PRO-REPRESENTABLE VIRTUAL DOUBLE CATEGORIES

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ROADMAP

Intro to VDCs:

Appendices:

Pro-representability:

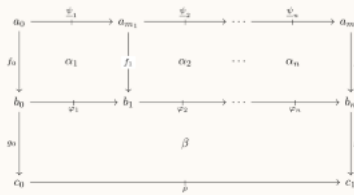**Conclusions and Future Work:**

Definition: VDCs

A virtual double category $\mathbb{D}$ consists of

1. A category $\mathrm{Tight}(\mathbb{D})$ of objects and tight, or vertical arrows
2. For every pair of objects a, b a collection of loose arrows $a \nrightarrow b$
3. For every boundary of loose and tight arrows a collection of cells

$$x_0 \xrightarrow{\varphi_1} x_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} x_n$$

f α_n g

$$y_0 \xrightarrow{\psi} y_1$$
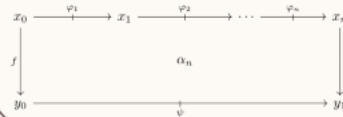
Along with an associative and unital composition of cells:

$$a_0 \xrightarrow{\psi_1} a_{m_1} \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_n} a_{m_n}$$

f_0 α_1 f_1 α_2 $\cdots$ α_n f_n

$$b_0 \xrightarrow{\varphi_1} b_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} b_n$$

g_0 β g_1

$$c_0 \xrightarrow{\rho} c_1$$

Note: Together with VD functors and tight transformations, VDC forms a (co)complete 2-category with finitely presentable underlying category

PSEUDO VS. VIRTUAL CONSTRUCTIONS

(Pseudo-)Double Categories:

- If $\mathcal{E}$ is a category with pushouts, we have a double category $\mathbb{C}\mathrm{ospan}(\mathcal{E})$

- If $(\mathcal{V}, \otimes, I)$ is a monoidal category with finite coproducts that are preserved by $\otimes$, then we have a double category $\mathcal{V}\mathbb{M}\mathrm{at}$

- If $\mathbb{D}$ is a double category with certain reflexive co-equalizers, we have a double category $\mathbb{M}\mathrm{od}(\mathbb{D})$ of monoids in $\mathbb{D}$

Virtual Double Categories:

- For any category $\mathcal{E}$, we have a virtual double category $\mathbb{C}\mathrm{ospan}(\mathcal{E})$

- For any virtual double category $\mathbb{D}$ we have a virtual double category $\mathbb{D}\mathbb{M}\mathrm{at}$

- For any virtual double category $\mathbb{D}$, we have a unital virtual double category $\mathbb{M}\mathrm{od}(\mathbb{D})$ of modules in $\mathbb{D}$

UNIVERSALITY OF VIRTUAL CONSTRUCTIONS

Universality of Constructions on Virtual Double Categories:

- For any category $\mathcal{E}$, the virtual double category $\mathbb{C}\mathrm{ospan}(\mathcal{E})$ is the free virtual equipment on $\mathcal{E}$ [DPP10]

- For any virtual double category $\mathbb{D}$ the virtual double category $\mathbb{D}\mathbb{M}\mathrm{at}$ is the free coproduct completion of $\mathbb{D}$ [Kaw25]

- For any virtual double category $\mathbb{D}$, $\mathbb{M}\mathrm{od}(\mathbb{D})$ is the cofree normal completion of $\mathbb{D}$ [CS10]

- For any virtual double category $\mathbb{D}$, $\mathbb{D}\mathbb{P}\mathrm{rof} = \mathbb{M}\mathrm{od}(\mathbb{D}\mathbb{M}\mathrm{at})$ is the free collage cocompletion of $\mathbb{D}$ [Kaw25]

Definition: Pro-representable VDCs

A virtual double category $\mathbb{D}$ is said to be pro-representable, or exponentiable, if the functor given by sending a virtual double category $\mathbb{A}$ to $\mathbb{A} \times \mathbb{D}$ admits a right adjoint:

$$
\text{VDC} \underset{-\times\mathbb{D}}{\overset{(-)^{\mathbb{D}}}{\rightleftarrows}} \text{VDC}
$$
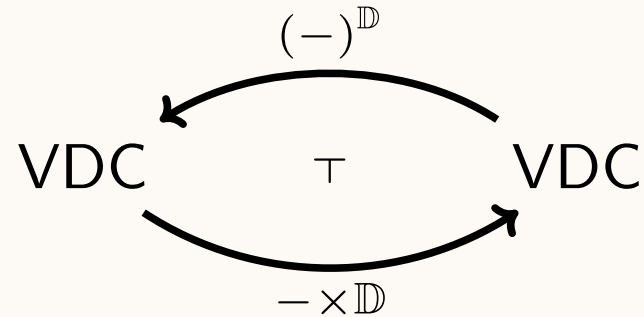
Explication: Exponential

If $\mathbb{D}$ and $\mathbb{E}$ are VDCs for which $\mathbb{E}^{\mathbb{D}}$ exists, then it must consist of the following data:
- Objects are functors Tight($\mathbb{D}$) →Tight($\mathbb{E}$)
- Tight arrows are natural transformations
- Loose arrows maps of spans

$$\begin{array}{ccccc}
\text{Tight}(\mathbb{D}) & \xleftarrow{\;s\;} & \text{Sq}(\mathbb{D}) & \xrightarrow{\;t\;} & \text{Tight}(\mathbb{D}) \\
\downarrow{\scriptstyle F_0} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F_1} \\
\text{Tight}(\mathbb{E}) & \xleftarrow{\;s\;} & \text{Sq}(\mathbb{E}) & \longrightarrow & \text{Tight}(\mathbb{E})
\end{array}$$

