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Across the Stars

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The first definition of metric space (under the name "(E) classes") is credited to Fréchet's thesis [Fré06]. We give the definition that is now standard (up to small variations).

Definition 1 (Metric space)**.** A **metric space** is a pair (X, d) comprising a set X and a function $d : X \times X \to [0, \infty)$ called the metric satisfying for all $x, y, z \in X$:

1. separation: $d(x, y) = 0 \Leftrightarrow x = y$,
2. symmetry: $d(x, y) = d(y, x)$, and
3. triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

0.3 Universal Quantitative Algebra

1 Universal Algebra

Concerto Al Andalus

Marcel Khalifé

For a comprehensive introduction to the concepts and themes explored in this chapter, please refer to §0.1. Here, we only give a brief overview.

In this chapter, we cover the content on universal algebra and monads that we will need in the rest of the thesis. This material has appeared many times in the literature,[0] but for completeness (and to be honest my own satisfaction) we take our time with it. In Chapter 3, we will follow the outline of the current chapter to generalize the definitions and results to sets equipped with a notion of distance. Thus, many choices in our notations and presentation are motivated by the needs of Chapter 3.[1]

[0] [Wec12] and [Bau19] are two of my favorite references on universal algebra, and both [Rie17, Chapter 5] and [BW05, Chapter 3] are great references for monads (the latter calls them *triples*).

[1] I hope this will not make this chapter too terse, but the payback of simply copy-pasting proofs to obtain the generalized results is worth it.

Outline: In §1.1, we define algebras, terms, and equations over a signature of finitary operation symbols. In §1.2, we explain how to construct the free algebras for a given signature and class of equations. In §1.3, we give the rules for equational logic to derive equations from other equations, and we show it is sound and complete. In §1.4, we define monads and algebraic presentations for monads. We give examples all throughout, some small ones to build intuition and some bigger ones that will be important later.

1.1 Algebras and Equations

We said in §0.1 that groups and rings are both examples of algebras we want to understand. Groups and rings allow different kinds of combinations of elements, you can do $x - y$ in a ring but not in a group. Essentially all of this chapter will be parametric over a signature Σ which determines what combinations are allowed.

Definition 2 (Signature)**.** A **signature** is a set Σ whose elements, called **operation symbols**, each come with an **arity** $n \in \mathbb{N}$. We write $\mathsf{op} : n \in \Sigma$ for a symbol op with arity n in Σ. With some abuse of notation, we also denote by Σ the functor $\Sigma : \mathbf{Set} \to \mathbf{Set}$ with the following action:[2]

$$\Sigma(A) := \coprod_{\mathsf{op}:n\in\Sigma} A^n \text{ on sets} \quad \text{and} \quad \Sigma(f) := \coprod_{\mathsf{op}:n\in\Sigma} f^n \text{ on functions.}$$

[2] The set $\Sigma(A)$ can be identified with the set containing $\mathsf{op}(a_1, \ldots, a_n)$ for all $\mathsf{op} : n \in \Sigma$ and $a_1, \ldots, a_n \in A$. Then, the function $\Sigma(f)$ sends $\mathsf{op}(a_1, \ldots, a_n)$ to $\mathsf{op}(f(a_1), \ldots, f(a_n))$.

An algebra for a signature Σ is a structure where each operation symbol in Σ is associated to a concrete way to combine elements.

Definition 3 (Σ-algebra)**.** A Σ-**algebra** (or just algebra) is a set A equipped with functions $[\![\mathsf{op}]\!]_A : A^n \to A$ for every $\mathsf{op} : n \in \Sigma$ called the **interpretation** of the symbol. We call A the **carrier** or **underlying** set, and when referring to an algebra, we will switch between using a single symbol $\mathbb{A}$[3] or the pair $(A, [\![-]\!]_A)$, where $[\![-]\!]_A : \Sigma(A) \to A$ is the function sending $\mathsf{op}(a_1, \dots, a_n)$ to $[\![\mathsf{op}]\!]_A(a_1, \dots, a_n)$ (it compactly describes the interpretations of all symbols).

[3] We will try to match the symbol for the algebra and the one for the underlying set only modifying the former with `mathbb`.

A **homomorphism** from $\mathbb{A}$ to $\mathbb{B}$ is a function $h : A \to B$ between the underlying sets of $\mathbb{A}$ and $\mathbb{B}$ that preserves the interpretation of all operation symbols in Σ, namely, for all $\mathsf{op} : n \in \Sigma$ and $a_1, \dots, a_n \in A$,[4]

$$h([\![\mathsf{op}]\!]_A(a_1, \dots, a_n)) = [\![\mathsf{op}]\!]_B(h(a_1), \dots, h(a_n)). \tag{1}$$

[4] Equivalently, h makes the following square commute:

$$\begin{tikzcd} \Sigma(A) \arrow[r, "\Sigma(f)"] \arrow[d, "{[\![-]\!]_A}"'] & \Sigma(B) \arrow[d, "{[\![-]\!]_B}"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{0}$$

This amounts to an equivalent and more concise definition of $\mathbf{Alg}(\Sigma)$: it is the category of algebras for the signature functor $\Sigma : \mathbf{Set} \to \mathbf{Set}$ [Awo10, Definition 10.8].

The identity maps $\mathrm{id}_A : A \to A$ and the composition of two homomorphisms are always homomorphisms, therefore we have a category whose objects are Σ-algebras and morphisms are Σ-algebra homomorphisms. We denote it by $\mathbf{Alg}(\Sigma)$.

This category is concrete over $\mathbf{Set}$ with the forgetful functor $U : \mathbf{Alg}(\Sigma) \to \mathbf{Set}$ which sends an algebra $\mathbb{A}$ to its carrier and a homomorphism to the underlying function between carriers.

Remark 4. In the sequel, we will rarely distinguish between the homomorphism $h : \mathbb{A} \to \mathbb{B}$ and the underlying function $h : A \to B$. Although, we may write Uh for the latter, when disambiguation is necessary.

Examples 5. 1. Let $\Sigma = \{\mathsf{p} : 0\}$ be the signature containing a single operation symbol p with arity 0. A Σ-algebra is a set A equipped with an interpretation of p as a function $[\![\mathsf{p}]\!]_A : A^0 \to A$. Since A^0 is the singleton $\mathbf{1}$, $[\![\mathsf{p}]\!]_A$ is just a choice of element in A,[5] so the objects of $\mathbf{Alg}(\Sigma)$ are pointed sets (sets with a distinguished element). Moreover, instantiating (1) for the symbol p, we find that a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ is a function $h : A \to B$ sending the distinguished point of A to the distinguished point of B. We conclude that $\mathbf{Alg}(\Sigma)$ is the category $\mathbf{Set}_*$ of pointed sets and functions preserving the points.

[5] For this reason, we often call 0-ary symbols **constants**.

2. Let $\Sigma = \{\mathsf{f} : 1\}$ be the signature containing a single unary operation symbol f. A Σ-algebra is a set A equipped with an interpretation of f as a function $[\![\mathsf{f}]\!]_A : A \to A$.

   For example, we have the Σ-algebra whose carrier is the set of integers $\mathbb{Z}$ and where f is interpreted as "adding 1", i.e. $[\![\mathsf{f}]\!]_{\mathbb{Z}}(k) = k + 1$. We also have the integers modulo 2, denoted by $\mathbb{Z}_2$, where $[\![\mathsf{f}]\!]_{\mathbb{Z}_2}(k) = k + 1 (\mathrm{mod}\ 2)$.

   The fact that a function $h : A \to B$ satisfies (1) for the symbol f is equivalent to the following commutative square.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "{[\![\mathsf{f}]\!]_A}"'] & B \arrow[d, "{[\![\mathsf{f}]\!]_B}"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

   We conclude that $\mathbf{Alg}(\Sigma)$ is the category whose objects are endofunctions and whose morphisms are commutative squares as above.[6] There is a homomor-

[6] For more categorical thinkers, we can also identify $\mathbf{Alg}(\Sigma)$ with the functor category $[\mathbf{B}\mathbb{N}, \mathbf{Set}]$ from the delooping of the (additive) monoid $\mathbb{N}$ to the category of sets. Briefly, it is because a functor $\mathbf{B}\mathbb{N} \to \mathbf{Set}$ is completely determined by where it sends $1 \in \mathbb{N}$.

phism is_odd from $\mathbb{Z}$ to $\mathbb{Z}_2$ that sends k to $k(\mathrm{mod}\ 2)$, that is, to 0 when it is even and to 1 when it is odd.

3. Let $\Sigma = \{\cdot : 2\}$ be the signature containing a single binary operation symbol. A Σ-algebra is a set A equipped with an interpretation $[\![\cdot]\!]_A : A \times A \to A$. Such a structure is often called a magma, and it is part of many more well-known algebraic structures like groups, rings, monoids, etc. While every group has an underlying Σ-algebra,[7] not every Σ-algebra underlies a group since $[\![\cdot]\!]_A$ is not required to be associative for example. The next definition will allow us to talk about certain classes of Σ-algebras with some properties like associativity.

   If we want to say that $\cdot$ is commutative, we could write

$$\forall a, b \in A, \quad [\![\cdot]\!]_A(a,b) = [\![\cdot]\!]_A(b,a).$$

To say that $\cdot$ is associative, we write

$$\forall a, b, c \in A, \quad [\![\cdot]\!]_A([\![\cdot]\!]_A(a,b),c) = [\![\cdot]\!]_A(a,[\![\cdot]\!]_A(b,c)),$$

and as you can see, it gets hard to read very quickly. We make our life easier by defining the interpretation of Σ-terms which are syntactic gadgets built by iterating the symbols in Σ.

[7] In fact, every group has an underlying algebra for the signature $\{\cdot : 2, e : 0, -^{-1} : 1\}$.

Definition 6 (Term)**.** Let Σ be a signature and A be a set.[8] We denote with $\mathcal{T}_\Sigma A$ the set of **Σ-terms** built syntactically from A and the operation symbols in Σ, i.e. the set inductively defined by

$$\frac{a \in A}{a \in \mathcal{T}_\Sigma A} \quad \text{and} \quad \frac{\mathsf{op} : n \in \Sigma \qquad t_1, \dots, t_n \in \mathcal{T}_\Sigma A}{\mathsf{op}(t_1, \dots, t_n) \in \mathcal{T}_\Sigma A}. \tag{2}$$

[8] In the sequel, unless otherwise stated, Σ will be an arbitrary signature.

We identify elements $a \in A$ with the corresponding terms $a \in \mathcal{T}_\Sigma A$, and we also identify (as outlined in Footnote 2) elements of $\Sigma(A)$ with terms in $\mathcal{T}_\Sigma A$ containing exactly one occurrence of an operation symbol.[9]

The assignment $A \mapsto \mathcal{T}_\Sigma A$ can be turned into a functor $\mathcal{T}_\Sigma : \mathbf{Set} \to \mathbf{Set}$ by inductively defining, for any function $f : A \to B$, the function $\mathcal{T}_\Sigma f : \mathcal{T}_\Sigma A \to \mathcal{T}_\Sigma B$ as follows:[10]

$$\frac{a \in A}{\mathcal{T}_\Sigma f(a) = f(a)} \quad \text{and} \quad \frac{\mathsf{op} : n \in \Sigma \qquad t_1, \dots, t_n \in \mathcal{T}_\Sigma A}{\mathcal{T}_\Sigma f(\mathsf{op}(t_1, \dots, t_n)) = \mathsf{op}(\mathcal{T}_\Sigma f(t_1), \dots, \mathcal{T}_\Sigma f(t_n))}. \tag{3}$$

[9] Note that any constant $\mathsf{p} : 0 \in \Sigma$ belongs to all $\mathcal{T}_\Sigma A$ by the second rule defining $\mathcal{T}_\Sigma A$.

[10] In words, $\mathcal{T}_\Sigma f$ replaces a with $f(a)$ and does nothing to operation symbols nor the structure of the term. In particular, $\mathcal{T}_\Sigma f$ acts as identity on constants.

Proposition 7. *We defined a functor $\mathcal{T}_\Sigma : \mathbf{Set} \to \mathbf{Set}$, namely, for any $A \xrightarrow{f} B \xrightarrow{g} C$, $\mathcal{T}_\Sigma \mathrm{id}_A = \mathrm{id}_{\mathcal{T}_\Sigma A}$ and $\mathcal{T}_\Sigma(g \circ f) = \mathcal{T}_\Sigma g \circ \mathcal{T}_\Sigma f$.*

Proof. We proceed by induction for both equations.[11] For any $a \in A$, we have $\mathcal{T}_\Sigma \mathrm{id}_A(a) = \mathrm{id}_A(a) = a$ and

$$\mathcal{T}_\Sigma(g \circ f)(a) = (g \circ f)(a) = \mathcal{T}_\Sigma g(\mathcal{T}_\Sigma f(a)).$$

For any $t = \mathsf{op}(t_1, \dots, t_n)$, we have

$$\mathcal{T}_\Sigma \mathrm{id}_A(\mathsf{op}(t_1, \dots, t_n)) \stackrel{(3)}{=} \mathsf{op}(\mathcal{T}_\Sigma \mathrm{id}_A(t_1), \dots, \mathcal{T}_\Sigma \mathrm{id}_A(t_n)) \stackrel{\text{I.H.}}{=} \mathsf{op}(t_1, \dots, t_n),$$

[11] Many proofs in this chapter are by induction until some point where we will have enough results to efficiently use commutative diagrams.

and

$$\begin{aligned}\mathcal{T}_\Sigma(g\circ f)(t) &= \mathcal{T}_\Sigma(g\circ f)(\mathsf{op}(t_1,\dots,t_n))\\ &= \mathsf{op}(\mathcal{T}_\Sigma(g\circ f)(t_1),\dots,\mathcal{T}_\Sigma(g\circ f)(t_n)) && \text{by (3)}\\ &= \mathsf{op}(\mathcal{T}_\Sigma g(\mathcal{T}_\Sigma f(t_1)),\dots,\mathcal{T}_\Sigma g(\mathcal{T}_\Sigma f(t_n))) && \text{I.H.}\\ &= \mathcal{T}_\Sigma g(\mathsf{op}(\mathcal{T}_\Sigma f(t_1),\dots,\mathcal{T}_\Sigma f(t_n))) && \text{by (3)}\\ &= \mathcal{T}_\Sigma g\mathcal{T}_\Sigma f(\mathsf{op}(t_1,\dots,t_n)). && \text{by (3)} \qquad \square\end{aligned}$$

Examples 8. 1. With $\Sigma = \{\mathsf{p}:0\}$, a Σ-term over A is either an element of A or the constant p. For a function $f : A \to B$, the function $\mathcal{T}_\Sigma f$ sends a to $f(a)$ and p to itself. The functor $\mathcal{T}_\Sigma$ is then naturally isomorphic to the maybe functor sending A to $A + \mathbf{1}$.

2. With $\Sigma = \{\mathsf{f}:1\}$, a Σ-term over A is either an element of A or a term $\mathsf{f}(\mathsf{f}(\cdots\mathsf{f}(a)))$ for some a and a finite number of iterations of f.[12] The functor $\mathcal{T}_\Sigma$ is then naturally isomorphic to the functor sending A to $\mathbb{N} \times A$.

3. With $\Sigma = \{\cdot:2\}$, a Σ-term is either an element of A or any expression formed by *multiplying* elements of A together like $a \cdot b$, $a \cdot (b \cdot c)$, $((a \cdot a) \cdot c) \cdot (b \cdot c)$ and so on when $a, b, c \in A$.[13]

[12] For a function $f : A \to B$, the function $\mathcal{T}_\Sigma f$ replaces a with $f(a)$ and does not change the number of iterations of f.

[13] We write $\cdot$ infix as is very common. The parentheses are formal symbols to help delimit which $\cdot$ is taken first. They are necessary because the interpretation of $\cdot$ is not necessarily associative so $a \cdot (b \cdot c)$ and $(a \cdot b) \cdot c$ can be interpreted differently in some Σ-algebras.

As we said above, any element in A is a term in $\mathcal{T}_\Sigma A$, we will denote this embedding with $\eta^\Sigma_A : A \to \mathcal{T}_\Sigma A$, in particular, we will write $\eta^\Sigma_A(a)$ to emphasize that we are dealing with the term a and not the element of A. For instance, the base case of the definition of $\mathcal{T}_\Sigma f$ in (3) becomes

$$\frac{a \in A}{\mathcal{T}_\Sigma f(\eta^\Sigma_A(a)) = \eta^\Sigma_B(f(a))}.$$

This is exactly what it means for the family of maps $\eta^\Sigma_A : A \to \mathcal{T}_\Sigma A$ to be natural in A,[14] in other words that $\eta^\Sigma : \mathrm{id}_{\mathbf{Set}} \Rightarrow \mathcal{T}_\Sigma$ is a natural transformation. We can mention now that it will be part of some additional structure on the functor $\mathcal{T}_\Sigma$ (a monad). The other part of that structure is a natural transformation $\mu^\Sigma : \mathcal{T}_\Sigma\mathcal{T}_\Sigma \Rightarrow \mathcal{T}_\Sigma$, that is more easily described using trees.

[14] As a commutative square:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta^\Sigma_A"'] & B \arrow[d, "\eta^\Sigma_B"] \\ \mathcal{T}_\Sigma A \arrow[r, "\mathcal{T}_\Sigma f"'] & \mathcal{T}_\Sigma B \end{tikzcd} \qquad (4)$$

For an arbitrary signature Σ, we can think of $\mathcal{T}_\Sigma A$ as the set of rooted trees whose leaves are labelled with elements of A and whose nodes with n children are labelled with n-ary operation symbols in Σ. This makes the action of a function $\mathcal{T}_\Sigma f$ fairly straightforward: it applies f to the labels of all the leaves as depicted in Figure 1.1.

This point of view is particularly helpful when describing the **flattening** of terms: there is a natural way to see a Σ-term over Σ-terms over A as a Σ-term over A. This is carried out by the map $\mu^\Sigma_A : \mathcal{T}_\Sigma\mathcal{T}_\Sigma A \to \mathcal{T}_\Sigma A$ which takes a tree T whose leaves are labelled with trees $T_1, \dots, T_n$ to the tree T where instead of the leaf labelled T_i, there is the root of T_i with all its children and their children and so on (we "glue" the tree T_i at the leaf labelled T_i). Figure 1.2 shows an example for $\Sigma = \{\cdot:2\}$. More formally, μ^Σ_A is defined inductively by:

$$\mu^\Sigma_A(\eta^\Sigma_{\mathcal{T}_\Sigma A}(t)) = t \text{ and } \mu^\Sigma_A(\mathsf{op}(t_1,\dots,t_n)) = \mathsf{op}(\mu^\Sigma_A(t_1),\dots,\mu^\Sigma_A(t_n)). \qquad (5)$$

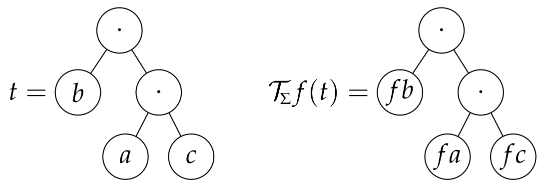


Figure 1.1: Applying $\mathcal{T}_\Sigma f$ to $b \cdot (a \cdot c)$ yields $f(b) \cdot (f(a) \cdot f(c))$.

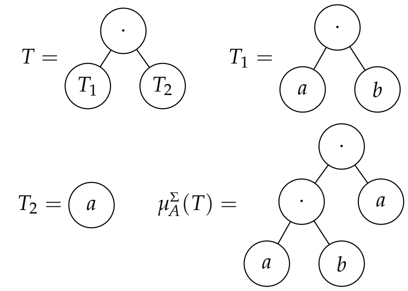


Figure 1.2: Flattening of a term

The use of the word "natural" above is not benign, μ^Σ is actually a natural transformation.

Proposition 9. *The family of maps $\mu^\Sigma_A : \mathcal{T}_\Sigma\mathcal{T}_\Sigma A \to \mathcal{T}_\Sigma A$ is natural in A.*

Proof. We need to prove that for any function $f : A \to B$, $\mathcal{T}_\Sigma f \circ \mu^\Sigma_A = \mu^\Sigma_B \circ \mathcal{T}_\Sigma\mathcal{T}_\Sigma f$.[15] It makes sense intuitively: we should get the same result when we apply f to all the leaves before or after flattening. Formally, we use induction.

For the base case (i.e. terms in the image of $\eta^\Sigma_{\mathcal{T}_\Sigma A}$), we have

$$\begin{aligned}
\mu^\Sigma_B(\mathcal{T}_\Sigma\mathcal{T}_\Sigma f(\eta^\Sigma_{\mathcal{T}_\Sigma A}(t))) &= \mu^\Sigma_B(\eta^\Sigma_{\mathcal{T}_\Sigma B}(\mathcal{T}_\Sigma f(t))) && \text{by (4)}\\
&= \mathcal{T}_\Sigma f(t) && \text{by (5)}\\
&= \mathcal{T}_\Sigma f(\mu^\Sigma_A(\eta^\Sigma_{\mathcal{T}_\Sigma A}(t))). && \text{by (5)}
\end{aligned}$$

For the inductive step, we have

$$\begin{aligned}
\mu^\Sigma_B(\mathcal{T}_\Sigma\mathcal{T}_\Sigma f(\mathsf{op}(t_1,\dots,t_n))) &= \mu^\Sigma_B(\mathsf{op}(\mathcal{T}_\Sigma\mathcal{T}_\Sigma f(t_1),\dots,\mathcal{T}_\Sigma\mathcal{T}_\Sigma f(t_n))) && \text{by (3)}\\
&= \mathsf{op}(\mu^\Sigma_B(\mathcal{T}_\Sigma\mathcal{T}_\Sigma f(t_1)),\dots,\mu^\Sigma_B(\mathcal{T}_\Sigma\mathcal{T}_\Sigma f(t_n))) && \text{by (5)}\\
&= \mathsf{op}(\mathcal{T}_\Sigma f(\mu^\Sigma_A(t_1)),\dots,\mathcal{T}_\Sigma f(\mu^\Sigma_A(t_n))) && \text{I.H.}\\
&= \mathcal{T}_\Sigma f(\mathsf{op}(\mu^\Sigma_A(t_1),\dots,\mu^\Sigma_A(t_n))) && \text{by (3)}\\
&= \mathcal{T}_\Sigma f(\mu^\Sigma_A(\mathsf{op}(t_1,\dots,t_n))) && \text{by (5)} \quad \square
\end{aligned}$$

By definition, we have that $\mu^\Sigma \cdot \eta^\Sigma\mathcal{T}_\Sigma$ is the identity transformation $\mathbb{1}_{\mathcal{T}_\Sigma} : \mathcal{T}_\Sigma \Rightarrow \mathcal{T}_\Sigma$.[16] In words, we say that seeing a term trivially as a term over terms then flattening it yields back the original term. Another similar property is that if we see all the variables in a term trivially as terms and flatten the resulting term over terms, the result is the original term. Formally:

Lemma 10. *For any set A, $\mu^\Sigma_A \circ \mathcal{T}_\Sigma\eta^\Sigma_A = \mathrm{id}_{\mathcal{T}_\Sigma A}$, hence $\mu^\Sigma \cdot \mathcal{T}_\Sigma\eta^\Sigma = \mathbb{1}_{\mathcal{T}_\Sigma}$.*

Proof. We proceed by induction. For the base case, we have

$$\mu^\Sigma_A(\mathcal{T}_\Sigma\eta^\Sigma_A(\eta^\Sigma_A(a))) \stackrel{(4)}{=} \mu^\Sigma_A(\eta^\Sigma_{\mathcal{T}_\Sigma A}(\eta^\Sigma_A(a))) \stackrel{(5)}{=} \eta^\Sigma_A(a).$$

For the inductive step, if $t = \mathsf{op}(t_1,\dots,t_n)$, we have

$$\mu^\Sigma_A(\mathcal{T}_\Sigma\eta^\Sigma_A(t)) = \mu^\Sigma_A(\mathcal{T}_\Sigma\eta^\Sigma_A(\mathsf{op}(t_1,\dots,t_n)))$$

[15] As a commutative square:

$$\begin{tikzcd}
\mathcal{T}_\Sigma\mathcal{T}_\Sigma A \arrow[r, "\mathcal{T}_\Sigma\mathcal{T}_\Sigma f"] \arrow[d, "\mu^\Sigma_A"'] & \mathcal{T}_\Sigma\mathcal{T}_\Sigma B \arrow[d, "\mu^\Sigma_B"] \\
\mathcal{T}_\Sigma A \arrow[r, "\mathcal{T}_\Sigma f"'] & \mathcal{T}_\Sigma B
\end{tikzcd} \tag{6}$$

[16] We write $\cdot$ to denote the vertical composition of natural transformations and juxtaposition (e.g. $F\phi$ or ϕF to denote the action of functors on natural transformations), namely, the component of $\mu^\Sigma \cdot \eta^\Sigma\mathcal{T}_\Sigma$ at A is $\mu^\Sigma_A \circ \eta^\Sigma_{\mathcal{T}_\Sigma A}$ which is $\mathrm{id}_{\mathcal{T}_\Sigma A}$ by (5).

$$\begin{aligned}
&= \mu_A^\Sigma(\mathsf{op}(\mathcal{T}_\Sigma\eta_A^\Sigma(t_1),\dots,\mathcal{T}_\Sigma\eta_A^\Sigma(t_n))) && \text{by (3)}\\
&= \mathsf{op}(\mu_A^\Sigma(\mathcal{T}_\Sigma\eta_A^\Sigma(t_1)),\dots,\mu_A^\Sigma(\mathcal{T}_\Sigma\eta_A^\Sigma(t_n))) && \text{by (5)}\\
&= \mathsf{op}(t_1,\dots,t_n) = t && \text{I.H.} \qquad \square
\end{aligned}$$

Trees also make the depth of a term a visual concept. A term $t \in \mathcal{T}_\Sigma A$ is said to be of **depth** $d \in \mathbb{N}$ if the tree representing it has depth d.[17] We give an inductive definition:

$$\mathrm{depth}(a) = 0 \text{ and } \mathrm{depth}(\mathsf{op}(t_1,\dots,t_n)) = 1 + \max\{\mathrm{depth}(t_1),\dots,\mathrm{depth}(t_n)\}.$$

[17] i.e. the longest path from the root to a leaf has d edges. In Figure 1.2, the depth of T and T_1 is 1, the depth of T_2 is 0 and the depth of $\mu_A^\Sigma T$ is 2.

A term of depth 0 is a term in the image of η_A^Σ. A term of depth 1 is an element of $\Sigma(A)$ seen as a term (recall Footnote 2).

In any Σ-algebra $\mathbb{A}$, the interpretations of operation symbols give us an element of A for each element of $\Sigma(A)$. Therefore, we get a value in A for all terms in $\mathcal{T}_\Sigma A$ of depth 0 or 1 (the value associated to $\eta_A^\Sigma(a)$ is a). Using the inductive definition of $\mathcal{T}_\Sigma A$, we can extend these interpretations to all terms: abusing notation, we define the function $[\![-]\!]_A : \mathcal{T}_\Sigma A \to A$ by[18]

$$\frac{a \in A}{[\![a]\!]_A = a} \quad \text{and} \quad \frac{\mathsf{op} : n \in \Sigma \qquad t_1,\dots,t_n \in \mathcal{T}_\Sigma A}{[\![\mathsf{op}(t_1,\dots,t_n)]\!]_A = [\![\mathsf{op}]\!]_A([\![t_1]\!]_A,\dots,[\![t_n]\!]_A)}. \tag{7}$$

[18] For categorical thinkers, $\mathcal{T}_\Sigma A$ is essentially defined to be the initial algebra for the endofunctor $\Sigma + A$: **Set** $\to$ **Set** sending X to $\Sigma(X) + A$. Any Σ-algebra $(A, [\![-]\!]_A)$ defines another algebra for that functor $[[\![-]\!]_A, \mathrm{id}_A] : \Sigma(A) + A \to A$. Then, the extension of $[\![-]\!]_A$ to terms is the unique algebra morphism drawn below.

$$\begin{tikzcd}
\Sigma(\mathcal{T}_\Sigma A) + A \arrow[r, dashed] \arrow[d] & \Sigma(A) + A \arrow[d, "{[[\![-]\!]_A, \mathrm{id}_A]}"] \\
\mathcal{T}_\Sigma A \arrow[r, dashed] & A
\end{tikzcd}$$

The vertical arrow on the left is basically (2).

This allows to further extend the interpretation $[\![-]\!]_A$ to all terms $\mathcal{T}_\Sigma X$ over some set of variables X, provided we have an assignment of variables $\iota : X \to A$, by precomposing with $\mathcal{T}_\Sigma\iota$. We denote this interpretation with $[\![-]\!]_A^\iota$:

$$[\![-]\!]_A^\iota = \mathcal{T}_\Sigma X \xrightarrow{\mathcal{T}_\Sigma\iota} \mathcal{T}_\Sigma A \xrightarrow{[\![-]\!]_A} A. \tag{8}$$

Example 11. In the signature $\Sigma = \{\mathsf{f} : 1\}$ and over the variables $X = \{x\}$, we have (amongst others) the terms $t = \mathsf{ff}x$ and $s = \mathsf{fff}x$. If we compute the interpretation of t and s in $\mathbb{Z}$ and $\mathbb{Z}_2$,[19] we obtain

$$[\![t]\!]_{\mathbb{Z}}^\iota = \iota(x) + 2 \quad [\![s]\!]_{\mathbb{Z}}^\iota = \iota(x) + 3 \quad [\![t]\!]_{\mathbb{Z}_2}^\iota = \iota(x) \quad [\![s]\!]_{\mathbb{Z}_2}^\iota = \iota(x) + 1 (\mathrm{mod}\ 2),$$

for any assignment $\iota : X \to \mathbb{Z}$ (resp. $\iota : X \to \mathbb{Z}_2$).

[19] Recall their Σ-algebra structure given in Example 5.

By definition, a homomorphism preserves the interpretation of operation symbols. We can prove by induction that it also preserves the interpretation of arbitrary terms. Namely, if $h : \mathbb{A} \to \mathbb{B}$ is a homomorphism, then the following square commutes.[20]

$$\begin{tikzcd}
\mathcal{T}_\Sigma A \arrow[r, "\mathcal{T}_\Sigma h"] \arrow[d, "{[\![-]\!]_A}"'] & \mathcal{T}_\Sigma B \arrow[d, "{[\![-]\!]_B}"] \\
A \arrow[r, "h"'] & B
\end{tikzcd} \tag{9}$$

[20] *Quick proof.* If $t = a \in A$, then both paths send it to $h(a)$. If $t = \mathsf{op}(t_1,\dots,t_n)$, then

$$\begin{aligned}
h([\![t]\!]_A) &= h([\![\mathsf{op}]\!]_A([\![t_1]\!]_A,\dots,[\![t_n]\!]_A))\\
&= [\![\mathsf{op}]\!]_B(h([\![t_1]\!]_A),\dots,h([\![t_n]\!]_A))\\
&= [\![\mathsf{op}]\!]_B([\![\mathcal{T}_\Sigma h(t_1)]\!]_B,\dots,[\![\mathcal{T}_\Sigma h(t_n)]\!]_B)\\
&= [\![\mathsf{op}(\mathcal{T}_\Sigma h(t_1),\dots,\mathcal{T}_\Sigma h(t_n))]\!]_B\\
&= [\![\mathcal{T}_\Sigma h(t)]\!]_B.
\end{aligned}$$

The converse is (almost trivially) true, if (9) commutes, then we can quickly see (0) commutes by embedding $\Sigma(A)$ into $\mathcal{T}_\Sigma A$ and $\Sigma(B)$ into $\mathcal{T}_\Sigma B$. It follows readily that for all homomorphisms $h : \mathbb{A} \to \mathbb{B}$ and all assignments $\iota : X \to A$,

$$h \circ [\![-]\!]_A^\iota = [\![-]\!]_B^{h\circ\iota}. \tag{10}$$

Coming back to associativity, instead of writing $[\![\cdot]\!]_A(a,[\![\cdot]\!]_A(b,c))$, we can now write $[\![a\cdot(b\cdot c)]\!]_A$, and it looks cleaner.[21] Moreover, instead of considering a different term for each choice of $a,b,c\in A$, we can consider the term $x\cdot(y\cdot z)$ over a set of variables $\{x,y,z\}$ and quantify over all the possible assignments $\{x,y,z\}\to A$. We obtain the following definition.

[21] Even cleaner since we are using the infix notation, but I still prefer $[\![a\cdot(b\cdot c)]\!]_A$ over $a\,[\![\cdot]\!]_A(b\,[\![\cdot]\!]_A\,c)$.

Definition 12 (Equation)**.** An **equation** over a signature Σ is a triple comprising a set X of variables called the **context**, and a pair of terms $s,t\in\mathcal{T}_\Sigma X$. We write these as $X\vdash s=t$.

A Σ-algebra $\mathbb{A}$ **satisfies** an equation $X\vdash s=t$ if for any assignment of variables $\iota: X\to A$, $[\![s]\!]^\iota_A=[\![t]\!]^\iota_A$. We use ϕ and ψ to refer to equations, and we write $\mathbb{A}\vDash\phi$ when $\mathbb{A}$ satisfies ϕ. We also write $\mathbb{A}\vDash^\iota\phi$ when the equality $[\![s]\!]^\iota_A=[\![t]\!]^\iota_A$ holds for a particular assignment $\iota: X\to A$ and not necessarily for all assignments.

