurring in the theories of the central nucleus. These hypotheses are implicitly contained also in the traditional theories, or in a general platform underlying them; at any rate they have nothing to do with impredicative definitions or other vicious circles occurring in the foundations of all traditional theories. The proto-demonstrations are the only available means of proof also in the extreme directions, and the gap between these and the prototheories may be a temporary one. If the foundations of ZF were the only task of the whole program I could say that the main problems of the extreme directions concerned with the substantiations of the mentioned hypotheses are essentially settled today.

6. Now I continue the list of traditional assumptions.

T6. The possibility of neglecting modalities and aims in foundations of mathematics.

T7. The possibility of neglecting tenses, voices and moods of verbs.

T8. The possibility of neglecting the rules of attention and neglecting.

T9. The hypothesis of potential feasibility.

T10. The division of theories into object theories and metatheories.

T11. The postulates of the intuitionistic predicate calculus.

The reasons for my criticism of T6 and T7 are connected with the previous analysis of traditional assumptions.

Evidently when one applies T9 (see (b) above) one confuses the feasibility of construction with the supposition that the construction is fulfilled. E.g. for an arbitrary m one concludes from (b) and (d) above that the deduction of $P(m)$ is feasible – but that does not suffice to obtain the deduction of $P(m')$ with the help of $P(m) \supset P(m')$, for in order to obtain it one has first to obtain the deduction of $P(m)$, and that is more than the assertion about its feasibility. Evidently there is a modal principle justifying this

transformation of the possible into the real in the foundation of mathematics – I shall call it the *principle of modal fulfilment* and analyse it more closely below. The exact formulation of this principle requires a theory of modalities.

There is another thing which makes modal investigations necessary in the ultra-intuitionistic program. I mentioned the necessity of making explicit the rules of collations; and this requires a systematic treatment of rules and aims, which leads to a development of a theory of modalities.

In virtue of the traditional intuitionistic criticism we have to consider the natural number series not as accomplished totalities but rather as infinite processes. But one cannot speak exactly about any process without using tenses in order to discern the past events from the future ones. For a natural number series, at each stage, its infiniteness belongs to the domain of the future and so one has to expose the rules of transforming the future tense into the present or the past in reasonings. The theory of tenses is closely connected with that of modalities, and in the course of the development of these theories voices and moods must also be taken in consideration.

I introduce *modal characteristics* in the notion of natural number series. I call such series *necessitary*, *real* or *eventual* according to the modality 'necessary', 'factually' or 'possible' with which the future appearance of n' after n is asserted. Similar characteristics are to be used in connection with the notion of closedness of a series with respect to a function φ, with the relation $N_1 \prec N_2$ (according to the modality with which the existence of $q \in N_2$ with $N_1 \prec q$ is asserted) and so on. The notion of finiteness I connect with that of an end (but not with natural numbers) and one can distinguish the modal characteristics of finiteness etc.

The theory of modalities will be closely connected with *semiotics*, i.e. the general theory of signs.

In semiotics I consider the act of *indication* as central. An indication is an action by which its *author shows* to the *addressee* a *point* with the help of a *sign*. There is a *denotational connection* between the sign X and every point x of an indication and I denote this connection by $X \to x$.

A *language* is a method or system of indications, it is defined by means of constructing signs and by denotational connections.

A *character* or *tactic* is a system of *rules*, i.e. of propositions expressing permissions and demands (see below). If a tactic is expressed in a language I call it a *method*. A *way* is a tactic or a method.

The notion of *connection* is a primitive one. As with all primitive notions, its use is to be described by some way, i.e. by some method or at least by a

tactic. There are many tactics for following connections – they correspond in particular to the different kinds of connections. There are *aim* and *denotational connections* and the tactics for following them I call the *tactics* of *attention*. Every other tactic for following connections is subject in a way to a tactic of attention.

To follow a tactic for following connections is to accept such sentences as '*a* is connected with *b*'. Every sentence of this form is always connected to such a tactic χ, and if some rule of χ prevents the acceptance of the sentence '*a* is connected with *b*' I say that '*a* is strange to *b* (relative to χ)'.

If a sign is made by means of a language I call it the *name* of its points in that language and the points I call the *senses* of that name.

Each sign is used with at least one tactic of attention permitting one to notice that sign, and with a tactic of collations permitting one to identify it in its occurrences. I call a sign *clear* if there is an accepted tactic for following denotational connections with it. All points strange to the sign (relative to this tactic) are always to be neglected as its points. So a sign (in particular a name) is always a sign for its non-strange points only. A clear sign I call *definite* if it is necessary that it has just one point. The definiteness of a sign depends on a tactic of collations.

A *class* is an indefinite name; its *elements* are the fixed values of its senses. (There are rules of using these explications. The idea is to identify the class defined by a name with that name; but that is not always convenient. But the use of the word 'class' is always to be connected with an indefinite name. Of course this use of the word 'class' has nothing to do with classical set-theoretical hypotheses on the existence of classes.)

I call *direct* those occurrences of a sign in which the rôle of the sign is confined to its playing a part as a sign for its points, and to the participation in following those tactics of attention and collations which make it a sign. (The *rôle* of an object is defined as a class of aims and obstacles to which the object is attached by a tactic for following connections.) The rule of interchangeability of synonyms is suitable only for direct occurrences of signs. So the interchangeability of the terms 'indefinite name' and 'class' applies to direct occurrences of these terms only, and can be limited by some further rules useful for a text. These rules may be replaced by rules for distinguishing somewhere the senses of the both terms. All this question is a typical example of how closely the senses of different terms may be interconnected. At the first sight even vicious circles must occur in these explications but one can avoid them by distinguishing the senses of the same term

in different occurrences, in the hope that the system of explications gives a reduction for these terms attached to their occurrences.

A *collection* is a finished class (i.e. a class whose elements are exhausted, which may be specified by means of collation theory).

There are the following three fundamental semiotical principles:

S1. Given a collection of indications $\{a_0 \to a_1, a_1 \to a_2, \ldots, a_{n-1} \to a_n\}$ in a language L the indication $a_0 \to a_n$ also belongs to L.

It is a rule for restricting languages. It enables us to follow the denotational connections. One uses this rule each time when the sense of word in a new occurrence is to be identified with its sense in a former one.

S2. If the connections $f \to F, a_1 \to A_1, \ldots, a_n \to A_n$ belong to a language L, F being an operation applicable to the objects $A_1, \ldots, A_n$, then the notation $f(a_1, \ldots, a_n) \to$ (the result of applying F to $A_1, \ldots, A_n$) also belongs to L; some specifications of the rôle of the order '$a_1, \ldots, a_n$' may be introduced, and '$f(a_1, \ldots, a_n)$' may be replaced by another record which enables one to reconstruct it. (*The principle of structural parallelism.*)

Parameters are indefinite indecomposable names.

S3. When a text is accepted the result of fixing some collection of parameters in it by their admissible values is also to be accepted; but this operation of fixation may require grammatical agreements in the text obtained by the fixation.

I have to draw special attention to the agreements in the verbal tenses. The fixed values may in turn depend on parameters or even be parameters.

7. Now I shall describe the main features of my modality theory.

First of all, it cannot be an axiomatic theory on the grounds of the vicious circle with T3 and T4. Also it must precede other logical theories, in particular that of the applicability of the axiomatic method, i.e. the genetic theory. So all proofs in it must be protodemonstrations.

I divide the modalities into three *categories* (necessary, real and possible) and four groups: *deontic* modalities connected with rules, e.g. permissions and demands, *aim* modalities connected with aims ('possible' means 'possible for an aim' i.e. the aim might be achieved with this means, and 'necessary' means 'needed for some aim'), and the two kinds of *alethic modalities* – the *organic modalities* connected with methods or ways (I can = I am able, I must = I am forced by rules of a way) and the *epistemic modalities* ('perhaps' or 'certainly'). The epistemic possibility of A may be interpreted as the organic possibility of the continuation of reasoning

after A is accepted. The epistemic necessity of A means the organic necessity of the acceptance of A forced by rules of a way. For each group there is the modality 'real' used, e.g. in the abstract consideration of an action in connection with rules in order to investigate the lawfulness of the action without any assumption about the alethic reality of the action.

A *situation* is a class of (senses of) sentences. Generally, a situation S *describes* some conditions indicated in a way. The conditions may be described by a situation with different degrees of precision, the degree being sufficient or not according to the aims of the description.

The modalities may apply to the situations and to the propositions – sentences or names of actions. The deontic, aim and organic modalities apply to the names of actions which may be propositions with one verb in the infinitive mood. The epistemic modalities apply to sentences (so the verb stands in the indicative mood).

When a modal operation is applied to a sentence or to a name of action it is always to be accompanied by the name S of one or more situations. (An occasion possible in one situation may not be possible in another, etc.) That is one of the characteristic features of my modality theory. Symbolically the situation index S is to be applied to the sign of the modal operation ($\diamond$, ! or $\Box$ for organic modalities, denoting respectively organic possibility, reality and necessity; $\dot{\diamond}$, !, $\dot{\Box}$ for aim modalities; P, F and O for deontic modalities; and M, R and N for epistemic modalities). So $M_S A$ means 'perhaps, in the situation S, A', etc. If the assertion $M_S A$, $\diamond_S A$ etc. is accepted for a class Σ of situations S, it is denoted by $M_\Sigma A$, $\diamond_\Sigma A$ etc. For aim modalities the sign T denoting the aim is to be added: $\dot{\diamond}{}^T_S A$ means 'in the situation S it is possible to achieve T by means of A' etc. But if the situation S or the class Σ is fixed in the context one can omit the index S or Σ in formulating the modal proposition. So one omits the class Σ of all situations obtainable in mathematics in the formulation of the hypothesis of potential feasibility.

There are several *degrees* of modalities. Modal words were used in the basic semiotic explications before the notions of class and situation were introduced. In those contexts it was impossible to use systematically the situation sign. That was the *first degree of modalities*. After the notion of situation appears, the *second degree* is obtained.