- n-Multi-cells assign to each n-multicell in $\mathbb{D}$ an n-multicell in $\mathbb{E}$ with the following boundary:

$$\begin{array}{ccc}
F_0 x_0 \xrightarrow{\;\Psi_1 H_1\;} \cdots \xrightarrow{\;\Psi_n H_n\;} F_n x_n \\
\gamma_0(!,f) \downarrow \qquad\qquad \gamma(\alpha) \qquad\qquad \downarrow \gamma_1(!,g) \\
G_0 y_0 \xrightarrow{\hspace{3cm}\Psi J\hspace{3cm}} G_1 y_1
\end{array}$$

... (cont. on next slide)

Explication: Exponential (cont.)

Where multicells are subject to functoriality with respect to vertical pasting:
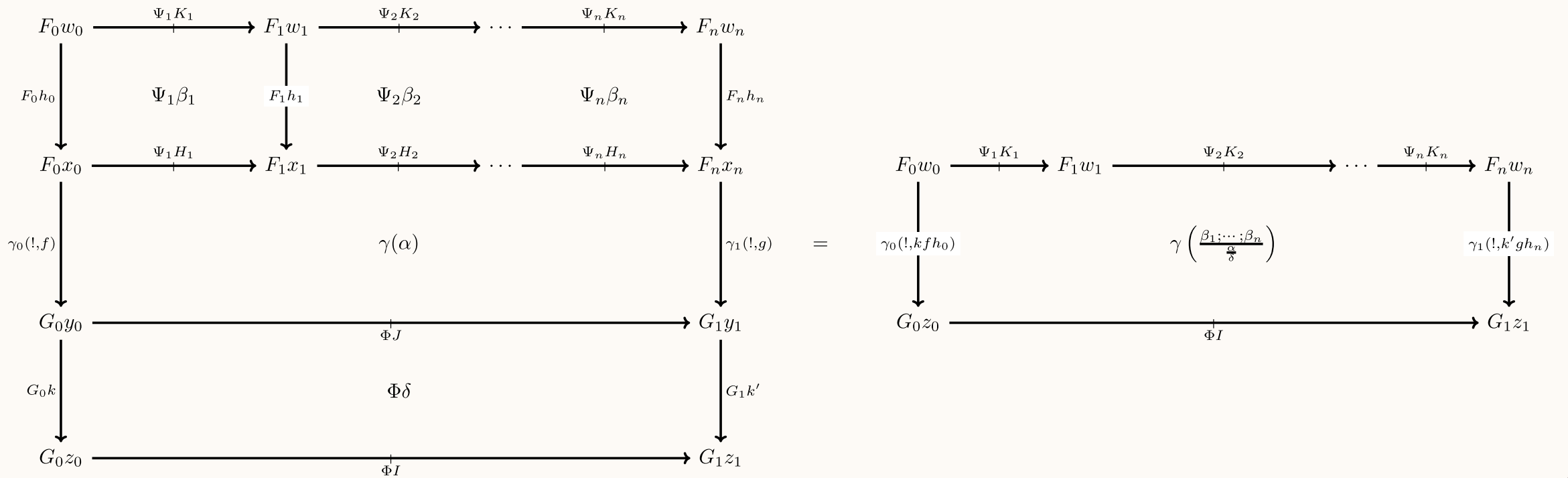
Theorem: Characterization of Pro-representable VDCs

Let $\mathbb{D}$ be a VDC, and write $\mathbb{D}(\varphi_1, \ldots, \varphi_n; \psi) =: \mathbb{D}(\underline{\varphi}; \psi)$ for the set of cells with loose source the sequence $\varphi_1, \ldots, \varphi_n$ and with loose target ψ. Vertical pasting can be encoded by functions

$$\int^{\varphi_i : \mathbb{D}} \mathbb{D}(\underline{\varphi}; \psi) \times \left(\mathbb{D}(\underline{\rho}_1; \varphi_1) \times_{\mathsf{Tight}(\mathbb{D})_1} \cdots \times_{\mathsf{Tight}(\mathbb{D})_1} \mathbb{D}(\underline{\rho}_n; \varphi_n) \right) \xrightarrow{\circ_{k_1, \ldots, k_n}} \mathbb{D}(\underline{\rho}; \psi)$$

out of co-ends, where $\left| \underline{\rho}_i \right| = k_i$. Then $\mathbb{D}$ is a pro-representable VDC if and only if all such functions are isomorphisms.