Remark 13. Our notation for equations is not standard because many authors do not bother writing the context of an equation and suppose it contains exactly the variables used in s and t. That is theoretically sound for universal algebra, but it will not remain so when we generalize to universal quantitative algebras. Thus, we make the context explicit in our equations as is done in [Wec12] or [Bau19] with the notations $\forall X.s=t$ and $X\mid s=t$ respectively.[22] We use the turnstile $\vdash$ to match the convention in the literature on quantitative algebras (e.g. [MPP16] and [FMS21]).

[22] Only finite contexts are used in [Wec12] and [Bau19]. We say a bit more on this in Remark 50

Example 14 (Associativity)**.** With the signature $\Sigma=\{\cdot:2\}$ and the context $X=\{x,y,z\}$, the equation $\phi=X\vdash x\cdot(y\cdot z)=(x\cdot y)\cdot z$[23] asserts that the interpretation of $\cdot$ is associative. Indeed, suppose $\mathbb{A}\vDash\phi$, we need to show that for any $a,b,c\in A$,

$$[\![\cdot]\!]_A(a,[\![\cdot]\!]_A(b,c))=[\![\cdot]\!]_A([\![\cdot]\!]_A(a,b),c). \tag{11}$$

[23] Alternatively, we may write ϕ omitting brackets:

$$x,y,z\vdash x\cdot(y\cdot z)=(x\cdot y)\cdot z.$$

Let $s=x\cdot(y\cdot z)$ and $t=(x\cdot y)\cdot z$. Observe that the `L.H.S.` is the interpretation of s under the assignment $\iota: X\to A$ sending x to a, y to b and z to c, that is, we have $[\![\cdot]\!]_A(a,[\![\cdot]\!]_A(b,c))=[\![s]\!]^\iota_A$. Under the same assignment, the interpretation of t is the `R.H.S.` Since $\mathbb{A}\vDash^\iota X\vdash s=t$, $[\![s]\!]^\iota_A=[\![t]\!]^\iota_A$, and we conclude (11) holds.

Examples 15. Here are some other simple examples of equations.

- $x,y\vdash x\cdot y=y\cdot x$ states that the binary operation $\cdot$ is commutative.
- $x\vdash x\cdot x=x$ states that the binary operation $\cdot$ is idempotent.
- $x\vdash \mathsf{f}x=\mathsf{ff}x$ states that the unary operation f is idempotent.
- $x\vdash \mathsf{p}=x$ states that the constant p is equal to all elements in the algebra (this means the algebra is a singleton).
- $x,y\vdash x=y$ states that all elements in the algebra are equal (this means the algebra is either empty or a singleton).

Using the fact that interpretations are preserved by homomorphisms (10), we can describe how satisfaction is also preserved. Very naively, one would want to say that if $h : \mathbb{A} \to \mathbb{B}$ is a homomorphism and $\mathbb{A} \vDash \phi$, then $\mathbb{B} \vDash \phi$. That is not true.[24] It is morally because there can be many more assignments into $\mathbb{B}$ than there are into $\mathbb{A}$. Nevertheless, the naive statement is true on a per-assignment basis.

[24] For any Σ which does not contain constants, there is an initial Σ-algebra $\mathbb{I}$ whose carrier is the empty set $\varnothing$ (the interpretation of operations is completely determined because there $\Sigma(\varnothing) = \varnothing$ and there is only one function $\varnothing^n \to \varnothing$). The unique function $\varnothing \to B$ is always a homomorphism $\mathbb{I} \to \mathbb{B}$ because (0) trivially commutes since $\Sigma(\varnothing) = \varnothing$. While $\mathbb{I}$ satisfies all equations (vacuously), it is clearly possible that $\mathbb{B}$ does not.

Lemma 16. *Let ϕ be a equation with context X. If $h : \mathbb{A} \to \mathbb{B}$ is a homomorphism and $\mathbb{A} \vDash^{\iota} \phi$ for an assignment $\iota : X \to A$, then $\mathbb{B} \vDash^{h \circ \iota} \phi$.*

Proof. Let ϕ be the equation $X \vdash s = t$, we have

$$\begin{aligned}
\mathbb{A} \vDash^{\iota} \phi &\iff [\![s]\!]^{\iota}_A = [\![t]\!]^{\iota}_A && \text{definition of } \vDash \\
&\implies h([\![s]\!]^{\iota}_A) = h([\![t]\!]^{\iota}_A) \\
&\implies [\![s]\!]^{h\circ\iota}_B = [\![t]\!]^{h\circ\iota}_B && \text{by (10)} \\
&\iff \mathbb{B} \vDash^{h\circ\iota} \phi. && \text{definition of } \vDash
\end{aligned}$$
□

Another neat fact is that flattening interacts well with interpreting in the following sense.

Lemma 17. *For any Σ-algebra $\mathbb{A}$, the following square commutes.*[25]

$$\begin{tikzcd}
\mathcal{T}_\Sigma\mathcal{T}_\Sigma A \arrow[r, "\mathcal{T}_\Sigma[\![-]\!]_A"] \arrow[d, "\mu^\Sigma_A"'] & \mathcal{T}_\Sigma A \arrow[d, "{[\![-]\!]_A}"] \\
\mathcal{T}_\Sigma A \arrow[r, "{[\![-]\!]_A}"'] & A
\end{tikzcd} \tag{12}$$

[25] In words, given a term in $\mathcal{T}_\Sigma\mathcal{T}_\Sigma A$, you obtain the same result if you interpret its flattening in $\mathbb{A}$, or if you interpret the term obtained by first interpreting all the "inner" terms.

This also generalizes to terms in $\mathcal{T}_\Sigma\mathcal{T}_\Sigma X$. Indeed, given an assignment, $\iota : X \to A$, we can either flatten a term and interpret it under ι, or we can interpret all the inner terms under ι, then interpret the result, as shown in (13).

$$\begin{tikzcd}
\mathcal{T}_\Sigma\mathcal{T}_\Sigma X \arrow[r, "\mathcal{T}_\Sigma\mathcal{T}_\Sigma\iota"] \arrow[d, "\mu^\Sigma_X"'] \arrow[rr, bend left, "{\mathcal{T}_\Sigma[\![-]\!]^{\iota}_A}"] \arrow[drr, phantom, "(6)"] & \mathcal{T}_\Sigma\mathcal{T}_\Sigma A \arrow[r, "\mathcal{T}_\Sigma[\![-]\!]_A"] \arrow[d, "\mu^\Sigma_A"'] \arrow[u, phantom, "(8)"] \arrow[dr, phantom, "(12)"] & \mathcal{T}_\Sigma A \arrow[d, "{[\![-]\!]_A}"] \\
\mathcal{T}_\Sigma X \arrow[r, "\mathcal{T}_\Sigma\iota"'] \arrow[rr, bend right, "{[\![-]\!]^{\iota}_A}"'] & \mathcal{T}_\Sigma A \arrow[r, "{[\![-]\!]_A}"'] \arrow[d, phantom, "(8)"] & A \\
& &
\end{tikzcd} \tag{13}$$

Proof. We proceed by induction. For the base case, we have

$$[\![\mu^\Sigma_A(\eta^\Sigma_A(t))]\!]_A \overset{(5)}{=} [\![t]\!]_A \overset{(7)}{=} [\![\eta^\Sigma_A([\![t]\!]_A)]\!]_A \overset{(4)}{=} [\![\mathcal{T}_\Sigma[\![-]\!]_A(\eta^\Sigma_A(t))]\!].$$

For the inductive step, if $t = \mathsf{op}(t_1, \dots, t_n)$, then

$$\begin{aligned}
[\![\mu^\Sigma_A(t)]\!]_A &= [\![\mathsf{op}(\mu^\Sigma_A(t_1), \dots, \mu^\Sigma_A(t_n))]\!]_A && \text{by (5)} \\
&= [\![\mathsf{op}]\!]_A\left([\![\mu^\Sigma_A(t_1)]\!]_A, \dots, [\![\mu^\Sigma_A(t_n)]\!]_A\right) && \text{by (7)} \\
&= [\![\mathsf{op}]\!]_A\left([\![\mathcal{T}_\Sigma[\![-]\!]_A(t_1)]\!]_A, \dots, [\![\mathcal{T}_\Sigma[\![-]\!]_A(t_n)]\!]_A\right) && \text{I.H.} \\
&= [\![\mathsf{op}(\mathcal{T}_\Sigma[\![-]\!]_A(t_1), \dots, \mathcal{T}_\Sigma[\![-]\!]_A(t_n))]\!]_A && \text{by (7)} \\
&= [\![\mathcal{T}_\Sigma[\![-]\!]_A(\mathsf{op}(t_1, \dots, t_n))]\!]_A && \text{by (3)} \\
&= [\![\mathcal{T}_\Sigma[\![-]\!]_A(t)]\!]_A.
\end{aligned}$$
□

Remark 18. To see Lemma 17 in another way, notice that (12) looks a lot like (9), but the map on the left is not the interpretation on an algebra. Except it is! Indeed, we can give a trivial (or syntactic) interpretation of $\mathsf{op} : n \in \Sigma$ on the set $\mathcal{T}_\Sigma A$ by letting $[\![\mathsf{op}]\!]_{\mathcal{T}_\Sigma A}(t_1, \dots, t_n) = \mathsf{op}(t_1, \dots, t_n)$. Then, we can verify by induction[26] that $[\![-]\!]_{\mathcal{T}_\Sigma A} : \mathcal{T}_\Sigma\mathcal{T}_\Sigma A \to \mathcal{T}_\Sigma A$ is equal to μ^Σ_A. We conclude that Lemma 17 says that for any algebra, $[\![-]\!]_A$ is a homomorphism from $(\mathcal{T}_\Sigma A, [\![-]\!]_{\mathcal{T}_\Sigma A})$ to $\mathbb{A}$.

[26] Or we can compare (5) and (7) to see they become the same inductive definition in this instance.

In light of this remark, we mention two very similar results: given a set A, μ^Σ_A is a homomorphism between $\mathcal{T}_\Sigma\mathcal{T}_\Sigma A$ and $\mathcal{T}_\Sigma A$, and given a function $f : A \to B$, $\mathcal{T}_\Sigma f$ is a homomorphism between $\mathcal{T}_\Sigma A$ and $\mathcal{T}_\Sigma B$.

Lemma 19. *For any function $f : A \to B$, the following squares commute.*[27]

$$\begin{tikzcd} \mathcal{T}_\Sigma\mathcal{T}_\Sigma\mathcal{T}_\Sigma A \arrow[r, "\mathcal{T}_\Sigma\mu^\Sigma_A"] \arrow[d, "\mu^\Sigma_{\mathcal{T}_\Sigma A}"'] & \mathcal{T}_\Sigma\mathcal{T}_\Sigma A \arrow[d, "\mu^\Sigma_A"] \\ \mathcal{T}_\Sigma\mathcal{T}_\Sigma A \arrow[r, "\mu^\Sigma_A"'] & \mathcal{T}_\Sigma \end{tikzcd} \quad (14) \qquad \begin{tikzcd} \mathcal{T}_\Sigma\mathcal{T}_\Sigma A \arrow[r, "\mathcal{T}_\Sigma\mathcal{T}_\Sigma B"] \arrow[d, "\mu^\Sigma_A"'] & \mathcal{T}_\Sigma\mathcal{T}_\Sigma B \arrow[d, "\mu^\Sigma_B"] \\ \mathcal{T}_\Sigma \arrow[r, "\mathcal{T}_\Sigma f"'] & \mathcal{T}_\Sigma B \end{tikzcd} \quad (15)$$

[27] *Proof.* We have already shown both these squares commute. Indeed, (14) is an instance of (12) where we identify μ^Σ_A with the interpretation $[\![-]\!]_{\mathcal{T}_\Sigma A}$ as explained in Remark 18, and (15) is the naturality square (6).

Another consequence of (14) is that if you have a term in $\mathcal{T}_\Sigma^n A$ for any $n \in \mathbb{N}$, there are $(n-1)!$ ways to flatten it[28] by successively applying an instance of $\mathcal{T}_\Sigma^i \mu^\Sigma_{\mathcal{T}_\Sigma^j A}$ with different i and j (i.e. flattening at different levels inside the term), but all these ways lead to the same end result in $\mathcal{T}_\Sigma A$. It is like when you have an expression built out of additions with possibly lots of nested bracketing, you can compute the sums in any order you want, and it will give the same result. That property of addition is a consequence of associativity, hence one also says μ^Σ is associative.

[28] There is 1 way to flatten a term in $\mathcal{T}_\Sigma^2 A$ to one in $\mathcal{T}_\Sigma A$, and there are $n-1$ ways to flatten from $\mathcal{T}_\Sigma^n A$ to $\mathcal{T}_\Sigma^{(n-1)} A$. By induction, we find $(n-1)!$ possible combinations of flattening $\mathcal{T}_\Sigma^n A \to \mathcal{T}_\Sigma A$.

While the categories $\mathbf{Alg}(\Sigma)$ for different signatures can be interesting to study on their own, the examples we wanted to generalize like **Grp** or **Ring** are not of that kind, they are special subcategories of some $\mathbf{Alg}(\Sigma)$ that are called varieties.

Definition 20 (Variety)**.** Given a class E of equations, we say $\mathbb{A}$ satisfies E and write $\mathbb{A} \vDash E$ if $\mathbb{A} \vDash \phi$ for all $\phi \in E$.[29] A (Σ, E)**-algebra** is a Σ-algebra that satisfies E. We define $\mathbf{Alg}(\Sigma, E)$, the category of (Σ, E)-algebras, to be the full subcategory of $\mathbf{Alg}(\Sigma)$ containing only those algebras that satisfy E. A **variety** is a category equal to $\mathbf{Alg}(\Sigma, E)$ for some class of equations E.

[29] Similarly for satisfaction under a particular assignment ι:

$$\mathbb{A} \vDash^\iota E \iff \forall \phi \in E, \mathbb{A} \vDash^\iota \phi.$$

There is an evident forgetful functor $U : \mathbf{Alg}(\Sigma, E) \to \mathbf{Set}$ which is the composition of the inclusion functor $\mathbf{Alg}(\Sigma, E) \to \mathbf{Alg}(\Sigma)$ and $U : \mathbf{Alg}(\Sigma) \to \mathbf{Set}$.[30]

[30] We will denote all the forgetful functors with the symbol U unless we need to emphasize the distinction. However, thanks to the `knowldege` package, you can click on (or hover) that symbol to check exactly which forgetful functor it is referring to.

It is never the case in practice that E is a proper class, it is usually a finite or countable set, even recursively enumerable. Still, nothing breaks when E is a class, and we will need this generality in one our main contributions (Theorem 207).

Examples 21. 1. With $\Sigma = \{\mathsf{p} : 0\}$, there are morally only four different equations:[31]

$$\vdash \mathsf{p} = \mathsf{p}, \quad x \vdash x = x, \quad x \vdash \mathsf{p} = x, \text{ and } \quad x, y \vdash x = y,$$

where we write nothing before the turnstile ($\vdash$) instead of the empty set $\emptyset$.

[31] Let us not formally argue about that here, but your intuition on equality and the fact that terms in $\mathcal{T}_\Sigma X$ are either $x \in X$ or p should be enough to convince you.

Any algebra $\mathbb{A}$ satisfies the first two equations because $[\![\mathsf{p}]\!]^\iota_A = [\![\mathsf{p}]\!]^\iota_A$, where $\iota : \emptyset \to A$ is the only possible assignment, and $[\![x]\!]^\iota_A = \iota(x) = [\![x]\!]^\iota_A$ for all $\iota : \{x\} \to A$. If $\mathbb{A}$ satisfies the third, it means that A is empty or a singleton because for any $a, b \in A$, the assignments $\iota_a = x \mapsto a$ and $\iota_b = x \mapsto b$ give us[32]

$$a = \iota_a(x) = [\![x]\!]^{\iota_a}_A = [\![\mathsf{p}]\!]^{\iota_a}_A = [\![\mathsf{p}]\!]^{\iota_b}_A = [\![x]\!]^{\iota_b}_A = \iota_b(x) = b.$$

[32] We find $a = b$ for any $a, b \in A$ and A contains at least one element, the interpretation of the constant p, so A is a singleton.

If $\mathbb{A}$ satisfies the fourth equation, it is also empty or a singleton because for any $a, b \in A$, the assignment ι sending x to a and y to b gives us

$$a = \iota(x) = [\![x]\!]^\iota_A = [\![y]\!]^\iota_A = \iota(y) = b.$$

Therefore,[33] there are only two varieties in that signature, either $\mathbf{Alg}(\Sigma, E)$ is all of $\mathbf{Alg}(\Sigma)$, or it contains only the empty set and the singletons.

[33] Modulo the argument about these being all the possible equations over Σ.

2. With $\Sigma = \{+:2, \mathsf{e}:0\}$, there are many more possible equations, but the following three are quite famous:

$$x,y,z \vdash x+(y+z) = (x+y)+z, \quad x,y \vdash x+y = y+x, \text{ and } \quad x \vdash x+\mathsf{e} = x. \tag{16}$$

We already saw in Example 14 that the first asserts associativity of the interpretation of $+$. With a similar argument, one shows that the second asserts $[\![+]\!]$ is commutative, and the third asserts $[\![\mathsf{e}]\!]$ is a neutral element (on the right) for $[\![+]\!]$.[34] Moreover, note that a homomorphism of Σ-algebras from $\mathbb{A}$ to $\mathbb{B}$ is any function $h : A \to B$ that satisfies

$$\forall a, a' \in A, \quad h([\![+]\!]_A(a,a')) = [\![+]\!]_B(h(a), h(a')) \text{ and } h([\![\mathsf{e}]\!]_A) = [\![\mathsf{e}]\!]_B.$$

Namely, a homomorphism preserves the addition and its neutral element. Thus, letting E be the set containing the equations in (16), we find that $\mathbf{Alg}(\Sigma, E)$ is the category **CMon** of commutative monoids and monoid homomorphisms.

3. We can add a unary operation symbol $-$ to get $\Sigma = \{+:2, \mathsf{e}:0, -:1\}$, and add the equation $x \vdash x + (-x) = \mathsf{e}$ to those in (16),[35] and we can show that $\mathbf{Alg}(\Sigma, E)$ is the category **Ab** of abelian groups and group homomorphisms.

4. We could very similarly develop signatures and equations to get **Grp** and **Ring** as varieties. Although we should note that it is possible for (Σ, E) and (Σ', E') to define the same variety (or isomorphic varieties).

Among different classes of equations over the same signature that define the same variety, there is a largest one.

Definition 22 (Algebraic theory)**.** Given a class E of equations over Σ, the **algebraic theory** generated by E, denoted by $\mathfrak{Th}(E)$, is the class of equations (over Σ) that are satisfied in all (Σ, E)-algebras:[36]

$$\mathfrak{Th}(E) = \{X \vdash s = t \mid \forall \mathbb{A} \in \mathbf{Alg}(\Sigma, E), \mathbb{A} \vDash X \vdash s = t\}.$$

Formulated differently, $\mathfrak{Th}(E)$ contains the equations that are semantically entailed by E, namely $\phi \in \mathfrak{Th}(E)$ if and only if

$$\forall \mathbb{A} \in \mathbf{Alg}(\Sigma), \quad \mathbb{A} \vDash E \implies \mathbb{A} \vDash \phi. \tag{17}$$

Of course, $\mathfrak{Th}(E)$ contains all of E,[37] but also many more equations like $x \vdash x = x$ which is satisfied by any algebra. We will see in §1.3 how to find which equations are entailed by others.

It is easy to see that $\mathbf{Alg}(\Sigma, E) = \mathbf{Alg}(\Sigma, E')$ implies $\mathfrak{Th}(E) = \mathfrak{Th}(E')$, $E \subseteq \mathfrak{Th}(E)$, and $\mathbf{Alg}(\Sigma, \mathfrak{Th}(E)) = \mathbf{Alg}(\Sigma, E)$. It follows that $\mathfrak{Th}(E)$ is the maximal class of equations defining the variety $\mathbf{Alg}(\Sigma, E)$.

Example 23. If E contains the equations in (16), then $\mathfrak{Th}(E)$ will contain all the equations that every commutative monoid satisfies. Here is a non-exhaustive list:

[34] i.e. if $\mathbb{A}$ satisfies $x \vdash x + \mathsf{e} = x$, then for all $a \in A$,

$$[\![a + \mathsf{e}]\!]_A = a.$$

By commutativity, we also get $[\![\mathsf{e} + a]\!]_A = a$.

[35] While the signature has changed between the two examples, the equations of (16) can be understood over both signatures because they concern terms constructed using the symbols common to both signatures.

[36] Note that, even if E is a set, there is no guarantee that $\mathfrak{Th}(E)$ is a set (in fact it never is) because the collection of all equations is a proper class (because the contexts can be any set).

[37] Because a (Σ, E)-algebra satisfies E by definition.

- $x \vdash \mathtt{e} + x = x$ says that $[\![\mathtt{e}]\!]$ is a neutral element on the left for $[\![+]\!]$ which is true because, by equations in (16), $[\![\mathtt{e}]\!]$ is neutral on the right and $[\![+]\!]$ is commutative.
- $z, w \vdash z + w = w + z$ also states commutativity of $[\![+]\!]$ but with different variable names.
- $x, y, z, w \vdash (x + w) + (x + z) + (x + y) = ((x + x) + x) + (y + (z + (\mathtt{e} + w)))$ is just a random equation that can be shown using the properties of commutative monoids.[38]

[38] We will see in §1.3 how to systematically generate all the equations in $\mathfrak{Th}(E)$.

1.2 Free Algebras

Morally, a free (Σ, E)-algebra is an algebra which satisfies the equations in E, those in $\mathfrak{Th}(E)$ (necessarily), and no more than that. We start with an example.

Example 24 (Words)**.** Let $\Sigma_{\mathbf{Mon}} = \{\cdot : 2, \mathtt{e} : 0\}$, $X = \{\mathtt{a}, \mathtt{b}, \cdots, \mathtt{z}\}$ be the set of (lowercase) letters in the Latin alphabet, and X^* be the set of finite words using only these letters.[39] There is a natural $\Sigma_{\mathbf{Mon}}$-algebra structure on X^* where $\cdot$ is interpreted as concatenation, i.e. $[\![\cdot]\!]_{X^*}(u, v) = uv$, and $\mathtt{e}$ as the empty word ε. This algebra satisfies the equations defining a monoid given in (18).[40]

$$E_{\mathbf{Mon}} = \{x, y, z \vdash x \cdot (y \cdot z) = (x \cdot y) \cdot z, \quad x \vdash x \cdot \mathtt{e} = x, \quad x \vdash \mathtt{e} \cdot x = x\}. \tag{18}$$

[39] We are talking about words in a mathematical sense, so X^* contains weird stuff like `aczlp` and the empty word ε.

[40] It does not satisfy $x, y \vdash x \cdot y = y \cdot x$ asserting commutativity because `ab` and `ba` are two different words.

In fact, X^* is the *free* monoid over X. This means that for any other $(\Sigma_{\mathbf{Mon}}, E_{\mathbf{Mon}})$-algebra $\mathbb{A}$ and any function $f : X \to A$, there exists a unique homomorphism $f^* : X^* \to \mathbb{A}$ such that $f^*(x) = f(x)$ for all $x \in X \subseteq X^*$.[41] This can be summarized in the following diagram.

[41] f^* sends $x_1 \cdots x_n$ to $[\![f(x_1) \cdot (f(x_2) \cdots f(x_n))]\!]_A$.

$$\begin{array}{ccc} \text{in } \mathbf{Set} & & \text{in } \mathbf{Alg}(\Sigma_{\mathbf{Mon}}, E_{\mathbf{Mon}}) \\ \begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & X^* \arrow[d, dashed, "f^*"] \\ & A \end{tikzcd} & \xleftarrow{\;U\;} & \begin{tikzcd} X^* \arrow[d, dashed, "f^*"] \\ \mathbb{A} \end{tikzcd} \end{array} \tag{19}$$

A consequence of (19) which makes the idea of freeness more concrete is that X^* satisfies an equation $X \vdash s = t$ if and only if all $(\Sigma_{\mathbf{Mon}}, E_{\mathbf{Mon}})$-algebras satisfy it.[42] In other words, X^* only satisfies the equations it *needs* to satisfy.

The free $(\Sigma_{\mathbf{Mon}}, E_{\mathbf{Mon}})$-algebra over any set is always[43] the set of finite words over that set with $\cdot$ and $\mathtt{e}$ interpreted as concatenation and the empty word respectively.

[42] The forward direction uses Lemma 16 with ι being the inclusion $X \hookrightarrow X^*$ and h being f^*. The converse direction is trivial since we know X^* belongs to $\mathbf{Alg}(\Sigma_{\mathbf{Mon}}, E_{\mathbf{Mon}})$.

[43] We have to say "up to isomorphism" here if we want to be fully rigorous. Let us avoid this bulkiness here and later in most places where it can be inferred.

At a first look, X^* does not seem correlated to the operation symbols in $\Sigma_{\mathbf{Mon}}$ and the equations in $E_{\mathbf{Mon}}$, so it may seem hopeless to generalize this construction of free algebra for an arbitrary Σ and E. It is possible however to describe the algebra X^* starting from $\Sigma_{\mathbf{Mon}}$ and $E_{\mathbf{Mon}}$.

Recall that $\mathcal{T}_{\Sigma_{\mathbf{Mon}}} X$ is the set of all terms constructed with the symbols in $\Sigma_{\mathbf{Mon}}$ and the elements of X.[44] Since we want the interpretation of $\mathtt{e}$ to be a neutral element for the interpretation of $\cdot$, we could identify many terms together like $\mathtt{e}$ and $\mathtt{e} \cdot \mathtt{e}$, in fact whenever a term has an occurrence of $\mathtt{e}$, we can remove it with no effect on its interpretation in a $(\Sigma_{\mathbf{Mon}}, E_{\mathbf{Mon}})$-algebra. Similarly, since we want $\cdot$ to

[44] For instance, it contains $\mathtt{e}$, $\mathtt{e} \cdot \mathtt{e}$, $\mathtt{a} \cdot \mathtt{a}$, $\mathtt{a} \cdot (\mathtt{r} \cdot (\mathtt{e} \cdot \mathtt{u}))$, and so on.

be interpreted as an associative operation, we could identify $\mathtt{r} \cdot (\mathtt{s} \cdot \mathtt{m})$ and $(\mathtt{r} \cdot \mathtt{s}) \cdot \mathtt{m}$, and more generally, we can rearrange the parentheses in a term with no effect on its interpretation in a $(\Sigma_{\mathbf{Mon}}, E_{\mathbf{Mon}})$-algebra.

Squinting a bit, you can convince yourself that a $\Sigma_{\mathbf{Mon}}$-term over X considered modulo occurrences of e and parentheses is the same thing as a finite word in X^*.[45] Under this correspondence, we find that the interpretation of $\cdot$ on X^* (which was concatenation) can be realized syntactically by the symbol $\cdot$. For example, the concatenation of the words corresponding to $\mathtt{r} \cdot \mathtt{r}$ and $\mathtt{u} \cdot \mathtt{p}$ is the word corresponding to $(\mathtt{r} \cdot \mathtt{r}) \cdot (\mathtt{u} \cdot \mathtt{p})$. The interpretation of e in X^* is the empty word which corresponds to e. We conclude that the algebra X^* could have been described entirely using the syntax of $\Sigma_{\mathbf{Mon}}$ and equations in $E_{\mathbf{Mon}}$.

[45] For instance, both $\mathtt{r} \cdot (\mathtt{s} \cdot \mathtt{m})$ and $(\mathtt{r} \cdot \mathtt{s}) \cdot \mathtt{m}$ become the word $\mathtt{rsm}$ and e, $\mathsf{e} \cdot \mathsf{e}$ and $\mathsf{e} \cdot (\mathsf{e} \cdot \mathsf{e})$ all become the empty word.

We promptly generalize this to other signatures and sets of equations. Fix a signature Σ and a class E of equations over Σ. For any set X, we can define a binary relation $\equiv_E$ on Σ-terms[46] that contains the pair (s,t) whenever the interpretation of s and t coincide in any (Σ, E)-algebra. Formally, we have for any $s, t \in \mathcal{T}_\Sigma X$,

[46] We omit the set X from the notation as it would be more bulky than illuminating.

$$s \equiv_E t \iff X \vdash s = t \in \mathfrak{Th}(E). \tag{20}$$

We now show $\equiv_E$ is a congruence relation on $\mathcal{T}_\Sigma X$.[47]

[47] A **congruence** on a Σ-algebra $\mathbb{A}$ is an equivalence relation $\sim \subseteq A \times A$ on the carrier satisfying for all $\mathsf{op} : n \in \Sigma$ and $a_1, \dots, a_n, b_1, \dots, b_n \in A$:

$$(\forall i, a_i \sim b_i) \implies [\![\mathsf{op}]\!]_A(a_1, \dots, a_n) \sim [\![\mathsf{op}]\!]_A(b_1, \dots, b_n).$$

Lemma 25. *For any set X, the relation $\equiv_E$ is reflexive, symmetric, transitive, and satisfies for any $\mathsf{op} : n \in \Sigma$ and $s_1, \dots, s_n, t_1, \dots, t_n \in \mathcal{T}_\Sigma X$,*

$$(\forall 1 \leq i \leq n, s_i \equiv_E t_i) \implies \mathsf{op}(s_1, \dots, s_n) \equiv_E \mathsf{op}(t_1, \dots, t_n). \tag{21}$$

Proof. Briefly, reflexivity, symmetry, and transitivity all follow from the fact that equality satisfies these properties, and (21) follows from the fact that operation symbols are interpreted as *deterministic* functions (a unique output for each input), so they preserve equality. We detail this below.

(*Reflexivity*) For any $t \in \mathcal{T}_\Sigma X$, and any Σ-algebra $\mathbb{A}$, $\mathbb{A} \vDash X \vdash t = t$ because it holds that $[\![t]\!]_A^\iota = [\![t]\!]_A^\iota$ for all $\iota : X \to A$.

(*Symmetry*) For any $s, t \in \mathcal{T}_\Sigma X$ and $\mathbb{A} \in \mathbf{Alg}(\Sigma)$, if $\mathbb{A} \vDash X \vdash s = t$, then $\mathbb{A} \vDash X \vdash t = s$. Indeed, if $[\![s]\!]_A^\iota = [\![t]\!]_A^\iota$ holds for all ι, then $[\![t]\!]_A^\iota = [\![s]\!]_A^\iota$ holds too. Symmetry follows because if all (Σ, E)-algebras satisfy $X \vdash s = t$, then they also satisfy $X \vdash t = s$.

(*Transitivity*) For any $s, t, u \in \mathcal{T}_\Sigma X$, if all (Σ, E)-algebras satisfy $X \vdash s = t$ and $X \vdash t = u$, then they also satisfy $X \vdash s = u$.[48] Transitivity follows.

[48] Just like for symmetry, it is because for any $\mathbb{A} \in \mathbf{Alg}(\Sigma)$ and $\iota : X \to A$, $[\![s]\!]_A^\iota = [\![t]\!]_A^\iota$ with $[\![t]\!]_A^\iota = [\![u]\!]_A^\iota$ imply $[\![s]\!]_A^\iota = [\![u]\!]_A^\iota$.

(21) For any $\mathsf{op} : n \in \Sigma$, $s_1, \dots, s_n, t_1, \dots, t_n \in \mathcal{T}_\Sigma X$, and $\mathbb{A} \in \mathbf{Alg}(\Sigma)$, if $\mathbb{A}$ satisfies $X \vdash s_i = t_i$ for all i, then for any assignment $\iota : X \to A$, we have $[\![s_i]\!]_A^\iota = [\![t_i]\!]_A^\iota$ for all i. Hence,

$$\begin{aligned}
[\![\mathsf{op}(s_1, \dots, s_n)]\!]_A^\iota &= [\![\mathsf{op}]\!]_A([\![s_1]\!]_A^\iota, \dots, [\![s_n]\!]_A^\iota) && \text{by (7)} \\
&= [\![\mathsf{op}]\!]_A([\![t_1]\!]_A^\iota, \dots, [\![t_n]\!]_A^\iota) && \forall i, [\![s_i]\!]_A^\iota = [\![t_i]\!]_A^\iota \\
&= [\![\mathsf{op}(s_1, \dots, s_n)]\!]_A^\iota && \text{by (7),}
\end{aligned}$$

which means $\mathbb{A} \vDash X \vdash \mathsf{op}(s_1, \dots, s_n) = \mathsf{op}(t_1, \dots, t_n)$. This was true for all Σ-algebras, so we can use the same arguments as above to conclude (21). $\square$

This lemma shows $\equiv_E$ is in particular an equivalence relation, so we can define terms modulo E. Given Σ, E and X, let $\mathcal{T}_{\Sigma,E}X = \mathcal{T}_\Sigma X/\equiv_E$ denote the set of Σ-**terms modulo** E. We will write $[-]_E : \mathcal{T}_\Sigma X \to \mathcal{T}_{\Sigma,E}X$ for the canonical quotient map, so $[t]_E$ is the equivalence class of t in $\mathcal{T}_{\Sigma,E}X$.