The theory of deontic modalities is essentially that of tactics or characters. These are systems of rules, a *rule* being a proposition of the form 'In the situation S it is permitted to do A' or 'In the situation S it is required to do

A'; the *proscription* of *A* is defined as the proposition of the form 'In the situation *S* it is required not to do *A*'. The atomic actions *A* for the foundations of mathematics are:

(I). preferences;
(II). acts of establishing or following connections, in particular acts of attention;
(III). collations;
(IV). indications;
(V). acceptances of *propositions*, i.e. sentences, rules and optional propositions expressing aims or sometimes desires or wishes; (the acceptance of a method is considered as the acceptance of all its rules);
(VI). perceptions;
(VII). the acts inverse to (II) and (V) – acts of neglecting and acts of refusal of accepted propositions;
(VIII). the abstention from the acts (I)–(VII).

In some theories related to the ultra-intuitionistic program, e.g. the theory of disputes, new items appear:

(IX). raising and answering questions (considered as actions similar to (V));
(X). addressing another person

and

(XI). including some text in the memory;

the abstentions (VIII) are extended to these new items.

The tactic expressed in a language is called a *method* (see above; now the notions of rule and tactics are specified). Tactics are *means* for aims (as well as materials which are usually considered as obtained by a tactic; in addition to them, the *means* are the rules occurring in tactics, and the actions performed for the achievement of the aims; in particular those fulfilled in the course of following the rules, etc.). By means of tactics new complicated deeds *A* may be introduced; and this leads to the extension of the class of rules.

In a natural way the notion of *following* a tactic is defined. A *discrete process** is the following of a method and I shall say that it is *described* by the method.

* Generally I take the word 'discrete' always to mean 'expressible in a language of words'. *Procedures* are processes which can be accomplished.

To follow a tactic χ generally presupposes another tactic ψ by means of which the actions A indicated by the rules of χ are fulfilled, the situations S are collated, and so on. The tactic ψ is called *external* or *deeper* with respect to χ or to a process described by χ. The actions made in following ψ but not χ are called *automatic* with respect to χ or a process described by χ. By the way, it happens often that a tactic χ or a process described by it uses many external tactics and the notion of automatic action applies to each of them.

All this is very important for my program. The ultra-intuitionistic program introduces tactics of acts (I)–(VII): but these acts envelop the profoundest acts of our thinking underlying the ultra-intuitionistic reasonings too. Even the reading of a text uses the acts (I)–(III) (e.g. one prefers to read from left to the right) and it would be completely impossible to investigate in the program the many tactics of attention and collations used in the beginning of the exposition of it. But this is superfluous. Most of these tactics turn out to be external to the tactics of the deeds we really need to investigate. So one can assume that many actions (I)–(III) are fulfilled automatically, and even apply such traditional expressions as 'the same word as before' etc. Only in certain cases especially indicated by the ultra-intuitionistic criticism shall I have to describe explicitly the rules of the tactics: and this criticism is ruthless enough, for those cases form a class sufficiently large for all important aims. Other cases may be treated with the traditional negligence to the explicitness of these actions which are considered as the automatic ones. However, the investigation of these external tactics are of great importance e.g. for cybernetics – but for the ultra-intuitionistic program it is not an urgent business.

Tactics may be *incomplete* for two reasons: (a) there may be situations for which the rules permit to make each of two or more incompatible actions (I call these situations the *Buridanian ones*) and (b) there may be *unforeseen* situations of which the rules say nothing. For both cases there are general ways of *completing* the tactics (though the completed tactic does not need to become complete). In case (a), generally, it is done by tactics of preference. In case (b) one chooses one of two fundamental preferences: according to the *principle of liberalism* one prefers permissions to proscriptions and according to the *principle of despotism* one prefers proscriptions to permissions. The acceptance of the first principle leads to the *liberal regime* characterized by the rule 'everything not proscribed by (the rules of) a tactic is permitted by the regime' and the acceptance of the

second principle leads to the *despotic regime* expressed by the rule 'everything not permitted by the tactic is proscribed by the regime'. One applies the principle of liberalism in cases one seeks for means for the accepted aims. One applies the principle of despotism in cases one tries to achieve aims with the help of accepted means, or to verify the proposed means with the help of accepted criteria.

Definitions are the rules of usage of names. Their usage is always governed by the despotic regime which leads to prohibiting the application of the term introduced by definition in any other sense. All inductive definitions, e.g. that of natural number, contain an 'indirect clause' which is the imposition of the despotic regime on the consideration of an arbitrary object as satisfying the definition.

The theory of methods and ways is a generalization of the traditional theory of algorithms. There are at least four respects in which the former is more general than the latter: (a) the primitive operations of the normal algorithms (the substitution of words) are performed in a way which is not itself considered as an algorithm; (b) the theory of algorithms depends on natural numbers, which are to be obtained in some way; (c) the methods can be applied to objects of a completely general nature; (d) the 'work' of a method or way (i.e. the following of it) may be an undetermined process.

Now I turn to the alethic modalities. A situation is called *real* if each element of it is accepted on the ground of a perception. So the void situation is real. The problem of perception is of course a very deep one (and it is connected with the consideration of active and passive voices) but in the domain of the foundations of mathematics we can restrict our attention to the perception of texts proposed in the exposition of exhaustions and of the atomic acts (I)–(V) and (VII)–(XI) without further analysis. Each real situation is also called *possible*.

The principle of modal fulfilment (pmf). If the situation S is possible and the action or event A is possible in S then the situation $S \cup \{A\}$ (obtained from S by adjoining A to it) is also possible.

This principle holds for all four groups of the modality 'possible' (all occurrences of which in the formulation of the principle have to belong to the same group). Symbolically it may be stated in the form

$$\bigcirc \frac{S \quad A}{S \cup \{A\}}$$

where '$\bigcirc$' denotes one of the modality signs $\Diamond$, $\Diamond^T$, M or P. It may be

specified that A stands in the future tense above the bar and in the present-past tense below it, symbolically

$$\bigcirc \frac{S \quad \Delta A}{S \cup \{\nabla A\}}$$

Δ denoting the future tense and ∇ the present-past tense. If $\bigcirc$ is $\Diamond$ or P then A in $\Diamond_S A$ is the name of an action, and can be expressed by a proposition with one verb in the infinitive mood, in the $\Diamond_S A$ or above the bar it stands in the active voice ('it is possible to do A') but in $S \cup \{A\}$ it stands in the passive voice. Below the bar A stands in the indicative mood. If $\bigcirc$ is M, then A in $M_S A$ is the name of an event, and the verb in A stands in the indicative mood both above and below the bar, and the voice is the same in both cases. So this principle contains also the rules for tenses, voices and moods.

The justification of pmf may consist in the definition of a possible situation as such which can be obtained from a real one by a string of applications of pmf. A closer examination shows that in this way pmf is reduced to its simplest case for which it is rather a tautology.

Perhaps a somewhat more explicit formulation of pmf is

$$\bigcirc \diagdown \frac{S \quad \bigcirc_s A}{S \cup \{A\}} \diagup$$

(or still more precisely with ΔA above the bar and ∇A below it etc.). Here one sees distinctly that A stands with the modality 'possible' attached to it above the bar and without this modality below it. That is the essence of the rule explaining its name of 'fulfilment', and it turns out that pmf is a formalization of the 'utilization' of possibilities. However the modality 'possible' does not disappear below the bar, but it is attached to the situation $S \cup \{A\}$ itself. Often one forgets to mention this occurrence of the modality and speaks of the situation $S \cup \{A\}$ as of a real one. So one obtains faith in $S \cup \{A\}$: and that is the general way of accepting-on-faith in the domains of religion and philosophy, and in traditional intuitionist or constructivist mathematics too. Such are the cases when one believes in mathematical 'reality' on the grounds that its objects *can* be obtained by a method. The ultra-intuitionist analysis reestablishes the correct use of modalities.

A deep idea of this program consists of representing the traditional mathematical 'real' situations only as possible ones obtained by iterations

of pmf. The consideration of mathematical situations as possible is certainly sufficient for the sake of consistency proofs.

Now I continue the review of my theory of modalities.

If a process D is described by a tactic χ and for a situation S occurring in the course of D the rules of χ demand an action or event A then for the continuation of D in S the fulfilment of A is necessary.

That is the *main principle of necessity*. It is in virtue of this principle that the definitions are observed in the course of a development of a theory and it seems that this principle itself can be based essentially on the definition of the word 'describe'. It is also in virtue of this principle that each move of a bishop in a chess-game necessarily is a move by a diagonal. This principle may seem quite tautological. But a closer examination shows that it depends on an elimination of a double negation. What *really* can be justified is that in the course of a process the rules of a tactic describing it cannot be violated. It is not the same as the 'fact' that the rules are necessarily observed. But generally I neglect the eliminations of double negations occurring in the deepest questions of that program because it is the matter of an extreme direction to deal with them.

If an event e of a process D is over and done with in a situation S of D and d precedes e in D then it is necessary that d is over in S. (*The principle of ordinal necessity*, p.o.n.) The 'being over and done with' is a form of expressing the past tense. So the principle deals with that tense too.

There are also such 'trivial' principles as 'everything necessary is real' and 'everything real is possible'. These are stated only for the two groups of alethic modalities and for possible situations only. (The reason of the impossibility of a situation may be precisely a violation of some trivial principle in it.) For deontic or aim modalities the observance of these principles is the characteristic feature of just or purposeful activities. There are also some relations between organic and epistemic modalities (e.g. that organic necessity implies the epistemic necessity) – these too are stated for possible situations only. Various other connections – the distributivity of conjunction and the equivalence between the impossibility of A and the necessity of not-A – can be established (with some restrictions) for possible situations.

The violation of the main principle of necessity is impossible in each possible situation. (The *principle of negative evidence*, p.n.e.)

For aim modalities I accept, also for possible situations S only:

If the aim T is not achieved in S, in order to achieve it in S it is necessary

to apply (a) some sufficient means and (b) each of the necessary means (the *inversion principle*).