Explication: Characterization of Pro-representable VDCs

In terms of pasting diagrams, a VDC $\mathbb{D}$ is pro-representable precisely when for any $N \geq 0$ and any partition $N = k_1 + \cdots + k_n$, N-multicells decompose as vertical pastings:
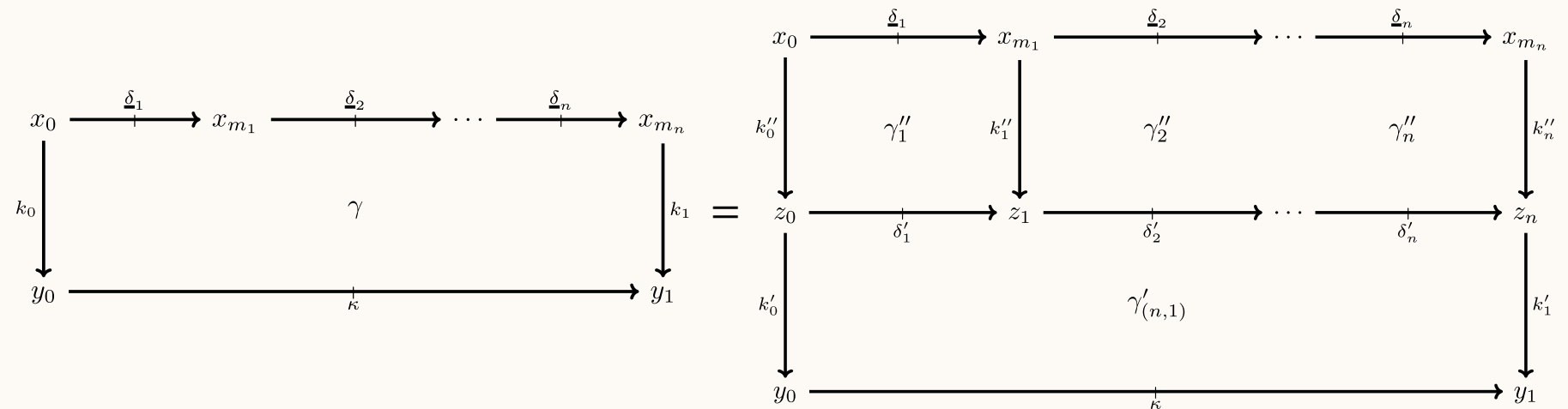
and any two decompositions are equivalent up to associativity of pasting with cells in the center of the decomposition

Corollary: Representable $\Longrightarrow$ Pro-representable

Representable VDCs (i.e. pseudo-double categories) are pro-representable.

Proof Idea: Let $\mathbb{D}$ be a representable VDC. Then any cell admits a canonical decomposition
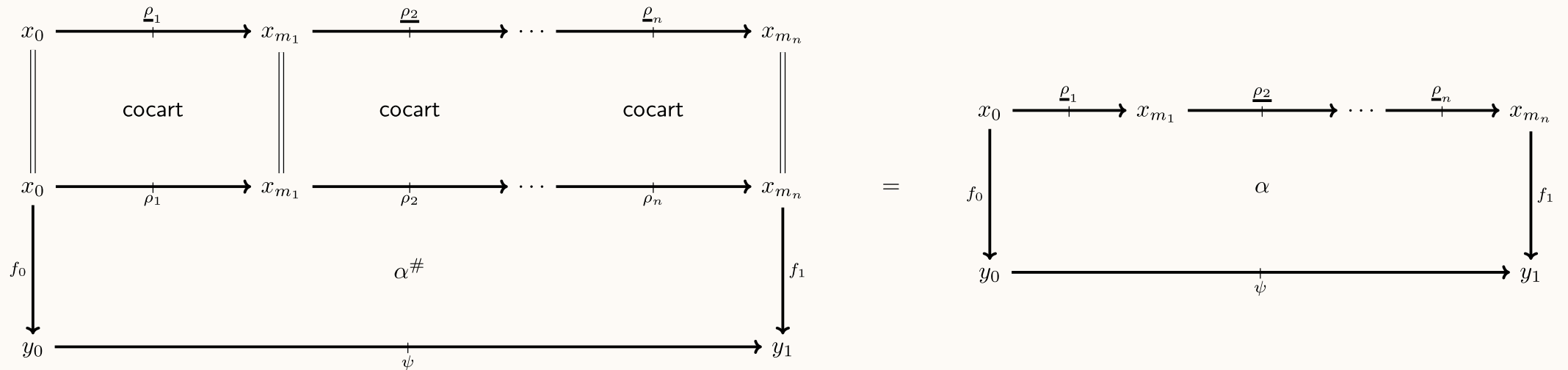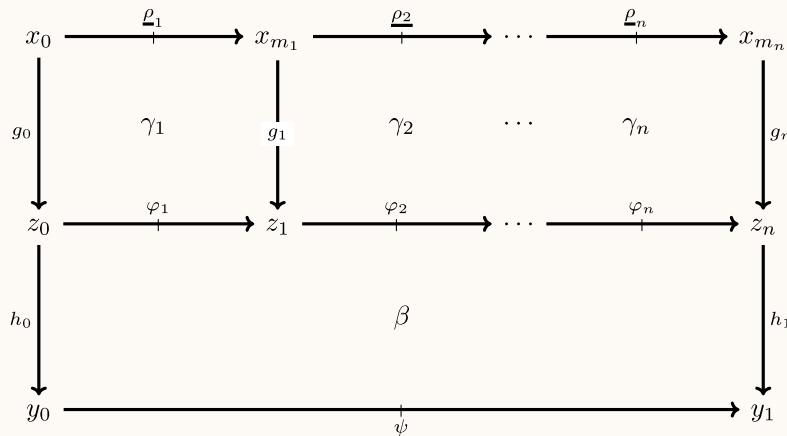
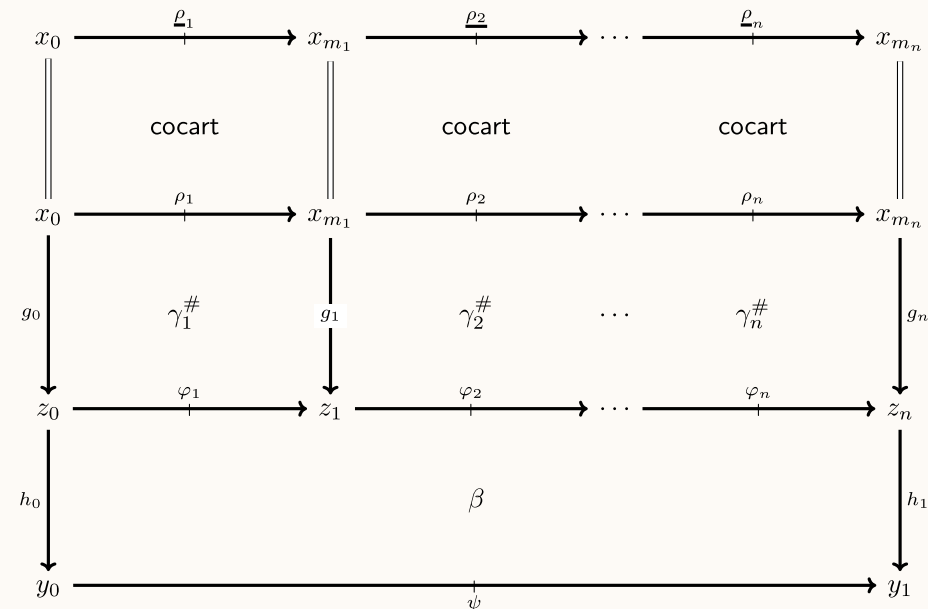**Corollary: Representable $\Longrightarrow$ Pro-representable**