This yields a functor $\mathcal{T}_{\Sigma,E} : \mathbf{Set} \to \mathbf{Set}$ which sends a function $f : X \to Y$ to the unique function $\mathcal{T}_{\Sigma,E}f$ making (22) commute, i.e. satisfying $\mathcal{T}_{\Sigma,E}f([t]_E) = [\mathcal{T}_\Sigma f(t)]_E$. By definition, $[-]_E$ is also a natural transformation from $\mathcal{T}_\Sigma$ to $\mathcal{T}_{\Sigma,E}$.

$$\begin{tikzcd} \mathcal{T}_\Sigma X \arrow[r, "{[-]_E}"] \arrow[d, "\mathcal{T}_\Sigma f"'] & \mathcal{T}_{\Sigma,E} X \arrow[d, dashed, "\mathcal{T}_{\Sigma,E} f"] \\ \mathcal{T}_\Sigma Y \arrow[r, "{[-]_E}"'] & \mathcal{T}_{\Sigma,E} Y \end{tikzcd} \tag{22}$$

Definition 26 (Term algebra, semantically)**.** The **term algebra** for (Σ, E) on X is the Σ-algebra whose carrier is $\mathcal{T}_{\Sigma,E}X$ and whose interpretation of $\mathsf{op} : n \in \Sigma$ is[49]

$$[\![\mathsf{op}]\!]_{\mathbb{T}X}([t_1]_E, \dots, [t_n]_E) = [\mathsf{op}(t_1, \dots, t_n)]_E. \tag{23}$$

We denote this algebra by $\mathbb{T}_{\Sigma,E}X$ or simply $\mathbb{T}X$.

[49] This is well-defined (i.e. invariant under change of representative) by (21).

A main motivation behind this definition is that it makes $[-]_E : \mathcal{T}_\Sigma X \to \mathcal{T}_{\Sigma,E}X$ a homomorphism,[50] namely, (24) commutes.

$$\begin{tikzcd} \mathcal{T}_\Sigma\mathcal{T}_\Sigma X \arrow[r, "\mathcal{T}_\Sigma{[-]_E}"] \arrow[d, "\mu^\Sigma_X"'] & \mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E} X \arrow[d, "{[\![-]\!]_{\mathbb{T}X}}"] \\ \mathcal{T}_\Sigma X \arrow[r, "{[-]_E}"'] & \mathcal{T}_{\Sigma,E} X \end{tikzcd} \tag{24}$$

[50] Indeed, (23) looks exactly like (1) with $h = [-]_E$, $\mathbb{A} = \mathcal{T}_\Sigma X$ and $\mathbb{B} = \mathbb{T}X$.

Remark 27*.* We can understand Definition 26 a bit more abstractly. If $\mathbb{A}$ is a Σ-algebra and $\sim \subseteq A \times A$ is a congruence, then the quotient $A/\sim$ inherits a Σ-algebra structure defined as in (23) ($[a]$ denotes the equivalence class of a in $A/\sim$):

$$[\![\mathsf{op}]\!]_{A/\sim}([a_1], \dots, [a_n]) = [[\![\mathsf{op}]\!]_A(a_1, \dots, a_n)].$$

Then, $\mathbb{T}_{\Sigma,E}X$ is the quotient of the algebra $\mathcal{T}_\Sigma X$ defined in Remark 18 by the congruence $\equiv_E$. From this point of view, one can give an equivalent definition of $\equiv_E$ as the smallest congruence on $\mathcal{T}_\Sigma X$ such that the quotient satisfies E.[51]

[51] Namely, if $\mathcal{T}_\Sigma X/\sim$ satisfies E, then $\equiv_E \subseteq \sim$.

It is very easy to *compute* in the term algebra because all operations are realized syntactically, that is, only by manipulating symbols. Let us first look at the interpretation of Σ-terms in $\mathbb{T}X$, i.e. the function $[\![-]\!]_{\mathbb{T}X} : \mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}X \to \mathcal{T}_{\Sigma,E}X$. It was defined inductively to yield[52]

$$[\![\eta^\Sigma_{\mathcal{T}_{\Sigma,E}X}([t]_E)]\!]_{\mathbb{T}X} = [t]_E \text{ and } [\![\mathsf{op}(t_1, \dots, t_n)]\!]_{\mathbb{T}X} = [\![\mathsf{op}]\!]_{\mathbb{T}X}([\![t_1]\!]_{\mathbb{T}X}, \dots, [\![t_n]\!]_{\mathbb{T}X}). \tag{25}$$

[52] where $t \in \mathcal{T}_\Sigma X$, $\mathsf{op} : n \in \Sigma$, and $t_1, \dots, t_n \in \mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}X$.

Remark 28*.* In particular, when E is empty, the set $\mathcal{T}_{\Sigma,\varnothing}X$ is $\mathcal{T}_\Sigma X$ quotiented by $\equiv_\varnothing$, and one can show that $\equiv_\varnothing$ is equal to equality ($=$), i.e. $\mathfrak{Th}(\varnothing)$ only contains equation of the form $X \vdash t = t$.[53] Therefore, $\mathcal{T}_{\Sigma,\varnothing}X = \mathcal{T}_\Sigma X$. Moreover, since $[-]_\varnothing$ is the identity map, we find that (23) becomes the definition of the interpretations given in Remark 18, so $\mathbb{T}_{\Sigma,\varnothing}X$ is the algebra on $\mathcal{T}_\Sigma X$ we had defined. Also, we find the interpretation of terms $[\![-]\!]_{\mathbb{T}_{\Sigma,\varnothing}X}$ is the flattening.[54]

[53] For any other equation $X \vdash s = t$ where s and t are not the same term, the Σ-algebra $\mathcal{T}_\Sigma X$ does not satisfy because the assignment $\eta^\Sigma_X : X \to \mathcal{T}_\Sigma X$ yields

$$[\![s]\!]^{\eta^\Sigma_X}_{\mathcal{T}_\Sigma X} = s \neq t = [\![t]\!]^{\eta^\Sigma_X}_{\mathcal{T}_\Sigma X}.$$

[54] By Remark 18 or by comparing (25) when $E = \varnothing$ and the definition of μ^Σ_X (5).

Example 29. Let $\Sigma = \Sigma_{\mathbf{Mon}}$ and $E = E_{\mathbf{Mon}}$ be the signature and equations defining monoids as explained in Example 24. We saw informally that $\mathcal{T}_{\Sigma,E}X$ is in correspondence with the set X^* of finite words over X, and we already have a monoid structure on X^*.[55] Thus, we may wonder whether the term algebra $\mathbb{T}X$ describes the same monoid. Let us compute the interpretation of $u \cdot (v \cdot w)$ where $u = \mathtt{uu}$, $v = \mathtt{vv}$ and $w = \mathtt{www}$ are words in $X^* \cong \mathcal{T}_{\Sigma,E}X$. First we use the inductive definition:

$$[\![u \cdot (v \cdot w)]\!]_{\mathbb{T}X} = [\![\cdot]\!]_{\mathbb{T}X}([\![u]\!]_{\mathbb{T}X}, [\![v \cdot w]\!]_{\mathbb{T}X}) = [\![\cdot]\!]_{\mathbb{T}X}([\![u]\!]_{\mathbb{T}X}, [\![\cdot]\!]_{\mathbb{T}X}([\![v]\!]_{\mathbb{T}X}, [\![w]\!]_{\mathbb{T}X})).$$

Next, we choose a representative for $u, v, w \in \mathcal{T}_{\Sigma,E}X$ and apply the base step of the inductive definition:

$$[\![u \cdot (v \cdot w)]\!]_{\mathbb{T}X} = [\![\cdot]\!]_{\mathbb{T}X}([\mathtt{u} \cdot \mathtt{u}]_E, [\![\cdot]\!]_{\mathbb{T}X}([\mathtt{v} \cdot \mathtt{v}]_E, [\mathtt{w} \cdot (\mathtt{w} \cdot \mathtt{w})]_E)).$$

Finally, we can apply (23) a couple of times to find

$$[\![u \cdot (v \cdot w)]\!]_{\mathbb{T}X} = [\![\cdot]\!]_{\mathbb{T}X}([\mathtt{u} \cdot \mathtt{u}]_E, [(\mathtt{v} \cdot \mathtt{v}) \cdot (\mathtt{w} \cdot (\mathtt{w} \cdot \mathtt{w}))]_E) = [(\mathtt{u} \cdot \mathtt{u}) \cdot ((\mathtt{v} \cdot \mathtt{v}) \cdot (\mathtt{w} \cdot (\mathtt{w} \cdot \mathtt{w})))]_E,$$

which means that the word corresponding to $[\![u \cdot (v \cdot w)]\!]_{\mathbb{T}X}$ is $\mathtt{uuvvwww}$, i.e. the concatenation of u, v and w.

In general (for other signatures), what happens when applying $[\![-]\!]_{\mathbb{T}X}$ to some big term in $\mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}X$ can be decomposed in three steps.

1. Apply the inductive definition until you have an expression built out of many $[\![\mathsf{op}]\!]_{\mathbb{T}X}$ and $[\![c]\!]_{\mathbb{T}X}$ where $\mathsf{op} \in \Sigma$ and c is an equivalence class of Σ-terms.
2. Choose a representative for each such classes (i.e. $c = [t]_E$).
3. Use (23) repeatedly until the result is just an equivalence class in $\mathcal{T}_{\Sigma,E}X$.

Working with terms in $\mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}X$ as trees whose leaves are labelled in $\mathcal{T}_{\Sigma,E}X$, $[\![-]\!]_{\mathbb{T}X}$ replaces each leaf by the tree corresponding to a representative for the equivalence class of the leaf's label, and then returns the equivalence class of the resulting tree. In this sense, $[\![-]\!]_{\mathbb{T}X}$ looks a lot like the flattening μ^Σ_X except it deals with equivalence classes of terms. This motivates the definition of $\mu^{\Sigma,E}_X$ to be the unique function making (26) commute.[56]

$$\begin{tikzcd} \mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}X \arrow[rr, "{[\![-]\!]_{\mathbb{T}X}}"] \arrow[dr, "{[-]_E}"'] & & \mathcal{T}_{\Sigma,E}X \\ & \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}X \arrow[ur, dashed, "\mu^{\Sigma,E}_X"'] & \end{tikzcd} \tag{26}$$

The first thing we showed when defining μ^Σ_X was that it yielded a natural transformation $\mu^\Sigma : \mathcal{T}_\Sigma\mathcal{T}_\Sigma \Rightarrow \mathcal{T}_\Sigma$. We can also do this for $\mu^{\Sigma,E}$.

Proposition 30. *The family of maps $\mu^{\Sigma,E}_X : \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}X \to \mathcal{T}_{\Sigma,E}X$ is natural in X.*

Proof. We need to prove that for any function $f : X \to Y$, the square below commutes.

$$\begin{tikzcd} \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}X \arrow[r, "\mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}f"] \arrow[d, "\mu^{\Sigma,E}_X"'] & \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}Y \arrow[d, "\mu^{\Sigma,E}_Y"] \\ \mathcal{T}_{\Sigma,E}X \arrow[r, "\mathcal{T}_{\Sigma,E}f"'] & \mathcal{T}_{\Sigma,E}Y \end{tikzcd} \tag{27}$$

[55] The interpretation of $\cdot$ and e is concatenation and the empty word.

[56] This guarantees $\mu^{\Sigma,E}_X$ satisfies the following equations that looks like the inductive definition of μ^Σ_X in (5): for any $t \in \mathcal{T}_\Sigma X$, $\mu^{\Sigma,E}_X([[t]_E]_E) = [t]_E$ and for any $\mathsf{op} : n \in \Sigma$ and $t_1, \ldots, t_n \in \mathcal{T}_\Sigma X$,

$$\mu^{\Sigma,E}_X([\mathsf{op}([t_1]_E, \ldots, [t_n]_E)]_E) = [\mathsf{op}(t_1, \ldots, t_n)]_E.$$

Thanks to Remark 28, we can immediately see that $\mu^{\Sigma,\emptyset}_X = \mu^\Sigma_X$ because $[-]_\emptyset$ is the identity and $[\![-]\!]_{\mathbb{T}_{\Sigma,\emptyset}X} = \mu^\Sigma_X$.

We can pave the following diagram.[57]

$$
\begin{tikzcd}
\mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}X \arrow[rr, "{[-]_E}"] \arrow[ddd, "{[-]_E}"'] \arrow[dddrr, "{[\![-]\!]_{\mathbb{T}X}}"] \arrow[drrr, "\mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}f"'] & & \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}X \arrow[rr, "\mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}f"] \arrow[d, phantom, "\text{(a)}"] & & \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}Y \arrow[ddd, "\mu_Y^{\Sigma,E}"] \\
& & {} & \mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}Y \arrow[ur, "{[-]_E}"'] \arrow[ddr, "{[\![-]\!]_{\mathbb{T}Y}}"'] \arrow[r, phantom, "\text{(d)}"] & {} \\
{} \arrow[r, phantom, "\text{(b)}"] & {} \arrow[rr, phantom, "\text{(c)}"] & & {} & \\
\mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}X \arrow[rr, "\mu_X^{\Sigma,E}"'] & & \mathcal{T}_{\Sigma,E}X \arrow[rr, "\mathcal{T}_{\Sigma,E}f"'] & & \mathcal{T}_{\Sigma,E}Y
\end{tikzcd}
$$

All of (a), (b) and (d) commute by definition. In more details, (a) is an instance of (22) with X replaced by $\mathcal{T}_{\Sigma,E}X$, Y by $\mathcal{T}_{\Sigma,E}Y$ and f by $\mathcal{T}_{\Sigma,E}f$, and both (b) and (d) are instances of (26). To show (c) commutes, we draw another diagram that looks like a cube with (c) as the front face. We can show all the other faces commute, and then use the fact that $\mathcal{T}_\Sigma[-]_E$ is surjective (i.e. epic) to conclude that the front face must also commute.[58]

$$
\begin{tikzcd}
\mathcal{T}_\Sigma\mathcal{T}_\Sigma X \arrow[rr, "\mathcal{T}_\Sigma\mathcal{T}_\Sigma f"] \arrow[dr, "{\mathcal{T}_\Sigma[-]_E}"'] \arrow[dd, "\mu_X^\Sigma"'] & & \mathcal{T}_\Sigma\mathcal{T}_\Sigma Y \arrow[dr, "{\mathcal{T}_\Sigma[-]_E}"] \arrow[dd, "\mu_Y^\Sigma", near start] & \\
& \mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}X \arrow[rr, crossing over, "\mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}f", near start] & & \mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}Y \arrow[dd, "{[\![-]\!]_{\mathbb{T}Y}}"] \\
\mathcal{T}_\Sigma X \arrow[rr, "\mathcal{T}_\Sigma f", near end] \arrow[dr, "{[-]_E}"'] & & \mathcal{T}_\Sigma Y \arrow[dr, "{[-]_E}"] & \\
& \mathcal{T}_{\Sigma,E}X \arrow[rr, "\mathcal{T}_{\Sigma,E}f"'] \arrow[from=uu, crossing over, "{[\![-]\!]_{\mathbb{T}X}}"', near start] & & \mathcal{T}_{\Sigma,E}Y
\end{tikzcd}
$$

The first diagram we paved implies (27) commutes because $[-]_E$ is epic. $\square$

The front face of the cube is interesting on its own, it says that for any function $f : X \to Y$, $\mathcal{T}_{\Sigma,E}f$ is a homomorphism from $\mathbb{T}_{\Sigma,E}X$ to $\mathbb{T}_{\Sigma,E}Y$. We redraw it below for future reference.

$$
\begin{tikzcd}
\mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}X \arrow[r, "\mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}f"] \arrow[d, "{[\![-]\!]_{\mathbb{T}X}}"'] & \mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}Y \arrow[d, "{[\![-]\!]_{\mathbb{T}Y}}"] \\
\mathcal{T}_{\Sigma,E}X \arrow[r, "\mathcal{T}_{\Sigma,E}f"'] & \mathcal{T}_{\Sigma,E}Y
\end{tikzcd} \tag{28}
$$

Stating it like this may remind you of Lemma 17 and Remark 18. We will need a variant of Lemma 17 for $\mathcal{T}_{\Sigma,E}$, but there is a slight obstacle due to types. Indeed, given a Σ-algebra $\mathbb{A}$ we would like to prove a square like in (29) commutes.

However, the arrows on top and bottom do not really exist, the interpretation $[\![-]\!]_A$ takes terms over A as input, not equivalence classes of terms. The quick fix is to assume that $\mathbb{A}$ satisfies the equations in E. This means that $[\![-]\!]_A$ is well-defined

[57] By paving a diagram, we mean to build a large diagram out of smaller ones, showing all the smaller ones commute, and then concluding the bigger must commute. We often refer parts of the diagram with letters written inside them, and explain how each of them commutes one at a time.

[58] In more details, the left and right faces commute by (24), the bottom and top faces commute by (22), and the back face commutes by (6).

The function $\mathcal{T}_\Sigma[-]_E$ is surjective (i.e. epic) because $[-]_E$ is (it is a canonical quotient map) and functors on **Set** preserve epimorphisms (if we assume the axiom of choice). Thus, it suffices to show that $\mathcal{T}_\Sigma[-]_E$ pre-composed with the bottom path or the top path of the front face gives the same result.

Now it is just a matter of going around the cube using the commutativity of the other faces. Here is the complete derivation (we write which face was used as justifications for each step).

$$
\begin{aligned}
&\mathcal{T}_{\Sigma,E}f \circ [\![-]\!]_{\mathbb{T}X} \circ \mathcal{T}_\Sigma[-]_E && \\
&= \mathcal{T}_{\Sigma,E}f \circ [-]_E \circ \mu_X^\Sigma && \text{left} \\
&= [-]_E \circ \mathcal{T}_\Sigma f \circ \mu_X^\Sigma && \text{bottom} \\
&= [-]_E \circ \mu_Y^\Sigma \circ \mathcal{T}_\Sigma\mathcal{T}_\Sigma f && \text{back} \\
&= [\![-]\!]_{\mathbb{T}Y} \circ \mathcal{T}_\Sigma[-]_E \circ \mathcal{T}_\Sigma\mathcal{T}_\Sigma f && \text{right} \\
&= [\![-]\!]_{\mathbb{T}Y} \circ \mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}f \circ \mathcal{T}_\Sigma[-]_E && \text{top}
\end{aligned}
$$

$$
\begin{tikzcd}
\mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}A \arrow[r, "{\mathcal{T}_\Sigma[\![-]\!]_A}"] \arrow[d, "{[\![-]\!]_{\mathbb{T}A}}"'] & \mathcal{T}_\Sigma A \arrow[d, "{[\![-]\!]_A}"] \\
\mathcal{T}_{\Sigma,E}A \arrow[r, "{[\![-]\!]_A}"'] & A
\end{tikzcd} \tag{29}
$$

on equivalence class of terms because if $[s]_E = [t]_E$, then $A \vdash s = t \in \mathfrak{Th}(E)$, so $\mathbb{A}$ satisfies that equation, and taking the assignment $\mathrm{id}_A : A \to A$, we obtain

$$\llbracket s \rrbracket_A = \llbracket s \rrbracket_A^{\mathrm{id}_A} = \llbracket t \rrbracket_A^{\mathrm{id}_A} = \llbracket t \rrbracket_A.$$

When $\mathbb{A}$ is a (Σ, E)-algebra, we abusively write $\llbracket - \rrbracket_A$ for the interpretation of terms and equivalence classes of terms as in (30).

$$\begin{tikzcd} \mathcal{T}_\Sigma A \arrow[rr, "{[-]_E}"] \arrow[dr, "\llbracket - \rrbracket_A"'] & & \mathcal{T}_{\Sigma,E} A \arrow[dl, "\llbracket - \rrbracket_A"] \\ & A & \end{tikzcd} \quad (30)$$

Lemma 31. *For any* (Σ, E)*-algebra* $\mathbb{A}$*, the square* (29) *commutes.*

Proof. Consider the following diagram that we can view as a triangular prism whose front face is (29). Both triangles commute by (30), the square face at the back and on the left commutes by (24), and the square face at the back and on the right commutes by (12). With the same trick as in the proof of Proposition 30 using the surjectivity of $\mathcal{T}_\Sigma[-]_E$, we conclude that the front face commutes.[59]

$$\begin{tikzcd} & \mathcal{T}_\Sigma \mathcal{T}_\Sigma A \arrow[dl, "{\mathcal{T}_\Sigma[-]_E}"'] \arrow[dr, "\mathcal{T}_\Sigma \llbracket - \rrbracket_A"] \arrow[dd, "\mu_A^\Sigma"] & \\ \mathcal{T}_\Sigma \mathcal{T}_{\Sigma,E} A \arrow[rr, "\mathcal{T}_\Sigma \llbracket - \rrbracket_A"] \arrow[dd, "\llbracket - \rrbracket_{\mathbb{T}A}"'] & & \mathcal{T}_\Sigma A \arrow[dd, "\llbracket - \rrbracket_A"] \\ & \mathcal{T}_\Sigma A \arrow[dl, "{[-]_E}"'] \arrow[dr, "\llbracket - \rrbracket_A"] & \\ \mathcal{T}_{\Sigma,E} A \arrow[rr, "\llbracket - \rrbracket_A"'] & & A \end{tikzcd}$$

□

[59] Here is the complete derivation.

$$\begin{aligned} & \llbracket - \rrbracket_A \circ \llbracket - \rrbracket_{\mathbb{T}A} \circ \mathcal{T}_\Sigma[-]_E \\ &= \llbracket - \rrbracket_A \circ [-]_E \circ \mu_A^\Sigma && \text{left} \\ &= \llbracket - \rrbracket_A \circ \mu_A^\Sigma && \text{bottom} \\ &= \llbracket - \rrbracket_A \circ \mathcal{T}_\Sigma \llbracket - \rrbracket_A && \text{right} \\ &= \llbracket - \rrbracket_A \circ \mathcal{T}_\Sigma \llbracket - \rrbracket_A \circ \mathcal{T}_\Sigma[-]_E && \text{top} \end{aligned}$$

Then, since $\mathcal{T}_\Sigma[-]_E$ is epic, we conclude that $\llbracket - \rrbracket_A \circ \llbracket - \rrbracket_{\mathbb{T}A} = \llbracket - \rrbracket_A \circ \mathcal{T}_\Sigma \llbracket - \rrbracket_A$.

An important consequence of Lemma 17 was (14) saying that flattening is a homomorphism from $\mathbb{T}_{\Sigma,\varnothing}\mathbb{T}_{\Sigma,\varnothing}A$ to $\mathbb{T}_{\Sigma,\varnothing}A$. This is also true when E is not empty, i.e. $\mu_A^{\Sigma,E}$ is a homomorphism from $\mathbb{T}\mathbb{T}A$ to $\mathbb{T}A$.

Lemma 32. *For any set* A*, the following square commutes.*

$$\begin{tikzcd} \mathcal{T}_\Sigma \mathcal{T}_{\Sigma,E} \mathcal{T}_{\Sigma,E} A \arrow[r, "\mathcal{T}_\Sigma \mu_A^{\Sigma,E}"] \arrow[d, "\llbracket - \rrbracket_{\mathbb{T}\mathbb{T}A}"'] & \mathcal{T}_\Sigma \mathcal{T}_{\Sigma,E} A \arrow[d, "\llbracket - \rrbracket_{\mathbb{T}A}"] \\ \mathcal{T}_{\Sigma,E} \mathcal{T}_{\Sigma,E} A \arrow[r, "\mu_A^{\Sigma,E}"'] & \mathcal{T}_{\Sigma,E} A \end{tikzcd} \quad (31)$$

Proof. We prove it exactly like Lemma 31 with the following diagram.[60]

$$\begin{tikzcd} & \mathcal{T}_\Sigma \mathcal{T}_\Sigma \mathcal{T}_{\Sigma,E} A \arrow[dl, "{\mathcal{T}_\Sigma[-]_E}"'] \arrow[dr, "\mathcal{T}_\Sigma \llbracket - \rrbracket_{\mathbb{T}A}"] \arrow[dd, "\mu_{\mathcal{T}_{\Sigma,E}A}^\Sigma"] & \\ \mathcal{T}_\Sigma \mathcal{T}_{\Sigma,E} \mathcal{T}_{\Sigma,E} A \arrow[rr, "\mathcal{T}_\Sigma \mu_A^{\Sigma,E}"] \arrow[dd, "\llbracket - \rrbracket_{\mathbb{T}\mathbb{T}A}"'] & & \mathcal{T}_\Sigma \mathcal{T}_{\Sigma,E} A \arrow[dd, "\llbracket - \rrbracket_{\mathbb{T}A}"] \\ & \mathcal{T}_\Sigma \mathcal{T}_{\Sigma,E} A \arrow[dl, "{[-]_E}"'] \arrow[dr, "\llbracket - \rrbracket_{\mathbb{T}A}"] & \\ \mathcal{T}_{\Sigma,E} \mathcal{T}_{\Sigma,E} A \arrow[rr, "\mu_A^{\Sigma,E}"'] & & \mathcal{T}_{\Sigma,E} A \end{tikzcd}$$

[60] The top and bottom faces commute by definition of $\mu_A^{\Sigma,E}$ (26), the back-left face by (24), and the back-right face by (12).

Then, $\mathcal{T}_\Sigma[-]_E$ is epic, so the following derivation suffices.

$$\begin{aligned} & \mu_A^{\Sigma,E} \circ \llbracket - \rrbracket_{\mathbb{T}\mathbb{T}A} \circ \mathcal{T}_\Sigma[-]_E \\ &= \mu_A^{\Sigma,E} \circ [-]_E \circ \mu_{\mathcal{T}_{\Sigma,E}A}^\Sigma && \text{left} \\ &= \llbracket - \rrbracket_{\mathbb{T}A} \circ \mu_{\mathcal{T}_{\Sigma,E}A}^\Sigma && \text{bottom} \\ &= \llbracket - \rrbracket_{\mathbb{T}A} \circ \mathcal{T}_\Sigma \llbracket - \rrbracket_{\mathbb{T}A} && \text{right} \\ &= \llbracket - \rrbracket_{\mathbb{T}A} \circ \mathcal{T}_\Sigma \mu_A^{\Sigma,E} \circ \mathcal{T}_\Sigma[-]_E && \text{top} \end{aligned}$$

□

In a moment, we will show that $\mathbb{T}_{\Sigma,E}X$ is not only a Σ-algebra, but also a (Σ, E)-algebra. This requires us to talk about satisfaction of equations, hence about the interpretation of terms in some $\mathcal{T}_\Sigma Y$ under an assignment $\sigma : Y \to \mathcal{T}_{\Sigma,E}X$.[61] By the definition $[\![-]\!]^\sigma_{\mathbb{T}X} = [\![-]\!]_{\mathbb{T}X} \circ \mathcal{T}_\Sigma\sigma$, and our informal description of $[\![-]\!]_{\mathbb{T}X}$, we can infer that $[\![t]\!]^\sigma_{\mathbb{T}X}$ is the equivalence class of the term t where all occurences of the variable y have been substituted by a representative of $\sigma(y)$.

In particular, this means that under the assignment $\sigma : X \to \mathcal{T}_{\Sigma,E}X$ that sends a variable x to its equivalence class $[x]_E$, the interpretation of a term $t \in \mathcal{T}_\Sigma X$ is $[t]_E$.[62] We prove this formally below.

[61] We used ι before for assignments, but when considering assignments into (equivalence classes of) terms, we prefer using σ because we will adopt a different attitude with them (see Definition 36).

[62] The representative chosen for $\sigma(x)$ is x so the term t is not modified.

Lemma 33. *Let $\sigma = X \xrightarrow{\eta^\Sigma_X} \mathcal{T}_\Sigma X \xrightarrow{[-]_E} \mathcal{T}_{\Sigma,E}X$ be an assignment. Then, $[\![-]\!]^\sigma_{\mathbb{T}X} = [-]_E$.*

Proof. We proceed by induction. For the base case, we have

$$\begin{aligned}
[\![\eta^\Sigma_X(x)]\!]^\sigma_{\mathbb{T}X} &= [\![\mathcal{T}_\Sigma\sigma(\eta^\Sigma_X(x))]\!]_{\mathbb{T}X} && \text{by (8)}\\
&= [\![\mathcal{T}_\Sigma[-]_E(\mathcal{T}_\Sigma\eta^\Sigma_X(\eta^\Sigma_X(x)))]\!]_{\mathbb{T}X} && \text{Proposition 7}\\
&= [\![\mathcal{T}_\Sigma[-]_E(\eta^\Sigma_{\mathcal{T}_\Sigma X}(\eta^\Sigma_X(x)))]\!]_{\mathbb{T}X} && \text{by (4)}\\
&= [\![\eta^\Sigma_{\mathcal{T}_{\Sigma,E}X}([\eta^\Sigma_X(x)]_E)]\!]_{\mathbb{T}X} && \text{by (4)}\\
&= [\eta^\Sigma_X(x)]_E && \text{by (25)}
\end{aligned}$$

For the inductive step, if $t = \mathsf{op}(t_1, \dots, t_n)$, we have

$$\begin{aligned}
[\![t]\!]^\sigma_{\mathbb{T}X} &= [\![\mathcal{T}_\Sigma\sigma(t)]\!]_{\mathbb{T}X} && \text{by (8)}\\
&= [\![\mathcal{T}_\Sigma\sigma(\mathsf{op}(t_1, \dots, t_n))]\!]_{\mathbb{T}X}\\
&= [\![\mathsf{op}(\mathcal{T}_\Sigma\sigma(t_1), \dots, \mathcal{T}_\Sigma\sigma(t_n))]\!]_{\mathbb{T}X} && \text{by (3)}\\
&= [\![\mathsf{op}]\!]_{\mathbb{T}X}\left([\![\mathcal{T}_\Sigma\sigma(t_1)]\!]_{\mathbb{T}X}, \cdots, [\![\mathcal{T}_\Sigma\sigma(t_n)]\!]_{\mathbb{T}X}\right) && \text{by (25)}\\
&= [\![\mathsf{op}]\!]_{\mathbb{T}X}\left([t_1]_E, \cdots, [t_n]_E\right) && \text{I.H.}\\
&= [\mathsf{op}(t_1, \dots, t_n)]_E. && \text{by (23)}
\end{aligned}$$

□

We will denote that special assignment $\eta^{\Sigma,E}_X = [-]_E \circ \eta^\Sigma_X : X \to \mathcal{T}_{\Sigma,E}X$.[63] A quick corollary of the previous lemma is that for any equation ϕ with context X, ϕ belongs to $\mathfrak{Th}(E)$ if and only if the algebra $\mathbb{T}_{\Sigma,E}X$ satisfies it under the assignment $\eta^{\Sigma,E}_X$. This comes back to Example 24 where we said that freeness of X^* means it satisfies all and only the equations in $\mathfrak{Th}(E_{\mathbf{Mon}})$. Instead here, we do not know yet that $\mathbb{T}X$ is free (we have not even proved it satisfies E yet), but we can already show it satisfies only the necessary equations, and freeness will follow.

[63] Note that $\eta^{\Sigma,E}$ becomes a natural transformation $\mathrm{id}_{\mathbf{Set}} \to \mathcal{T}_{\Sigma,E}$ because it is the vertical composition $[-]_E \cdot \eta^\Sigma$.

Lemma 34. *Let $s, t \in \mathcal{T}_\Sigma X$, $X \vdash s = t \in \mathfrak{Th}(E)$ if and only if $\mathbb{T}_{\Sigma,E}X \vDash^{\eta^{\Sigma,E}_X} X \vdash s = t$.*[64]

[64] *Proof.* By Lemma 33, we have

$$[\![s]\!]^{\eta^{\Sigma,E}_X}_{\mathbb{T}X} = [s]_E \text{ and } [\![t]\!]^{\eta^{\Sigma,E}_X}_{\mathbb{T}X} = [t]_E,$$

then by definition of $\equiv_E$, $X \vdash s = t \in \mathfrak{Th}(E)$ if and only if $[s]_E = [t]_E$.

The interaction between μ^Σ and η^Σ is mimicked by $\mu^{\Sigma,E}$ and $\eta^{\Sigma,E}$.

Lemma 35. *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{T}_{\Sigma,E}X \arrow[r, "\eta^{\Sigma,E}_{\mathcal{T}_{\Sigma,E}X}"] \arrow[dr, "\mathrm{id}_{\mathcal{T}_{\Sigma,E}X}"'] & \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}X \arrow[d, "\mu^{\Sigma,E}_X"] & \mathcal{T}_{\Sigma,E}X \arrow[l, "\mathcal{T}_{\Sigma,E}\eta^{\Sigma,E}_X"'] \arrow[dl, "\mathrm{id}_{\mathcal{T}_{\Sigma,E}X}"] \\
& \mathcal{T}_{\Sigma,E}X &
\end{tikzcd}$$

Proof. For the triangle on the left, we pave the following diagram.

$$\begin{tikzcd}
\mathcal{T}_{\Sigma,E}X \arrow[r, "\eta^{\Sigma}_{\mathcal{T}_{\Sigma,E}X}"] \arrow[rr, bend left=50, "\eta^{\Sigma,E}_{\mathcal{T}_{\Sigma,E}X}"] \arrow[drr, bend right=30, "\mathrm{id}_{\mathcal{T}_{\Sigma,E}X}"'] & \mathcal{T}_{\Sigma}\mathcal{T}_{\Sigma,E}X \arrow[r, "{[-]_E}"] \arrow[dr, "\llbracket - \rrbracket_{\mathbb{T}X}"'] \arrow[u, phantom, "\text{(a)}"] \arrow[d, phantom, "\text{(b)}"] & \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}X \arrow[d, "\mu^{\Sigma,E}_X"] \arrow[dl, phantom, "\text{(c)}"] \\
& & \mathcal{T}_{\Sigma,E}X
\end{tikzcd} \tag{32}$$

Showing (32) commutes:

(a) Definition of $\eta^{\Sigma,E}_X$.