Each system of actions is sufficient for the achievement of each of its necessary result.

For the future tense I deal primarily with the tense 'A will have been' which I denote $\Delta_S A$ or simply ΔA if the situation S to which this 'will' refers is clearly indicated in the context. I accept:

(a). If A has been in S it will have been in every situation which is later than S.

(b). If A is necessary in every situation of a class Σ of situations, A is necessary also in each future situation of Σ.

(c). In each possible situation an appearing event is always later than events already entering in the situation, and this order relation will necessarily be observed in every later situation.

Also the rules occurring in pmf and pon are very important for the theory of tenses. The following rule is of a semiotical nature:

(d). In order that a proposition be meaningful it is necessary that its predicate agree in tense with the objects to which it refers. This means e.g. that the phrase 'tomorrow *is* August 12' is meaningless if pronounced August 11; a meaningful phrase is on that date 'tomorrow *will be* August 12'.

(d) is very important for the analysis of deductoids exposed in the ordinary logico-mathematical language having no future tense. A sentence $A(\boldsymbol{m}, \boldsymbol{n})$, containing the names $\boldsymbol{m}$, $\boldsymbol{n}$ of the events m, n of a process, can be meaningful only on the supposition that m and n have already occurred. (It could be otherwise if the predicate in $A(\boldsymbol{m}, \boldsymbol{n})$ stands in a future tense.) The deductoids may contain the rule (Ca) and so the sentences occurring in them constitute an infinite process. The identifications required for the substantiations of the applications of the rules of inference etc. may introduce a Zenonian situation (even in a narrower sense), and therefore it is not evident that the deductoid must possess an establishment of convincingness.

So a sentence $A(\boldsymbol{m}_1, \ldots, \boldsymbol{m}_k)$ may occur in a deductoid only on the supposition that $m_1, \ldots, m_k$ have already occurred. That is the *first genetic principle* for ultra-intuitionistic proofs. The *second genetic principle* is that if $n_1, \ldots, n_p$ are events of the same process I shall say they have all ensued only if a later event n of the process has ensued (cf. p.o.n.). The *third genetic principle* is connected with the inversion principle: e.g. I cannot consider an occurrence of A in B (or even an occurrence of A depending on an occurrence of B) before this occurrence of B is considered as already

having occurred. These three genetical principles constitute a *genetic constitution* which is to be observed in the ontological and especially in the genetic theory. (In the ontological theory as well as in prototheories the sentences $A(m, n)$ containing the future tense may occur; and then the first genetic principle may be violated. E.g. in the definition of a natural number series there is the location 'n' will be ensued', to which a modality may be attached; the application of pmf to this location belongs to the ontological theory. Cases of such a nature with the future tense are only ones dealt with in the ontological theory.)

8. Now I return to the consideration of the natural number series and the assumptions T8–T11.

First of all, I have considered a series F without 10^{12} but it does not mean a number q has to be so great as 10^{12} in order it could be longer than some natural number series. If a series N contains a 20th event this does not mean that another series M has to contain *its* 20th event also. In the notion of series there is no proviso about this.

So let us consider two necessitary series N_1 and N_2, whose numbers I denote $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ respectively. Let N_2 contain b_{20} and let N_1 be shorter than b_{20}, which means that there never shall be such entity as a_{20} in N_1. (Later I shall prove the possibility that N_1 does not contain even a_2.) Let $N_1 N_2$ be the series $a_0, b_0, a_1, b_1, \ldots, a_i, b_i, a_{i+1}, \ldots$ obtained from N_1 and N_2 by alternation. It really is a series because if a_i has occurred then $i < 20$ and there is b_i in N_2 and a_{i+1} necessarily will have occurred; if b_i has occurred in $N_1 N_2$ this means that a_i has occurred in N_1 and so a_{i+1} necessarily will have occurred. Now let us consider the task: to find the occurrence of b_{20} in $N_1 N_2$. This task requires only 42 steps each of them being alethically possible when the preceding steps are accomplished. Nevertheless the fulfilment of the task is impossible because if b_{20} is found in $N_1 N_2$ then a_{20} also is found in it, and so a_{20} is in N_1, which contradicts the condition.

I say there is an *obstacle* to the fulfilment of this task, and I call this obstacle the *catching* of this task *on* N_1. Of course catchings may arise also from more complicated alternation, e.g. the task of finding b_{1000} in the series $a_0, b_0, a_1, b_1, b_2, a_2, b_3, b_4, b_5, a_6, b_6, b_7, b_8, b_9, a_7, \ldots$ (the numbers of b's after a_{i+1} is one greater than after a_i) also catches on N_1. Generally catching is (roughly speaking) an obstacle to a task consisting in the fact that its fulfilment requires the exhaustion of an infinite process.

In the case of a Zenonian situation (in the narrower sense) with the field $\{0, 1, \ldots, 10^{12}\}$ of N-numbers the task of the structural identification of that field in its two occurrences catches on the series F of feasible numbers. So every Zenonian situation in the narrower sense is impossible on the grounds of a catching. Conversely, to every catching it is possible to associate a Zenonian situation (e.g. obtained by the identification of b_i's in $N_1 N_2$ and in N_2, the first one's forming the infinite series; this situation is one in the narrower sense if one requires the structural identification of the segment $\{b_0, \ldots, b_{20}\}$ of N_2 in its occurrences before and after this introduction of infinity into it).

The introduction of the notion of obstacle is a most important feature of the ultra-intuitionistic program. Of course the constructivists were also aware of some 'trivial' obstacles for their constructions, such as the absence of necessary means or a vicious circle in a task. But these obstacles always have been the *barriers for a single step* in the fulfilment of the task. The catching is a *new* kind of obstacle arising from the possibility of a finite sequence being longer than an infinite series. Each 'finite' task whose fulfilment is destroyed by a catching I shall call the *field* of this catching.

The possibility of a catching frequently appears in the genetic theory, whose principal aim is to obtain a list of conditions sufficient for a deductoid to possess a semiotic substantiation undamaged by catchings. In this theory catchings are represented by sequences whose members are just the occurrences like those of b_i in $N_1 N_2$; and these sequences are represented by functions φ, whose values are those occurrences and whose arguments are the numbers of a short natural number series on which the occurrences depend. E.g. $b_i = \varphi(a_i)$ where a_i is in N_1 and b_i is in $N_1 N_2$. I call these functions *metafunctions* (because the occurrences of the members of the sequences are traditionally considered as metatheoretical objects). Each catching in the genetic theory may be represented by a metafunction given for an infinite process (containing the arguments of the metafunction) and all of whose values must belong to the field of the catching and be distinct – these are called *unbounded metafunctions*.

9. Now I pass to my criticism of T8. I wish to introduce a new branch of logic – *relevancy theory* – into the foundations of mathematics. It is a theory about how to take something into attention or to neglect it.

These problems appear at each step in our thinking but the traditional logic worries too little about them. Everyone knows the syllogism: all men

are mortal, Socrates is a man, therefore Socrates is mortal. But another argument is no worse: All men can die only once, Socrates has died, therefore now Socrates cannot die and Socrates is not mortal. (I suppose that the word 'mortal' refers only to beings who can die.) That is a paradox. Its explanation I see in the fact that the word 'man' had different senses in this text, one time denoting all men – living and dead – and one time denoting only living men. In order to eliminate all paradoxes of this sort one has to specify the way of following the denotational connections. Every way of following connections I call here a *tactic of attention* (but I mean mainly following the aim or denotational connections).

Perhaps one feels the problem of this tactic is strange to the foundations of mathematics, but even if it is so this fact belongs to relevancy theory and one has to prove it, which already requires some acquaintance with this theory. That is already a criticism of T8.

The main rôle of relevancy theory is attached to the fact that one always. may neglect strange objects, i.e. objects strange to the aims of considerations. With some trivial provisos I insist upon the inverse principle: *only* strange objects may be neglected, which I call the *main principle of relevantism*. This means: first prove the strangeness of an object and only then neglect it. This presupposes that the way of establishing or following connections, i.e. the tactic of attention is sufficiently well known.

In traditional mathematical logic one has never argued about tactics of attention and rules of connection and neglection. One could think that the rigid system of definitions makes that superfluous. But the rigidity of this system depends on the notion of natural numbers and so it cannot be more rigid than this notion is. The rejection of T1 is already the rejection of this rigidity: and so the problem of introducing the tactics of attention into consideration becomes a very urgent one.

But the means of traditional theories are not sufficient in order to fix in a unique way the supposed tactic of attention. The existence of consistent but ω-inconsistent systems may be interpreted as revealing this fact even by traditional means.

I shall apply the relevancy theory to the study of metafunctions. Strange objects are neglected and therefore instead of proving the impossibility of a metafunction ρ destroying a construction C I can prove that each such metafunction ρ must be strange. This I prove chiefly with the help of the following *principle of strangeness* which can be established in the relevancy theory:

If an object $\mathfrak{E}$ is indispensable for any consideration of ρ but one considers C without taking $\mathfrak{E}$ into attention then ρ is strange to C.

The importance of this principle is due to the fact that when an ultra-intuitionistic theory T considers two natural number series F and N, $F \prec N$, one always can intentionally introduce the function $\rho(x)$ on F such that $\rho(m)$ is always the N-number equivalent to m and if $F \prec q \in \mathrm{N}$ then this ρ enables one to introduce the Zenonian situation with the field $\hat{q} = \{0, 1, \ldots, q\}$ where all of $0, 1, \ldots, q$ are N-numbers. As explained before one can further consider a catching destroying an appearance Oc_q of q. But this does not mean one has to worry about it because it is possible that Oc_q has nothing to do with T. $\hat{q}$ is one thing which may be really necessary for the study of N but Oc_q may be strange to the theory T or to a part T′ of it in which this question arises. If ρ creates an obstacle for Oc_q but not for the consideration of $\hat{q}$, then one may neglect this obstacle as a strange one provided one is interested only in $\hat{q}$ but not in Oc_q. One can neglect ρ in such cases even if the obstacle created by ρ impedes a construction of $\hat{q}$ considered in connection with Oc_q. Let $\mathfrak{E}$ be the equivalence relation between F- and N-numbers. Although $\mathfrak{E}$ is a typical ultra-intuitionistic object it may happen that in T or a part T′ of it there is no need for the consideration of it: and then one can suppose that T or at least T′ is really being considered, without taking $\mathfrak{E}$ into attention. Let $\hat{q}$ be considered in T or T′ respectively. Then in T (or T′ respectively) ρ is strange to $\hat{q}$ because $\mathfrak{E}$ is indispensable for any consideration of ρ.