Representable VDCs (i.e. pseudo-double categories) are pro-representable.

Proof Idea: (cont.)

Any other decomposition below left can be canonically factored through the composition cells:
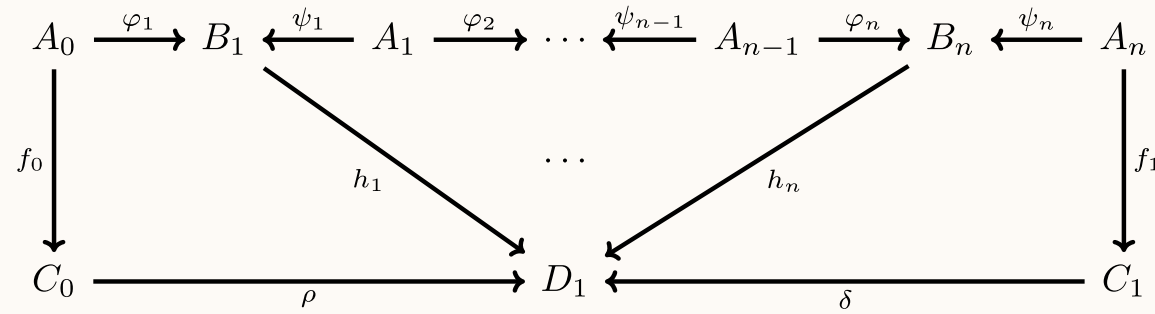
Proposition: Cospan VDCs are Pro-representable

For any category $\mathcal{E}$ the VDC $\mathbb{C}\mathrm{ospan}(\mathcal{E})$ is pro-representable, and it is representable if and only if $\mathcal{E}$ has finite pushouts.

Proof Idea: An arbitrary multicell:

$$A_0 \xrightarrow{\varphi_1} B_1 \xleftarrow{\psi_1} A_1 \xrightarrow{\varphi_2} \cdots \xleftarrow{\psi_{n-1}} A_{n-1} \xrightarrow{\varphi_n} B_n \xleftarrow{\psi_n} A_n$$

with f_0, h_1, ..., h_n, f_1, ρ, δ over $C_0 \to D_1 \leftarrow C_1$.

admits a canonical decomposition:

$$A_0 \xrightarrow{\varphi_1} B_1 \xleftarrow{\psi_1} \cdots \xrightarrow{\varphi_{k_1}} B_{k_1} \xleftarrow{\psi_{k_1}} A_{k_1} \xrightarrow{\varphi_{k_1}} \cdots \xleftarrow{\psi_{n-1}} A_{m_{n-1}} \xrightarrow{\varphi_{m_{n-1}+1}} B_{m_{n-1}+1} \xleftarrow{\psi_{m_{n-1}+1}} \cdots \xrightarrow{\varphi_{m_n}} B_{m_n} \xleftarrow{\psi_{m_n}} A_{m_n}$$

with f_0, h_1, h_{k_1}, α_{k_1}, ..., $\alpha_{m_{n-1}}$, $h_{m_{n-1}+1}$, h_{m_n}, f_1 over $C_0 \xrightarrow{\rho} D_1 = D_1 = \cdots = D_1 = D_1 \xleftarrow{\delta} C_1$, and $C_0 \xrightarrow{\rho} D_1 \xleftarrow{\delta} C_1$.