(b) Definition of $\llbracket - \rrbracket_{\mathbb{T}X}$ (25).

(c) Definition of $\mu^{\Sigma,E}_X$ (26).

For the triangle on the right, we show that $[-]_E = \mu^{\Sigma,E}_X \circ \mathcal{T}_{\Sigma,E}\eta^{\Sigma,E}_X \circ [-]_E$ by paving (33), and we can conclude since $[-]_E$ is epic that $\mathrm{id}_{\mathcal{T}_{\Sigma,E}X} = \mu^{\Sigma,E}_X \circ \mathcal{T}_{\Sigma,E}\eta^{\Sigma,E}_X$.

$$\begin{tikzcd}
\mathcal{T}_{\Sigma}X \arrow[r, "{[-]_E}"] \arrow[drr, "\mathcal{T}_{\Sigma}\eta^{\Sigma}_X"'] \arrow[ddrr, "\mathrm{id}_{\mathcal{T}_{\Sigma}X}"'] & \mathcal{T}_{\Sigma,E}X \arrow[r, "\mathcal{T}_{\Sigma,E}\eta^{\Sigma}_X"] \arrow[rrr, bend left=40, "\mathcal{T}_{\Sigma,E}\eta^{\Sigma,E}_X"] \arrow[d, phantom, "\text{(b)}"] & \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma}X \arrow[rr, "{\mathcal{T}_{\Sigma,E}[-]_E}"] \arrow[r, phantom, "\text{(a)}"] & & \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}X \arrow[dd, "\mu^{\Sigma,E}_X"] \\
& \text{(d)} & \mathcal{T}_{\Sigma}\mathcal{T}_{\Sigma}X \arrow[u, "{[-]_E}"] \arrow[r, "{\mathcal{T}_{\Sigma}[-]_E}"'] \arrow[d, "\mu^{\Sigma}_X"] \arrow[ur, phantom, "\text{(c)}"] \arrow[dr, phantom, "\text{(e)}"] & \mathcal{T}_{\Sigma}\mathcal{T}_{\Sigma,E}X \arrow[ur, "{[-]_E}"'] \arrow[dr, "\llbracket - \rrbracket_{\mathbb{T}X}"'] \arrow[r, phantom, "\text{(f)}"] & \\
& & \mathcal{T}_{\Sigma}X \arrow[rr, "{[-]_E}"'] & & \mathcal{T}_{\Sigma,E}X
\end{tikzcd} \tag{33}$$

Showing (33) commutes:

(a) Definition of $\eta^{\Sigma,E}_X$ and functoriality of $\mathcal{T}_{\Sigma,E}$.

(b) Naturality of $[-]_E$ (22).

(c) Naturality of $[-]_E$ again.

(d) Definition of μ^{Σ}_X (5).

(e) By (24).

(f) By (26).

□

We single out another special case of interpretation in a term algebra when E is empty (recall from Remark 28 that $\mathbb{T}_{\Sigma,\emptyset}X$ is the algebra on $\mathcal{T}_{\Sigma}X$ whose interpretation of op applies op syntactically).

Definition 36 (Substitution)**.** Given a signature Σ, an empty set of equations, and an assignment $\sigma : Y \to \mathcal{T}_{\Sigma}X$,[65] we call $\llbracket - \rrbracket^{\sigma}_{\mathbb{T}X}$ the **substitution** map, and we denote it by $\sigma^* : \mathcal{T}_{\Sigma}Y \to \mathcal{T}_{\Sigma}X$. We saw in Remark 28 that $\llbracket - \rrbracket_{\mathbb{T}X} = \mu^{\Sigma}_X$, thus substitution is

$$\sigma^* = \mathcal{T}_{\Sigma}Y \xrightarrow{\mathcal{T}_{\Sigma}\sigma} \mathcal{T}_{\Sigma}\mathcal{T}_{\Sigma}X \xrightarrow{\mu^{\Sigma}_X} \mathcal{T}_{\Sigma}X. \tag{34}$$

In words, σ^* replaces the occurrences of a variable y by $\sigma(y)$.[66]

[65] We can identify $\mathcal{T}_{\Sigma}X$ with $\mathcal{T}_{\Sigma,\emptyset}X$ because $\equiv_{\emptyset}$ is the equality relation.

[66] You may be more familiar with the notation $t[\sigma(y)/y]$ (e.g. from substitution in the λ-calculus). An inductive definition can also be given: for any $y \in Y$, $\sigma^*(\eta^{\Sigma}_Y(y)) = \sigma(y)$, and

$$\sigma^*(\mathsf{op}(t_1, \ldots, t_n)) = \mathsf{op}(\sigma^*(t_1), \ldots, \sigma^*(t_n)).$$

That simple description makes substitution a little special, and the following result has even deeper implications. It morally says that substitution preserves the satisfaction of equations.[67]

[67] We will give more intuition on Lemma 37 when we define equational logic.

Lemma 37. *Let $Y \vdash s = t$ be an equation, $\sigma : Y \to \mathcal{T}_\Sigma X$ an assignment, and $\mathbb{A}$ a Σ-algebra. If $\mathbb{A}$ satisfies $Y \vdash s = t$, then it also satisfies $X \vdash \sigma^*(s) = \sigma^*(t)$.*

Proof. Let $\iota : X \to A$ be an assignment, we need to show $[\![\sigma^*(s)]\!]_A^\iota = [\![\sigma^*(t)]\!]_A^\iota$. Define the assignment $\iota_\sigma : Y \to A$ that sends $y \in Y$ to $[\![\sigma(y)]\!]_A^\iota$, we claim that $[\![-]\!]_A^{\iota_\sigma} = [\![\sigma^*(-)]\!]_A^\iota$. The lemma then follows because by hypothesis, $[\![s]\!]_A^{\iota_\sigma} = [\![t]\!]_A^{\iota_\sigma}$. The following derivation proves our claim.

$$\begin{aligned}
[\![-]\!]_A^{\iota_\sigma} &= [\![-]\!]_A \circ \mathcal{T}_\Sigma(\iota_\sigma) && \text{by (8)}\\
&= [\![-]\!]_A \circ \mathcal{T}_\Sigma([\![\sigma(-)]\!]_A^\iota) && \text{definition of } \iota_\sigma\\
&= [\![-]\!]_A \circ \mathcal{T}_\Sigma\left([\![-]\!]_A \circ \mathcal{T}_\Sigma\iota \circ \sigma\right) && \text{by (8)}\\
&= [\![-]\!]_A \circ \mathcal{T}_\Sigma[\![-]\!]_A \circ \mathcal{T}_\Sigma\mathcal{T}_\Sigma\iota \circ \mathcal{T}_\Sigma\sigma && \text{Proposition 7}\\
&= [\![-]\!]_A \circ \mu_A^\Sigma \circ \mathcal{T}_\Sigma\mathcal{T}_\Sigma\iota \circ \mathcal{T}_\Sigma\sigma && \text{by (12)}\\
&= [\![-]\!]_A \circ \mathcal{T}_\Sigma\iota \circ \mu_Y^\Sigma \circ \mathcal{T}_\Sigma\sigma && \text{by (6)}\\
&= [\![-]\!]_A \circ \mathcal{T}_\Sigma\iota \circ \sigma^* && \text{by (34)}\\
&= [\![\sigma^*(-)]\!]_A^\iota. && \text{by (8)} \qquad \square
\end{aligned}$$

We are finally ready to show that $\mathbb{T}_{\Sigma,E}A$ is a (Σ, E)-algebra.[68]

[68] All the work we have been doing finally pays off.

Proposition 38. *For any set A, the term algebra $\mathbb{T}_{\Sigma,E}A$ satisfies all the equations in E.*

Proof. Let $X \vdash s = t$ belong to E and $\iota : X \to \mathcal{T}_{\Sigma,E}A$ be an assignment. We need to show that $[\![s]\!]_{\mathbb{T}A}^\iota = [\![t]\!]_{\mathbb{T}A}^\iota$. We factor ι into[69]

[69] This factoring is correct because

$$\begin{aligned}
\iota &= \mathrm{id}_{\mathcal{T}_{\Sigma,E}A} \circ \iota\\
&= \mu_A^{\Sigma,E} \circ \eta_{\mathcal{T}_{\Sigma,E}A}^{\Sigma,E} \circ \iota && \text{Lemma 35}\\
&= \mu_A^{\Sigma,E} \circ \mathcal{T}_{\Sigma,E}\iota \circ \eta_X^{\Sigma,E}. && \text{naturality of } \eta^{\Sigma,E}
\end{aligned}$$

$$\iota = X \xrightarrow{\eta_X^{\Sigma,E}} \mathcal{T}_{\Sigma,E}X \xrightarrow{\mathcal{T}_{\Sigma,E}\iota} \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}A \xrightarrow{\mu_A^{\Sigma,E}} \mathcal{T}_{\Sigma,E}A.$$

Now, Lemma 34 says that the equation is satisfied in $\mathbb{T}X$ under the assignment $\eta_X^{\Sigma,E}$, i.e. that $[\![s]\!]_{\mathbb{T}X}^{\eta_X^{\Sigma,E}} = [\![t]\!]_{\mathbb{T}X}^{\eta_X^{\Sigma,E}}$. We also know by Lemma 16 that homomorphisms preserve satisfaction, so we can apply it twice using the facts that $\mathcal{T}_{\Sigma,E}\iota$ and $\mu_A^{\Sigma,E}$ are homomorphisms (by (28) and (31) respectively) to conclude that

$$[\![s]\!]_{\mathbb{T}A}^\iota = [\![s]\!]_{\mathbb{T}A}^{\mu_A^{\Sigma,E}\circ\mathcal{T}_{\Sigma,E}\iota\circ\eta_X^{\Sigma,E}} = [\![t]\!]_{\mathbb{T}A}^{\mu_A^{\Sigma,E}\circ\mathcal{T}_{\Sigma,E}\iota\circ\eta_X^{\Sigma,E}} = [\![t]\!]_{\mathbb{T}A}^\iota. \qquad \square$$

We now know that $\mathbb{T}_{\Sigma,E}X$ belongs to $\mathbf{Alg}(\Sigma, E)$. In order to tie up the parallel with Example 24, we will show that $\mathbb{T}_{\Sigma,E}X$ is the free (Σ, E)-algebra over X.

Definition 39 (Free object)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories, $U : \mathbf{D} \to \mathbf{C}$ be a functor between them, and $X \in \mathbf{C}_0$. A **free object** on X (with respect to U) is an object $Y \in \mathbf{D}_0$ along with a morphism $i \in \mathrm{Hom}_{\mathbf{C}}(X, UY)$ such that for any object $A \in \mathbf{D}_0$ and morphism $f \in \mathrm{Hom}_{\mathbf{C}}(X, UA)$, there exists a unique morphism $f^* \in \mathrm{Hom}_{\mathbf{D}}(Y, A)$ such that $Uf^* \circ i = f$. This is summarized in the following diagram.[70]

[70] This is almost a copy of (19).

$$
\begin{tikzcd}
\text{in } \mathbf{C} & & & \text{in } \mathbf{D} \\
X \arrow[r, "i"] \arrow[dr, "f"'] & UY \arrow[d, dashed, "Uf^*"] & & Y \arrow[d, dashed, "f^*"] \\
& UA & \arrow[l, "U"'] & A
\end{tikzcd}
\tag{35}
$$

Proposition 40. *Free objects are unique up to isomorphism, namely, if Y and Y' are free objects on X, then $Y \cong Y'$.*[71]

Proposition 41. *For any set X, the term algebra $\mathbb{T}_{\Sigma,E}X$ is the free (Σ, E)-algebra on X.*

Proof. Let $\mathbb{A}$ be another (Σ, E)-algebra and $f : X \to A$ a function. We claim that $f^* = [\![-]\!]_A \circ \mathcal{T}_{\Sigma,E} f$ is the unique homomorphism making the following commute.

$$
\begin{tikzcd}
\text{in } \mathbf{Set} & & & \text{in } \mathbf{Alg}(\Sigma,E) \\
X \arrow[r, "\eta_X^{\Sigma,E}"] \arrow[dr, "f"'] & \mathcal{T}_{\Sigma,E}X \arrow[d, dashed, "f^*"] & & \mathbb{T}X \arrow[d, dashed, "f^*"] \\
& A & \arrow[l, "U"'] & \mathbb{A}
\end{tikzcd}
$$

First, f^* is a homomorphism because it is the composite of two homomorphisms $\mathcal{T}_{\Sigma,E} f$ (by (28)) and $[\![-]\!]_A$ (by Lemma 31 since $\mathbb{A}$ satisfies E). Next, the triangle commutes by the following derivation.

$$
\begin{aligned}
[\![-]\!]_A \circ \mathcal{T}_{\Sigma,E} f \circ \eta_X^{\Sigma,E} &= [\![-]\!]_A \circ \eta_A^{\Sigma,E} \circ f && \text{naturality of } \eta^{\Sigma,E} \\
&= [\![-]\!]_A \circ [-]_E \circ \eta_A^{\Sigma} \circ f && \text{definition of } \eta^{\Sigma,E} \\
&= [\![-]\!]_A \circ \eta_A^{\Sigma} \circ f && \text{by (30)} \\
&= f && \text{definition of } [\![-]\!]_A \text{ (7)}
\end{aligned}
$$

Finally, uniqueness follows from the inductive definition of $\mathbb{T}X$ and the homomorphism property. Briefly, if we know the action of a homomorphism on equivalence classes of terms of depth 0, we can infer all of its action because all other classes of terms can be obtained by applying operation symbols.[72] $\square$

Once we have free objects, we have an adjunction, and once we have an adjunction, we have a monad, the most wonderful mathematical object in the world (objectively). Unfortunately, our universal algebra spiel is not finished yet, we will get back to monads shortly.

1.3 Equational Logic

We were happy that interpretations in the term algebra are computed syntactically, but there is a big caveat. Everything is done modulo $\equiv_E$ which was defined in (20) to basically contain all the equations in $\mathfrak{Th}(E)$, that is, all the equations semantically entailed by E. Thanks to Lemma 34, if we want to know whether $X \vdash s = t$ is in $\mathfrak{Th}(E)$, it is enough to check if the free (Σ, E)-algebra $\mathbb{T}X$ satisfies it, but that is a circular argument since the carrier $\mathcal{T}_{\Sigma,E}X$ is defined via $\equiv_E$.

[71] Very abstractly: a free object on X is the same thing as an initial object in the comma category $\Delta(X) \downarrow U$, and initial objects are unique up to isomorphism.

[72] Formally, let $f, g : \mathbb{T}X \to \mathbb{A}$ be two homomorphisms such that for any $x \in X$, $f[x]_E = g[x]_E$, then, we can show that $f = g$. For any $t \in \mathcal{T}_\Sigma X$, we showed in Lemma 33 that $[t]_E = [\![t]\!]_{\mathbb{T}X}^{\eta_X^{\Sigma,E}}$. Then using (10), we have

$$f[t]_E = [\![t]\!]_A^{f \circ \eta_X^{\Sigma,E}} = [\![t]\!]_A^{g \circ \eta_X^{\Sigma,E}} = g[t]_E,$$

where the second inequality follows by hypothesis that f and g agree on equivalence classes of terms of depth 0.

Equational logic is a deductive system which produces an alternative definition of the free algebra, relying only on syntax. In short, the rules of equational logic allow to syntactically derive all of $\mathfrak{Th}(E)$ starting from E.

In Lemma 25, we proved that $\equiv_E$ is a congruence (i.e. reflexive, symmetric, transitive, and invariant under operations), and in Lemma 37 we showed $\equiv_E$ is also preserved by substitutions. This can help us syntactically derive $\mathfrak{Th}(E)$ because, for instance, if we know $X \vdash s = t \in E$, we can conclude $X \vdash t = s \in \mathfrak{Th}(E)$ by symmetry. If we know $x, y \vdash x = y \in E$, then we can conclude $X \vdash s = t \in \mathfrak{Th}(E)$, i.e. all terms are equal modulo E, by substituting x with s and y with t. This can be summarized with the inference rules of **equational logic** in Figure 1.3.

$$\frac{}{X \vdash t = t}\ \textsc{Refl} \qquad \frac{X \vdash s = t}{X \vdash t = s}\ \textsc{Symm} \qquad \frac{X \vdash s = t \qquad X \vdash t = u}{X \vdash s = u}\ \textsc{Trans}$$

$$\frac{\mathsf{op} : n \in \Sigma \qquad \forall 1 \leq i \leq n, X \vdash s_i = t_i}{X \vdash \mathsf{op}(s_1, \dots, s_n) = \mathsf{op}(t_1, \dots, t_n)}\ \textsc{Cong}$$

$$\frac{\sigma : Y \to \mathcal{T}_\Sigma X \qquad Y \vdash s = t}{X \vdash \sigma^*(s) = \sigma^*(t)}\ \textsc{Sub}$$

Figure 1.3: Rules of equational logic over the signature Σ, where X and Y can be any set, and s, t, u, s_i and t_i can be any term in $\mathcal{T}_\Sigma X$ (or $\mathcal{T}_\Sigma Y$ for Sub). As indicated in the premises of the rules Cong and Sub, they can be instantiated for any n-ary operation symbol, and for any function σ respectively.

The first four rules are fairly simple, and they essentially say that equality is an equivalence relation that is preserved by operations. The Sub rule looks a bit more complicated, it is named after the function σ^* used in the conclusion which we called substitution. Intuitively, it reflects the fact that variables in the context Y are universally quantified. If you know $Y \vdash s = t$ holds, then you can replace each variable $y \in Y$ by $\sigma(y)$ (which may even be a complex term using new variables in X), and you can prove that $X \vdash \sigma^*(s) = \sigma^*(t)$ holds. We did this in Lemma 37, and the argument to extract from there is that the interpretation of $\sigma^*(t)$ under some assignment $\iota : X \to A$ is equal to the interpretation of t under the assignment ι_σ sending $y \in Y$ to the interpretation of $\sigma(y)$ under ι. Since satisfaction of $Y \vdash s = t$ means satisfaction under any assignment (this is where universal quantification comes in), we conclude that $X \vdash \sigma^*(s) = \sigma^*(t)$ must be satisfied.

If you have written sequences of computations to solve a mathematical problem, you are already familiar with the essence of doing proofs in equational logic. The rigorous details of such proofs can be formalized with the following definition.

Definition 42 (Derivation)**.** A **derivation**[73] of $X \vdash s = t$ in equational logic with axioms E (a class of equations) is a finite rooted tree such that:

[73] Many other definitions of derivations exist, and our treatment of them will not be 100% rigorous.

- all nodes are labelled by equations,
- the root is labelled by $X \vdash s = t$,

- if an internal node (not a leaf) is labelled by ϕ and its children are labelled by $\phi_1, \dots, \phi_n$, then there is a rule in Figure 1.3 which concludes ϕ from $\phi_1, \dots, \phi_n$, and
- all the leaves are either in E or instances of Refl, i.e. an equation $Y \vdash u = u$ for some set Y and $u \in \mathcal{T}_\Sigma Y$.

Example 43. We write a derivation with the same notation used to specify the inference rules in Figure 1.3. Consider the signature $\Sigma = \{+:2, \mathsf{e}:0\}$ with E containing the equations defining commutative monoids in (16). Here is a derivation of $x, y, z \vdash x + (y + z) = z + (x + y)$ in equational logic with axioms E.

$$\dfrac{\dfrac{}{x,y,z \vdash x + (y+z) = (x+y) + z} \in E \qquad \dfrac{\sigma = \begin{array}{l} x \mapsto x+y \\ y \mapsto z \end{array} \qquad \dfrac{}{x,y \vdash x+y = y+x} \in E}{x,y,z \vdash (x+y) + z = z + (x+y)} \textsc{Sub}}{x,y,z \vdash x + (y+z) = z + (x+y)} \textsc{Trans}$$

Given any class of equations E, we denote by $\mathfrak{Th}'(E)$ the class of equations that can be proven from E in equational logic, i.e. $\phi \in \mathfrak{Th}'(E)$ if and only if there is a derivation of ϕ in equational logic with axioms E.

Our goal now is to prove that $\mathfrak{Th}'(E) = \mathfrak{Th}(E)$. We say that equational logic is sound and complete for (Σ, E)-algebras. Less concisely, soundness means that whenever equational logic proves an equation ϕ with axioms E, ϕ is satisfied by all (Σ, E)-algebras, and completeness says that whenever an equation ϕ is satisfied by all (Σ, E)-algebras, there is a derivation of ϕ in equational logic with axioms E.

Soundness is a straightforward consequence of earlier results.[74]

[74] In the story we are telling, the rules of equational logic were designed to be sound because we knew some properties of $\equiv_E$ already. In general when defining rules of a logic, we may use intuitions and later prove soundness to confirm them, or realize that soundness does not hold and infirm them.

Theorem 44 (Soundness)**.** *If $\phi \in \mathfrak{Th}'(E)$, then $\phi \in \mathfrak{Th}(E)$.*

Proof. In the proof of Lemma 25, we proved that each of Refl, Symm, Trans, and Cong are sound rules for a fixed arbitrary algebra. Namely, if $\mathbb{A} \in \mathbf{Alg}(\Sigma)$ satisfies the equations on top, then it satisfies the one on the bottom. Lemma 37 states the same soundness property for Sub. This implies a weaker property: if all (Σ, E)-algebras satisfy the equations on top, then they satisfy the one on the bottom.[75]

[75] This is a classical theorem of first order logic:

$$(\forall A.(PA \Rightarrow QA)) \Rightarrow (\forall A.PA \Rightarrow \forall A.QA)$$

Now, if $\phi \in \mathfrak{Th}'(E)$ was proven using equational logic and the axioms in E, then since all $\mathbb{A} \in \mathbf{Alg}(\Sigma, E)$ satisfy all the axioms, by repeatedly applying the weaker property above for each rule in the derivation, we find that all $\mathbb{A} \in \mathbf{Alg}(\Sigma, E)$ satisfy ϕ, i.e. $\phi \in \mathfrak{Th}(E)$. □

Completeness is the harder direction, and there are many ways to prove it.[76] We will define an algebra exactly like $\mathbb{T}X$ but using the equality relation induced by $\mathfrak{Th}'(E)$ instead of $\equiv_E$ which was induced by $\mathfrak{Th}(E)$. We then show that algebra is a (Σ, E)-algebra, and by construction, it will imply $\mathfrak{Th}(E) \subseteq \mathfrak{Th}'(E)$.

[76] The original proof of Birkhoff [Bir35, Theorem 10] relies on constructing free algebras. Several later proofs (e.g. [Wec12, Theorem 29]) rely on a theory of congruences.

Fix a signature Σ and a class E of equations over Σ. For any set X, we can define a binary relation $\equiv'_E$ on Σ-terms[77] that contains the pair (s, t) whenever $X \vdash s = t$ can be proven in equational logic. Formally, we have for any $s, t \in \mathcal{T}_\Sigma X$ (c.f. (20)),

[77] Again, we omit the set X from the notation.

$$s \equiv'_E t \iff X \vdash s = t \in \mathfrak{Th}'(E). \tag{36}$$

We can show $\equiv'_E$ is a congruence relation.

Lemma 45. *For any set X, the relation $\equiv'_E$ is reflexive, symmetric, transitive, and for any $\mathsf{op}: n \in \Sigma$ and $s_1, \dots, s_n, t_1, \dots, t_n \in \mathcal{T}_\Sigma X$,*[78]

$$(\forall 1 \leq i \leq n, s_i \equiv'_E t_i) \implies \mathsf{op}(s_1, \dots, s_n) \equiv'_E \mathsf{op}(t_1, \dots, t_n). \tag{37}$$

[78] i.e. $\equiv'_{\hat{E}}$ is a congruence on the Σ-algebra $\mathcal{T}_\Sigma X$ defined in Remark 18.

Proof. This is immediate from the presence of Refl, Symm, Trans, and Cong in the rules of equational logic. □

We write $\wr - \int_E : \mathcal{T}_\Sigma X \to \mathcal{T}_\Sigma X/\equiv'_E$ for the canonical quotient map, so $\wr t \int_E$ is the equivalence class of t modulo the congruence $\equiv'_E$ induced by equational logic.

Definition 46 (Term algebra, syntactically)**.** The *new* term algebra for (Σ, E) on X is the Σ-algebra whose carrier is $\mathcal{T}_\Sigma X/\equiv'_E$ and whose interpretation of $\mathsf{op}: n \in \Sigma$ is defined by[79]

$$[\![\mathsf{op}]\!]_{\mathbb{T}'X}(\wr t_1 \int_E, \dots, \wr t_n \int_E) = \wr \mathsf{op}(t_1, \dots, t_n) \int_E. \tag{38}$$

[79] This is well-defined (i.e. invariant under change of representative) by (37).

We denote this algebra by $\mathbb{T}'_{\Sigma,E}X$ or simply $\mathbb{T}'X$.

With soundness (Theorem 44) of equational logic, completeness would mean this alternative definition of the term algebra coincides with $\mathbb{T}X$. First, we have to show that $\mathbb{T}'X$ belongs to $\mathbf{Alg}(\Sigma, E)$ like we did for $\mathbb{T}X$ in Proposition 38, and we prove a technical lemma before that.

Lemma 47. *Let $\iota : Y \to \mathcal{T}_\Sigma X/\equiv'_E$ be an assignment. For any function $\sigma : Y \to \mathcal{T}_\Sigma X$ satisfying $\wr \sigma(y) \int_E = \iota(y)$ for all $y \in Y$, we have $[\![-]\!]^\iota_{\mathbb{T}'X} = \wr \sigma^*(-) \int_E$.*[80]

[80] This result looks like a stronger version of Lemma 33 for $\mathbb{T}'X$. Morally, they are both saying that interpretation of terms in $\mathbb{T}X$ or $\mathbb{T}'X$ is just a syntactical matter.

Proof. We proceed by induction. For the base case, we have by definition of the interpretation of terms (7), definition of σ, and definition of σ^* (34),

$$[\![\eta^\Sigma_Y(y)]\!]^\iota_{\mathbb{T}'X} \overset{(7)}{=} \iota(y) = \wr \sigma(y) \int_E \overset{(34)}{=} \wr \sigma^*(\eta^\Sigma_Y(y)) \int_E.$$

For the inductive step, we have

$$\begin{aligned}
[\![\mathsf{op}(t_1, \dots, t_n)]\!]^\iota_{\mathbb{T}'X} &= [\![\mathsf{op}]\!]_{\mathbb{T}'X}([\![t_1]\!]^\iota_{\mathbb{T}'X}, \dots, [\![t_n]\!]^\iota_{\mathbb{T}'X}) && \text{by (7)}\\
&= [\![\mathsf{op}]\!]_{\mathbb{T}'X}(\wr \sigma^*(t_1) \int_E, \dots, \wr \sigma^*(t_n) \int_E) && \text{I.H.}\\
&= \wr \mathsf{op}(\sigma^*(t_1), \dots, \sigma^*(t_n)) \int_E && \text{by (38)}\\
&= \wr \sigma^*(\mathsf{op}(t_1, \dots, t_n)) \int_E. && \text{definition of } \sigma^*
\end{aligned}$$
□

Proposition 48. *For any set X, $\mathbb{T}'X$ satisfies all the equations in E.*

Proof. Let $Y \vdash s = t$ belong to E and $\iota : Y \to \mathcal{T}_\Sigma X/\equiv'_E$ be an assignment. By the axiom of choice,[81] there is a function $\sigma : Y \to \mathcal{T}_\Sigma X$ satisfying $\wr \sigma(y) \int_E = \iota(y)$ for all $y \in Y$. Thanks to Lemma 47, it is enough to show $\wr \sigma^*(s) \int_E = \wr \sigma^*(t) \int_E$.[82] Equivalently, by definition of $\wr - \int_E$ and $\mathfrak{Th}'(E)$, we can just exhibit a derivation of $X \vdash \sigma^*(s) = \sigma^*(t)$ in equational logic with axioms E. This is rather simple because that equation can be proven with the Sub rule instantiated with $\sigma : Y \to \mathcal{T}_\Sigma X$ and the equation $Y \vdash s = t$ which is an axiom. □

[81] Choice implies the quotient map $\wr - \int_E$ has a right inverse $r : \mathcal{T}_\Sigma X/\equiv'_E \to \mathcal{T}_\Sigma X$, and we can then set $\sigma = r \circ \iota$.

[82] By Lemma 47, it implies

$$[\![s]\!]^\iota_{\mathbb{T}'X} = \wr \sigma^*(s) \int_E = \wr \sigma^*(t) \int_E = [\![t]\!]^\iota_{\mathbb{T}'X},$$

and since ι was an arbitrary assignment, we conclude that $\mathbb{T}'X \vDash Y \vdash s = t$.

Completeness of equational logic readily follows.

Theorem 49 (Completeness). *If $\phi \in \mathfrak{Th}(E)$, then $\phi \in \mathfrak{Th}'(E)$.*

Proof. Write $\phi = X \vdash s = t \in \mathfrak{Th}(E)$. By Proposition 48 and definition of $\mathfrak{Th}(E)$, we know that $\mathbb{T}'X \vDash \phi$. In particular, $\mathbb{T}'X$ satisfies ϕ under the assignment

$$\iota = X \xrightarrow{\eta^\Sigma_X} \mathcal{T}_\Sigma X \xrightarrow{\wr - \int_E} \mathcal{T}_\Sigma X/{\equiv'_E},$$

namely, $[\![s]\!]^\iota_{\mathbb{T}'X} = [\![t]\!]^\iota_{\mathbb{T}'X}$. Moreover with $\sigma = \eta^\Sigma_X$, we can show σ satisfies the hypothesis of Lemma 47 and $\sigma^* = \mathrm{id}_{\mathcal{T}_\Sigma X}$,[83] thus we conclude

$$\wr s \int_E = [\![s]\!]^\iota_{\mathbb{T}'X} = [\![t]\!]^\iota_{\mathbb{T}'X} = \wr t \int_E.$$

This implies $s \equiv'_E t$ which in turn means $X \vdash s = t$ belongs to $\mathfrak{Th}'(E)$. □

[83] We defined ι precisely to have $\wr \sigma(x) \int_E = \iota(x)$. To show $\sigma^* = \eta^{\Sigma *}_X$ is the identity, use (34) and the fact that $\mu^\Sigma \cdot \mathcal{T}_\Sigma \eta^\Sigma = \mathbb{1}_{\mathcal{T}_\Sigma}$ (Lemma 10).

Note that because $\mathbb{T}X$ and $\mathbb{T}'X$ were defined in the same way in terms of $\mathfrak{Th}(E)$ and $\mathfrak{Th}'(E)$ respectively, and since we have proven the latter to be equal, we obtain that $\mathbb{T}X$ and $\mathbb{T}'X$ are the same algebra.[84]

[84] It is good to keep in mind these two equivalent definitions of the free (Σ, E)-algebra on X. It means you can prove s equals t in $\mathbb{T}X$ by exhibiting a derivation of $X \vdash s = t$ in equational logic, or you can prove $s \neq t$ by exhibiting an algebra that satisfies E but not $X \vdash s = t$.

Remark 50. We have used the axiom of choice in proving completeness of equational logic, but that is only an artifact of our presentation that deals with arbitrary contexts. Since terms are finite and operation symbols have finite arities, we can make do with only finite contexts (which removes the need for choice). Formally, one can prove by induction on the derivation that a proof of $X \vdash s = t$ can be transformed into a proof of $\mathrm{FV}\{s,t\} \vdash s = t$ which uses only equations with finite contexts.[85] You can also verify semantically that $\mathbb{A}$ satisfies $X \vdash s = t$ if and only if it satisfies $\mathrm{FV}\{s,t\} \vdash s = t$ essentially because the extra variables have no effect on the quantification of the free variables in s and t nor on the interpretation.

[85] We denoted by $\mathrm{FV}\{s,t\}$ the set of **free variables** used in s and t. This can be defined inductively as follows:

$$\begin{aligned} \mathrm{FV}\{\eta^\Sigma_X(x)\} &= \{x\} \\ \mathrm{FV}\{\mathsf{op}(t_1,\dots,t_n)\} &= \mathrm{FV}\{t_1\} \cup \cdots \cup \mathrm{FV}\{t_n\} \\ \mathrm{FV}\{t_1,\dots,t_n\} &= \mathrm{FV}\{t_1\} \cup \cdots \cup \mathrm{FV}\{t_n\}. \end{aligned}$$

Note that $\mathrm{FV}\{-\}$ applied to a finite set of terms is always finite.

We mention now two related results for the sake of comparison when we introduce quantitative equational logic. First, for any set X and variable y, the following inference rules are derivable in equational logic.

$$\frac{X \vdash s = t}{X \cup \{y\} \vdash s = t}\ \textsc{Add} \qquad\qquad \frac{X \vdash s = t \qquad y \notin \mathrm{FV}\{s,t\}}{X \setminus \{y\} \vdash s = t}\ \textsc{Del}$$

In words, Add says that you can always add a variable to the context, and Del says you can remove a variable from the context when it is not used in the terms of the equations. Both these rules are instances of Sub. For the first, take σ to be the inclusion of X in $X \cup \{y\}$ (it may be the identity if $y \in X$). For the second, let σ send y to whatever element of $X \setminus \{y\}$ and all the other elements of X to themselves[86], then since y is not in the free variables of s and t, $\sigma^*(s) = s$ and $\sigma^*(t) = t$.