Of course there may be cases of much more complicated metafunctions than this ρ. Typical objects $\mathfrak{E}$ are e.g. fixations of parameters. Usually any ultra-intuitionistic consideration T′ depends on some chosen natural number series $N_1', \ldots, N_k'$ which may be considered as the fixed values of parameters $N_1, \ldots, N_k$ for the series. Then T′ may be considered as the result of assigning these values to these parameters in a more general consideration T. Let $N_1'', \ldots, N_k''$ be another system of values for $N_1, \ldots, N_k$ not mentioned in T′. Then if ρ is a metafunction whose definition uses $N_1'', \ldots, N_k''$ or N_j''-numbers q_j'' such that $N_j' \prec q_j''$ $(j = 1, \ldots, k)$ then ρ is strange to T′. The nature of $\mathfrak{E}$ may be completely arbitrary, e.g. $\mathfrak{E}$ may be using of a tactic of attention χ in a situation where there is no given tactic of collations for objects considered with the help of χ. With the help of such a $\mathfrak{E}$ one can prove the strangeness of a metafunction ρ defined in a manner independent of any tactic of collation; and with the help of a suitable $\mathfrak{E}$ one can do the same, if the tactic of collation of ρ essentially differs from that

used for the elements of the supposed field of the catching represented by ρ. In such a manner I prove that a sufficiently symmetrical object cannot be a field of catching – this idea is well known in traditional mathematical logic (cf. e.g. Mostowski's proof of the independence of the axiom of choice).

There are some principles of preference for the tactics of attention in connection with obstacles. When one seeks means for the accepted aims one can prefer the attention to a means to the attention to an obstacle for the application of the means. But when one has chosen a means in order to achieve an aim, e.g. when one carries out a construction, one has before all to take into attention every obstacle for the aim. Nevertheless even here the word 'obstacle' cannot refer to any strange obstacle.

The choice of a tactic of attention may depend (and usually does) on the accepted aims. But for the most part this choice is made implicitly and automatically with respect to the considered activity. Even in my own program, the cases in which I explicitly consider the rules of such tactics are comparatively rare (but important).

A natural number series N is defined as a discrete process having an initial event 0 and a unary *leading operation* n' with an evident (external) tactic of collations for its events and a despotic regime imposed on its events. This regime belongs to the tactic of attention which one has to accept in order to follow the notion of an N-number but it does not define this tactic uniquely. In order to see this, I shall consider the construction of a very short natural number series.

Generally for constructing a natural number series N one can take an accomplished event – say, the indication of a void place – as its zero and take the indication of n as the leading operation n'. If one wishes to receive in this way a necessitary series N, one has to accept the rule requiring the indication of n in the situation that n has occurred and is distinguished from all preceding N-numbers. The necessitary series is also a real one. If one wishes to achieve an eventual series, one can accept two permissions (a) and (b) in place of this requirement: (a) it is permitted to indicate n; (b) it is permitted not to do so. In all cases the events, as they appear, are required to be distinguished from all preceding ones; and the fulfilment of this requirement is to be given by a single action with a parameter m for the preceding numbers. Then the sentence $m \neq n$ may be accepted for each value n of the N-numbers and the m's preceding that n, and by the fixing of m with m preceding n one can obtain the sentence $m \neq n$ as an accepted one (see S3 above). But N is not fixed till the tactic of attention to its

numbers is specified and the notion of N-number remains not clear. Now suppose (by S3) this tactic is fixed in some manner. Then this notion becomes clear though depending on the undefinite value of this fixation. Let this tactic of attention be denoted by χ.

Now I introduce a new tactic τ of attention to the N-numbers. At each stage n of N (i.e. after the appearance of n before the appearance of n') τ permits one to take into attention only n' and the N-numbers $\leqslant n$. (The word 'only' is an abbreviation for the introduction of the despotic regime.) I call τ the *short-sighted* tactic of attention. The series N considered with this tactic τ I denote by N_τ.

At each stage n of N_τ only the N_τ-numbers $\leqslant n'$ are considered as N_τ-numbers and there is no such entity as n'' amongst all possible N-numbers. Evidently N_τ is not closed with respect to the function n''. If $n = 0$ then (at the stage 0) N_τ is to be considered as a series shorter than 2 (the number 2 being taken from another series, perhaps from N). As a matter of fact, after the sentence S_0^τ asserting the appearance of $0'$ in N_τ is accepted, then (at stage $0'$) the name 'N_τ-number' will denote just the entities 0, $0'$ and $0''$ but not $0'''$, and so on. At each stage of every reasoning one has to consider which are the accepted sentences of the form S_n^τ. Here it is useful that the word 'accepted' stands in its passive form.

That is the proof of the possibility of very short natural number series, and this proof is independent of all empirical considerations like that of unfeasibility of the number 10^{12}. Of course it is as yet too vague, because still I have not expounded the ultra-intuitionistic reconstruction of logic; and without this one does not know how to deal with N_τ. I criticize T4 but at present I have not secured even the simple applicability of the postulates of intuitionistic logic. So I have to reconstruct the substantiations of these postulates (see T11) in order to see that the substantiation is independent of the chosen tactic of attention, or at least that the short-sighted tactic is a suitable one for the acceptance of the postulates. This follows essentially from the fact that a tactic of attention has always underlain the consideration of the traditional intuitionistic postulates, a tactic deeper than its substantiation. Its properties have never been studied: and this may be explained by the fact they are almost inessential. The only important thing was that one can follow a chosen tactic and consider it as a fixed one. But the short-sighted tactic for N_τ contains even less requirements of action than the original tactic χ for N and so one can follow it. There can be no obstacle to combining two tactics χ and τ so as to follow the first in considerations of

N-numbers and the second in considerations of N_τ-numbers: and with this combined tactic of attention one can consider the pair of these series. Besides this one has to remark that in purely logical reasonings about the given objects and processes nothing forces us to accept such a sentence as S_0^τ. This fact belongs to the contemplations of voices, and leads immediately to the possibility of consideration of several natural number series at once. Of course, from the series N_τ without 2 one can obtain a series with the arbitrary given number k from another series K but without $k+2$: it suffices to add N_τ to the accomplished beginning $\{0, \ldots, k\}$ of K.

It is useful to remark that the rules of tenses prevent the acceptance of the appearance of $0'$ in N. I can assume that N and N_τ are necessitary. According to the definition of a series 0 has occurred (= is) in N_τ, and $0'$ *will have* been, but there is no rule enabling to obtain from that the sentence that $0'$ *has* been (= is). After $0'$ will have been in N, $0''$ will have been also, but a future relative to another future is not yet a future. So one cannot obtain a sentence about the appearance in N_τ of its future event $0''$.

10. Hitherto I spoke only about destructive critical ideas insofar as the possibility of constructing a sufficiently long natural number series seems to be diminished by this criticism. But the theory of collations leads to a contrary result, it restores the possibility of considering great finished totalities of objects. The operation of *duplication* of a given finished totality $P = \{a_1, \ldots, a_{n_p}\}$ is accomplished in the following way: given one occurrence Oc_p of P, form another one Oc'_p simply by indicating Oc_p once more and distinguishing P and a_i $(i = 1, \ldots, n_p)$ in their old occurrences (through Oc_p) and the new ones (through Oc'_p); then it is sufficient to indicate the pair of indications on Oc_p and on Oc'_p and obtain with the help of S1 the indications on $2n_p$ objects $0a_1, \ldots, 0a_{n_p}$, $1a_1, \ldots, 1a_{n_p}$ ($0a_i$ denoting the a_i in its occurrence through Oc_p and $1a_i$ the occurrence of a_i through Oc'_p). So with the help of indications and distinctions one obtains the duplicated totality $2P = 0a_i, \ldots, 0a_{n_p}, 1a_1, \ldots, 1a_{n_p}$. It is then natural to identify $0a_i$ with a_i $(i = 1, \ldots, n_p)$. Starting with $P = \{0\}$ one obtains after 40 steps of duplication a totality with 2^{40} $(> 10^{12})$ members. S1 is used for indications of these members.

This is still not a natural number series with 10^{12}. In order to go further I have to consider a hypothesis related to T9:

The central ontological hypothesis (c.o.h.). Let E be an accomplished procedure (called the *basis* of c.o.h.). Let for each step e of E l_e be a finite

procedure feasible if the l_d's are accomplished for all steps d preceding to e in E. (I call l_e the *load* of c.o.h.). Then a procedure E_l is feasible which consists in the reproduction of E with l_e following each step e (before the later steps f of E are accomplished).

C.o.h. is a very strong hypothesis which practically implies all applications of T9. It is a hypothesis to the effect that the applications of pmf may be iterated; and this is the only kind of iteration of pmf allowed in my program.

C.o.h. deals only with the feasibility, i.e. the organic possibility, of l_e and E_l. There is no counterpart of c.o.h. for epistemic possibility in my theory.

C.o.h. asserts only the feasibility of E_l. Feasibility is a kind of possibility – the possibility of a construction – and if one wishes to assert the presence of E_l one has to accompany the application of c.o.h. by that of pmf.

The acceptance of c.o.h. leads to a *third degree* of modalities and the strength of c.o.h. consists in the extension of the old rules to it. But one can show that c.o.h. is not a sufficient means for obtaining an impossible situation from possible ones and so one can justify the applications of c.o.h. in the consistency-proof for ZF. For the leading idea of this proof is, that if ZF were inconsistent then a violation of the main principle of necessity would hold in a possible situation obtained with the help of c.o.h. (This violation would consist e.g. in the fulfilling of an identification of 0 with 0′ in the course of an activity prohibiting this identification.)