for any partition of n (the case where $k_1, k_n \geq 1$ is shown for simplicity).
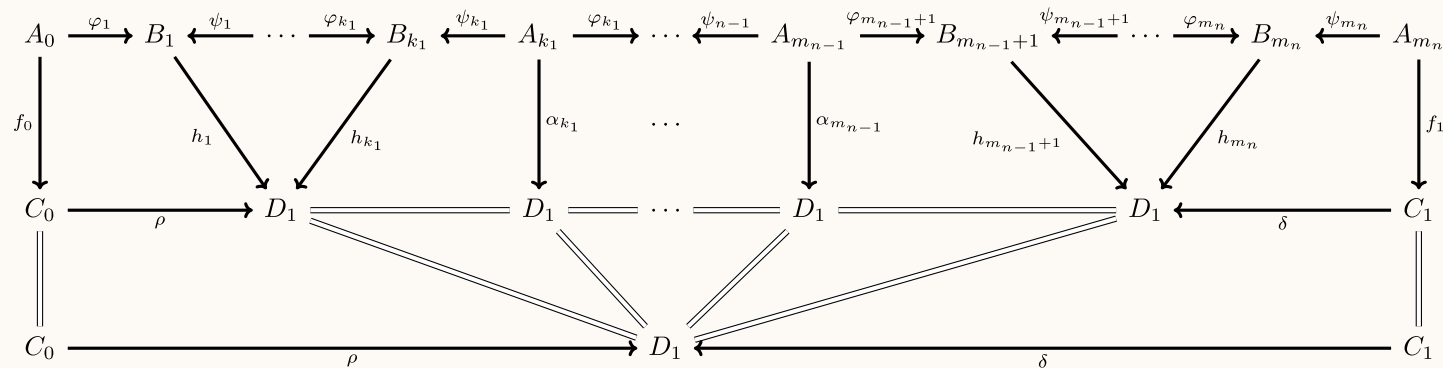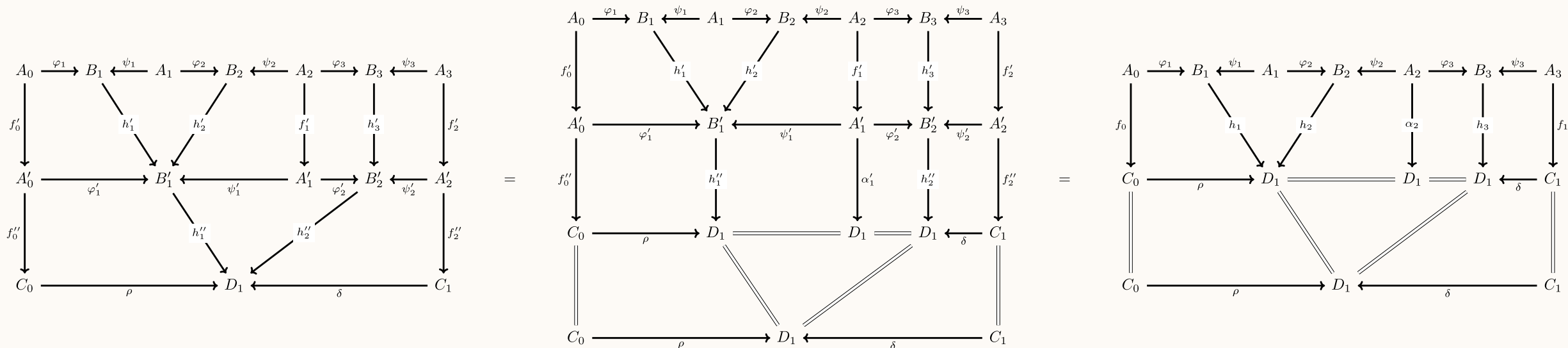
Proposition: Cospan VDCs are Pro-representable

For any category $\mathcal{E}$ the VDC $\mathbb{C}\text{ospan}(\mathcal{E})$ is pro-representable, and it is representable if and only if $\mathcal{E}$ has finite pushouts.

Proof Idea: For uniqueness consider the case where $n = 3, k_1 = 2, k_2 = 1$ as an example. Then an arbitrary decomposition, below left, can be seen to be equivalent to the canonical decomposition via sliding cells:

Non-example: Non-unital Walking Loose Arrow

The VDC $\mathbb{L}$oose consisting of two objects 0 and 1 and a single loose arrow $0 \nrightarrow 1$ is not pro-representable. Consider the diagram and colimit $\mathbb{C}$ in VDC depicted below:
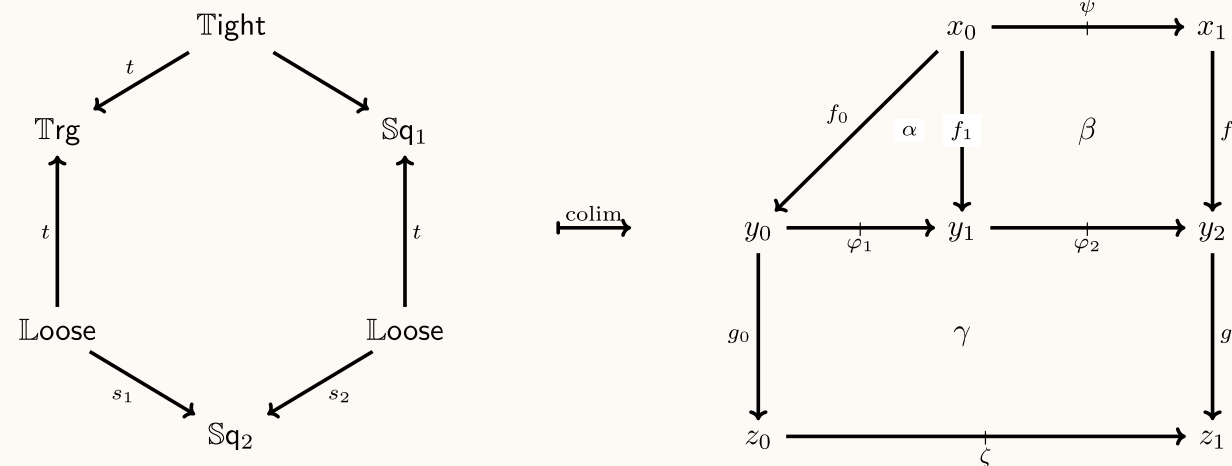
Then $\mathbb{C} \times \mathbb{L}$oose has two non-identity cells, while the VDC obtained by applying $- \times \mathbb{L}$oose to the diagram before taking the colimit only has one.

APPENDIX

Modules in exponential VDCs:

MODULES IN EXPONENTIAL VDC 17

Explication: Modules in Exponential

If $\mathbb{D}$ and $\mathbb{E}$ are VDCs for which $\mathbb{E}^{\mathbb{D}}$ exists, then the VDC $\mathbb{Mod}(\mathbb{E}^{\mathbb{D}})$ consists of the following data:

- An object is a VD functor $F : \mathbb{D} \to \mathbb{E}$
- A tight arrow is a tight transformation between VD functors
- A loose arrow is a VD functor $F : \mathbb{Loose}_{\mathrm{u}} \times \mathbb{D} \to \mathbb{E}$, where $\mathbb{Loose}_{\mathrm{u}}$ is the unital walking loose arrow
- An n-multicell is a diagram with boundary as below, satisfying certain equivariance identities

$$
\begin{array}{ccc}
F_0 & \xrightarrow{\ \Psi_1\ }\cdots\xrightarrow{\ \Psi_n\ } & F_n \\
\alpha_0 \downarrow & \Gamma & \downarrow \alpha_1 \\
G_0 & \xrightarrow[\ \Phi\ ]{} & G_1
\end{array}
$$

Roadmap:

Explication: Modules in Exponential

If $\mathbb{D}$ and $\mathbb{E}$ are VDCs for which $\mathbb{E}^{\mathbb{D}}$ exists, then the VDC $\mathbb{Mod}(\mathbb{E}^{\mathbb{D}})$ consists of the following data:
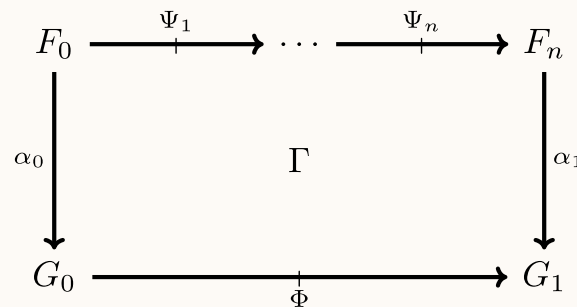
- An object is a VD functor $F \colon \mathbb{D} \to \mathbb{E}$
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$$
\begin{array}{ccccc}
F_0 & \xrightarrow{\Psi_1} & \cdots & \xrightarrow{\Psi_n} & F_n \\
\alpha_0 \downarrow & & \Gamma & & \downarrow \alpha_1 \\
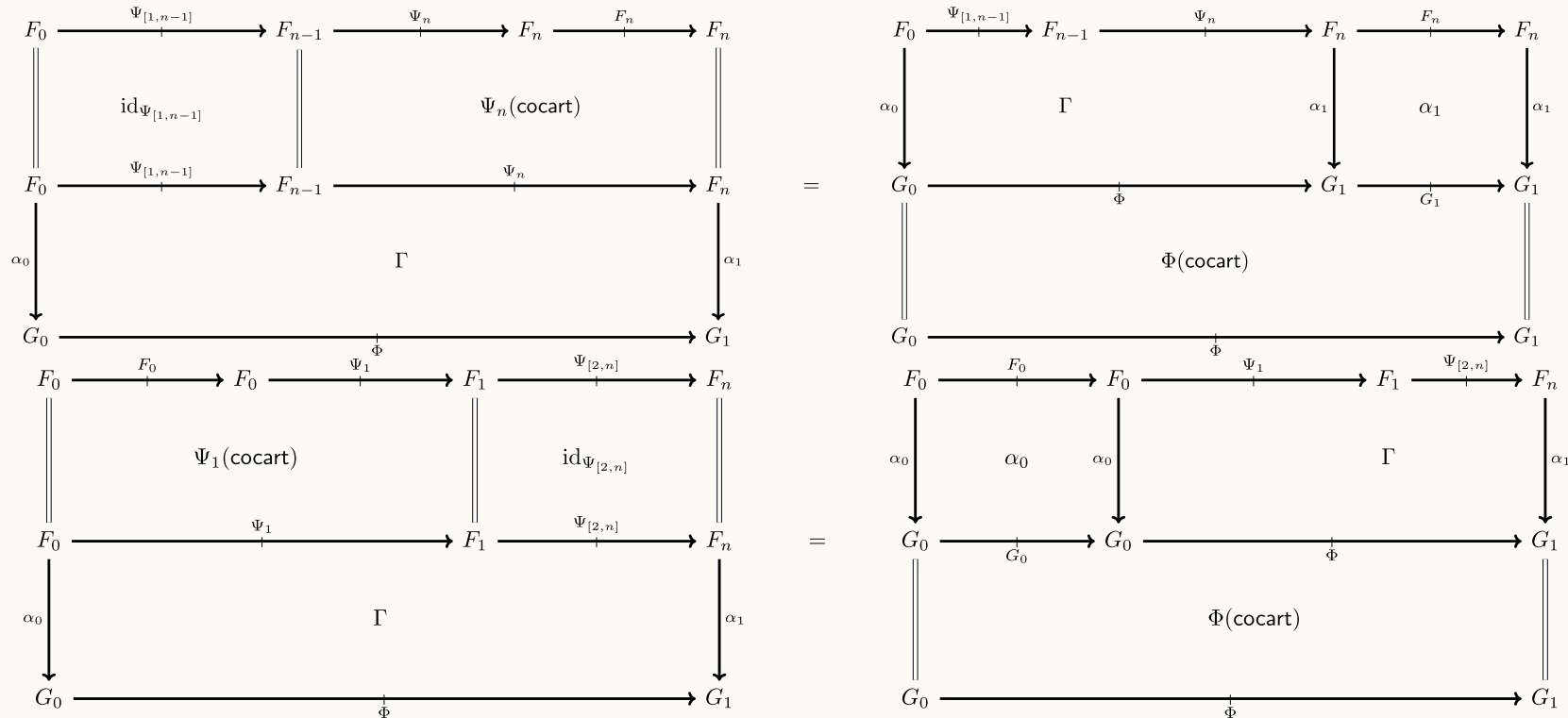G_0 & & \xrightarrow{\Phi} & & G_1
\end{array}
$$