[86] When X is empty, the equations on the top and bottom of Del coincide, so the rule is derivable.

Second, we allowed the collection of equations E generating an algebraic theory $\mathfrak{Th}(E)$ to be a proper class, and that is really not common. Oftentimes, a countable set of variables $\{x_1, x_2, \dots\}$ is assumed, and equations are defined only when with a context contained in that set. With this assumption, the collection of all equations, E, and $\mathfrak{Th}(E)$ are all sets. This has no effect on expressiveness since for any equation $X \vdash s = t$, there is an equivalent equation $X' \vdash s' = t'$ with $X' \subseteq \{x_1, x_2, \dots\}$.[87]

[87] We already know $X \vdash s = t$ is equivalent to $\mathrm{FV}\{s,t\} \vdash s = t$, and since the context of the latter is finite, we have a bijection $\sigma : \mathrm{FV}\{s,t\} \cong \{x_1, \dots, x_n\}$. Then the Sub rule instantiated with σ and σ^{-1} proves the desired equivalence.

1.4 Monads

Our presentation of universal algebra used the language of category theory, e.g. functors, natural transformations, commutative diagrams. Both these fields of mathematics were born within a decade of each other[88] with a similar goal: abstracting the way mathematicians use mathematical objects in order to apply one general argument to many specific cases.[89] One could argue (looking at today's practicing mathematicians) that category theory was more successful. This is why a portion of this manuscript is spent on monads, a more categorical formulation of the content in universal algebra which became popular in computer science after Moggi's work [Mog89, Mog91] using monads to abstract computational effects.

[88] [Bir35] and [EM45] were the seminal papers for universal algebra and category theory respectively. Birkhoff and MacLane even wrote an undergraduate textbook together [MB99].

[89] This is very close to a goal of mathematics as a whole: abstracting the way nature works in order to apply one general argument to many specific cases, c.f. Cheng calling category theory the "mathematics of mathematics" [Che16].

There is another categorical approach to universal algebra introduced by Lawvere [Law63] and first popularized in the computer science community by Hyland, Plotkin, and Power [PP01, HPP06, HP07]. We will stick to monads because most of the literature on quantitative algebras does, and because I am not sure yet how the generalizations we contributed port to Lawvere's approach.[90]

[90] In the paper introducing quantitative algebra [MPP16], the authors already mentioned enriched Lawvere theories [Pow99]. The work of Lucyshyn-Wright and Parker [Luc15, LP23] is also relevant.

Definition 51 (Monad)**.** A **monad** on a category $\mathbf{C}$ is a triple (M, η, μ) made up of an endofunctor $M : \mathbf{C} \to \mathbf{C}$ and two natural transformations $\eta : \mathrm{id}_{\mathbf{C}} \Rightarrow M$ and $\mu : M^2 \Rightarrow M$ called the **unit** and **multiplication** respectively that make (39) and (40) commute in $[\mathbf{C}, \mathbf{C}]$.[91]

$$\begin{tikzcd} M \arrow[r, "M\eta"] \arrow[dr, "\mathbb{1}_M"'] & M^2 \arrow[d, "\mu"] & M \arrow[l, "\eta M"'] \arrow[dl, "\mathbb{1}_M"] \\ & M & \end{tikzcd} \tag{39}$$

$$\begin{tikzcd} M^3 \arrow[r, "\mu M"] \arrow[d, "M\mu"'] & M^2 \arrow[d, "\mu"] \\ M^2 \arrow[r, "\mu"'] & M \end{tikzcd} \tag{40}$$

We often refer to the monad (M, η, μ) simply with M.

[91] I also recommend Marsden's series of blog posts on monads for a relatively light and comprehensive survey: `https://stringdiagram.com/2022/05/17/hello-monads/`.

In this chapter we will mostly talk about monads on **Set**, but it is good to keep some arguments general for later. Here are some very important examples (for computer scientists and especially for this manuscript).

Example 52 (Maybe)**.** Suppose $\mathbf{C}$ has (binary) coproducts and a terminal object $\mathbf{1}$, then $(- + \mathbf{1}) : \mathbf{C} \to \mathbf{C}$ is a monad. It is called the **maybe monad** (the name "option monad" is also common).[92] We write inl^{X+Y} (resp. inr^{X+Y}) for the coprojection of X (resp. Y) into $X + Y$.[93] First, note that for a morphism $f : X \to Y$,

$$f + \mathbf{1} = [\mathsf{inl}^{Y+\mathbf{1}} \circ f, \mathsf{inr}^{Y+\mathbf{1}}] : X + \mathbf{1} \to Y + \mathbf{1}.$$

[92] It is also called the lift monad in [Jac16, Example 5.1.3.2].

[93] These notations are common in the community of programming language research, they stand for *injection left* (resp. *right*). We may omit the superscript.

The components of the unit are given by the coprojections, i.e. $\eta_X = \mathsf{inl}^{X+\mathbf{1}} : X \to X + \mathbf{1}$, and the components of the multiplication are

$$\mu_X = [\mathsf{inl}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}] : X + \mathbf{1} + \mathbf{1} \to X + \mathbf{1}.$$

Checking that (39) and (40) commute is an exercise in reasoning with coproducts. It is much more interesting to give the intuition in **Set** where $+$ is the disjoint union and $\mathbf{1}$ is the singleton $\{*\}$:[94]

- $X + \mathbf{1}$ is the set X with an additional (fresh) element $*$,

[94] This intuition should carry over well to many categories where the coproduct and terminal objects have similar behaviors.

- the function $f + \mathbf{1}$ acts like f on X and sends the new element $* \in X$ to the new element $* \in Y$,
- the unit $\eta_X : X \to X + \mathbf{1}$ is the injection (sending $x \in X$ to itself),
- the multiplication μ_X acts like the identity on X and sends the two new elements of $X + \mathbf{1} + \mathbf{1}$ to the single new element $X + \mathbf{1}$,
- one can check (39) and (40) commute by hand because (briefly) $x \in X$ is always sent to $x \in X$ and $*$ is always sent to $*$.

More often than not, the fresh element $*$ is seen as a terminating state, so the maybe monad models the most basic computational effect. Even when no other observation can be made on states of a program, one can distinguish between states by looking at their execution traces which may or may not contain $*$.[95]

[95] This was already known to Moggi who used different terminology in [Mog91, Example 1.1].

Example 53 (Powerset)**.** The covariant **non-empty finite powerset** functor $\mathcal{P}_{\mathrm{ne}} : \mathbf{Set} \to \mathbf{Set}$ sends a set X to the set of non-empty finite subsets of X which we denote by $\mathcal{P}_{\mathrm{ne}}X$. It acts on functions just like the usual powerset functor, i.e. given a function $f : X \to Y$, $\mathcal{P}_{\mathrm{ne}}f$ is the direct image function, it sends $S \subseteq X$ to $f(S) = \{f(x) \mid x \in S\}$.[96]

[96] It is clear that $f(S)$ is non-empty and finite when S is non-empty and finite.

One can show $\mathcal{P}_{\mathrm{ne}}$ is a monad with the following unit and multiplication:[97]

[97] Note that $\{x\}$ is non-empty and finite, and so is $\cup_{s\in F} s$ whenever F and all $s \in F$ are non-empty and finite. Thus, we can define $\mathcal{P}_{\mathrm{ne}}$ as a submonad of the *full* powerset monad in, e.g., [Jac16, Example 5.1.3.1].

$$\eta_X : X \to \mathcal{P}_{\mathrm{ne}}(X) = x \mapsto \{x\} \text{ and } \mu_X : \mathcal{P}_{\mathrm{ne}}(\mathcal{P}_{\mathrm{ne}}(X)) \to \mathcal{P}_{\mathrm{ne}}(X) = F \mapsto \bigcup_{s\in F} s.$$

Again, as early as in Moggi's papers, the powerset monad was used to model non-deterministic computations (see also [VW06, KS18, BSV19, GPA21]). A set $S \in \mathcal{P}_{\mathrm{ne}}X$ is seen as all the possible states at a point in the execution. We assume that S is finite for convenience, and that it is non-empty because an empty set of possible states would mean termination which can already be modelled with the maybe monad.[98]

[98] Also, the maybe monad can be *combined* with any other monad, see for example [MSV21, Corollary 5].

Example 54 (Distributions)**.** The functor $\mathcal{D} : \mathbf{Set} \to \mathbf{Set}$ sends a set X to the set of **finitely supported distributions** on X:[99]

[99] We will simply call them distributions.

$$\mathcal{D}(X) := \{\varphi : X \to [0,1] \mid \sum_{x\in X} \varphi(x) = 1 \text{ and } \varphi(x) \neq 0 \text{ for finitely many } x\text{'s}\}.$$

We call $\varphi(x)$ the **weight** of φ at x and let $\mathrm{supp}(\varphi)$ denote the **support** of φ, that is, $\mathrm{supp}(\varphi)$ contains all the elements $x \in X$ such that $\varphi(x) \neq 0$.[100] On morphisms, $\mathcal{D}$ sends a function $f : X \to Y$ to the function between sets of distributions defined by

[100] We often write $\varphi(S)$ for the total weight of φ on all of $S \subseteq X$.

$$\mathcal{D}f : \mathcal{D}X \to \mathcal{D}Y = \varphi \mapsto \left(y \mapsto \sum_{x\in X, f(x)=y} \varphi(x)\right).$$

In words, the weight of $\mathcal{D}f(\varphi)$ at y is equal to the total weight of φ on the preimage of y under f.[101]

[101] The distribution $\mathcal{D}f(\varphi)$ is sometimes called the **pushforward** of φ.

One can show that $\mathcal{D}$ is a monad with unit $\eta_X = x \mapsto \delta_x$, where δ_x is the **Dirac** distribution at x (the weight of δ_x is 1 at x and 0 everywhere else), and multiplication

$$\mu_X = \Phi \mapsto \left(x \mapsto \sum_{\varphi\in\mathrm{supp}(\Phi)} \Phi(\varphi)\varphi(x)\right).$$

In words, the weight $\mu_X(\Phi)$ at x is the average of $\varphi(x)$ weighted by $\Phi(\varphi)$ for all distributions in the support of Φ.[102]

Moggi only hinted at the distribution monad being a good model for computations that rely on random/probabilistic choices. For fleshed out research see, e.g., [VW06, SW18, BSV19].

[102] It was Giry [Gir82] who first studied probabilities through the categorical lens with a monad with inspiration from Lawvere [Law62], $\mathcal{D}$ is a discrete version of Giry's original construction. (See [Jac16, Example 5.1.3.4].)

Monads have been a popular categorical approach to universal algebra[103] thanks to a result of Linton [Lin66, Proposition 1] stating that any algebraic theory gives rise to a monad. Given a signature Σ and a class E of equations, we already implicitly described the monad Linton constructed, it is the triple $(\mathcal{T}_{\Sigma,E}, \eta^{\Sigma,E}, \mu^{\Sigma,E})$.

[103] See [HP07] for a thorough survey on categorical approaches to universal algebra.

Proposition 55. *The functor $\mathcal{T}_{\Sigma,E} : \mathbf{Set} \to \mathbf{Set}$ defines a monad on $\mathbf{Set}$ with unit $\eta^{\Sigma,E}$ and multiplication $\mu^{\Sigma,E}$. We call it the **term monad** for (Σ, E).*

Proof. We have done most of the work already.[104] We showed that $\eta^{\Sigma,E}$ and $\mu^{\Sigma,E}$ are natural transformations of the right type in Footnote 63 and Proposition 30 respectively, and we showed the appropriate instance of (39) commutes in Lemma 35. It remains to prove (40) commutes which, instantiated here, means proving the following diagram commutes for every set A.

$$\begin{tikzcd}
\mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}A \arrow[r, "\mathcal{T}_{\Sigma,E}\mu_A^{\Sigma,E}"] \arrow[d, "\mu^{\Sigma,E}_{\mathcal{T}_{\Sigma,E}A}"'] & \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}A \arrow[d, "\mu_A^{\Sigma,E}"] \\
\mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}A \arrow[r, "\mu_A^{\Sigma,E}"'] & \mathcal{T}_{\Sigma,E}A
\end{tikzcd}$$

[104] In fact, we have done it twice because we showed that $\mathbb{T}_{\Sigma,E}A$ is the free (Σ, E)-algebra on A for every set A, and that automatically yields (through abstract categorical arguments) a monad sending A to the carrier of $\mathbb{T}_{\Sigma,E}A$, i.e. $\mathcal{T}_{\Sigma,E}A$.

It follows from the following paved diagram.[105]

$$\begin{tikzcd}
\mathcal{T}_{\Sigma}\mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}A \arrow[rrrr, "\mathcal{T}_{\Sigma}\mu_A^{\Sigma,E}"] \arrow[dr, "{[-]_E}"'] \arrow[dd, "{[\![-]\!]_{\mathbb{T}\mathbb{T}A}}"'] & & & & \mathcal{T}_{\Sigma}\mathcal{T}_{\Sigma,E}A \arrow[dl, "{[-]_E}"] \arrow[dd, "{[\![-]\!]_{\mathbb{T}A}}"] \\
 & \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}A \arrow[rr, "\mathcal{T}_{\Sigma,E}\mu_A^{\Sigma,E}"'] \arrow[dl, "\mu^{\Sigma,E}_{\mathcal{T}_{\Sigma,E}A}"] & & \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}A \arrow[dr, "\mu_A^{\Sigma,E}"'] & \\
\mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}A \arrow[rrrr, "\mu_A^{\Sigma,E}"'] & & & & \mathcal{T}_{\Sigma,E}A
\end{tikzcd}$$

(a) (b) (c) (d)

Note that when E is empty, we get a monad $(\mathcal{T}_{\Sigma}, \eta^{\Sigma}, \mu^{\Sigma})$.[106] □

[105] We know that (a), (b) and (c) commute by (26), (22), and (26) respectively. This means that (d) precomposed by the epimorphism $[-]_E$ yields the outer square. Moreover, we know the outer square commutes by (31), therefore, (d) must also commute.

[106] Here is an alternative proof that $\mathcal{T}_{\Sigma}$ is a monad. We showed η^{Σ} and μ^{Σ} are natural in (4) and (6) respectively. The right triangle of (39) commutes by definition of μ^{Σ} (5), the left triangle commutes by Lemma 10, and the square (40) commutes by (14).

Linton also showed that from a monad M, you can build a theory whose corresponding term monad is isomorphic to M [Lin69, Lemma 10.1]. This however relied on a more general notion of theory. We will not go over the details here, rather we will introduce the necessary concepts to talk about our main examples on $\mathbf{Set}$: $(- + \mathbf{1})$, $\mathcal{P}_{\text{ne}}$, and $\mathcal{D}$. First, just like (Σ, E)-algebras are models of the theory (Σ, E), we can define models for a monad, which we also call algebras.

Definition 56 (M–algebra)**.** Let (M, η, μ) be a monad on $\mathbf{C}$, an M**-algebra** is a pair (A, α) comprising an object $A \in \mathbf{C}_0$ and a morphism $\alpha : MA \to A$ such that (41) and (42) commute.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_A"'] & MA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \quad (41) \qquad \begin{tikzcd} MMA \arrow[r, "\mu_A"] \arrow[d, "M\alpha"'] & MA \arrow[d, "\alpha"] \\ MA \arrow[r, "\alpha"'] & A \end{tikzcd} \quad (42)$$

We call A the carrier and we may write only α to refer to an M-algebra.

Definition 57 (Homomorphism)**.** Let (M, η, μ) be a monad and (A, α) and (B, β) be two M-algebras. An M-algebra **homomorphism** or simply M-homomorphism from α to β is a morphism $h : A \to B$ in $\mathbf{C}$ making (43) commute.

$$\begin{tikzcd} MA \arrow[r, "Mh"] \arrow[d, "\alpha"'] & MB \arrow[d, "\beta"] \\ A \arrow[r, "h"'] & B \end{tikzcd} \quad (43)$$

The composition of two M-homomorphisms is an M-homomorphism and id_A is an M-homomorphism from (A, α) to itself, thus we get a category of M-algebras and M-homomorphisms called the **Eilenberg–Moore category** of M and denoted by $\mathbf{EM}(M)$.[107] Since $\mathbf{EM}(M)$ was built from objects and morphisms in $\mathbf{C}$, there is an obvious forgetful functor $U^M : \mathbf{EM}(M) \to \mathbf{C}$ sending an M-algebra (A, α) to its carrier A and an M-homomorphism to its underlying morphism.

[107] Named after the authors of the article introducing that category [EM65].

Example 58. We will see some more concrete examples in a bit, but we can mention now that the similarities between the squares in the definitions of a monad (40), of an algebra (42), and of a homomorphism (43) have profound consequences. First, for any A, the pair (MA, μ_A) is an M-algebra because (44) and (45) commute by the properties of a monad.[108]

[108] (44) is the component at A of the right triangle in (39), and (45) is the component at A of (40).

$$\begin{tikzcd} MA \arrow[r, "\eta_{MA}"] \arrow[dr, "\mathrm{id}_{MA}"'] & MMA \arrow[d, "\mu_A"] \\ & MA \end{tikzcd} \quad (44) \qquad \begin{tikzcd} MMMA \arrow[r, "\mu_{MA}"] \arrow[d, "M\mu_A"'] & MMA \arrow[d, "\mu_A"] \\ MMA \arrow[r, "\mu_A"'] & MA \end{tikzcd} \quad (45)$$

Furthermore, for any M-algebra $\alpha : MA \to A$, (42) (reflected through the diagonal) precisely says that α is a M-homomorphism from (MA, μ_A) to (A, α). After a bit more work[109] we conclude that (MA, μ_A) is the free M-algebra (with respect to $U^M : \mathbf{EM}(M) \to \mathbf{Set}$).

[109] Given an M-algebra (A', α') and a function $f : A \to A'$, we can show $\alpha' \circ Mf$ is the unique M-homomorphism such that $\alpha' \circ Mf \circ \eta_A = f$.

The terminology suggests that (Σ, E)-algebras and $\mathcal{T}_{\Sigma,E}$-algebras are the same thing.[110] Let us check this, obtaining a large family of examples at the same time.

[110] Also, Example 58 starts to confirm this if we compare it with Remark 18, and Lemma 19.

Proposition 59. *There is an isomorphism* $\mathbf{Alg}(\Sigma, E) \cong \mathbf{EM}(\mathcal{T}_{\Sigma,E})$.

Proof. Given a (Σ, E)-algebra $\mathbb{A}$, we already explained in (30) how to obtain a function $[\![-]\!]_A : \mathcal{T}_{\Sigma,E}A \to A$ which sends $[t]_E$ to the interpretation of the term t under the trivial assignment $\mathrm{id}_A : A \to A$.[111] Let us verify that $[\![-]\!]_A$ is a $\mathcal{T}_{\Sigma,E}$-algebra. We need to show the following instances of (41) and (42) commutes.

[111] That is well-defined because $\mathbb{A}$ satisfies all the equations in $\mathfrak{Th}(E)$.

$$\begin{tikzcd} A \arrow[r, "\eta^{\Sigma,E}_A"] \arrow[dr, "\mathrm{id}_A"'] & \mathcal{T}_{\Sigma,E}A \arrow[d, "\llbracket - \rrbracket_A"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}A \arrow[r, "\mu^{\Sigma,E}_A"] \arrow[d, "\mathcal{T}_{\Sigma,E}\llbracket - \rrbracket_A"'] & \mathcal{T}_{\Sigma,E}A \arrow[d, "\llbracket - \rrbracket_A"] \\ \mathcal{T}_{\Sigma,E}A \arrow[r, "\llbracket - \rrbracket_A"'] & A \end{tikzcd}$$

The triangle commutes by definitions,[112] and the square commutes by the following diagram.

[112] We have $\llbracket \eta^{\Sigma,E}_A(a) \rrbracket_A = \llbracket [a]_E \rrbracket_A = \llbracket a \rrbracket_A = a$.

$$\begin{tikzcd} \mathcal{T}_\Sigma\mathcal{T}_{\Sigma,E}A \arrow[rrr, "\mathcal{T}_\Sigma\llbracket - \rrbracket_A"] \arrow[dr, "{[-]_E}"] \arrow[dd, "\llbracket - \rrbracket_{\mathbb{T}A}"'] & & & \mathcal{T}_\Sigma A \arrow[dl, "{[-]_E}"'] \arrow[dd, "\llbracket - \rrbracket_A"] \\ \text{(b)} & \mathcal{T}_{\Sigma,E}\mathcal{T}_{\Sigma,E}A \arrow[r, "\mathcal{T}_{\Sigma,E}\llbracket - \rrbracket_A"] \arrow[dl, "\mu^{\Sigma,E}_A"] \arrow[u, phantom, "\text{(a)}"] \arrow[dr, phantom, "\text{(c)}"] & \mathcal{T}_{\Sigma,E}A \arrow[dr, "\llbracket - \rrbracket_A"] & \text{(d)} \\ \mathcal{T}_{\Sigma,E}A \arrow[rrr, "\llbracket - \rrbracket_A"'] & & & A \end{tikzcd}$$

Since the outer rectangle commutes by Lemma 31, (a) commutes by naturality of $[-]_E$ (22), (b) commutes by definition of $\mu^{\Sigma,E}_A$ (26), and (d) commutes by (30), we can conclude that (c) commutes because $[-]_E$ is epic.

We also already explained in Footnote 20 that any homomorphism $h : \mathbb{A} \to \mathbb{B}$ makes the outer rectangle below commute.

$$\begin{tikzcd} \mathcal{T}_\Sigma A \arrow[rrr, "\mathcal{T}_\Sigma h"] \arrow[dr, "{[-]_E}"] \arrow[dd, "\llbracket - \rrbracket_A"'] & & & \mathcal{T}_\Sigma B \arrow[dl, "{[-]_E}"'] \arrow[dd, "\llbracket - \rrbracket_B"] \\ \text{(b)} & \mathcal{T}_{\Sigma,E}A \arrow[r, "\mathcal{T}_{\Sigma,E}h"] \arrow[dl, "\llbracket - \rrbracket_A"] \arrow[u, phantom, "\text{(a)}"] \arrow[dr, phantom, "\text{(c)}"] & \mathcal{T}_{\Sigma,E}A \arrow[dr, "\llbracket - \rrbracket_A"'] & \text{(d)} \\ A \arrow[rrr, "h"'] & & & B \end{tikzcd}$$

Since (a), (b), and (d) commute by naturality of $[-]_E$, (30), and (30) respectively, we conclude that (c) commutes again because $[-]_E$ is epic. This means h is a $\mathcal{T}_{\Sigma,E}$-homomorphism.

We obtain a functor[113] $P : \mathbf{Alg}(\Sigma, E) \to \mathbf{EM}(\mathcal{T}_{\Sigma,E})$ sending $\mathbb{A} = (A, \llbracket - \rrbracket_A)$ to $(A, \alpha_{\mathbb{A}})$ where $\alpha_{\mathbb{A}} = \llbracket - \rrbracket_A : \mathcal{T}_{\Sigma,E}A \to A$ (we give it a different name to make the sequel easier to follow).

[113] Checking functoriality is trivial because P acts like the identity on morphisms.

In the other direction, given an algebra $\alpha : \mathcal{T}_{\Sigma,E}A \to A$, we define an algebra $\mathbb{A}_\alpha$ with the interpretation of $\mathsf{op} : n \in \Sigma$ given by

$$\llbracket \mathsf{op} \rrbracket_\alpha(a_1, \dots, a_n) = \alpha[\mathsf{op}(a_1, \dots, a_n)]_E, \tag{46}$$

and we can prove by induction that $\llbracket t \rrbracket_\alpha = \alpha[t]_E$ for any Σ-term t over A (note that we use the $\mathcal{T}_{\Sigma,E}$-algebra properties of α).[114] Now, if $h : (A, \alpha) \to (B, \beta)$ is a $\mathcal{T}_{\Sigma,E}$-homomorphism, then h is a homomorphism from $\mathbb{A}_\alpha$ to $\mathbb{B}_\beta$ because for any $\mathsf{op} : n \in \Sigma$ and $a_1, \dots, a_n \in A$, we have

$$h(\llbracket \mathsf{op} \rrbracket_\alpha(a_1, \dots, a_n)) = h(\alpha[\mathsf{op}(a_1, \dots, a_n)]_E) \qquad \text{by (46)}$$

[114] For the base case, we have

$$\llbracket a \rrbracket_\alpha \overset{(7)}{=} a \overset{(41)}{=} \alpha[\eta^\Sigma_A(a)]_E = \alpha[a]_E.$$

For the inductive step, let $t = \mathsf{op}(t_1, \dots, t_n) \in \mathcal{T}_\Sigma A$:

$$\begin{aligned} \llbracket t \rrbracket_\alpha &= \llbracket \mathsf{op}(t_1, \dots, t_n) \rrbracket_\alpha \\ &= \llbracket \mathsf{op} \rrbracket_\alpha(\llbracket t_1 \rrbracket_\alpha, \dots, \llbracket t_n \rrbracket_\alpha) && (7) \\ &= \llbracket \mathsf{op} \rrbracket_\alpha(\alpha[t_1]_E, \dots, \alpha[t_n]_E) && \text{I.H.} \\ &= \alpha[\mathsf{op}(\alpha[t_1]_E, \dots, \alpha[t_n]_E)]_E && (46) \\ &= \alpha[\mathcal{T}_\Sigma\alpha(\mathsf{op}([t_1]_E, \dots, [t_n]_E))]_E && (3) \\ &= \alpha(\mathcal{T}_{\Sigma,E}\alpha[\mathsf{op}([t_1]_E, \dots, [t_n]_E)]_E) && (22) \\ &= \alpha(\mu^{\Sigma,E}_A[\mathsf{op}([t_1]_E, \dots, [t_n]_E)]_E) && (41) \\ &= \alpha[\mathsf{op}(t_1, \dots, t_n)]_E && (26) \\ &= \alpha[t]_E. \end{aligned}$$

$$\begin{aligned}
&= \beta(\mathcal{T}_{\Sigma,E} h[\mathsf{op}(a_1,\dots,a_n)]_E) && \text{by (43)}\\
&= \beta[\mathcal{T}_\Sigma h(\mathsf{op}(a_1,\dots,a_n))]_E && \text{by (22)}\\
&= \beta[\mathsf{op}(h(a_1),\dots,h(a_n))]_E && \text{by (3)}\\
&= [\![\mathsf{op}]\!]_\beta(h(a_1),\dots,h(a_n)). && \text{by (46)}
\end{aligned}$$

We obtain a functor $P^{-1} : \mathbf{EM}(\mathcal{T}_{\Sigma,E}) \to \mathbf{Alg}(\Sigma, E)$ sending (A,α) to $\mathbb{A}_\alpha$.

Finally, we need to check that P and P^{-1} are inverses to each other, i.e. that $\alpha_{\mathbb{A}_\alpha} = \alpha$ and $\mathbb{A}_{\alpha_{\mathbb{A}}} = \mathbb{A}$. For the former, $\alpha_{\mathbb{A}_\alpha}$ is defined to be the interpretation $[\![-]\!]_\alpha$ extended to terms modulo E, which we showed in Footnote 114 acts just like α. For the latter, we need to show that $[\![-]\!]_{\alpha_{\mathbb{A}}}$ and $[\![-]\!]_A$ coincide. Using Footnote 114 for the first equation and the definition of $\alpha_{\mathbb{A}}$ for the second, we have

$$[\![t]\!]_{\alpha_{\mathbb{A}}} = \alpha_{\mathbb{A}}[t]_E = [\![t]\!]_A.$$

Therefore, P and P^{-1} are inverses, thus $\mathbf{Alg}(\Sigma, E)$ and $\mathbf{EM}(\mathcal{T}_{\Sigma,E})$ are isomorphic.[115] □

[115] Observe that the functors P and P^{-1} commute with the forgetful functors because they do not change the carriers of the algebras.

Remark 60. This result (along with the construction of free (Σ, E)-algebras in Proposition 41) means that $U : \mathbf{Alg}(\Sigma, E) \to \mathbf{Set}$ is a (strictly) **monadic** functor. I decided not to define or discuss monadic functors in this document in order to have less prerequisites,[116] and because I like to exhibit the explicit isomorphism between categories of algebras. MacLane proves Proposition 59 using a monadicity theorem in [Mac71, §VI.8, Theorem 1].

[116] I became comfortable with monadicity relatively late into my PhD, so I think avoiding them keeps things more accessible. Speaking of accessibility, I am still not comfortable with accessible functors, so we will not work with them here.

What about algebras for other monads? Are they algebras for some signature Σ and equations E?

Example 61 (Maybe)**.** In **Set**, a $(- + \mathbf{1})$-algebra is a function $\alpha : A + \mathbf{1} \to A$ making the following diagrams commute.

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_A"'] & A+\mathbf{1} \arrow[d, "\alpha"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
A+\mathbf{1}+\mathbf{1} \arrow[r, "\mu_A"] \arrow[d, "\alpha+\mathbf{1}"'] & A+\mathbf{1} \arrow[d, "\alpha"] \\
A+\mathbf{1} \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

Reminding ourselves that η_A is the inclusion in the left component, the triangle commuting enforces α to act like the identity function on all of A. We can also write $\alpha = [\mathrm{id}_A, \alpha(*)]$.[117] The square commuting adds no constraint. Thus, an algebra for the maybe monad on **Set** is just a set with a distinguished point. Let $h : A \to B$ be a function, commutativity of (47) is equivalent to $h(\alpha(*)) = \beta(*)$. Hence, a $(- + \mathbf{1})$-homomorphism is a function that preserves the distinguished point.

[117] We identify the element $\alpha(*) \in A$ with the function $\alpha(*) : \mathbf{1} \to A$ picking out that element.

$$\begin{tikzcd}
A+\mathbf{1} \arrow[r, "h+\mathbf{1}"] \arrow[d, "{[\mathrm{id}_A,\alpha(*)]}"'] & B+\mathbf{1} \arrow[d, "{[\mathrm{id}_B,\beta(*)]}"] \\
A \arrow[r, "h"'] & B
\end{tikzcd} \tag{47}$$

Seeing the distinguished point of a $(- + \mathbf{1})$-algebra as the interpretation of a constant, we recognize that the category $\mathbf{EM}(- + \mathbf{1})$ is isomorphic to the category $\mathbf{Alg}(\Sigma)$ where $\Sigma = \{\mathsf{p} : 0\}$ contains a single constant.[118]

[118] Notice, again, that this isomorphism would commute with the forgetful functors to **Set** because the carriers are unchanged.

Another option to recognize $\mathbf{EM}(- + \mathbf{1})$ as a category of algebras is via monad isomorphisms.

Definition 62 (Monad morphism)**.** Let (M, η^M, μ^M) and (N, η^N, μ^N) be two monads on $\mathbf{C}$. A **monad morphism** from M to N is a natural transformation $\rho : M \Rightarrow N$

making (48) and (49) commute.[119]

$$\begin{tikzcd} \mathrm{id}_{\mathbf{C}} \arrow[d, "\eta^M"'] \arrow[dr, "\eta^N"] & \\ M \arrow[r, "\rho"'] & N \end{tikzcd} \qquad (48) \qquad\qquad \begin{tikzcd} MM \arrow[r, "\rho\diamond\rho"] \arrow[d, "\mu^M"'] & NN \arrow[d, "\mu^N"] \\ M \arrow[r, "\rho"'] & N \end{tikzcd} \qquad (49)$$

[119] Recall that $\rho \diamond \rho$ denotes the horizontal composition of ρ with itself, i.e.

$$\rho \diamond \rho = \rho N \cdot M\rho = N\rho \cdot \rho M.$$

As expected ρ is called a monad isomorphism when there is a monad morphism $\rho^{-1} : N \Rightarrow M$ satisfying $\rho \cdot \rho^{-1} = \mathbb{1}_N$ and $\rho^{-1} \cdot \rho = \mathbb{1}_M$. In fact, it is enough that all the components of ρ are isomorphisms in $\mathbf{C}$ to guarantee ρ is a monad isomorphism.[120]

[120] One checks that natural isomorphisms are precisely the natural transformations whose components are all isomorphisms, and that the inverse of a monad morphism is a monad morphism.

Example 63. For the signature $\Sigma = \{\mathsf{p}:0\}$, the term monad $\mathcal{T}_\Sigma$ is isomorphic to $- + \mathbf{1}$. Indeed, recall that a Σ-term over A is either an element of A or p, this yields a bijection $\rho_A : \mathcal{T}_\Sigma A \to A + \mathbf{1}$ that sends any element of A to itself and p to $* \in \mathbf{1}$. To verify that ρ is a monad morphism, we check these diagrams commute.[121]

$$\begin{tikzcd} \mathcal{T}_\Sigma A \arrow[r, "\rho_A"] \arrow[d, "\mathcal{T}_\Sigma f"'] & A+\mathbf{1} \arrow[d, "f+\mathbf{1}"] \\ \mathcal{T}_\Sigma B \arrow[r, "\rho_B"'] & B+\mathbf{1} \end{tikzcd} \ (50) \quad \begin{tikzcd} A \arrow[d, "\eta^\Sigma_A"'] \arrow[dr, "\eta_A"] & \\ \mathcal{T}_\Sigma A \arrow[r, "\rho_A"'] & A+\mathbf{1} \end{tikzcd} \ (51) \quad \begin{tikzcd} \mathcal{T}_\Sigma\mathcal{T}_\Sigma A \arrow[r, "\rho_{\mathcal{T}_\Sigma A}\circ(\rho_A+\mathbf{1})"] \arrow[d, "\mu^\Sigma_A"'] & A+\mathbf{1}+\mathbf{1} \arrow[d, "\mu_A"] \\ \mathcal{T}_\Sigma A \arrow[r, "\rho_A"'] & A+\mathbf{1} \end{tikzcd} \ (52)$$

[121] All of them commute essentially because ρ_A and both multiplications act like the identity on A.