There is another principal hypothesis in my program: *All* (non-trivial) *obstacles other than catchings are strange.* This *hypothesis on obstacles* essentially is not new to the foundations of mathematics because in traditional mathematical logic even catchings are considered as strange. With the help of c.o.h. I can essentially show a protodemonstration of this hypothesis. But the detailed justification of c.o.h. and this hypothesis belongs to some extreme directions of the program.

11. In connection with T10 I say that already the introduction of tactics of attention and collations makes the structure of logical theories more complicated than is provided for by T10. Often it is inconvenient to consider the collations used in a text at the same level as the text, and one has to return from the collations to the collated signs. The consideration of aims introduces pragmatics as a part of the semiotics under investigation. I don't exclude the importance of the two levels considered in T10 but I find it insufficient for my aims. I deal essentially with semantical and pragmatic considerations connected with metatheories. (Following Curry I prefer

the term 'epitheory' reserving the term 'metatheory' for a well-defined epitheory.) It is hard to answer whether the rules of inference belong to the object theories or to the metatheories in traditional logic. This question becomes still more difficult if one analyzes these rules and studies the means of analysis in a third theory. Certainly the new means do not belong to the object theory but the metatheory then splits. A fortiori it splits when we introduce considerations of an essentially new kind dealing with the establishment of convincingness. The structure of the new program is to be described in terms of methods, tactics and aims rather than in terms of the traditional division into object and metatheories.*

As concerns T11, I say here only that for most of the intuitionistic postulates one can repeat a way of substantiation available in the traditional logic. There are several methods, and the substantiation is always connected with the interpretation chosen for the ordinary logical operations. I have to carry out such a substantiation in order to see that it is compatible with the requirements of my prototheories: relevancy theory, collation theory, and the theories of the modalities, tenses and voices. Here I note that the traditional postulate $\neg A \supset (A \supset B)$ was reduced by A. Heyting to $(A \vee B) \,\&\, \neg A \supset B$. In order to substantiate the latter I use the permission to identify with b 'the member which is distinguished from a of the pair $\{a, b\}$ of distinguished objects'. It is not difficult to see that this permission is compatible with the requirements of collation theory.

In connection with Bernays' axioms $\forall x A(x) \supset A(t)$ and $A(t) \supset \exists x A(x)$ I note that they presuppose the denotational connection $x \to t$. These connections I call simply 'arrows'. For a single axiom this is merely a trivial remark: but if a great totality of such axioms enters in a deductoid the study of these arrows becomes a large part of the establishment of convincingness.

Consider two of Bernays' axioms $\forall x A(x) \supset A(t(y))$ and $\forall y B(y) \supset B(s(z))$ entering in a deductoid. They are to be accompanied by the arrows $x \to t(y)$ and $y \to s(z)$ respectively. The first of these arrows shows that for each value of y $t(y)$ is an admissible value of x, and the second shows the same thing for z, $s(z)$ and y. Then if the variable y is to be identi-

* The notion of 'formalization' is now to be enlarged; 'to formalize' the use of a notion means for me 'to expose a method of using its name'. Perhaps in this deviation from the traditional understanding of formalization by means of a formal system lies the essence of my criticism of T10. (N.B. A method = a system of rules (expressed in a language). Formalization is to be understood as any establishment of a system of rules.)

fied in its occurrences in both axioms it is natural to suppose that $t(s(z))$ should be (for each value of z) an admissible value of x. It is denoted by the arrow $x \to t(s(z))$ which is also to be inscribed in connection with the deductoid, I say this arrow *refers to* it. If a third Bernays' axiom, say $C(r(w)) \supset \exists z\, C(z)$, accompanied by the arrow $z \to r(w)$, occurs in the deductoid, and the variable z is to be identified here with the same variable in its former occurrences, then in the same way the arrow $x \to t(s(r(w)))$ is to be considered as referring to the deductoid; and this means that (for each value of w) $t(s(r(w)))$ is an admissible value of x. This fact in its general form is justified by means of S1–S3. But it must not be assumed that the composition of functions (with admissible domains and ranges) always leads to a well-defined function. E.g. every natural number series must be closed relative to n' but N_τ is not closed relative to the function n''. So the semantical conditions expressed by arrows may be violated.

One of the principal concerns of the establishment of convincingness is to show the correctness of all arrows referring to the deductoid. There is a strict system of rules governing the notion of an arrow referring to a deductoid. They are in agreement with the requirements of collation theory, e.g. if an arrow $x \to t$ refers to a deductoid, and t in this occurrence is to be identified with s somewhere in the deductoid then the arrow $x \to s$ also has to refer to the deductoid. (This arrow $x \to s$ is called the *identificational arrow*.) Many complications arise from the fact that rule (Ca) (instead of T3) is allowed as a rule of inference. But the main difficulties resulting from the use of (Ca) are connected with catchings. The latter are studied as unbounded metafunctions, and the second principal concern of the establishment of convincingness is to show that every such metafunction is strange to the deductoid. There are several ways to do it, e.g. the method of symmetry, analogous to that used by A. Mostowski in his proof of the independence of the axiom of choice. This method is applicable when one has to show that a symmetrical part of a deductoid cannot be the field of a catching. The underlying idea, that one cannot define a single-valued function taking infinitely many values in a sufficiently symmetrical field, is made more precise by means of the principle of strangeness from relevancy theory.

There is a kind of identification which is dangerous in the sense that it can very easily create a Zenonian situation. Suppose that φ is a function symbol entering in a deductoid (φ denotes a function defined in a studied process and taking its values also in a studied process) and that there is an infinite class of identifications of the values of φ with different parts of a

finite object E (e.g. a term or formula entering into the deductoid). Then E becomes the field of a Zenonian situation. I call such identifications *dangerous*; this danger is of great importance, especially in the case when φ is defined on a short natural number series (which is shorter than E), but theoretically even a long series may be cofinal with a very short subsequence of it. From now on I suppose that all deductoids considered are free from dangerous identifications unless the contrary is allowed explicitly.

There is a protodemonstration of the fact that a deductoid without arrows always possesses an establishment of its convincingness.* The same is easily proved for deductoids without the use of (Ca). But this assertion depends on the fact that all deductoids are considered in a form not making use of open formulas; the axioms of Bernays mentioned are to be replaced by their closures.

In order to say more about the form of deductoids I add here that only closed formulas are admitted in them; this affects the list of axioms for quantifiers. So only closed formulas of the type $\forall x A(x) \supset A(t)$ are admitted as axioms and in the arrow $x \to t$ the term t is a constant. But for each application of (Ca) with the premises $P(m)$ whose deductoids use an arrow $x \to \varphi(m)$ the conclusion $\forall m P(m)$ is accompanied by the arrow $x \to \varphi(m)$ which is said to *absorb* the arrows $x \to \varphi(m)$. Thus the closure of a formula $\forall x A(x) \supset A(t(y, z))$ is proved with the help of two applications of (Ca), and is accompanied by the absorbing arrow $x \to t(y, z)$. Each arrow $x \to t$ accompanying a premise of a rule of inference accompanies also its conclusion; and if x occurs in the other premise the arrow accompanies the latter premise too, provided x is identified in both premises. Thus arrows are repeated and I call them in their new occurrences the *induced* arrows. Now the rules for obtaining the *composition* $x \to t(s(z))$ from two arrows $x \to t(y)$ and $y \to s(z)$ (and for the analogous compositions involving terms $t(y, u)$ with two or more variables) are formulated in a syntactically natural manner. (If an arrow is obtained by several acts of forming compositions of $x \to t(y)$ and $y \to s(z)$ etc., then the different applications of these acts are connected to distinct acts of identification of the variable y in the formulas of the deductoid.) If x enters into a formula with the arrow $x \to t$ through a term $\varphi(x)$, it is required that $\varphi(t)$ should be an admissible value of a variable. A special *interpretational* occurrence of $\varphi(t)$ is introduced in such

* Besides the absence of dangerous identifications, this prototheorem presupposes that all applications of (Ca) are of a nature really encountered in my investigations; cf. the second paragraph on page 31 or the third on page 35.

cases. (The word 'term' is replaced by 'termoid' if it is not certain that for each system of values of its variables the term denotes an admissible value of a variable.)

One can reduce even the first principal objective in regards to arrows to the problem of proving the impossibility of non-strange catchings; a fortiori it is true for the second objective introduced especially in connection with catchings. The latter being the only (non-trivial) obstacles, one can say that the establishment of convincingness is the verification of the absence of (non-strange) obstacles for the construction needed in connection with the substantiation of the deductoid.

Recall now the well-known theorem of Bernays and Gödel concerning the expressibility in the ordinary arithmetic of every prf φ. If one analyzes the occurrences of arrows in the proof of this theorem (represented in a form where T3 is excluded in favour of (Ca)), one notices that the correctness of the condition on arrows presupposes that the natural number series is closed relative to φ.