Explication: Modules in Exponential (Inner Equivariance)

Inner equivariance requires that the pasting diagrams below are equal for any $1 < i < n$:
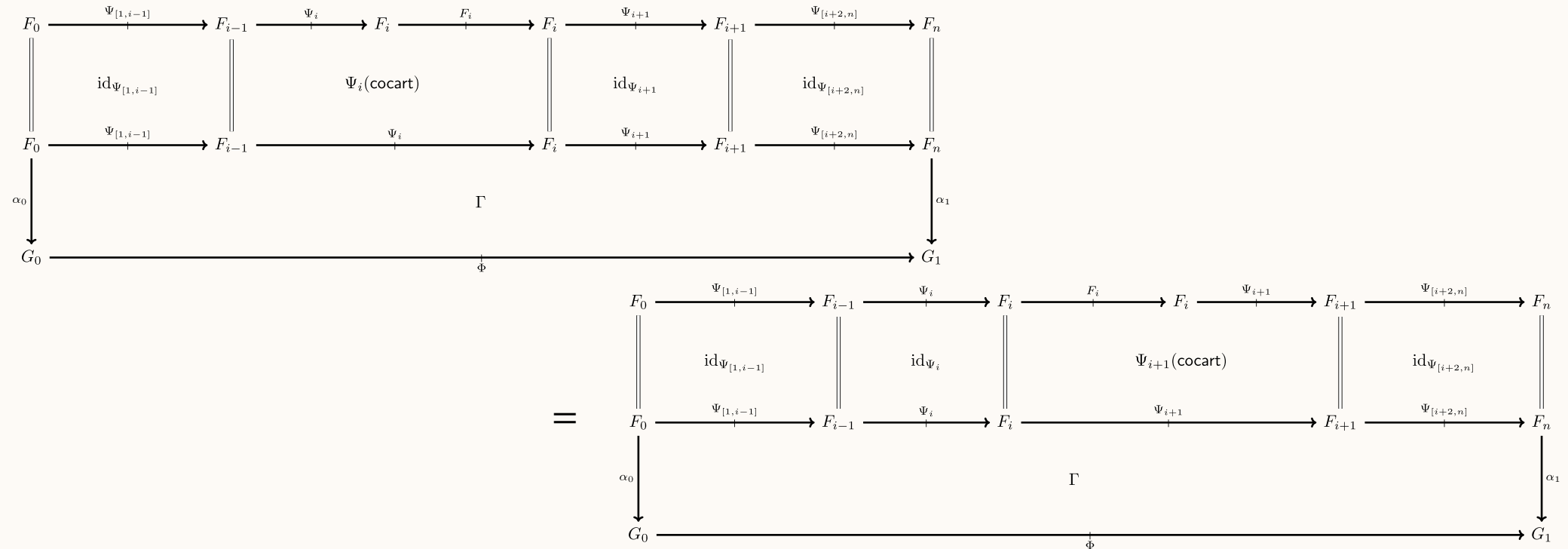
Explication: Modules in Exponential (Outer Equivariance)

Outer equivariance requires we have the two pasting equalities below:

Definition: Loose Bimodules Between VDCs

A loose bimodule between virtual double categories $\mathbb{D}$ and $\mathbb{E}$, is a virtual double category $\mathbb{M}$ equipped a VD functor $M \colon \mathbb{M} \to \mathbb{Loose}_u$ into the unital loose arrow such that the fibers over 0 and 1 are $\mathbb{D}$ and $\mathbb{E}$, respectively:

$$
\begin{array}{ccccc}
\mathbb{D} & \longrightarrow & \mathbb{M} & \longleftarrow & \mathbb{E} \\
\downarrow & {}_{\llcorner}{}^{\lrcorner} & \downarrow{\scriptstyle M} & {}_{\llcorner}{}^{\lrcorner} & \downarrow \\
[0]_u & \underset{0}{\longrightarrow} & \mathbb{Loose}_u & \underset{1}{\longleftarrow} & [0]_u
\end{array}
$$

Remark: VDC of Loose Bimodules

We have a virtual double category $\mathbb{B}\text{imodVDC}$ with
- Objects: virtual double categories
- Tight arrows: virtual double functors
- Loose arrows: loose bimodules
- Cells: maps of cospans

Explicitly, $\mathbb{B}\text{imodVDC}$ is the sub-virtual double category of $\mathbb{C}\text{ospan(VDC)}$ whose objects, tight arrows, and cells are all the same, but whose loose arrows are the codiscrete cofibrations

$$\mathbb{D} \longrightarrow \mathbb{D} +_M \mathbb{E} \longleftarrow \mathbb{E}$$

KEY TAKEAWAYS

- VDCs provide the necessary flexibility to characterize universal properties of double categorical constructions
- Exponentiable VDCs are those admitting essentially unique cell decompositions

FUTURE DIRECTIONS

- Characterize exponentiable maps between VDCs
- Determine what (co)limits and construction pro-representable VDCs are closed under
- Explore properties of enrichment using loose bimodules
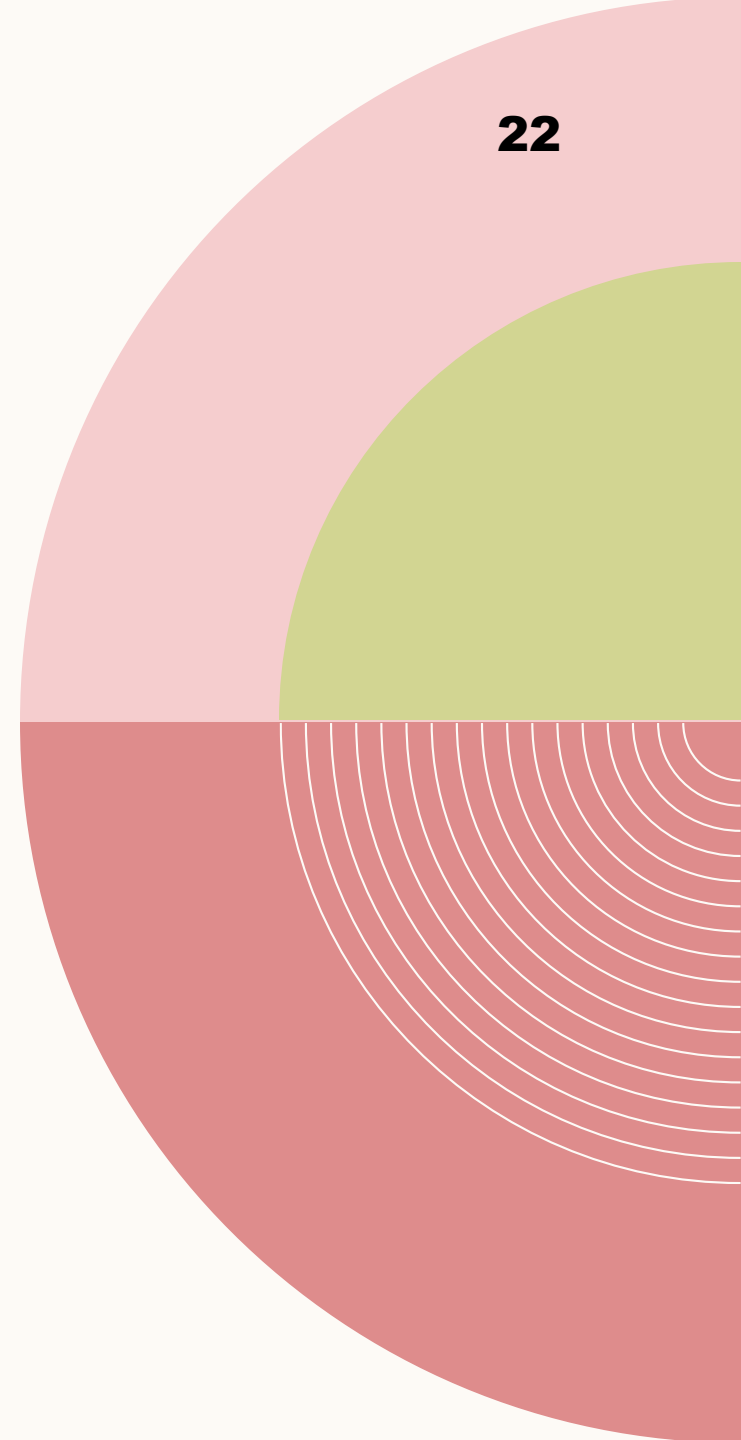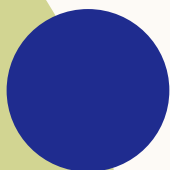
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