We obtain a monad isomorphism between the maybe monad and the term monad for the signature $\Sigma = \{\mathsf{p}:0\}$. We can recover the isomorphism between the categories of algebras from Example 61 with the following result.

Proposition 64. *If $\rho : M \Rightarrow N$ is a monad morphism, then there is a functor $-\rho$: $\mathbf{EM}(N) \to \mathbf{EM}(M)$. If ρ is a monad isomorphism, then $-\rho$ is also an isomorphism.*

Proof. Given an N-algebra $\alpha : NA \to A$, we show that $\alpha \circ \rho_A : MA \to A$ is an M-algebra by paving the following diagrams.

$$\begin{tikzcd} A \arrow[r, "\eta^M_A"] \arrow[dr, "\eta^N_A"] \arrow[ddr, "\mathrm{id}_A"'] & MA \arrow[d, "\rho_A"] \\ & NA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} MMA \arrow[rr, "\mu^M_A"] \arrow[d, "M\rho_A"'] & \text{(c)} & MA \arrow[d, "\rho_A"] \\ MNA \arrow[r, "\rho_{NA}"] \arrow[d, "M\alpha"'] & NNA \arrow[r, "\mu^N_A"] \arrow[d, "N\alpha"] & NA \arrow[d, "\alpha"] \\ MA \arrow[r, "\rho_A"'] & NA \arrow[r, "\alpha"'] & A \end{tikzcd} \qquad (53)$$

Showing (53) commutes:

(a) By (48).

(b) By (41) for $\alpha : NA \to A$.

(c) By (49), noting that $(\rho \diamond \rho)_A = \rho_{NA} \circ M\rho_A$.

(d) Naturality of ρ.

(e) By (42) for $\alpha : NA \to A$.

Moreover, if $h : A \to B$ is an N-homomorphism from α to β, then it is also a M-homomorphism from $\alpha \circ \rho_A$ to $\beta \circ \rho_B$ by the paving below.[122]

$$\begin{tikzcd} MA \arrow[r, "Mh"] \arrow[d, "\rho_A"'] & MB \arrow[d, "\rho_B"] \\ NA \arrow[r, "Nh"'] \arrow[d, "\alpha"'] & NB \arrow[d, "\beta"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

[122] The top square commutes by naturality of ρ and the bottom square commutes because h is an N-homomorphism (43).

We obtain a functor $-\rho : \mathbf{EM}(N) \to \mathbf{EM}(M)$ taking an algebra (A, α) to $(A, \alpha \circ \rho_A)$ and a homomorphism $h : (A, \alpha) \to (B, \beta)$ to $h : (A, \alpha \circ \rho_A) \to (B, \beta \circ \rho_B)$.

Furthermore, it is easy to see that $-\rho = \mathrm{id}_{\mathbf{EM}(M)}$ when $\rho = \mathbb{1}_M$ is the identity monad morphism, and that for any other monad morphism $\rho' : N \Rightarrow L$, $-(\rho' \cdot \rho) = (-\rho) \circ (-\rho')$.[123] Thus, when ρ is a monad isomorphism with inverse ρ^{-1}, $-\rho^{-1}$ is the inverse of $-\rho$, so $-\rho$ is an isomorphism. $\square$

[123] In other words, the assignments $M \mapsto \mathbf{EM}(M)$ and $\rho \mapsto -\rho$ becomes a functor from the category of monads on $\mathbf{C}$ and monad morphisms to the category of categories (ignoring size issues).

With the monad isomorphism $\mathcal{T}_\Sigma \cong - + \mathbf{1}$ of Example 63, we obtain an isomorphism $\mathbf{EM}(- + \mathbf{1}) \cong \mathbf{EM}(\mathcal{T}_\Sigma)$, and composing it with the isomorphism of Proposition 59 $\mathbf{EM}(\mathcal{T}_\Sigma) \cong \mathbf{Alg}(\Sigma)$ (instantiating $E = \varnothing$), we get back the result from Example 61 that algebras for the maybe monad are the same thing as algebras for the signature with a single constant.

In general, we now know that $\mathcal{T}_{\Sigma,E} \cong M$ implies $\mathbf{EM}(M) \cong \mathbf{Alg}(\Sigma, E)$, but constructing a monad isomorphism (and showing it is one) is not always the easiest thing to do.[124] There is a converse implication, but it requires a restriction to isomorphisms of categories that commute with the forgetful functors to $\mathbf{Set}$. Anyways, that is a mild condition we foreshadowed.

[124] For instance, the isomorphism of categories of algebras in Example 61 is definitely clearer than the isomorphism of monads in Example 63.

Proposition 65. *If $P : \mathbf{EM}(N) \to \mathbf{EM}(M)$ is a functor such that $U^M \circ P = U^N$, then there is a monad morphism $\rho : M \to N$. If P is an isomorphism, then so is ρ.*

Proof. Quick corollary of [BW05, Chapter 3, Theorem 6.3]. $\square$

This motivates the following definition which states that a monad M is presented by (Σ, E) when it is isomorphic to the term monad $\mathcal{T}_{\Sigma,E}$ or, thanks to Proposition 65 and Proposition 59, when M-algebras on A and (Σ, E)-algebras on A are identified.

Definition 66 (**Set** presentation)**.** Let M be a monad on **Set**, an **algebraic presentation** of M is signature Σ and a class of equations E along with a monad isomorphism $\rho : \mathcal{T}_{\Sigma,E} \cong M$. We also say M is presented by (Σ, E).

We chose to state the definition with the monad isomorphism it makes some arguments in §3.4 quicker. Showing that a monad is presented by (Σ, E) can be done in many ways that are equivalent to building a monad isomorphism.[125]

[125] We already gave one with Proposition 65, and you can also read some great discussions in Remark 3.6 and §4.2 in [BSV22].

We have proven in Example 63 that $\Sigma = \{\mathsf{p} : 0\}$ and $E = \varnothing$ is an algebraic presentation for the maybe monad on **Set**. Here is a couple of additional examples.

Example 67 (Powerset)**.** The powerset monad $\mathcal{P}_{\mathrm{ne}}$ is presented by the theory of **semilattices** $(\Sigma_{\mathbf{S}}, E_{\mathbf{S}})$,[126] where $\Sigma_{\mathbf{S}} = \{\oplus : 2\}$ and $E_{\mathbf{S}}$ contains the following equations stating that $\oplus$ is idempotent, commutative and associative respectively.

[126] Usually, when we say "theory of X", we mean that Xs are the algebras for that theory. For instance, semilattices are the $(\Sigma_{\mathbf{S}}, E_{\mathbf{S}})$-algebras. After some unrolling, we get the more common definition of a semilattice, that is, a set with a binary operation that is idempotent, commutative, and associative.

$$x \vdash x = x \oplus x \qquad x, y \vdash x \oplus y = y \oplus x \qquad x, y, z \vdash x \oplus (y \oplus z) = (x \oplus y) \oplus z$$

This means there is a monad isomorphism $\mathcal{T}_{\Sigma_{\mathbf{S}}, E_{\mathbf{S}}} \cong \mathcal{P}_{\mathrm{ne}}$.

Another thing we obtain from this isomorphism is that for any set X, interpreting $\oplus$ as union on $\mathcal{P}_{\mathrm{ne}} X$ (i.e. $(S, T) \mapsto S \cup T$) yields the free semilattice on X.[127]

[127] It is relatively easy to show that union is idempotent, commutative, and associative, freeness is more difficult but follows from the algebraic presentation, and the fact that $(\mathcal{P}_{\mathrm{ne}} X, \mu_X)$ is the free $\mathcal{P}_{\mathrm{ne}}$-algebra (recall Example 58).

Example 68 (Distributions)**.** The distribution monad $\mathcal{D}$ is presented by the theory of **convex algebras** $(\Sigma_{\mathbf{CA}}, E_{\mathbf{CA}})$ where $\Sigma_{\mathbf{CA}} = \{+_p : 2 \mid p \in (0,1)\}$ and $E_{\mathbf{CA}}$ contains the following equations for all $p, q \in (0,1)$.

$$x \vdash x = x +_p x \qquad x, y \vdash x +_p y = y +_{1-p} x$$
$$x, y, z \vdash (x +_p y) +_q z = x +_{pq} + (y +_{\frac{p(1-q)}{1-pq}} z)$$

The free convex algebra on X can now be seen as $\mathcal{D}X$ with $+_p$ interpreted as the usual convex combination, that is,[128]

$$[\![\varphi +_p \psi]\!]_{\mathcal{D}X} = p\varphi + (1-p)\psi = (x \mapsto p\varphi(x) + (1-p)\psi(x)). \tag{54}$$

[128] For later, we will write $\overline{p}$ for $1 - p$.

Remark 69*.* Not all monads on **Set** have an algebraic presentation.[129] The monads that can be presented by a signature with finitary operation symbols are aptly called **finitary monads**. They can be characterized as the monads whose underlying functor preserve limits of a certain shape and size, see e.g. [Bor94, Proposition 4.6.2].

[129] For example, the *full* powerset monad does not, although it still has an algebraic flavor as its algebras are in correspondence with complete sup-lattices, see e.g. [Bor94, Proposition 4.6.5].

In Chapter 3, we will need to relate monads on different categories, we give some background on that here.

Definition 70 (Monad functor)**.** Let (M, η^M, μ^M) be a monad on $\mathbf{C}$, and (T, η^T, μ^T) be a monad on $\mathbf{D}$. A **monad functor** from M to T is a pair (F, λ) comprising a functor $F : \mathbf{C} \to \mathbf{D}$, and a natural transformation $\lambda : TF \Rightarrow FM$ making (55) and (56) commute.[130]

[130] Note the similarities with Definition 62, monad functors generalize monad morphisms to monads on different base categories.

$$\begin{tikzcd} F \arrow[d, "\eta^T F"'] \arrow[dr, "F\eta^M"] & \\ TF \arrow[r, "\lambda"'] & FM \end{tikzcd} \tag{55}$$

$$\begin{tikzcd} TTF \arrow[r, "T\lambda"] \arrow[d, "\mu^T F"'] & TFM \arrow[r, "\lambda M"] & FMM \arrow[d, "F\mu^M"] \\ TF \arrow[rr, "\lambda"'] & & FM \end{tikzcd} \tag{56}$$

Proposition 71. *If $(F, \lambda) : M \to T$ is a monad functor, then there is a functor $F - \circ\lambda$: $\mathbf{EM}(M) \to \mathbf{EM}(T)$ sending an M-algebra $\alpha : MA \to A$ to $F\alpha \circ \lambda_A : TFA \to A$, and an M-homomorphism $h : A \to B$ to $Fh : FA \to FB$.*[131]

[131] By definition, the functor $F - \circ\lambda$ lifts F along the forgetful functors, namely, it makes (57) commute.

$$\begin{tikzcd} \mathbf{EM}(M) \arrow[r, "F-\circ\lambda"] \arrow[d, "U^M"'] & \mathbf{EM}(T) \arrow[d, "U^T"] \\ \mathbf{C} \arrow[r, "F"'] & \mathbf{D} \end{tikzcd} \tag{57}$$

Proof. We need to show that $F\alpha \circ \lambda$ is a T-algebra whenever α is an M-algebra. We pave the following diagrams showing (41) and (42) commute respectively.

$$\begin{tikzcd} FA \arrow[r, "\eta^T_{FA}"] \arrow[dr, "F\eta^M_A"] \arrow[ddr, "\mathrm{id}_{FA}"'] & TFA \arrow[d, "\lambda_A"] \arrow[dl, phantom, "\text{(a)}"] \\ & FMA \arrow[d, "F\alpha"] \arrow[dl, phantom, "\text{(b)}"] \\ & FA \end{tikzcd} \qquad \begin{tikzcd} TTFA \arrow[rr, "\mu^T_{FA}"] \arrow[d, "T\lambda_A"'] & \text{(c)} & TFA \arrow[d, "\lambda_A"] \\ TFMA \arrow[r, "\lambda_{MA}"] \arrow[d, "TF\alpha"'] & FMMA \arrow[r, "F\mu^M_A"] \arrow[d, "FM\alpha"] \arrow[l, phantom, "\text{(d)}", shift right=6] & FMA \arrow[d, "F\alpha"] \arrow[l, phantom, "\text{(e)}", shift right=6] \\ TFA \arrow[r, "\lambda_A"'] & FMA \arrow[r, "F\alpha"'] & FA \end{tikzcd} \tag{58}$$

Showing (58) commutes:

(a) By (55).

(b) Apply F to (41).

(c) By (56).

(d) Naturality of λ.

(e) Apply F to (42).

Next, we need to show that when $h : A \to B$ is an M-homomorphism from α to β, then Fh is a T-homomorphism from $F\alpha \circ \lambda_A$ to $F\alpha \circ \lambda_B$. We pave the following diagram where (a) commutes by naturality of λ and (b) by applying F to (43).

$$\begin{tikzcd}
TFA \arrow[r, "TFh"] \arrow[d, "\lambda_A"'] \arrow[dr, phantom, "\text{(a)}"] & TFB \arrow[d, "\lambda_B"] \\
FMA \arrow[r, "FMh"] \arrow[d, "F\alpha"'] \arrow[dr, phantom, "\text{(b)}"] & FMB \arrow[d, "F\beta"] \\
FA \arrow[r, "Fh"'] & FB
\end{tikzcd}$$

□

There are two special cases of monad functors. When M and T are on the same category $\mathbf{C}$ and $F = \mathrm{id}_{\mathbf{C}}$, a monad functor is just a monad morphism,[132] and then the proof above reduces to the proof of Proposition 64. When λ_A is an identity morphism for every A, i.e. $TF = FM$, we say that M is a monad lifting of T along F. That notion is central to §3.4, where we redefine it in a more specific setting.

[132] Sometimes, authors introduce monad functors with the name monad morphism, and take our notion of monad morphism as a particular instance. Some authors also use the name monad map for either notion.

Our goal for the next two chapters is to make all the results here more general by considering carriers to be generalized metric spaces, i.e. sets with a notion of distance. In Chapter 2 we define what we mean by distance, and in Chapter 3, we define quantitative algebras, quantitative equational logic, and quantitative algebraic presentations analogously to the definitions above.

2 Generalized Metric Spaces

The Homeless Wanderer

Emahoy Tsegué-Maryam Guèbrou

For a comprehensive introduction to the concepts and themes explored in this chapter, please refer to §0.2. Here, we only give a brief overview.

In this chapter, we give our definition of generalized metric spaces which is different from the many (pairwise different) definitions already in the literature.[133] Once again, we take our time with this material in preparation for the next chapter, introducing many examples and disseminating some insights along the way. While the content of Chapter 1 can safely be skipped before reading the current chapter, our main point here is the definition of quantitative equation (Definition 93) as an answer to the question "How do we impose constraints on distances with the familiar syntax of equations?", thus it makes sense to be comfortable with equational reasoning before reading what follows.

[133] e.g. [BvBR98, Bra00]

Outline: In §2.1, we define complete lattices and relations valued in a complete lattice, we also give an equivalent definition that justifies the syntax of quantitative equations. In §2.2, we defined quantitative equations and the categories of generalized metric spaces which are defined by collections of quantitative equations. In §2.3, we study the properties that all categories of generalized metric spaces have.

2.1 L-Spaces

Chapter 1 is titled *Universal Algebra* and Chapter 3 is titled *Universal Quantitative Algebra*. In order to go from the former to the latter, we will explain what we mean by *quantitative*. In the original paper on quantitative algebras [MPP16], and in many other works on quantitative program semantics,[134] the **quantities** considered are, more often than not, real numbers. In [MSV22, MSV23], we worked with quantities inside $[0,1]$. In this document, we will abstract away from real numbers, thinking of quantities as things you can compare and say whether one is bigger or smaller than another. You can do that with real numbers thanks to the usual ordering $\leq$, but it has a crucial property that we exploit, it is *complete* in the (informal) sense that you can always find the smallest quantity of a set of real numbers. We say it is a complete lattice.[135]

[134] e.g. [Kwi07, vBW01, KyKK+21, ZK22].

[135] Small caveat: we need to add ∞ to the real numbers or work with an upper bound (see Example 74).

Definition 72 (Complete lattice)**.** A **complete lattice** is a partially ordered set $(\mathsf{L}, \leq$

)[136] where all subsets $S \subseteq \mathsf{L}$ have an infimum and a supremum denoted by $\inf S$ and $\sup S$ respectively. In particular, L has a **bottom element** $\bot = \sup \varnothing$ and a **top element** $\top = \inf \varnothing$ that satisfy $\bot \leq \varepsilon \leq \top$ for all $\varepsilon \in \mathsf{L}$. We use L to refer to the lattice and its underlying set, and we call its elements **quantities**.

Let us describe two central (for this thesis) examples of complete lattices.

Example 73 (Unit interval)**.** The **unit interval** $[0, 1]$ is the set of real numbers between 0 and 1. It is a poset with the usual order $\leq$ ("less than or equal") on numbers. It is usually an axiom in the definition of $\mathbb{R}$[137] that all non-empty bounded subsets of real numbers have an infimum and a supremum. Since all subsets of $[0, 1]$ are bounded (by 0 and 1), we conclude that $([0, 1], \leq)$ is a complete lattice with $\bot = 0$ and $\top = 1$.

Later in this section, we will see elements of $[0, 1]$ as distances between points of some space. It would make sense, then, to extend the interval to contain values bigger than 1. Still because a complete lattice must have a top element there must be a number above all others. We could either stop at some arbitrary $0 \leq B \in \mathbb{R}$ and consider $[0, B]$, or we can consider ∞ to be a number as done below.[138]

Example 74 (Extended interval)**.** Similarly to the unit interval, the **extended interval** is the set $[0, \infty]$ of positive real numbers extended with ∞, and it is a poset after asserting $\varepsilon \leq \infty$ for all $\varepsilon \in [0, \infty]$. It is also a complete lattice because non-empty bounded subsets of $[0, \infty)$ still have an infimum and supremum, and if a subset is not bounded above or contains ∞, then its supremum is ∞. We find that 0 is bottom and ∞ is top.

It is the prevailing custom to consider distances valued in the extended interval.[139] However, in our research, we preferred to use the unit interval, and in almost all cases, there is no difference. Since $[0, 1]$ and $[0, \infty]$ are isomorphic as complete lattices,[140] one might think that switching between $[0, 1]$ and $[0, \infty]$ is entirely benign. That is not true because in practice $[0, 1]$ and $[0, \infty]$ are not just seen as complete lattices. For instance, we are often interested in adding quantities together in $[0, 1]$ or $[0, \infty]$ or doing a convex combination.

Remark 75*.* The first two examples are both **quantales** [HST14, §II.1.10], informally, complete lattices where quantities can be added together in a way that preserves the order and the "smallest" quantities. It is also quite common in the literature on quantitative programming semantics to generalize from real numbers to elements of a quantale.[141] Since none of the results we establish require dealing with addition, we will work at the level of generality complete lattices (absolutely no difficulty arises from this abstraction), even though many of the following examples are quantales.

There are many other interesting complete lattices, although (unfortunately) they are more rarely viewed as possible places to value distances.

Example 76 (Booleans)**.** The **Boolean lattice** B is the complete lattice containing only two elements, bottom and top. Its name comes from the interpretation of $\bot$ as

136 i.e. L is a set and $\leq \subseteq \mathsf{L} \times \mathsf{L}$ is a binary relation on L that is reflexive, transitive and antisymmetric.

137 Or possibly a theorem proven after constructing $\mathbb{R}$.

138 If one needs negative distances, it is also possible to work with any interval $[A, B]$ with $A \leq B \in \mathbb{R}$, or even $[-\infty, \infty]$. We will stick to $[0, 1]$ and $[0, \infty]$.

139 In fact, $[0, \infty]$ is also famous under the name *Lawvere quantale* because of Lawvere's seminal paper [Law02]. In that work, he used the quantale structure on $[0, \infty]$ to give a categorical definition very close to that of a metric.

140 Take the mapping $x \mapsto \frac{1}{1-x} - 1$ from $[0, 1]$ to $[0, \infty]$ with $\frac{1}{0} = \infty$. It is monotone and preserves infimums.

141 e.g. [DGY19, GP21, GD23, FSW+23].

a `false` value and $\top$ as a `true` value which makes the infimum act like an `AND` and the supremum like an `OR`.

Example 77 (Extended natural numbers)**.** The set $\mathbb{N}_\infty$ of natural numbers extended with ∞ is a sublattice of $[0,\infty]$.[142] Indeed, it is a poset with the usual order and the infimum and supremum of a subset of natural numbers is either itself a natural number or ∞ (when the subset is empty or unbounded respectively).

[142] As expected, a **sublattice** of $(\mathsf{L},\leq)$ is a set $\mathsf{S} \subseteq \mathsf{L}$ closed under taking infimums and supremums. Note that the top and bottom of S need not coincide with those of L. For instance $[0,1]$ is a sublattice of $[0,\infty]$, but $\top = 1$ in the former and $\top = \infty$ in the latter.

Example 78 (Powerset lattice)**.** For any set X, we denote the powerset of X by $\mathcal{P}(X)$. The inclusion relation $\subseteq$ between subsets of X makes $\mathcal{P}(X)$ a poset. The infimum of a family of subsets $S_i \subseteq X$ is the intersection $\cap_{i\in I} S_i$, and its supremum is the union $\cup_{i\in I} S_i$. Hence, $\mathcal{P}(X)$ is a complete lattice. The bottom element is $\emptyset$ and the top element is X.

It is well-known that subsets of X correspond to functions $X \to \{\bot,\top\}$.[143] Endowing the two-element set with the complete lattice structure of B is what yields the complete lattice structure on $\mathcal{P}(X)$. The following example generalizes this construction.

[143] A subset $S \subseteq X$ is sent to the characteristic function χ_S, and a function $f : X \to \mathsf{B}$ is sent to $f^{-1}(\top)$. We say that $\{\bot,\top\}$ is the subobject classifier of **Set**.

Example 79 (Function space)**.** Given a complete lattice $(\mathsf{L},\leq)$, for any set X, we denote the set of functions from X to L by L^X. The pointwise order on functions defined by

$$f \leq_* g \iff \forall x \in X, f(x) \leq g(x)$$

is a partial order on L^X. The infimums and supremums of families of functions are also computed pointwise. Namely, given $\{f_i : X \to \mathsf{L}\}_{i\in I}$, for all $x \in X$:

$$(\inf_{i\in I} f_i)(x) = \inf_{i\in I} f_i(x) \quad \text{and} \quad (\sup_{i\in I} f_i)(x) = \sup_{i\in I} f_i(x).$$

This makes L^X a complete lattice. The bottom element is the function that is constant at $\bot$ and the top element is the function that is constant at $\top$.

Taking $\mathsf{L} = \mathsf{B}$, we find that $\mathcal{P}(X)$ and B^X are isomorphic as complete lattices under the usual correspondence. Namely, pointwise infimums and supremums become intersections and unions respectively. For example, if $\chi_S, \chi_T : X \to \mathsf{B}$ are the characteristic functions of $S, T \subseteq X$, then

$$\begin{aligned} \inf\{\chi_S,\chi_T\}(x) = \top &\Leftrightarrow \chi_S(x) = \chi_T(x) = \top \\ &\Leftrightarrow x \in S \text{ and } x \in T \\ &\Leftrightarrow x \in S \cap T. \end{aligned}$$

As a special case of function spaces, it is easy to show that when X is a set with two elements, L^X is isomorphic (as complete lattices) to the product $\mathsf{L} \times \mathsf{L}$.

Example 80 (Product)**.** Let $(\mathsf{L},\leq_{\mathsf{L}})$ and $(\mathsf{K},\leq_{\mathsf{K}})$ be two complete lattices. Their **product** is the poset $(\mathsf{L}\times\mathsf{K},\leq_{\mathsf{L}\times\mathsf{K}})$ on the Cartesian product of L and K with the order defined by

$$(\varepsilon,\delta) \leq_{\mathsf{L}\times\mathsf{K}} (\varepsilon',\delta') \iff \varepsilon \leq_{\mathsf{L}} \varepsilon' \text{ and } \delta \leq_{\mathsf{K}} \delta'. \tag{59}$$

It is a complete lattice where the infimums and supremums are computed coordinatewise, namely, for any $S \subseteq \mathsf{L}\times\mathsf{K}$,[144]

$$\begin{aligned} \inf S &= (\inf\{\pi_{\mathsf{L}}(c) \mid c \in S\}, \inf\{\pi_{\mathsf{K}}(c) \mid c \in S\}) \text{ and} \\ \sup S &= (\sup\{\pi_{\mathsf{L}}(c) \mid c \in S\}, \sup\{\pi_{\mathsf{K}}(c) \mid c \in S\}). \end{aligned}$$

[144] Where π_{L} and π_{K} are the projections from $\mathsf{L}\times\mathsf{K}$ to L and K respectively.

The bottom (resp. top) element of $\mathsf{L}\times\mathsf{K}$ is the pairing of the bottom (resp. top) elements of L and K. i.e. $\bot_{\mathsf{L}\times\mathsf{K}} = (\bot_{\mathsf{L}},\bot_{\mathsf{K}})$ and $\top_{\mathsf{L}\times\mathsf{K}} = (\top_{\mathsf{L}},\top_{\mathsf{K}})$.

The following example is also based on functions, and it appears in several works on generalized notions of distances, e.g. [Fla97, HR13].

Example 81 (CDF)**.** A **cumulative distribution function**[145] (or CDF for short) is a function $f : [0,\infty] \to [0,1]$ that is monotone (i.e. $\varepsilon \leq \delta \implies f(\varepsilon) \leq f(\delta)$) and satisfies

$$f(\delta) = \sup\{f(\varepsilon) \mid \varepsilon < \delta\}. \tag{60}$$

Intuitively, (60) says that f cannot abruptly change value at some $x \in [0,\infty]$, but it can do that "after" some x.[146] For instance, out of the two functions below, only $f_{>1}$ is a CDF.

$$f_{\geq 1} = x \mapsto \begin{cases} 0 & x < 1 \\ 1 & x \geq 1 \end{cases} \qquad f_{>1} = x \mapsto \begin{cases} 0 & x \leq 1 \\ 1 & x > 1 \end{cases}$$

We denote by $\mathsf{CDF}([0,\infty])$ the subset of $[0,1]^{[0,\infty]}$ containing all CDFs, it inherits a poset structure (pointwise ordering), and we can show it is a complete lattice.[147]

Let $\{f_i : [0,\infty] \to [0,1]\}_{i \in I}$ be a family of CDFs. We will show the pointwise supremum $\sup_{i \in I} f_i$ is a CDF, and that is enough since having all supremums implies having all infimums [DP02, Theorem 2.31].

- If $\varepsilon \leq \delta$, since all f_is are monotone, we have $f_i(\varepsilon) \leq f_i(\delta)$ for all $i \in I$ which implies
$$(\sup_{i\in I} f_i)(\varepsilon) = \sup_{i \in I} f_i(\varepsilon) \leq \sup_{i \in I} f_i(\delta) = (\sup_{i \in I} f_i)(\delta).$$
- For any $\delta \in [0,\infty]$, we have
$$(\sup_{i\in I} f_i)(\delta) = \sup_{i\in I} f_i(\delta) = \sup_{i \in I}\sup_{\varepsilon<\delta} f_i(\varepsilon) = \sup_{\varepsilon<\delta}\sup_{i\in I} f_i(\varepsilon) = \sup_{\varepsilon<\delta}(\sup_{i\in I} f_i)(\varepsilon).$$

Nothing prevents us from defining CDFs on other domains, and we will write $\mathsf{CDF}(\mathsf{L})$ for the complete lattice of functions $\mathsf{L} \to [0,1]$ that are monotone and satisfy (60).

Definition 82 (L-space)**.** Given a complete lattice L and a set A, an L**-relation** on A is a function $d : A \times A \to \mathsf{L}$. We call the pair (A, d) an L**-space**, and A its **carrier** or **underlying** set. We will also use a single bold-face symbol $\mathbf{A}$ to refer to an L-space with underlying set A and L-relation $d_{\mathbf{A}}$.[148]

A **nonexpansive** map from $\mathbf{A}$ to $\mathbf{B}$ is a function $f : A \to B$ between the underlying sets of $\mathbf{A}$ and $\mathbf{B}$ that satisfies

$$\forall x, x' \in A, \quad d_{\mathbf{B}}(f(x), f(x')) \leq d_{\mathbf{A}}(x, x'). \tag{61}$$

The identity maps $\mathrm{id}_A : A \to A$ and the composition of two nonexpansive maps are always nonexpansive[149], therefore we have a category whose objects are L-spaces and morphisms are nonexpansive maps. We denote it by $\mathsf{L}\mathbf{Spa}$.

This category is concrete over $\mathbf{Set}$ with the forgetful functor $U : \mathsf{L}\mathbf{Spa} \to \mathbf{Set}$ which sends an L-space $\mathbf{A}$ to its carrier and a morphism to the underlying function between carriers.

[145] Although cumulative *sub*distribution function might be preferred.

[146] This property is often called *right-continuity*.

[147] Note however that $\mathsf{CDF}([0,\infty])$ is not a sublattice of $[0,1]^{[0,\infty]}$ because the infimums are not always taken pointwise. For instance, given $0 < n \in \mathbb{N}$, define f_n by (see them on Desmos)

$$f_n(x) = \begin{cases} 0 & x \leq 1 - \frac{1}{n} \\ nx & 1 - \frac{1}{n} < x < 1 \\ 1 & 1 \leq x \end{cases}.$$

The pointwise infimum of $\{f_n\}_{n\in\mathbb{N}}$ clearly sends everything below 1 to 0 and everything above and including 1 to 1, so it does not satisfy $f(1) = \sup_{\varepsilon<1} f(\varepsilon)$. We can find the infimum with the general formula that defines infimums in terms of supremums:

$$\inf_{n>0} f_n = \sup\{f \in \mathsf{CDF}([0,\infty]) \mid \forall n > 0, f \leq_* f_n\}.$$

We find that $\inf_{n>0} f_n = f_{>1}$.

[148] We will often switch between referring to spaces with $\mathbf{A}$ or $(A, d_{\mathbf{A}})$, and we will try to match the symbol for the space and the one for its underlying set only modifying the former with `mathbf`.

[149] Fix three L-spaces $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ with two nonexpansive maps $f : A \to B$ and $g : B \to C$, we have by nonexpansiveness of g then f:

$$\begin{aligned} d_{\mathbf{C}}(gf(a), gf(a')) &\leq d_{\mathbf{B}}(f(a), f(a')) \\ &\leq d_{\mathbf{A}}(a, a'). \end{aligned}$$

Remark 83. In the sequel, we will not distinguish between the morphism $f : \mathbf{A} \to \mathbf{B}$ and the underlying function $f : A \to B$. Although, we may write Uf for the latter, when disambiguation is necessary.

Instantiating L for different complete lattices, we can get a feel for what the categories $\mathsf{L}\mathbf{Spa}$ look like. We also give concrete examples of L-spaces.

Examples 84 (Binary relations)**.** When $\mathsf{L} = \mathsf{B}$, a function $d : A \times A \to \mathsf{B}$ is the same thing as a subset of $A \times A$, which is the same thing as a binary relation on A.[150] Then, a B-space is a set equipped with a binary relation and we choose to have, as a convention, $d(a, a') = \bot$ when a and a' are related and $d(a, a') = \top$ when they are not.[151] A nonexpansive map from $\mathbf{A}$ to $\mathbf{B}$ is a function $f : A \to B$ such that for any $a, a' \in A$, $f(a)$ and $f(a')$ are related when a and a' are. When a and a' are not related, $f(a)$ and $f(a')$ might still be related.[152] The category $\mathsf{B}\mathbf{Spa}$ is well-known under different names, **EndoRel** in [Vig23], **Rel** in [AHS06] (although that name is more commonly used for the category where relations are morphisms) and 2**Rel** in my book. Here are a couple of fun examples of B-spaces:

[150] Hence, the choice of terminology L-relation.

[151] This convention might look backwards, but it makes sense with the morphisms.

[152] Note that this interpretation of nonexpansiveness depends on our just chosen convention. Swapping the meaning of $d(a, a') = \top$ and $d(a, a') = \bot$ is the same thing as taking the opposite order on B (i.e. $\top \leq \bot$), namely, morphisms become functions $f : A \to B$ such that for any $a, a' \in A$, $f(a)$ and $f(a')$ are *not* related when neither are a and a'.