12. The establishment of convincingness of a deductoid consists mainly in considering a class called the *envelope* of the deductoid. The deductoid itself is a finite object: the applications of (Ca) are represented in it by their tasks. (Two tasks of (Ca) are to be identified if for every m they lead to the identified deductoids of the premise $B(m)$.) If an application of (Ca) with the premises $B(m)$ enters into a deductoid P then the deductoids for the premises $B(m)$ are not presupposed to appear before the conclusion $\forall m\, B(m)$ of that (Ca). This is connected with the fact that the truth of $\forall m\, B(m)$ is interpreted as the truth, i.e. the feasibility of a proof, for each $B(m)$ (cf.(f) on p. 10), but not as its presence. I say that the deductoids for $B(m)$ *refer to* P but not *directly* (whereas the task of (Ca) and the formula of P *refer to* P *directly*). The envelope of P, called eP, consists of all objects referring to P; these are the formulas of P and of deductoids of premises of (Ca) referring to P, the arrows belonging to formulas, the tasks of (Ca) and the applications of the other rules of inference, the interpretational occurrences of termoids and all collations required for these objects as well as the establishments of exhaustion refering directly to their structural procedures. For each process studied, I introduce the corresponding *semantical process*. The events of this are the names obtained for the events of the studied process, according to S2. Each such name appears in the semantical process after its sense has occurred in the process studied; it is called a *semantical event*. These events

refer to P and for each reference of a constant termoid t to P the structural identification of t in this occurrence with the synonym of t occurring as a semantical event refers to P. (Structural identifications for two synonyms are defined in a natural manner with the help of identifications for more simple synonyms occurring as their parts.) These *semantical identifications* represent the interpretations of constant terms occurring in the envelope eP. Also for each m of an application of (Ca) a semantical identification of that m in the semantical process and in $B(m)$ is required, in order to guarantee the coincidence of the studied process mentioned in connection with different applications of (Ca) related to the process. All objects of the envelope eP are considered as appearing in accordance with the requirements of genetical constitution. This entails that Zenonian situations destroying the feasibility of the tasks of the elements of eP may appear, and one of the requirements of the establishment of convincingness is to guarantee that this will not be the case. The constructions involved in the semiotical justifications of the intuitionistic postulates don't require the appearance of any event except those which have to have occurred in order that the postulate be meaningful (in its reference to P). So these constructions are always feasible on the grounds of c.o.h., and they cannot introduce new Zenonian situations or catchings. On these grounds these constructions are not introduced in eP (but semantical identifications have been included in eP so that this introduction might be possible without any essential change in the feasibility of the other elements of eP).

There are two principal kinds of arrows:

(a_1). those accompanying the Bernays axioms, the absorbing arrows accompanying the conclusions of (Ca) and the identificational arrows;

(a_2). the arrows obtained from the arrows (a_1) by composition.

For (a_2) as well as for (a_1) there are induced occurrences; and compositions are to be made only from arrows belonging to the same occurrence of a formula. The rule for obtaining the identificational arrows is as follows: Let an arrow $x \to t_1$ belong to a formula F and let $\mathrm{Oc}_1, \ldots, \mathrm{Oc}_n$ be the occurrences of terms $t_1, \ldots, t_n$ such that for each two Oc_i, Oc_{i+1} $(1 \leqslant i < n)$ either the identification of t_i and t_{i+1} in these occurrences refers to P or the occurrences Oc_i and Oc_{i+1} are parts of an equality $t_i = t_{i+1}$ (or of a synonym $\mathrm{Eq}(t_i, t_{i+1})$ of such an equality). Then the *identificational* arrow $x \to t_n$ also belongs to the formula F in the same occurrence of the latter. The identificational arrows refer to P. But it is allowed to neglect them if $x \to t_n$ already has an occurrence belonging to the same occurrence of F

or if there is a protodemonstration that the string $Oc_1, \ldots, Oc_n$ is strange to the construction of elements of eP. (The equality $t_i = t_{i+1}$ may be proscribed in the construction of these strings if it is certain that it does not express an identification of t_i and t_{i+1}, e.g. when t_i and t_{i+1} are the events of different studied processes.) When it is not clear if an identificational arrow belongs to a formula, I prefer to count it as belonging to it. (Superfluous arrows can only damage the establishment of convincingness.) The rules for identifications related to parts of arrows are formulated in a natural manner.

There is an important requirement called the requirement of *termoidal completeness*. When a task $\tau(x)$ of a name $t(x)$ of an event of a studied process refers directly to the task of a deductoid (e.g. through a task of (Ca)) for indefinitely many x's, then it is required that there is a functional symbol $t(x)$ representing this τ in the (envelope of the) deductoid. It is required that for each of those x's an identification of $t(x)$ with $\tau(x)$ refers to a deductoid or at least that there should be a string of such identifications uniting $t(x)$ with $\tau(x)$. This requirement is in accordance with those of collation theory. It guarantees that one cannot prevent the appearance of the absorbing arrow $x \to \varphi(m)$ by replacing the $\varphi(m)$ by their synonyms not containing 'φ'.

The tasks of (Ca) may be, generally, of a completely arbitrary nature. But in practice I need for my aims only several simple cases of these tasks requiring no semiotic pathology. (The most difficult of them are connected with the representation of some variants of T3.) If all (Ca) are restricted to these simple cases there is a protodemonstration of the fact that the requirement of the termoidal completeness is fulfilled if the non-absorbed arrows (of type a_1: cf. p. 34) are finite in number. The latter condition I call that *of the finiteness of the arrows*.

For every element Π of eP I denote by $\hat{\Pi}$ the finite set of objects preceding Π in order of genetic constitution. (All these objects are to belong to eP.)

Now I can say that the establishment of convincingness of a deductoid P consists in three requirements imposed on P and on deductoids P' referring to P.

I. every termoid referring to P shall be a term;

II. the requirement of termoidal completeness;

III. for every Π in eP' each unbounded metafunction whose values belong to $\hat{\Pi}$ is strange to the considered task of P'.

In accordance with S3 the deductoids as well as their establishments of convincingness may depend on various parameters. These may be fixed in

an admissible manner through all occurrences of them; the establishment of convincingness remains an establishment thereafter.

There are many reductions of I–III to sufficient conditions of a finitistic type. E.g. I observe for condition I that for the arrows $x \to t(y)$ occurring in deductoids of formulas $\forall n(P(n) \supset P(n'))$, if a variable y occurs in $t(y)$ it does not occur as the left part of an arrow, which saves the envelope from an accumulation of functional symbols in the right parts of arrows obtained by the composition of $x \to t(y)$ with the arrows $y \to s(z)$. Note also that from l arrows of the form $x \to t$ where t is another variable y or y' one cannot obtain an arrow $x \to r$ where r contains more than 2^{2^l} strokes. Of course if K_l is a natural number series such that for each α in it there must be a K_l-number equivalent to $\alpha + 2^{2^l}$ this leads to some simple finitistic conditions sufficient for I. Requirement II is mostly replaced by the condition of finiteness of arrows. The fulfilment of III is guaranteed by a number of conditions dealing with certain symmetries of different parts of deductoids etc.

I mentioned above that the fulfilment of the condition on the arrows can be reduced to the impossibility of some non-strange catchings. Now I can say that the field of those catchings consists of the *numeroids*, i.e. the termoids obtained by c.o.h. as a result of calculation of given termoids; the obstacle is an obstacle to the semantical interpretation of the numeroid and consists in the fact that the parts of the numeroid catch on a semantical process (or on a studied one).

Now I shall describe briefly the semantical interpretation of the usual logical operators preferred by me in connection with T11. $A \supset B$ is interpreted as the presence of a proof of B from A (this proof being of course of a somewhat more simple nature than that wherein the $A \supset B$ occurs; e.g., this proof may depend on a more simple value of a parameter or on a more simple way of fixing it). $\neg A$ is short for $A \supset f$ where f acts like $0 = 0'$. It is important to remark that this concerns only $\neg$ in the usual contexts of mathematical logic, because the appearances of 'not' in the names of (VIII) on p. 16 cannot be reduced to implications. The quantifiers are introduced before the connectives and afterwards, e.g. conjunction is defined as a universal quantifier with a finite range. This provides me with connectives symmetrical from the beginning. (x) is called the *universal* quantifier: $(x)A(x)$ means the presence of a method enabling one to accept $A(x)$ for every value x of x that has occurred. (Ex) is called the *particular* quantifier and $(Ex)A(x)$ means that there may occur (with an arbitrary

modal characteristic and tense) an x for which $A(x)$ can be accepted. Here the modality of x in both quantifiers is arbitrary, but if it is 'real' these quantifiers become the usual $\forall x$ and $\exists x$ (used in the explication of the connectives). Some tactic of acceptance of sentences is here borne in mind.

It is not difficult to construct a deductoid for two formulas A and $\neg A$ and even to find an establishment of convincingness for each of them in such a manner that if one forms a further deductoid for $A \,\&\, \neg A$ one cannot establish its convincingness. More precisely, this impossibility holds only if the structural identification of both A's in $A \,\&\, \neg A$ is to be fulfilled. There are cases when this identification leads to a violation of the condition on the arrows, and there are also other cases when this identification causes a Zenonian situation or catching (e.g. with the field A). These are the *apparent contradictions* or *contradictoids* and they don't disturb the program. Some of them arise if one tries to prove with the help of T3 the isomorphism of two different natural number series: after replacing T3 by (Ca) and analyzing the arrows one finds a violation of the condition on the arrows. There are also contradictoids arising from considerations of the ultra-intuitionistic model for ZF (see below). E.g. one can in many ways indicate a set x and choose as A the sentence 'x is finite' (in some set-theoretical meaning of finiteness). Another contradictoid arises if one finds in this model a set y violating the axiom of choice and takes as A the sentence 'y is well-ordered'. (Besides the consistency of ZF I obtain ultra-intuitionistically Cohen's result about the independence of the axiom of choice and the continuum-hypothesis; but not his result about the independence of the continuum-hypothesis from the axiom of choice in ZF.) It is worth noting that contradictoids disappear if one excludes the identification of A in the text of $A \,\&\, \neg A$. Then the formula $A \,\&\, \neg A$ ceases to express a contradiction. In connection with T11 I must say that conjunctions $A \,\&\, B$ may differ according to the possibilities that parts of A and B are or are not collated to one another. If one has a proof of A and another one of $\neg A$ then the collations of different parts of A within the first A belong to the first proof, and those within $\neg A$ to the second; but if one merely combines both proofs one makes by this action no identification of both A's in the theorems A and $\neg A$. On similar grounds it is possible that there are two theorems A and $A \supset B$ such that B is not a theorem.

Sometimes I use the possibility of considering the texts of conjunctions $A_1 \,\&\, \ldots \,\&\, A_n$ with no identifications of the parts of the different A_j's. This enables me to consider some important parts of deductoids as sym-

metrical. E.g. in the axioms $A_1 \,\&\, \ldots \,\&\, A_n \supset A_j$ $(1 \leqslant j \leqslant n)$ the identification of both A_j's destroys the symmetry of the conjunction. But if one analyzes the aims of collations one remarks that this identification is unnecessary for establishing the fact that implication is an axiom, provided that one knows that A_j in the right-hand part of it is a member of the conjunction in the left-hand part of it. So one can exclude some of the customary identifications as useless for logical aims and reestablish the symmetry of many parts of deductoids. This symmetry is used in proving that these parts cannot be fields of catchings. It is only one of many similar devices of my program. E.g. I use the fact that in applications of (Ca) there is no need to collate the parts of different premises $P(m)$ and $P(n)$. Of course the exclusion of the customary identifications leads to a revision of T11. But as I have mentioned I can reestablish the substantiations of the traditional axioms of the intuitionistical predicate calculus.

This belongs to a brief description of my genetic theory. This theory is essentially more complicated than the ontological theory. (That is natural because the notion of a proof must be more complicated than that of a natural number series.)

13. Now I return to a brief discussion of the ontological theory.

Let K_0 be a natural number series containing a number k but not $k+2$ (see above, p. 27). Then K_0 does not contain 2^k.

For each K_0-number p I denote by $\hat{p}$ the totality of all K_0-numbers $\leqslant p$. (I call $\hat{p}$ the *genesis* of the K_0-number p and similarly I call the *genesis* $\hat{z}$ of an event z of an arbitrary discrete process D the totality of all those events which have necessarily occurred provided z has.) For each K_0-number p I consider now the application of c.o.h. with the basis p and the load consisting, for $0 \in \hat{p}$, in the formation of $\{0\}$ (i.e. in the indication of a void place) and for each $i+1 \in \hat{p}$ in the duplicating of the object obtained as the result of the ith load. So for each $i+1 \in \hat{p}$ I obtain the totality of strings $a_0 \ldots a_{i+1}$, where each a_j $(j \leqslant i+1)$ is 0 or 1. I apply the tactic of collations which enables one to identify $00 \ldots 0a_j \ldots a_{i+1}$ of the $(i+1)$st load with the $a_j \ldots a_{i+1}$ of the $(i-j)$th load. Now I apply c.o.h. once more with the basis $\hat{p}$ and with the $(i+1)$st load consisting in looking over the strings $a_0 \ldots a_{i+1}$ in their natural lexicographical order. This load is feasible if the ith such load is because the $(i+1)$st load consists in repeating the ith load two times (the first time in the form $a_1 \ldots a_{i+1}$ or $0a_1 \ldots a_{i+1}$ of the strings, and the second time in the form $1a_1 \ldots a_{i+1}$ of the strings with

the same ordering of $a_1 \ldots a_{i+1}$); here the step $10 \ldots 0$ is to follow the step $01 \ldots 1$, which causes a fold. Then I apply c.o.h. with the basis $\hat{p}$ and the load consisting for each $i+1 \in \hat{p}$ in the repeating of the foregoing $(i+1)$st load without taking the fold into attention (the result of this load being the consideration of the strings $a_0 \ldots a_{i+1}$ as appearing in lexicographical order). Now I consider the class K_0^* of all appearances of such strings $a_0 \ldots a_{i+1}$ with $i+1 \in \hat{p}$ where p is a K_0-number, the strings with different i's being collated according to the tactic described. The operation $x+1$ or x' is introduced for these appearances in the natural manner; and then the class K_0^* becomes a class of events occurring in the course of following a method which defines a natural number series. In the theory of collations I establish that this class K_0^* may be identified with this course itself, i.e. with a natural number series. (At least I show that the class of texts involving those wherein this identification is allowed is sufficiently large to include the texts I need in the central nucleus of my program.)

So given a natural number series K_0 with a K_0-number k such that 2^k does not belong to K_0 I obtain a new natural number series K_0^* containing numbers $\leqslant 2^i$ for each K_0-number i, and no other numbers. (Of course when I say this, I identify the equivalent K_0- and K_0^*-numbers, which is not always allowed.) From the ontological standpoint the fact that K_0^* is a natural number series means nothing but that the appearance of its events can be described by a method of the kind used in the definition of the notion of a natural number series and the fact of the possibility of K_0^* with the indicated property means nothing but the possibility of obtaining by means of c.o.h. the feasibility of each of the events of K_0^*. Further it is evident that if $K_0 \prec q \in K_1$ then $K_0^* \prec 2^q \in K_1^*$. Also (b) K_0^* is closed with respect to the sum-operation $a+b$ defined in a natural way. If $q = 2^k$ then $q \in K_0^* \prec 2^q$, so (c) there is a natural number series K_0^* closed with respect to $a+b$ but *explicitly not closed* with respect to 2^a and (d) one can construct many series $K_0^* \prec K_1^{**} \prec \ldots$ each of them satisfying the property (c).

Now I fix a natural number series L having property (c), e.g. $L = K_0^*$, and I introduce the series $M_0 = K_0^*$, $M_1 = M_0^*, \ldots M_{i+1} = M_i^*, \ldots$ where $i \in L$. For each L-number i the introduction of M_i is made by means of c.o.h. with the basis $\hat{\imath}$ and the load consisting, for $0 \in \hat{\imath}$, in obtaining the introduction of K_0^*, and for each $e+1 \in \hat{\imath}$, in obtaining K^* from the series K obtained in the result of the eth load. (So by c.o.h. the names of these series are feasible and their events are only the points of indications allowed by S1.) Finally I obtain a natural number series M_L whose events are those

belonging to some M_i, $i \in L$ with the natural tactic of collations identifying equivalent numbers of different M_i's. Now it is evident that (e) M_L is closed with respect to 2^a and (f) if $L_1 \prec L_2$ then $M_{L_1} \prec M_{L_2}$. Also it is easy to show (g) $L \leqslant M_L$. For the considered series L it is easy to introduce an operation $\varphi(L)$ such that $L \prec \varphi(L)$ and $\varphi(L)$ is closed with respect to $a+b$ (e.g. $\varphi(K_0) = K_0^*$, $\varphi(L) = L^*$ †, $\varphi(M_i) = M_{i+1}$, $\varphi(M_L) = M_{\varphi(L)}$). Let $\overline{M}_L$ be $M_{\varphi(L)}$. Then (e) and (f) hold with M_L replaced by $\overline{M}_L$ and (g) can be strengthened to (g*) $L \prec \overline{M}_L$.

If $m \in M_i^*$, then

$$2^{\cdot^{\cdot^{\cdot^{2^m}}}},$$

where 2 occurs g times belongs to M_{i+g}^* provided $g \in L$. (This follows from $i+g \in L$.) I denote by $2^{(a)}$ the ath *superpower* of 2 defined by the equations $2^{(0)} = 1$, $2^{(a')} = 2^{2^{(a)}}$. If $m = 1$ then the result obtained shows that $a \in L$ entails $2^{(a)} \in M_L$ (and a fortiori $2^{(a)} \in \overline{M}_L$). $q \in K_0^* \prec 2^q$ gives $q \in M_0 \prec 2^q$ and further

$$2^q \in M_1 \prec 2^{2^q} \in M_2 \prec 2^{2^{2^q}} \in M_3 \quad \text{etc.},$$

the number

$$2^{\cdot^{\cdot^{\cdot^{2^q}}}} \quad (\in M_i)$$

with i 2's being obtainable by means of c.o.h.; and $L = K_0^* \prec 2^q$ gives for each $j \in L$

$$M_j \prec 2^{\cdot^{\cdot^{\cdot^{2^q}}}},$$

where 2 occurs 2^q times whence

$$M_L \prec 2^{\cdot^{\cdot^{\cdot^{2^q}}}}$$

with 2^q 2's and $2^q < 2^{(q)}$ gives $M_L \prec 2^{(2^q+q)} < 2^{(2^{q+1})}$. So $M_L \prec 2^{(2^{q+1})}$; but $q \in K_0^* = M_0$ gives $2^{q+1} \in M_2 \subseteq M_L$, so $2^{q+1} \in M_L \prec 2^{(2^{q+1})}$. (In a similar way one can obtain an r such that $r \in \overline{M}_L \prec 2^{(r)}$.) Now the operation M_L over L behaves itself towards $2^{(a)}$ just as K^* does towards 2^a (with M_L in the rôle of K_0^* and 2^{q+1} in the rôle of k); therefore I can repeat the

† Here it is supposed that $L \prec L^*$.

construction used for K_0^* and 2^a and obtain a series closed with respect to $2^{(a)}$ and (with the help of (g^*)) even many such series $N_1, N_2, \ldots$.

One can go further and obtain series closed with respect to the operations $2_i^{(m)}$ defined by $2_4^{(m)} = 2^{(m)}$ and $2_{i+1}^{(0)} = 1$, $2_{i+1}^{(m')} = 2_i^{(2_{i+1}^{(m)})}$. One can even replace here the equation $2_{i+1}^{(0)} = 1$ by $2_{i+1}^{(0)} = 2_i^{(1)}$. For each i from a series L one can obtain another series $N_1 \prec N_2 \prec \ldots$ closed with respect to $2_i^{(m)}$. One can even obtain series closed with respect to $2_i^{(m)}$ as a function of two variables i and m. According to a well-known result of R. Peter, for each prf $\varphi(n)$ one can find an i_φ such that $\varphi(m) < 2_{i_\varphi}^{(m)}$; so one can obtain a natural number series N closed with respect to any prf and even many such series $N_0, N_1 \ldots$ (The index i in N_i may belong to any previously constructed series.)