1. **Chess.** Let P be the set of positions on a chessboard (`a2`, `d6`, `f3`, etc.) and $d_B : P \times P \to \mathsf{B}$ send a pair (p, q) to $\bot$ if and only if q is accessible from p in one bishop's move. The pair (P, d_B) is an object of $\mathsf{B}\mathbf{Spa}$. Let d_Q be the B-relation sending (p, q) to $\bot$ if and only if q is accessible from p in one queen's move. The pair (P, d_Q) is another object of $\mathsf{B}\mathbf{Spa}$. The identity function $\mathrm{id}_P : P \to P$ is nonexpansive from (P, d_B) to (P, d_Q) because whenever a bishop can go from p to q, a queen can too. However, it is not nonexpansive from (P, d_Q) to (P, d_B) because e.g. a queen can go from `a1` to `a2` but a bishop cannot.[153] One can check that any rotation of the chessboard is nonexpansive from (P, d_B) to itself and from (P, d_Q) to itself. And since nonexpansive maps compose, any rotation is also nonexpansive from (P, d_B) to (P, d_Q).

[153] In other words, the set of valid moves for a bishop is included in the set of valid moves for a queen, but not vice versa.

2. **Siblings.** Let H be the set of all humans (me, Paul Erdős, my brother Paul, etc.) and $d_S : H \times H \to \mathsf{B}$ send (h, k) to $\bot$ if and only if h and k are full siblings.[154] The pair (H, d_S) is an object of $\mathsf{B}\mathbf{Spa}$. Let $d_=$ be the B-relation sending (h, k) to $\bot$ if and only if h and k are the same person. The pair $(H, d_=)$ is another object of $\mathsf{B}\mathbf{Spa}$. The function $f : H \to H$ sending h to their biological mother is nonexpansive from (H, d_S) to $(H, d_=)$ because whenever h and k are full siblings, they have the same biological mother.

[154] Full siblings share the same biological parents.

Examples 85 (Distances)**.** The main examples of L-spaces in this thesis are $[0, 1]$-spaces or $[0, \infty]$-spaces. These are sets A equipped with a function $d : A \times A \to [0, 1]$ or $d : A \times A \to [0, \infty]$, and we can usually understand $d(a, a')$ as the distance between two points $a, a' \in A$. With this interpretation, a function is nonexpansive when applying it never increases the distances between points.[155] Let us give several examples of $[0, 1]$- and $[0, \infty]$-spaces:

[155] This is a justification for the term nonexpansive. In the setting of distances being real-valued, another popular term is 1-Lipschitz.

1. **Euclidean.** Probably the most famous distance in mathematics is the **Euclidean distance** on real numbers $d : \mathbb{R} \times \mathbb{R} \to [0, \infty] = (x, y) \mapsto |x - y|$. The distance

between any two points is unbounded, but it is never ∞. The pair $(\mathbb{R}, d)$ is an object of $[0,\infty]\mathbf{Spa}$.[156] Multiplication by $r \in \mathbb{R}$ is a nonexpansive function $r \cdot - : (\mathbb{R}, d) \to (\mathbb{R}, d)$ if and only if r is between -1 and 1. Intuitively, a function $f : (\mathbb{R}, d) \to (\mathbb{R}, d)$ is nonexpansive when its derivative at any point is between -1 and 1.[157]

2. **Collaboration.** Let H be the set of humans again. A **collaboration chain** between two humans h and k is a sequence of scientific papers $P_1, \dots, P_n$ such that h is a coauthor of P_1, k is a coauthor of P_n and P_i and P_{i+1} always have at least one common coauthor. The collaboration distance d between two humans h and k is the length of a shortest collaboration chain.[158] For instance $d(\text{me}, \text{Paul Erdős}) = 4$ as computed by `csauthors.net` on February 20th 2024:

$$\text{me} \xrightarrow{\text{[PS21]}} \text{D. Petrişan} \xrightarrow{\text{[GPR16]}} \text{M. Gehrke} \xrightarrow{\text{[EGP07]}} \text{M. Erné} \xrightarrow{\text{[EE86]}} \text{P. Erdős}$$

The pair (H, d) is a $[0,\infty]$-space, but it could also be seen as a $\mathbb{N}_\infty$-space (because the length of a chain is always an integer).

3. **Hamming.** Let W be the set of words of the English language. If two words u and v have the same number of letters, the Hamming distance $d(u, v)$ between u and v is the number of positions in u and v where the letters do not match.[159] When u and v are of different lengths, we let $d(u, v) = \infty$, and we obtain a $[0,\infty]$-space (W, d). (It is also a $\mathbb{N}_\infty$-space.)

Remark 86. As Example 85 come with many important intuitions, we will often call an L-relation $d : X \times X \to \mathsf{L}$ a **distance function** and $d(x, y)$ the **distance** from x to y,[160] even when L is neither $[0,1]$ nor $[0,\infty]$.

Examples 87. We give more examples of L-spaces to showcase the potential of our abstract framework.

1. **Diversion.**[161] Let J be the set of products available to consumers inside a vending machine (including a "no purchase" option), the second-choice diversion $d(p, q)$ from product p to product q is the fraction of consumers that switch from buying p to buying q when p is removed (or out of stock) from the machine. That fraction is always contained between 0 and 1, so we have a function $d : J \times J \to [0,1]$ which makes (J, d) an object of $[0,1]\mathbf{Spa}$.[162]

2. **Rank.** Let P be the set of web pages available on the internet. In [BP98], the authors introduce an algorithm to measure the importance of a page $p \in P$ giving it a rank $R(p) \in [0,1]$. This data can be organized in a function $d_R : P \times P \to [0,1]$ which assigns $R(p)$ to a pair (p, p) and 0 (or 1) to a pair (p, q) with $p \neq q$.[163] This yields a $[0,1]$-space (P, d_R).

   The rank of a page varies over time (it is computed from the links between all web pages which change quite frequently), so if we let T be the set of instants of time, we can define $d'_R(p, p)$ to be the function of type $T \to [0,1]$ which sends t to the rank $R(p)$ computed at time t.[164] This makes (P, d'_R) into a $[0,1]^T$-space.

[156] It is also very common to study subsets of $\mathbb{R}$, like $\mathbb{Q}$ or $[0,1]$, with the Euclidean distance appropriately restricted. We say that $(\mathbb{Q}, d)$ and $([0,1], d)$ are subspaces of $(\mathbb{R}, d)$. In general, a **subspace** of a L-space $\mathbf{A}$ is a subset $B \subseteq A$ equipped with the L-relation $d_{\mathbf{A}}$ restricted to B, i.e. $d_{\mathbf{B}} = B \times B \hookrightarrow A \times A \xrightarrow{d_{\mathbf{A}}} \mathsf{L}$.

[157] The derivatives might not exist, so this is just an informal explanation.

[158] As conventions, the length of a chain is the number of papers, not humans. Also, $d(h, k) = \infty$ when no such chain exists between h and k, except when $h = k$, then $d(h, h) = 0$ (or we could say it is the length of the empty chain from h to h).

[159] For instance $d(\texttt{carrot}, \texttt{carpet}) = 2$ because these words differ only in two positions, the second and third to last ($\texttt{r} \neq \texttt{p}$ and $\texttt{o} \neq \texttt{e}$).

[160] The asymmetry in the terminology "distance from x to y" is justified because, in general, nothing guarantees $d(x, y) = d(y, x)$. Since language is processed in a sequential order, we cannot even get rid of this asymmetry, but I feel like "distance *between* x and y" would be more appropriate if we required $d(x, y) = d(y, x)$.

[161] This example takes inspiration from the diversion matrices in [CMS23], where the authors consider the automobile market in the U.S.A. instead of a vending machine.

[162] Even though d is valued in $[0,1]$, calling it a distance function does not fit our intuition because when $d(p, q)$ is big, it means the products p and q are probably very similar.

[163] The values $d_R(p, q)$ when $p \neq q$ are considered irrelevant, so they are filled with an arbitrary value, e.g. 0 or 1.

[164] Again, $d_R(p, q)$ can be set to some unimportant constant value.

In order to create a search engine, we also need to consider the input of the user looking for some web page.[165] If U is the set of possible user inputs, we can define $d''_R(p,p)$ to depend on U and T, so that (P, d''_R) is a $[0,1]^{U\times T}$-space.

[165] The rank of a Wikipedia page about ramen will be lower when the user inputs "`Genre Humaine`" than when they input "`Ramen_Lord`".

3. **Collaboration (bis).** In Example 85, we defined the collaboration distance $d : H \times H \to \mathbb{N}_\infty$ that measures how far two people are from collaborating on a scientific paper. We can define a finer measure by taking into account the total number of people involved in the collaboration. It allows us to say you are closer to Erdös if you wrote a paper with him and no one else than if you wrote a paper with him and two additional coauthors. The distance d' is now valued in $\mathbb{N}_\infty \times \mathbb{N}_\infty$, the first coordinate of $d'(h,k)$ is $d(h,k)$ the length of the shortest collaboration chain between h and k, and the second coordinate of $d'(h,k)$ is the smallest total number of authors in a collaboration chain of length $d(h,k)$. For instance, according to `csauthors.net` on February 20th 2024, there are only two chains of length four between me and Erdös, both involving (the same) seven people, hence $d'(\text{me}, \text{Paul Erdös}) = (4,7)$.

There may be cases where $d'(h,k) = (4,7)$ (a long chain with few authors) and $d'(h,k') = (2,16)$ (a short chain with many authors). Then, with the product of complete lattices defined in Example 80, we could not compare the two distances. This is unfortunate in this application, so we may want to consider a different kind of product of complete lattices. The **lexicographical order** on $\mathbb{N}_\infty \times \mathbb{N}_\infty$ is

$$(\varepsilon,\delta) \leq_{\text{lex}} (\varepsilon',\delta') \Leftrightarrow \varepsilon \leq \varepsilon' \text{ or } (\varepsilon = \varepsilon' \text{ and } \delta \leq \delta').$$

In words, you use the order on the first coordinates, and only when they are equal, you use the order on the second coordinates.

If L and K are complete lattices, $(\mathsf{L}\times\mathsf{K}, \leq_{\text{lex}})$ is a complete lattice where the infimum is not computed pointwise, but rather

$$\inf S = (\inf \pi_{\mathsf{L}} S, \sup\{\varepsilon \mid \forall s \in S, (\inf \pi_{\mathsf{L}} S, \varepsilon) \leq s\}).$$

4. **Bisimulation for CTS.** A conditional transition system (CTS) [ABH+12, Example 2.5] is a labelled transition system with a semantics different than the usual one. Instead of following transitions when the label matches an input, some label is chosen before the execution, and only those transitions which have the chosen label remain possible. Formulated differently, it is a family of transition systems on the same set of states indexed by a set of labels. If X is the set of states, and L is the set of labels, we can define a $\mathcal{P}(L)$-relation $d : X \times X \to \mathcal{P}(L)$ by[166]

$$d(x,y) = \{\ell \in L \mid x \text{ and } y \text{ are not bisimilar when } \ell \text{ is chosen}\}.$$

[166] More details in [ABH+12, §Definitions C.1 and C.2].

Here is one last example further making the case for working over an abstract complete lattice.

Example 88 (Hausdorff distance)**.** Given an L-relation d on a set X, we define the L-relation $d^\uparrow$ on non-empty finite subsets of X:

$$\forall S,T \in \mathcal{P}_{\text{ne}}X, \quad d^\uparrow(S,T) = \sup\left\{\sup_{x\in S}\inf_{y\in T} d(x,y), \sup_{y\in T}\inf_{x\in S} d(x,y)\right\}.$$

This distance is a variation of a metric defined by Hausdorff in [Hau14].[167] It measures how far apart two subsets are in three steps. First, we postulate that a point $x \in S$ and T are as far apart as x and the closest point $y \in T$. Then, the distance from S to T is as big as the distance between the point $x \in S$ furthest from T. Finally, to obtain a symmetric distance, we take the maximum of the distance from S to T and from T to S. As we expect from any interesting optimization problem, there is a dual formulation given by the L-relation $d^\downarrow$.[168]

[167] Hausdorff considered positive real valued distances and compact subsets.

$$\forall S,T \in \mathcal{P}_{\text{ne}}X,\ d^\downarrow(S,T) = \inf\left\{\sup_{(x,y)\in C} d(x,y) \mid C \subseteq X\times X, \pi_1(C) = S, \pi_2(C) = T\right\}$$

[168] The notation was inspired by [BBKK18]. We write $\pi_S(C)$ for $\{x \in S \mid \exists (x,y) \in C\}$ and similarly for π_T. (We should really write $\mathcal{P}_{\text{ne}}\pi_S(C)$ and $\mathcal{P}_{\text{ne}}\pi_T(C)$.)

To compare two sets with the second method, you first need a binary relation C on X that covers all and only the points of S and T in the first and second coordinate respectively. Borrowing the terminology from probability theory, we call C a **coupling** of S and T, it is a subset of $X \times X$ whose *marginals* are S and T. According to a coupling C, the distance between S and T is the biggest distance between a pair in C. Amongst all couplings of S and T, we take the one achieving the smallest distance to define $d^{\downarrow}(S,T)$.

The first punchline of this example is that the two L-relations $d^{\uparrow}$ and $d^{\downarrow}$ coincide.

Lemma 89. *For any* $S, T \in \mathcal{P}_{\mathrm{ne}}X$, $d^{\uparrow}(S,T) = d^{\downarrow}(S,T)$.[169]

[169] Hardly adapted from [Mé11, Proposition 2.1].

Proof. ($\leq$) For any coupling $C \subseteq X \times X$, for each $x \in S$, there is at least one $y_x \in T$ such that $(x, y_x) \in C$ (because $\pi_1(C) = S$) so

$$\sup_{x\in S} \inf_{y \in T} d(x,y) \leq \sup_{x \in S} d(x, y_x) \leq \sup_{(x,y)\in C} d(x,y).$$

After a symmetric argument, we find that $d^{\uparrow}(S,T) \leq \sup_{(x,y)\in C} d(x,y)$ for all couplings, the first inequality follows.

($\geq$) For any $x \in S$, let $y_x \in T$ be a point in T that attains the infimum of $d(x,y)$,[170] and note that our definition ensures $d(x,y_x) \leq d^{\uparrow}(S,T)$. Symmetrically define x_y for any $y \in T$ and let $C = \{(x,y_x) \mid x \in S\} \cup \{(x_y, y) \mid y \in T\}$. It is clear that C is a coupling of S and T, and by our choices of y_x and x_y, we ensured that

[170] It exists because T is non-empty and finite.

$$\sup_{(x,y)\in C} d(x,y) \leq d^{\uparrow}(S,T),$$

therefore we found a coupling witnessing that $d^{\downarrow}(S,T) \leq d^{\uparrow}(S,T)$ as desired. □

The second punchline of this example comes from instantiating it with the complete lattice B. Recall that a B-relation d on X corresponds to a binary relation $R_d \subseteq X \times X$ where x and y are related if and only if $d(x,y) = \bot$. This seemingly backwards convention makes it so that nonexpansive functions are those that preserve the relation. Let us be careful about it while describing $R_{d^{\uparrow}}$ and $R_{d^{\downarrow}}$.

Given $S, T \in \mathcal{P}_{\mathrm{ne}}X$ and $x \in S$, notice that $\inf_{y\in T} d(x,y) = \bot$ if and only if $d(x,y) = \bot$ for at least one y, or equivalently, if x is related by R_d to at least one $y \in T$. This means the infimum behaves like an existential quantifier. Dually, the supremum acts like a universal quantifier yielding[171]

[171] Symmetrically,
$$\sup_{y\in T} \inf_{x \in S} d(x,y) = \bot \Leftrightarrow \forall y \in T, \exists x \in S, (x,y) \in R_d.$$

$$\sup_{x\in S} \inf_{y\in T} d(x,y) = \bot \iff \forall x \in S, \exists y \in T, (x,y) \in R_d.$$

Combining with its symmetric counterpart, and noting that a binary universal quantification is just an `AND`, we find that (S,T) belongs to $R_{d^{\uparrow}}$ if and only if

$$\forall x \in S, \exists y \in T, (x,y) \in R_d \text{ and } \forall y \in T, \exists x \in S, (x,y) \in R_d. \quad (62)$$

We call $R_{d^{\uparrow}}$ the Egli–Milner extension of R_d as in, e.g., [WS20, GPA21].

Given a coupling C of S and T, $\sup_{(x,y)\in C} d(x,y)$ can only equal $\perp$ when all pairs $(x,y) \in C$ are related by R_d. Then, if a coupling $C \subseteq R_d$ exists, the infimum of $d^{\downarrow}$ will be $\perp$. Therefore, S and T are related by $R_{d^{\downarrow}}$ if and only if

$$\exists C \subseteq R_d, \pi_S(C) = S \text{ and } \pi_T(C) = T. \tag{63}$$

The relation $R_{d^{\downarrow}}$ is sometimes called the Barr lifting of R_d [Bar06].

Our proof above yields the equivalence between (62) and (63).[172]

[172] That equivalence is folklore and has probably been given as exercise to many students in a class on bisimulation or coalgebras.

While the categories B**Spa**, $[0,1]$**Spa** and $[0,\infty]$**Spa** are interesting on their own, they contain subcategories which are more widely studied. For instance, the category **Poset** of posets and monotone maps is a full subcategory of B**Spa** where we only keep B-spaces (X,d) where the binary relation corresponding to d is reflexive, transitive and antisymmetric. Similarly, a $[0,\infty]$-space (X,d) where the distance function satisfies the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$ and reflexivity $d(x,x) \leq 0$ is known as a Lawvere metric space [Law02].

The next section lays out the language we will use to state conditions as those above on L-spaces. The syntax there is heavily inspired by the syntax of equations in universal algebra, the binary predicate $=$ for equality is joined by a family of binary predicates $=_\varepsilon$ indexed by the quantities in L. That clever idea comes from the original work of Mardare, Panangaden, and Plotkin on quantitative algebras [MPP16], and it implicitly relies on the following equivalent definition of L-spaces.

Definition 90 (L-structure)**.** Given a complete lattice L, an L**-structure**[173] is a set X equipped with a family of binary relations $R_\varepsilon \subseteq X \times X$ indexed by $\varepsilon \in \mathsf{L}$ satisfying

- **monotonicity** in the sense that if $\varepsilon \leq \varepsilon'$, then $R_\varepsilon \subseteq R_{\varepsilon'}$, and
- **continuity** in the sense that for any I-indexed family of elements $\varepsilon_i \in \mathsf{L}$,[174]

$$\bigcap_{i\in I} R_{\varepsilon_i} = R_\delta, \text{ where } \delta = \inf_{i\in I} \varepsilon_i.$$

[173] We borrow the name "structure" from model theorists. Closer to home, the more general notion of relational structure is used in [FMS21, Par22, Par23]. Our L-structures are both more and less general than the $\mathcal{L}_S$-structures of [Con17].

[174] By monotonicity, $R_\delta \subseteq R_{\varepsilon_i}$ so the inclusion $R_\delta \subseteq \cap_{i\in I} R_{\varepsilon_i}$ always holds. Also, continuity implies monotonicity because $\varepsilon \leq \varepsilon'$ implies

$$R_\varepsilon \cap R_{\varepsilon'} = R_{\inf\{\varepsilon,\varepsilon'\}} = R_\varepsilon,$$

which means $R_\varepsilon \subseteq R_{\varepsilon'}$. Still, we keep monotonicity explicit for better exposition.

Intuitively $(x,y) \in R_\varepsilon$ should be interpreted as bounding the distance from x to y above by ε. Then, monotonicity means the points that are at a distance below ε are also at a distance below ε' when $\varepsilon \leq \varepsilon'$. Continuity means the points that are at a distance below a bunch of bounds ε_i are also at a distance below the infimum of those bounds $\inf_{i\in I} \varepsilon_i$.

The names for these conditions come from yet another equivalent definition.[175] Organizing the data of an L-structure into a function $R : \mathsf{L} \to \mathcal{P}(X \times X)$ sending ε to R_ε, we can recover monotonicity and continuity by seeing $\mathcal{P}(X \times X)$ as a complete lattice like in Example 78. Indeed, monotonicity is equivalent to R being a monotone function between the posets $(\mathsf{L}, \leq)$ and $(\mathcal{P}(X \times X), \subseteq)$, and continuity is equivalent to R preserving infimums. Seeing L and $\mathcal{P}(X \times X)$ as posetal categories, we can simply say that R is a continuous functor.[176]

[175] This time more directly equivalent.

[176] Limits in a posetal category are always computed by taking the infimum of all the points in the diagram, so preserving limits and preserving infimums is the same thing.

A morphism between two L-structures $(X, \{R_\varepsilon\})$ and $(Y, \{S_\varepsilon\})$ is a function $f : X \to Y$ satisfying

$$\forall \varepsilon \in \mathsf{L}, \forall x, x' \in X, (x,x') \in R_\varepsilon \implies (f(x), f(x')) \in S_\varepsilon. \tag{64}$$

This should feel similar to nonexpansive maps.[177] Let us call $\mathsf{L}\mathbf{Str}$ the category of L-structures.

[177] In words, (64) reads as: if x and x' are at a distance below ε' then so are $f(x)$ and $f(x')$.

We give one trivial example, before proving that L-structures are just L-spaces.

Example 91. A consequence of continuity (take $I = \varnothing$) is that $R_\top$ is the full binary relation $X \times X$. Therefore, taking $\mathsf{L} = 1$ to be a singleton where $\bot = \top$, a 1-structure is only a set (there is no choice for R), and a morphism is only a function (the implication in (64) is always true because $S_\varepsilon = Y \times Y$). In other words, $1\mathbf{Str}$ is isomorphic to **Set**. Instantiating the next result (Proposition 92) means that $1\mathbf{Spa}$ is also isomorphic to **Set**, this is clear because there is only one function $d : X \times X \to 1$ for any set X. This example is relatively important because it means the theory we develop later over an arbitrary category of L-spaces specializes to the case of **Set**.[178]

[178] See Example 181.

Proposition 92. *For any complete lattice L, the categories $\mathsf{L}\mathbf{Spa}$ and $\mathsf{L}\mathbf{Str}$ are isomorphic.*[179]

[179] This result is a stripped down version of [MPP17, Theorem 4.3]. A more general version also appears in [FMS21, Example 3.5.(4)]. Another similar result is shown in [Par22, Appendix]. The core idea here ((65) and (66)) also appears in [Con17, Theorem A].

Proof. Given an L-relation (X, d), we define the binary relations $R^d_\varepsilon \subseteq X \times X$ by

$$(x, x') \in R^d_\varepsilon \iff d(x, x') \leq \varepsilon. \tag{65}$$

This family satisfies monotonicity because for any $\varepsilon \leq \varepsilon'$ we have

$$(x, x') \in R^d_\varepsilon \overset{(65)}{\iff} d(x, x') \leq \varepsilon \implies d(x, x') \leq \varepsilon' \overset{(65)}{\iff} (x, x') \in R^d_{\varepsilon'}.$$

It also satisfies continuity because if $(x, x') \in R_{\varepsilon_i}$ for all $i \in I$, then $d(x, x') \leq \varepsilon_i$ for all $i \in I$. By definition of infimum, we must have $d(x, x') \leq \inf_{i \in I} \varepsilon_i$, hence $(x, x') \in R_{\inf_{i\in I} \varepsilon_i}$. We conclude the forward inclusion ($\subseteq$) of continuity holds, the converse ($\supseteq$) follows from monotonicity.

Taking $\mathsf{L} = \mathsf{B}$, Proposition 92 gives back our interpretation of $\mathsf{B}\mathbf{Spa}$ as the category $2\mathbf{Rel}$ from Example 84. Indeed, a B-structure is just a set X equipped with a binary relation $R_\bot \subseteq X \times X$ (because $R_\top$ is required to equal $X \times X$), and morphisms of B-structures are functions that preserve that binary relation. This also justifies our weird choice of $d(x, y) = \bot$ meaning x and y are related.

Any nonexpansive map $f : (X, d) \to (Y, \Delta)$ in $\mathsf{L}\mathbf{Spa}$ is also a morphism between the L-structures $(X, \{R^d_\varepsilon\})$ and $(Y, \{R^\Delta_\varepsilon\})$ because for all $\varepsilon \in \mathsf{L}$ and $x, x' \in X$,

$$(x, x') \in R^d_\varepsilon \overset{(65)}{\iff} d(x, x') \leq \varepsilon \overset{(61)}{\implies} \Delta(f(x), f(x')) \leq \varepsilon \overset{(65)}{\iff} (f(x), f(x')) \in R^\Delta_\varepsilon.$$

It follows that the assignment $(X, d) \mapsto (X, \{R^d_\varepsilon\})$ is a functor $F : \mathsf{L}\mathbf{Spa} \to \mathsf{L}\mathbf{Str}$ acting trivially on morphisms.

Given an L-structure $(X, \{R_\varepsilon\})$, we define the function $d_R : X \times X \to \mathsf{L}$ by

$$d_R(x, x') = \inf \left\{ \varepsilon \in \mathsf{L} \mid (x, x') \in R_\varepsilon \right\}.$$

Note that monotonicity and continuity of the family $\{R_\varepsilon\}$ imply[180]

$$d_R(x, x') \leq \varepsilon \iff (x, x') \in R_\varepsilon. \tag{66}$$

[180] The converse implication ($\Leftarrow$) is by definition of infimum. For ($\Rightarrow$), continuity says that

$$R_{d_R(x,x')} = \bigcap_{\varepsilon \in \mathsf{L}, (x,x') \in R_\varepsilon} R_\varepsilon,$$

so $R_{d_R(x,x')}$ contains (x, x'), then by monotonicity, $d_R(x, x') \leq \varepsilon$ implies R_ε also contains (x, x').

This allows us to prove that a morphism $f : (X, \{R_\varepsilon\}) \to (Y, \{S_\varepsilon\})$ is nonexpansive from (X, d_R) to (Y, d_S) because for all $\varepsilon \in \mathsf{L}$ and $x, x' \in X$, we have

$$d_R(x, x') \leq \varepsilon \overset{(66)}{\iff} (x, x') \in R_\varepsilon \overset{(64)}{\implies} (f(x), f(x')) \in S_\varepsilon \overset{(66)}{\iff} d_S(f(x), f(x')) \leq \varepsilon,$$

hence putting $\varepsilon = d_R(x,x')$, we obtain $d_S(f(x), f(x')) \leq d_R(x,x')$. It follows that the assignment $(X, \{R_\varepsilon\}) \mapsto (X, d_R)$ is a functor $G : \mathsf{L}\mathbf{Str} \to \mathsf{L}\mathbf{Spa}$ acting trivially on morphisms.

Observe that (65) and (66) together say that $R_\varepsilon^{d_R} = R_\varepsilon$ and $d_{R^d} = d$, so F and G are inverses to each other on objects. Since both functors do nothing to morphisms, we conclude that F and G are inverses to each other, and that $\mathsf{L}\mathbf{Spa} \cong \mathsf{L}\mathbf{Str}$. $\square$

This result is central in our treatment of L-spaces because it allows us to specify an L-relation through the (binary) truth value of a family of predicates $=_\varepsilon$. In other words, we can reason equationally about L-spaces.

2.2 Equational Constraints

It is often the case one wants to impose conditions on the L-spaces they consider. For instance, recall that when L is $[0,1]$ or $[0,\infty]$, L-spaces are sets with a notion of distance between points. Starting from our intuition on the distance between points of the space we live in, people have come up with several abstract conditions to enforce on distance functions. For example, we can restate (with a slight modification[181]) the axioms defining metric spaces (Definition 1).

[181] The separation axiom is now divided in two, (68) and (69).

First, symmetry says that the distance from x to y is the same as the distance from y to x:

$$\forall x, y \in X, \quad d(x,y) = d(y,x). \tag{67}$$

Reflexivity, also called indiscernibility of identicals, says that the distance between x and itself is 0 (i.e. the smallest distance possible):

$$\forall x \in X, \quad d(x,x) = 0. \tag{68}$$

Identity of indiscernibles, also called Leibniz's law, says that if two points x and y are at distance 0, then x and y must be the same:

$$\forall x, y \in X, \quad d(x,y) = 0 \implies x = y. \tag{69}$$

Finally, the triangle inequality says that the distance from x to z is always smaller than the sum of the distances from x to y and from y to z:

$$\forall x, y, z \in X, \quad d(x,z) \leq d(x,y) + d(y,z). \tag{70}$$

There are also very famous axioms on B-spaces (X, d) that arise from viewing the binary relation corresponding to d as some kind of order on elements of X.

First, reflexivity says that any element x is related to itself.[182] Translating back to the B-relation, this is equivalent to:

[182] We abstract orders that look like the "smaller or equal" order $\leq$ on say real numbers rather than the strict order $<$.

$$\forall x \in X, \quad d(x,x) = \bot. \tag{71}$$

Antisymmetry says that if both (x,y) and (y,x) are in the order relation, then they must be equal:

$$\forall x, y \in X, \quad d(x,y) = \bot = d(y,x) \implies x = y. \tag{72}$$

Finally, transitivity says that if (x, y) and (y, z) belong to the order relation, then so does (x, z):

$$\forall x, y, z \in X, \quad d(x, y) = \bot = d(y, z) \implies d(x, z) = \bot. \tag{73}$$

We can immediately notice that all the axioms (67)–(73) start with a universal quantification of variables. A harder thing to see is that we never actually needed to talk about equality between distances. For instance, the equation $d(x, y) = d(y, x)$ in the axiom of symmetry (67) can be replaced by two inequalities $d(x, y) \leq d(y, x)$ and $d(y, x) \leq d(x, y)$, and moreover since x and y are universally quantified, only one of these inequalities is necessary:

$$\forall x, y \in X, \quad d(x, y) \leq d(y, x). \tag{74}$$

If we rely on the equivalence between L-spaces and L-structures (Proposition 92), we can transform (74) into a family of implications indexed by all $\varepsilon \in \mathsf{L}$:[183]

$$\forall x, y \in X, \quad (y, x) \in R^d_\varepsilon \implies (x, y) \in R^d_\varepsilon. \tag{75}$$

Starting from the triangle inequality (70) and applying the same transformations that got us from (67) to (75), we obtain a family of implications indexed by two values $\varepsilon, \delta \in \mathsf{L}$:[184]

$$\forall x, y, z \in X, \quad (x, y) \in R^d_\varepsilon \text{ and } (y, z) \in R^d_\delta \implies (x, z) \in R^d_{\varepsilon+\delta}. \tag{76}$$

The last conceptual step is to make the L.H.S. of the implication part of the universal quantification. That is, instead of saying "for all x and y, if P then Q", we say "for all x and y such that P, Q". We do this by introducing a syntax very similar to the equations of universal algebra. We fix a complete lattice $(\mathsf{L}, \leq)$, but you can keep in mind the examples $\mathsf{L} = [0, 1]$ and $\mathsf{L} = [0, \infty]$.

Definition 93 (Quantitative equation)**.** A **quantitative equation** (over L) is a tuple comprising an L-space $\mathbf{X}$ called the **context**, two elements $x, y \in X$ and optionally a quantity $\varepsilon \in \mathsf{L}$. We write these as $\mathbf{X} \vdash x = y$ when no ε is given or $\mathbf{X} \vdash x =_\varepsilon y$ when it is given.

An L-space **A satisfies** a quantitative equation

- $\mathbf{X} \vdash x = y$ if for any nonexpansive assignment $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, $\hat{\imath}(x) = \hat{\imath}(y)$.
- $\mathbf{X} \vdash x =_\varepsilon y$ if for any nonexpansive assignment $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, $d_{\mathbf{A}}(\hat{\imath}(x), \hat{\imath}(y)) \leq \varepsilon$.[185]

We use ϕ and ψ to refer to a quantitative equation, and we sometimes call them simply equations. We write $\mathbf{A} \vDash \phi$ when $\mathbf{A}$ satisfies ϕ,[186] and we also write $\mathbf{A} \vDash^{\hat{\imath}} \phi$ when the equality $\hat{\imath}(x) = \hat{\imath}(y)$ or the bound $d_{\mathbf{A}}(\hat{\imath}(x), \hat{\imath}(y)) \leq \varepsilon$ holds for a particular assignment $\hat{\imath} : \mathbf{X} \to \mathbf{A}$ (and not necessarily for all assignments).

Let us illustrate this definition with an example.

[183] Recall that $(x, y) \in R^d_\varepsilon$ is the same thing as $d(x, y) \leq \varepsilon$. Hence, (74) and (75) are equivalent because requiring $d(x, y)$ to be smaller than $d(y, x)$ is equivalent to requiring all upper bounds of $d(y, x)$ (in particular $d(y, x)$ itself) to also be upper bounds of $d(x, y)$.

[184] You can try proving how (70) and (76) are equivalent if the process of going from the former to the latter was not clear to you.

[185] Viewing it in the L-structure $(A, \{R^{d_{\mathbf{A}}}_\varepsilon\})$, we want that $\hat{\imath}(x)\ R^{d_{\mathbf{A}}}_\varepsilon\ \hat{\imath}(y)$ which looks a lot like $x =_\varepsilon y$.

[186] Of course, satisfaction generalizes straightforwardly to sets of quantitative equations, i.e. if $\hat{E}$ is a class of quantitative equations, $\mathbf{A} \vDash \hat{E}$ means $\mathbf{A} \vDash \phi$ for all $\phi \in \hat{E}$.

Example 94 (Symmetry)**.** We want to translate (75) into a quantitative equation. A first approximation would be replacing the relation R^d_ε with our new syntax $=_\varepsilon$ to obtain something like

$$x, y \vdash y =_\varepsilon x \implies x =_\varepsilon y.$$

We are not allowed to use implications like this, so we have implement the last step mentioned above by putting the premise $y =_\varepsilon x$ into the context. This means we need to quantify over variables x and y with a bound ε on the distance from y to x.