Of course ultra-intuitionistically the notion of prf is to be relativized and generalized. Let prf be defined as in Kleene's book 'Introduction to metamathematics'. Then each prf can be obtained from some initial prf's in a number of applications of two functionals: (a) the schema of composition, and (b) the schema of primitive recursion. This number is now a N-number, N being a parameter. (One of the initial functions also depends on such a parameter; but that is not very essential.) Further one can now consider prf's defined on different series and taking their values in others. To the two mentioned functionals I propose to add still two functionals consisting merely in replacing an argument or value by its equivalent in another series: (c) continuation and (d) restriction. The rôle of these functionals is to replace one series by another in the trivial manner, the resulting function being undefined if one cannot obtain its value in this way. This is the most essential ultra-intuitionistical generalization of the notion of prf; so one obtains the notion of prf defined by means of various natural number series. The notion of partial recursive function remains essentially the same, insofar as it depends on the calculations, where the latter are to be feasible on the ground of c.o.h. It is easy to show that each partial recursive function $\chi(x_1, \ldots x_n)$ is a prf in the sense just introduced. Namely, let $\psi(\mu y[\chi(x_1, \ldots x_n, y) = 0])$ be its traditional Post normal form – by a continuation of the series considered one gets its applicability (i.e. closure with respect to ψ and χ). Now for the operator $\mu y[\chi(x_1, \ldots, x_n, y) = 0]$ I consider its representation in terms of the function τ defined by a single equation $\tau(z', 0, y) = y$ (see Kleene's book § 57). Let N denote the series on which these functions are now considered and $N \prec q \in \bar{N}$. (For the series considered one always can find such an $\bar{N}$ and q.) Let $\bar{\tau}$ be defined

on $\bar{N}$ by the equations $\bar{\tau}(z', 0, y) = y$, $\bar{\tau}(0, u, y) = \bar{\tau}(v, u', y) = q$. Then $\mu y[\chi(x_1, \ldots, x_n, y) = 0]$ can be defined on N as the restriction to N of the prf

$$\mu\, y_{y<q}\,[\bar{\tau}(\prod_{s<y}\chi(x_1, \ldots, x_n, s), \prod_{s\leqq y}\chi(x_1, \ldots, x_n, s), y) = y]$$

and so $\chi(x_1, \ldots, x_n)$ is a prf.

Of course the result obtained before about the possibility of series N closed with respect to every prf is to be understood as referring to prf's defined by means of N-numbers only. Even this notion is now relativized and one can find series $N_1 \prec N_2 \prec \ldots$ with this property. It may be that these prf's on N_2 majorize Ackerman's function on N_1, and so on.

14. But in order to prove the consistency of ZF with k inaccessible cardinals one needs only the $k+2$ series $K \prec N_0 \prec N_1 \prec \ldots \prec N_k$, the N_i's being closed with respect to superpower and the possibility of every N_k-number k' of such $N_0' \prec N_i' \prec \ldots \prec N_{k'}'$. For the pure system ZF it is sufficient to consider $K \prec N_0$ with possibility of varying N_0 by choosing its values $N_0^1 \prec N_0^2 \prec \ldots \prec N_0^m$ for each N_0-number m. By means of N_0^i I construct a model D^i for the system $\widetilde{\mathrm{ZF}}_i$ equiconsistent with ZF without the axiom of infinity and I have to introduce the latter into D^i *. In this model D^i there is a set m_l of l 'urelemente', and I choose K such that $m \in K$ entails $m+2^{2^l} \in K$. Then I introduce intentionally a Zenonian situation with the help of a 1-1 function defined on K and taking its values in m_l, so that the set m_l becomes infinite. After this I have to verify the axioms of $\widetilde{\mathrm{ZF}}_i$ (in-

* The main differences between $\widetilde{\mathrm{ZF}}_i$ and ZF are the following:

(a). the axiom of extensionality is missing in $\widetilde{\mathrm{ZF}}_i$ and a new functional symbol $q(x)$ with axioms $x \subseteq q(x)$, and $\forall u(u \in x \backsim u \in y) \supset q(x) = q(y)$ is introduced in order to obtain the equiconsistency with ZF;

(b). the logic of $\widetilde{\mathrm{ZF}}_i$ is intuitionistic and all $\exists x$ in axioms are replaced by $\neg\forall x\neg$ (likewise for all disjunctions);

(c). for the atomic formulas $x \in y$ and $x = y$ the law $\neg\neg A \supset A$ is accepted.

(c) enables one to prove the consistency of $\widetilde{\mathrm{ZF}}_i$ with classical logic. But (b) entails that for the set-theoretic functions (and even for those representing prf's in ZF) instead of existence only the double negation of existence is provable in $\widetilde{\mathrm{ZF}}_i$. This explains the fact that the closedness of N_0 with respect to superdegree enables us to construct a model for $\widetilde{\mathrm{ZF}}_i$.

(d). The axiom of infinity is replaced by the introduction of a new alphabet $\alpha, \beta, \ldots$ of variables for the elements of m_l and of the arithmetical signs 0 and $'$ with Peano's axioms $\forall\alpha\alpha' \neq 0$ and $\forall\alpha\forall\beta(\alpha' = \beta' \supset \alpha = \beta)$.

cluding the axiom of infinity) for the resulting model. I construct a deductoid for each of them, and I show that if there were a contradiction C in $\widetilde{\mathrm{ZF}}_i$ containing $\leqslant l$ formulas then by replacing in C the axioms by their deductoids I obtain a deductoid D for a contradiction (not only a contradictoid!*) and an establishment of convincingness for D. Here I use essentially the fact that the sets of the model D^i are symmetrical, and so the main parts of the deductoids may be constructed as symmetrical also: in this way the most disturbing catchings are avoided. (More precisely, for $k = 0$, I construct deductoids in such a way that all objects taking part in it and depending on l are symmetrical. These cases are the most difficult of all.)

Of course all this is but a very short sketch, in which only a very small part of the consistency proof for ZF is indicated. All technical parts of this proof are investigated by me in complete detail and exposed in many manuscripts. The most urgent of them is ready for publication. But the size of all these manuscripts exceeds that of Kleene's book and that prevents me from publishing them quickly. In this most urgent text, the demonstroids are written down, as well as an exact description of the establishment of their convincingness and the chief ideas needed in carrying it out. After this text becomes available to my colleagues I shall be able to answer further questions with a suitable degree of exactness. But all this technically is a very large program and my requirements as to rigour surpass my ability of writing quickly. So many questions will remain and I hope they will be settled in the course of further study. If I have a success even today, it consists in the fact that these questions are essentially deeper than all questions known independently of my program.

Of course, from the ultra-intuitionistic standpoint the traditional axiomatic set theory ZF is but a poor fragment of our thinking.† It is to be supplemented by the theory of modality and the whole domain of research of my prototheories, and this can essentially be done with the help of my method for founding it. But the resulting theory is of course not an axiomatic one.

15. The ultra-intuitionistic program is also an ultra-pedantic one. In order to fulfil its requirements completely it is necessary to construct theories

* More exactly C is considered as a formal proof of $0 = 0'$ in $\widetilde{\mathrm{ZF}}_i$ and D as a demonstroid for $0 = 0'$.

† Also I have to note that ZF is only one way of formalization of Cantor's set theory and therefore its formalization must not close the domain of research in foundations of set theory. E.g. one has to investigate the consistency of Quine's system 'New Foundations'.

about each grammatical category unavoidable in the exposition of this program. That is possible, but still not fulfilled in all detail. So today there are questions answered only intuitively. There is a kind of intuition unavoidable in this research. To this intuition belongs the understanding of aims of each use of a sign or of each other elementary act of thinking. I aspire to describe also this intuition by some methods but then new elementary acts are required and I am forced to indicate some external tactic followed only intuitively. That is unavoidable. But one of the extreme directions of the program consists in reduction of all uses of intuition to the intuitive understanding that the signs used correspond to their aims. I call this direction *pragma-ultra-intuitionism.*

At any rate nobody can dispense with a kind of confidence to his own memory in his study of a theory. Of course, that is true for my program, too. I argue that this kind of confidence is different from faith in the ordinary sense, because it is not used as an argument in a proof or deduction. Nevertheless the *logic of confidence* governing this kind of confidence is required in my program in order to justify confidence in memory too. This logic deals with the transition from '*A* asserts *B*' and 'I believe *A*' to 'I accept *B*', and with many rules of preference of one source of confidence to another one. I recall that the ethical term 'fair' enters in the general explication of 'proof', and I aim to establish an ethical theorem that it is better to accept memory as a source of confidence than to reject it. (If I prefer *A* to *B*, I say *A* is *better* than *B*; this explication shows why I have always to prefer the better; and if sometimes I say I refuse to do it the reason is a play on words based on the presence of several tactics of preference.) The principle of tautology is considered as the best source of confidence.

I introduce the extreme directions in order to state that today all unsolved questions essentially belong to the extreme directions. So in order to show today's state of affairs in my program I can list these extreme directions. They too have been studied by me in part. I can indicate today the following seven directions (in their names I use the abbreviation 'uism' for 'ultra-intuitionism'):

(1). *ultra-ultra-intuitionism* (briefly: uduism) – the construction of the theory of disputes – the logic of confidence and the relevant parts of ethics;

(2). *extra-ultra-intuitionism* (briefly: eduism) or *relevantism* – the substantiation of the hypothesis on obstacles, or its abolishment from our considerations;

(3). *trans-ultra-intuitionism* (briefly: teduism) – the same for c.o.h.;

(4). *pragma-ultra-intuitionism* (briefly: peduism);

(5). *lega-ultra-intuitionism* (briefly: eluism) – the foundations of the primary permissions and the deontic relations between different extreme directions; the relevant parts of ethics;

(6). *nega-ultra-intuitionism* (briefly: neguism) – the substantiation of the principle of the negative evidence;

(7). *bi-nega-ultra-intuitionism* (briefly: bineguism) – the substantiation of the elimination of double negations in the foundations of prototheories. Some questions of the same type are still remaining in the foundations of the genetic theory, but here there is evidence that they can be settled essentially by means of Gödel's imbedding operations. In many cases this elimination requires the consideration of the double negation of a possibility as a new kind of possibility, for which pmf and other modal principles can be proved. Generally to this direction belong only such questions that can be dissolved in the considerations of modalities.