Note that when defining satisfaction of a quantitative equation, the quantification happens at the level of assignments $\hat{\imath} : \mathbf{X} \to \mathbf{A}$. Hence, we have to find a context $\mathbf{X}$ such that nonexpansive assignments $\mathbf{X} \to \mathbf{A}$ correspond to choices of two elements in $\mathbf{A}$ with the same bound ε on their distance.

Let the context $\mathbf{X}_\varepsilon$ be the L-space with two elements x and y such that $d_{\mathbf{X}_\varepsilon}(y, x) = \varepsilon$ and all other distances are $\top$. A nonexpansive assignment $\hat{\imath} : \mathbf{X}_\varepsilon \to \mathbf{A}$ is just a choice of two elements $\hat{\imath}(x), \hat{\imath}(y) \in A$ satisfying $d_{\mathbf{A}}(\hat{\imath}(y), \hat{\imath}(x)) \leq \varepsilon$.[187] For all of these, we have to impose the condition $d_{\mathbf{A}}(\hat{\imath}(x), \hat{\imath}(y)) \leq \varepsilon$. Therefore, our quantitative equation is

$$\mathbf{X}_\varepsilon \vdash x =_\varepsilon y. \tag{77}$$

[187] Indeed, since $\top$ is the top element of L, the other values of $d_{\mathbf{X}}$ being $\top$ means that they impose no further condition on $d_{\mathbf{A}}$.

For a fixed $\varepsilon \in \mathsf{L}$, an L-space $\mathbf{A}$ satisfies (77) if and only if it satisfies (75). Hence,[188] if $\mathbf{A}$ satisfies that quantitative equation for all $\varepsilon \in \mathsf{L}$, then it satisfies (67), i.e. the distance $d_{\mathbf{A}}$ is symmetric.

[188] Recall our argument in Footnote 183.

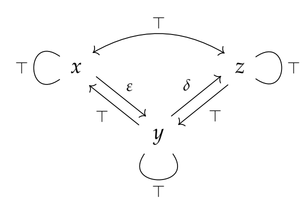
In practice, defining the context like this is more cumbersome than need be, so we will define some syntactic sugar to remedy this. Before that, we take the time to do another example.

Example 95 (Triangle inequality)**.** With $\mathsf{L} = [0, 1]$ or $\mathsf{L} = [0, \infty]$, let the context $\mathbf{X}_{\varepsilon,\delta}$ be the L-space with three elements x, y and z such that $d_{\mathbf{X}_{\varepsilon,\delta}}(x, y) = \varepsilon$ and $d_{\mathbf{X}_{\varepsilon,\delta}}(y, z) = \delta$, and all other distances are $\top$. [189] A nonexpansive assignment $\hat{\imath} : \mathbf{X}_{\varepsilon,\delta} \to \mathbf{A}$ is just a choice of three elements $a = \hat{\imath}(x), b = \hat{\imath}(y), c = \hat{\imath}(z) \in A$ such that $d_{\mathbf{A}}(a, b) \leq \varepsilon$ and $d_{\mathbf{A}}(b, c) \leq \delta$. Hence, if $\mathbf{A}$ satisfies

$$\mathbf{X}_{\varepsilon,\delta} \vdash x =_{\varepsilon+\delta} z, \tag{78}$$

it means that for any such assignment, $d_{\mathbf{A}}(a, c) \leq \varepsilon + \delta$ also holds. We conclude that $\mathbf{A}$ satisfies (76). If $\mathbf{A}$ satisfies $\mathbf{X}_{\varepsilon,\delta} \vdash x =_{\varepsilon+\delta} z$ for all $\varepsilon, \delta \in \mathsf{L}$, then $\mathbf{A}$ satisfies the triangle inequality (70).

[189] Here is a depiction of $\mathbf{X}_{\varepsilon,\delta}$, where the label on an arrow is the distance from the source to the target of that arrow:



Remark 96. There is a small caveat above. If we are in $\mathsf{L} = [0, 1]$ and $\varepsilon = 1$ and $\delta = 1$, then $\varepsilon + \delta = 2 \notin [0, 1]$, so the predicate $x =_{\varepsilon+\delta} z$ is not allowed. There are two easy fixes that we never explicit. You can either define a truncated addition so that $\varepsilon + \delta = 1$ whenever their sum is really above 1, or you can quantify over ε and δ such that $\varepsilon + \delta \leq 1$. Indeed, every $[0, 1]$-space satisfies $\mathbf{X}_{\varepsilon,\delta} \vdash x =_1 z$ because 1 is a global upper bound for the distance between points, thus there is no difference between having that equation or not as an axiom.

Notice that in the contexts $\mathbf{X}_\varepsilon$ and $\mathbf{X}_{\varepsilon,\delta}$, we only needed to set one or two distances and all the others where the maximum they could be $\top$. In our **syntactic sugar** for quantitative equations, we will only write the distances that are important (using the syntax $=_\varepsilon$), and we understand the underspecified distances to be as high as they can be. For instance, (77) will be written[190]

$$y =_\varepsilon x \vdash x =_\varepsilon y, \tag{79}$$

and (78) will be written

$$x =_\varepsilon y, y =_\delta z \vdash x =_{\varepsilon+\delta} z. \tag{80}$$

In this syntax, we call **premises** everything on the left of the turnstile $\vdash$ and **conclusion** what is on the right.

More generally, when we write $\{x_i =_{\varepsilon_i} y_i\}_{i\in I} \vdash x =_\varepsilon y$ (resp. $\{x_i =_{\varepsilon_i} y_i\}_{i\in I} \vdash x = y$), it corresponds to the quantitative equation $\mathbf{X} \vdash x =_\varepsilon y$ (resp. $\mathbf{X} \vdash x = y$), where the context $\mathbf{X}$ contains the variables in[191]

$$X = \{x, y\} \cup \{x_i \mid i \in I\} \cup \{y_i \mid i \in I\},$$

and the L-relation is defined for $u, v \in X$ by[192]

$$d_{\mathbf{X}}(u, v) = \inf\{\varepsilon \mid u =_\varepsilon v \in \{x_i =_\varepsilon y_i\}_{i\in I}\}.$$

Remark 97. The definition of quantitative equations in [MPP16] and most subsequent papers on quantitative algebras follows our syntactic sugar rather than our presentation with contexts. We showed the two approaches are formally equivalent in [MSV23, Lemma 8.4], but there is a special case we want to discuss.

In [MPP16, Definition 2.1], one axiom of their logic is (almost)

$$\{x =_{\varepsilon_i} y \mid i \in I\} \vdash x =_{\inf_{i\in I} \varepsilon_i} y.$$

Now, if we apply our translation to obtain a quantitative equation as in Definition 93, we get $\mathbf{X} \vdash x =_\varepsilon y$, where $d_{\mathbf{X}}(x, y) = \varepsilon = \inf_{i\in I} \varepsilon_i$ and all other distances are $\top$. This quantitative equation is obviously always satisfied,[193] so it makes sense to have it as an axiom, but it seems we are loosing a bit of information. That is, the original axiom looks like it ensures the continuity property of Definition 90. In fact, that axiom has several names in different papers, one of which is Cont. In the version of quantitative equational logic we propose in this thesis (Figure 3.1), there is an inference rule (rather than an axiom) that ensures continuity.

Here are some more translations of famous properties into quantitative equations written with the syntactic sugar:

- reflexivity (of a metric) (68) becomes $x \vdash x =_0 x$,[194]
- Leibniz's law (69) becomes $x =_0 y \vdash x = y$,
- reflexivity (of an order) (71) becomes $x \vdash x =_\perp x$,
- antisymmetry (72) becomes $x =_\perp y, y =_\perp x \vdash x = y$, and

[190] We can understand this syntax as putting back the information in the context into an implication. For instance, you can read (79) as "**if** the distance from y to x is bounded above by ε, **then** so is the distance from x to y". You can read (80) as "if the distance from x to y is bounded above by ε **and** the distance from y to z is bounded above by δ, then the distance from x to z is bounded above by $\varepsilon + \delta$".

[191] Note that the x_is, y_is, x and y need not be distinct. In fact, x and y almost always appear in the x_is and y_is.

[192] In words, the distance from u to v is the smallest value ε such that $u =_\varepsilon v$ was a premise. If no such premise occurs, the distance from u to v is $\top$. It is rare that u and v appear several times together (because $u =_\varepsilon v$ and $u =_\delta v$ can be replaced with $u =_{\inf\{\varepsilon,\delta\}} v$), but our definition allows it.

[193] For any nonexpansive assignment $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, $d_{\mathbf{A}}(\hat{\imath}(x), \hat{\imath}(y)) \leq d_{\mathbf{X}}(x, y) = \varepsilon$.

[194] As further sugar, we also write x instead of $x =_\top x$ to the left of the turnstile $\vdash$ to say that the variable x is in the context without imposing any constraint. For instance, the context of $x, y \vdash x = y$ has two variables x and y and all distances are $\top$. Thus, if $\mathbf{A}$ satisfies $x, y \vdash x = y$, then $\mathbf{A}$ is either empty or a singleton.

- transitivity (73) becomes $x =_\perp y, y =_\perp z \vdash x =_\perp z$.

Remark 98. The translations of (68) and (71) look very close. In fact, noting that 0 is the bottom element of $[0,1]$ and $[0,\infty]$, the quantitative equation $x \vdash x =_\perp x$ can state the reflexivity of a distance in $[0,1]$ or $[0,\infty]$ or the reflexivity of a binary relation.

Similarly, in the translation of the triangle inequality (80), if we let ε and δ range over B and interpret $+$ as an `OR`, we get three vacuous quantitative equations[195] and the translation of (73) above. So transitivity and triangle inequality are the same under this abstract point of view.[196]

Let us emphasize one thing about contexts of quantitative equations: they only give constraints that are upper bounds for distances.[197] In particular, it can be very hard to operate on the quantities in L non-monotonically. For instance, we will see (after Definition 108) that we cannot read $x =_{\varepsilon_1} y, y =_{\varepsilon_2} z, y =_{\varepsilon_3} y \vdash x =_{\varepsilon_1+\varepsilon_2-\varepsilon_3} z$ as saying that $d(x,z) \leq d(x,y) + d(y,z) - d(y,y)$, and one quick explanation is that subtraction is not a monotone operation on $[0,\infty] \times [0,\infty]$.[198] Another consequence is that an equation ϕ will always entail ψ when the latter has a *stricter* context (i.e. when the upper-bounds in the premises are smaller).[199] We prove a more general version of this below.

[195] When either ε or δ equals $\top$, $\varepsilon + \delta = \top$, but when the conclusion of a quantitative equation is $x =_\top z$, it must be satisfied.

[196] These observations were probably folkloric since at least the original publication of [Law02] in 1973.

[197] Well, if you consider the opposite order on L, they now give lower bounds. What is important is that they only speak about one of them.

[198] Assume $\mathsf{L} = [0,\infty]$ and $d(y,y)$ may be non-zero.

[199] For example, if $\mathbf{A}$ satisfies $x =_{1/2} y \vdash x = y$, then it satisfies $x =_{1/3} y \vdash x = y$. This says that if all distances between distinct points are above $1/2$, then they are also above $1/3$.

Lemma 99. *Let $f : \mathbf{X} \to \mathbf{Y}$ be a nonexpansive map. If $\mathbf{A}$ satisfies $\mathbf{X} \vdash x = y$ (resp. $\mathbf{X} \vdash x =_\varepsilon y$), then $\mathbf{A}$ satisfies $\mathbf{Y} \vdash f(x) = f(y)$ (resp. $\mathbf{Y} \vdash f(x) =_\varepsilon f(y)$).*

Proof. Any nonexpansive assignment $\hat{\iota} : \mathbf{Y} \to \mathbf{A}$ yields a nonexpansive assignment $\hat{\iota} \circ f : \mathbf{X} \to \mathbf{A}$. By hypothesis, we have

$$\mathbf{A} \vDash^{\hat{\iota}\circ f} \mathbf{X} \vdash x = y \qquad (\text{resp. } \mathbf{A} \vDash^{\hat{\iota}\circ f} \mathbf{X} \vdash x =_\varepsilon y),$$

which means $\hat{\iota}(f(x)) = \hat{\iota}(f(y))$ (resp. $d_{\mathbf{A}}(\hat{\iota}(f(x)), \hat{\iota}(f(y))) \leq \varepsilon$). Thus, we conclude

$$\mathbf{A} \vDash^{\hat{\iota}} \mathbf{Y} \vdash f(x) = f(y) \qquad (\text{resp. } \mathbf{A} \vDash^{\hat{\iota}} \mathbf{Y} \vdash f(x) =_\varepsilon f(y)). \qquad \square$$

Let us continue this list of examples for a while, just in case it helps a reader that is looking to translate an axiom into a quantitative equation. We will also give some results later which could imply that reader's axiom cannot be translated in this language.

Examples 100. For any complete lattice L.

1. The **strong triangle inequality** states that $d(x,z) \leq \max\{d(x,y), d(y,z)\}$,[200] it is equivalent to the satisfaction of the following family of quantitative equations

$$\forall \varepsilon, \delta \in \mathsf{L}, \quad x =_\varepsilon y, y =_\delta z \vdash x =_{\sup\{\varepsilon,\delta\}} z. \tag{81}$$

2. We can impose that all distances are below a **global upper bound** $\varepsilon \in \mathsf{L}$ (i.e. $d(x,y) \leq \varepsilon$) with the quantitative equation[201]

$$x, y \vdash x =_\varepsilon y. \tag{82}$$

[200] This property is used in defining ultrametrics [Rut96].

[201] For instance $[0,1]$-spaces are $[0,\infty]$-spaces that satisfy $x, y \vdash x =_1 y$.

3. We can *almost* impose a **global lower bound** $\varepsilon \in \mathsf{L}$ on distances. What we can do instead is impose a strict lower bound on distances that are not self-distances (i.e. $\forall x \neq y, d(x,y) > \varepsilon$).[202] To achieve this with an equation, we ensure the equivalent property that whenever $d(x,y)$ is smaller than ε, then $x = y$:

$$x =_\varepsilon y \vdash x = y. \tag{83}$$

[202] We can also do a non-strict lower bound (i.e. $\forall x \neq y, d(x,y) \geq \varepsilon$) by considering the family of equations $x =_\delta y \vdash x = y$ for all $\delta < \varepsilon$.

Let $\mathsf{L} = [0,1]$ or $\mathsf{L} = [0,\infty]$.

1. Given a positive number $b > 0$, the **b-triangle inequality** states that $d(x,z) \leq b(d(x,y) + d(y,z))$,[203] it is equivalent to the satisfaction of

$$\forall \varepsilon, \delta \in \mathsf{L}, \quad x =_\varepsilon y, y =_\delta z \vdash x =_{b(\varepsilon+\delta)} z. \tag{84}$$

[203] This property is used in defining b-metrics [KP22, Definition 1.1].

2. The **rectangle inequality** states that $d(x,w) \leq d(x,y) + d(y,z) + d(z,w)$,[204] it is equivalent to the satisfaction of

$$\forall \varepsilon_1, \varepsilon_2 \in \mathsf{L}, \quad x =_{\varepsilon_1} y, y =_{\varepsilon_2} z, z =_{\varepsilon_3} w \vdash x =_{\varepsilon_1+\varepsilon_2+\varepsilon_3} w. \tag{85}$$

[204] This property is used in defining g.m.s. in [Bra00, Definition 1.1].

Let $\mathsf{L} = \mathsf{B}$.

1. A binary relation R on $X \times X$ is said to be **functional** if there are no two distinct $y, y' \in X$ such that $(x,y) \in R$ and $(x,y') \in R$ for a single $x \in X$. This is equivalent to satisfying

$$x =_\perp y, x =_\perp y' \vdash y = y'. \tag{86}$$

2. We say $R \subseteq X \times X$ is **injective** if there are no two distinct $x, x' \in X$ such that $(x,y) \in R$ and $(x',y) \in R$ for a single $y \in X$.[205] This is equivalent to satisfying

$$x =_\perp y, x' =_\perp y \vdash x = x'. \tag{87}$$

[205] Equivalently, the opposite (or converse) of R is functional. You may want to formulate totality or surjectivity of a binary relation with quantitative equations, but you will find that difficult. We show in Example 116 that it is not possible.

3. We say $R \subseteq X \times X$ is **circular** if whenever (x,y) and (y,z) belong to R, then so does (z,x) (compare with transitivity (73)). This is equivalent to satisfying

$$x =_\perp y, y =_\perp z \vdash z =_\perp x. \tag{88}$$

We now turn to the study of subcategories of L**Spa** that are defined via (sets of) quantitative equations. Given a class $\hat{E}$ of quantitative equations, we can define a full subcategory of L**Spa** that contains only those L-spaces that satisfy $\hat{E}$, this is the category $\mathbf{GMet}(\mathsf{L}, \hat{E})$ whose objects we call generalized metric spaces or spaces for short. We also write $\mathbf{GMet}(\hat{E})$ or **GMet** when the complete lattices L or the class $\hat{E}$ are fixed or irrelevant. There is an evident forgetful functor $U : \mathbf{GMet} \to \mathbf{Set}$ which is the composition of the inclusion functor $\mathbf{GMet} \to \mathsf{L}\mathbf{Spa}$ and $U : \mathsf{L}\mathbf{Spa} \to \mathbf{Set}$.[206]

[206] Recall that while we use the same symbol for both forgetful functors, you can disambiguate them with the hyperlinks.

The terminology generalized metric space appears quite a lot in the literature with different meanings [BvBR98, Bra00], so I expect many will navigate to this definition before reading what is above. Catering to these readers, let us redefine what we mean by generalized metric space in a more concrete (but informal) form.

Definition 101 (Generalized metric space)**.** A **generalized metric space** or **space** is a set X along with a function $d : X \times X \to \mathsf{L}$ into a complete lattice L such that (X, d) satisfies some constraints expressed by quantitative equations.

When $\mathsf{L} = [0, \infty]$, examples include metrics [Fré06], ultrametrics [Rut96], pseudometrics, quasimetrics [Wil31a], semimetrics [Wil31b], b-metrics [KP22], the generalized metric spaces of [Bra00], dislocated metrics [HS00] also called diffuse metrics in [CKPR21], the generalized metric spaces of [BvBR98] which are the metric spaces of [Law02], and probably much more.[207]

When $\mathsf{L} = \mathsf{B}$ (the Boolean lattice), examples include posets...

[207] The literature is too vast to give an exhaustive list.

The most notable examples of generalized metric spaces are posets and metric spaces, they form the categories **Poset** and **Met**.

- **Poset** is the full subcategory of B**Spa** with all B-spaces satisfying reflexivity, antisymmetry, and transitivity stated as quantitative equations:[208]

$$\hat{E}_{\mathbf{Poset}} = \{ x \vdash x =_{\perp} x, x =_{\perp} y, y =_{\perp} x \vdash x = y, x =_{\perp} y, y =_{\perp} z \vdash x =_{\perp} z \}.$$

[208] Examples of posets include any set of numbers (e.g. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$) equipped with the usual (non-strict) order $\leq$, and $\mathcal{P}_{\text{ne}}X$ with the inclusion order.

- **Met** is the full subcategory of $[0,1]$**Spa** (taking $[0, \infty]$ works just as well) with all **metric spaces**, namely, $[0,1]$-spaces satisfying symmetry, reflexivity, identity of indiscernibles and triangle inequality stated as quantitative equations:[209] $\hat{E}_{\mathbf{Met}}$ contains all the following

$$\begin{aligned} \forall \varepsilon \in [0,1], \quad y =_{\varepsilon} x &\vdash x =_{\varepsilon} y \\ &\vdash x =_0 x \\ x =_0 y &\vdash x = y \\ \forall \varepsilon, \delta \in [0,1], \quad x =_{\varepsilon} y, y =_{\delta} z &\vdash x =_{\varepsilon + \delta} z. \end{aligned}$$

[209] Examples of metric spaces include $[0,1]$ with the Euclidean distance from Example 85, and the total variation distance from Example 102.

Example 102. The **total variation** distance is a metric defined on probability distributions. For any X, we define $\mathsf{tv} : \mathcal{D}X \times \mathcal{D}X \to [0,1]$ by, for any $\varphi, \psi \in \mathcal{D}X$,[210]

$$\mathsf{tv}(\varphi, \psi) = \sup_{S \subseteq X} |\varphi(S) - \psi(S)|.$$

Let us show tv is indeed a metric (it is more natural to show the properties equivalent to the equations in $\hat{E}_{\mathbf{Met}}$ hold rather than proving tv satisfies $E_{\mathbf{Met}}$).

[210] Since φ and ψ have finite support, we can restrict the quantification of the supremum to finite subsets of X, or even to subsets of the union of the supports of φ and ψ. Also, both $\varphi(S)$ and $\psi(S)$ are at most in $[0,1]$, so $\mathsf{tv}(\varphi, \psi)$ also takes values in $[0,1]$.

Proof. Symmetry is clear from the definition ($\forall r, s \in \mathbb{R}, |r - s| = |s - r|$). We can prove reflexivity and identity of indiscernibles at once by[211]

$$\begin{aligned} \mathsf{tv}(\varphi, \psi) = 0 &\Leftrightarrow \sup_{S \subseteq X} |\varphi(S) - \psi(S)| = 0 \\ &\Leftrightarrow \forall S \subseteq X, |\varphi(S) - \psi(S)| = 0 \\ &\Leftrightarrow \forall S \subseteq X, \varphi(S) = \psi(S) \\ &\Leftrightarrow \forall x \in X, \varphi(x) = \psi(x) \\ &\Leftrightarrow \varphi = \psi. \end{aligned}$$

[211] For the second to last equivalence, take $S = \{x\}$ for the forward direction, and for the converse use

$$\varphi(S) = \sum_{x \in S \cap (\mathrm{supp}(\varphi) \cup \mathrm{supp}(\psi))} \varphi(x).$$

For the triangle inequality, let $\varphi, \psi, \tau \in \mathcal{D}X$, we have[212]

$$\begin{aligned}
\mathsf{tv}(\varphi,\psi) + \mathsf{tv}(\psi,\tau) &= \sup_{S\subseteq X} |\varphi(S) - \psi(S)| + \sup_{S\subseteq X} |\psi(S) - \tau(S)| \\
&\geq \sup_{S\subseteq X} |\varphi(S) - \psi(S)| + |\psi(S) - \tau(S)| \\
&\geq \sup_{S\subseteq X} |\varphi(S) - \psi(S) + \psi(S) - \tau(S)| \\
&= \sup_{S\subseteq X} |\varphi(S) - \tau(S)| \\
&= \mathsf{tv}(\varphi,\tau). \qquad \square
\end{aligned}$$

[212] Using standard properties of supremums and absolute values.

Posets are not the only kind of interesting B-relations, by imposing a different set of equations, we can get different subcategories of B that we depict in a Hasse diagram.

$$\begin{tikzcd}
 & \mathbf{Pre} \arrow[dl] & \mathbf{Poset} \arrow[l] & \\
\mathsf{B}\mathbf{Spa} & \mathbf{Gph} \arrow[l] & & \mathbf{Set} \arrow[ul] \arrow[dl] \\
 & \mathbf{PER} \arrow[ul] & \mathbf{Equiv} \arrow[l] \arrow[ul] \arrow[uul] &
\end{tikzcd} \tag{89}$$

We can do the same thing for different subcategories of $[0,1]\mathbf{Spa}$.

$$\begin{tikzcd}
 & \mathbf{DMet} \arrow[ddl] & \mathbf{MMet} \arrow[l] & & \\
 & & \mathbf{PMet} \arrow[ul] \arrow[dl] \arrow[dddl] & & \\
{[0,1]}\mathbf{Spa} & \mathbf{PSMet} \arrow[l] & & \mathbf{Met} \arrow[uul] \arrow[ul] \arrow[dl] \arrow[ddl] & \mathbf{UMet} \arrow[l] \\
 & & \mathbf{SMet} \arrow[ul] & & \\
 & \mathbf{PQMet} \arrow[uul] & \mathbf{QMet} \arrow[l] & &
\end{tikzcd} \tag{90}$$

2.3 The Categories GMet

In this section, we prove some basic results about the categories of generalized metric spaces. We fix a complete lattice L and a class of quantitative equations $\hat{E}$ throughout, and denote by **GMet** the category of L-spaces that satisfy $\hat{E}$. The goal here is mainly to become familiar with L-spaces and quantitative equations, so not everything will be useful later. This also means we will avoid using abstract results (that we prove later) which can (sometimes drastically) simplify some proofs.[213]

We also take some time to identify some (well-known) conditions on L-spaces that cannot be expressed via quantitative equations.[214] These proofs are always in

[213] For instance, we will see that $U : \mathbf{GMet} \to \mathbf{Set}$ is a right adjoint, so it has many nice properties which we could use in this section.

[214] Again, we cannot make an exhaustive list.

the same vein, we know **GMet** has some property, we show the class of L-spaces with a condition does not have that property, hence that condition is not expressible as a class of quantitative equations.

In order to keep all the information about **GMet** in the same place, we will quickly summarize at the end the things we know about these categories (including things that will come from results in Chapter 3).

Products

The category **GMet** has all products. We prove this in three steps. First, we find the terminal object, second we show L**Spa** has all products, and third we show the products of L-spaces which all satisfy some quantitative equation also satisfies that quantitative equation.

Proposition 103. *The category* **GMet** *has a terminal object.*

Proof. The terminal object **1** in L**Spa** is relatively easy to find,[215] it is a singleton $\{*\}$ with the L-relation $d_{\mathbf{1}}$ sending $(*,*)$ to $\bot$. Indeed, for any L-space **X**, we have a function $!: X \to *$ that sends any x to $*$, and because $d_{\mathbf{1}}(*,*) = \bot \leq d_{\mathbf{X}}(x,x')$ for any $x, x' \in X$, ! is nonexpansive. We obtain a morphism $!: \mathbf{X} \to \mathbf{1}$, and since any other morphism $\mathbf{X} \to \mathbf{1}$ must have the same underlying function[216], ! is the unique morphism of this type.

[215] Again, many abstract results could help guide our search, but it is enough to have a bit of intuition about L-spaces.

[216] Because $\{*\}$ is terminal in **Set**.

Since **GMet** is a full subcategory of L**Spa**, it is enough to show **1** is in **GMet** to conclude it is the terminal object in this subcategory. We can do this by showing **1** satisfies absolutely all quantitative equations, and in particular those of $\hat{E}$.[217] Let **X** be any L-space, $x, y \in X$ and $\varepsilon \in \mathsf{L}$. As we have seen above, there is only one assignment $\hat{\iota}: \mathbf{X} \to \mathbf{1}$, and it sends x and y to $*$. This means

[217] Which defined **GMet** at the start of this section.

$$\hat{\iota}(x) = * = \hat{\iota}(y) \quad \text{and} \quad d_{\mathbf{1}}(\hat{\iota}(x), \hat{\iota}(y)) = d_{\mathbf{1}}(*,*) = \bot \leq \varepsilon.$$

Therefore, **1** satisfies both $\mathbf{X} \vdash x = y$ and $\mathbf{X} \vdash x =_\varepsilon y$. We conclude $\mathbf{1} \in \mathbf{GMet}$. ☐

Proposition 104. *The category* L**Spa** *has all products.*

Proof. Let $\{\mathbf{A}_i = (A_i, d_i) \mid i \in I\}$ be a family of L-spaces indexed by I. We define the L-space $\mathbf{A} = (A, d)$ with carrier $A = \prod_{i \in I} A_i$ (the Cartesian product of the carriers) and L-relation $d : A \times A \to \mathsf{L}$ defined by the following supremum:[218]

[218] For $a \in A$, let a_i be the ith coordinate of a.

$$\forall a, b \in A, \quad d(a,b) = \sup_{i \in I} d_i(a_i, b_i). \tag{91}$$

For each $i \in I$, we have the evident projection $\pi_i : \mathbf{A} \to \mathbf{A}_i$ sending $a \in A$ to $a_i \in A_i$, and it is nonexpansive because, by definition, for any $a, b \in A$,

$$d_i(a_i, b_i) \leq \sup_{i \in I} d_i(a_i, b_i) = d(a,b).$$

We will show that **A** with these projections is the product $\prod_{i \in I} \mathbf{A}_i$.

Let **X** be some L-space and $f_i : \mathbf{X} \to \mathbf{A}_i$ be a family of nonexpansive maps. By the universal property of the product in **Set**, there is a unique function $\langle f_i \rangle : X \to A$

satisfying $\pi_i \circ \langle f_i \rangle = f_i$ for all $i \in I$. It remains to show $\langle f_i \rangle$ is nonexpansive from $\mathbf{X}$ to $\mathbf{A}$. For any $x, x' \in X$, we have[219]

$$d(\langle f_i \rangle(x), \langle f_i \rangle(x')) = \sup_{i \in I} d_i(f_i(x), f_i(x')) \leq d_{\mathbf{X}}(x, x').$$

[219] The equation holds because the ith coordinate of $\langle f_i \rangle(x)$ is $f_i(x)$ by definition of $\langle f_i \rangle$, and the inequality holds because for all $i \in I$, $d_i(f_i(x), f_i(x')) \leq d_{\mathbf{X}}(x, x')$ by nonexpansiveness of f_i.

Note that a particular case of this construction for I being empty is the terminal object $\mathbf{1}$ from Proposition 103. Indeed, the empty Cartesian product is the singleton, and the empty supremum is the bottom element $\perp$. $\square$

In order to show that satisfaction of a quantitative equation is preserved by the product of L-spaces, we first prove a simple lemma.[220]

[220] It may remind you of Lemma 16 which states the same result for homomorphism and non-quantitative equations.

Lemma 105. *Let ϕ be a quantitative equation with context $\mathbf{X}$. If $f : \mathbf{A} \to \mathbf{B}$ is a nonexpansive map and $\mathbf{A} \vDash^{\hat{\imath}} \phi$ for a nonexpansive assignment $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, then $\mathbf{B} \vDash^{f \circ \hat{\imath}} \phi$.*

Proof. There are two very similar cases. If ϕ is of the form $\mathbf{X} \vdash x = y$, we have[221]

$$\mathbf{A} \vDash^{\hat{\imath}} \phi \iff \hat{\imath}(x) = \hat{\imath}(y) \implies f\hat{\imath}(x) = f\hat{\imath}(y) \iff \mathbf{B} \vDash^{f \circ \hat{\imath}} \phi.$$

[221] The equivalences hold by definition of $\vDash$.

If ϕ is of the form $\mathbf{X} \vdash x =_\varepsilon y$, we have[222]

$$\mathbf{A} \vDash^{\hat{\imath}} \phi \iff d_{\mathbf{A}}(\hat{\imath}(x), \hat{\imath}(y)) \leq \varepsilon \implies d_{\mathbf{B}}(f\hat{\imath}(x), f\hat{\imath}(y)) \leq \varepsilon \iff \mathbf{B} \vDash^{f \circ \hat{\imath}} \phi. \qquad \square$$

[222] The equivalences hold by definition of $\vDash$, and the implication holds by nonexpansiveness of f.

Proposition 106. *If all L-spaces $\mathbf{A}_i$ satisfy a quantitative equation ϕ, then $\prod_{i \in I} \mathbf{A}_i \vDash \phi$.*

Proof. Let $\mathbf{A} = \prod_{i \in I} \mathbf{A}_i$ and $\mathbf{X}$ be the context of ϕ. It is enough to show that for any assignment $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, the following equivalence holds:[223]

$$\left(\forall i \in I, \mathbf{A}_i \vDash^{\pi_i \circ \hat{\imath}} \phi\right) \iff \mathbf{A} \vDash^{\hat{\imath}} \phi. \tag{92}$$

[223] When I is empty, the `L.H.S.` of (92) is vacuously true, and the `R.H.S.` is true since $\mathbf{A}$ is the terminal L-space which we showed satisfies all quantitative equations in Proposition 103.

The proposition follows because if $\mathbf{A}_i \vDash \phi$ for all $i \in I$, then the `L.H.S.` holds for any $\hat{\imath}$, hence the `R.H.S.` does too, and we conclude $\mathbf{A} \vDash \phi$. Let us prove (92).

($\Rightarrow$) Consider the case $\phi = \mathbf{X} \vdash x = y$. The satisfaction $\mathbf{A}_i \vDash^{\pi_i \circ \hat{\imath}} \phi$ means $\pi_i\hat{\imath}(x) = \pi_i\hat{\imath}(y)$. If it is true for all $i \in I$, then we must have $\hat{\imath}(x) = \hat{\imath}(y)$ by universality of the product, thus we get $\mathbf{A} \vDash^{\hat{\imath}} \phi$. In case $\phi = \mathbf{X} \vdash x =_\varepsilon y$, the satisfaction $\mathbf{A}_i \vDash^{\pi_i \circ \hat{\imath}} \phi$ means $d_{\mathbf{A}_i}(\pi_i\hat{\imath}(x), \pi_i\hat{\imath}(y)) \leq \varepsilon$. If it is true for all $i \in I$, we get $\mathbf{A} \vDash \phi$ because

$$d_{\mathbf{A}}(\hat{\imath}(x), \hat{\imath}(y)) = \sup_{i \in I} d_{\mathbf{A}_i}(\pi_i\hat{\imath}(x), \pi_i\hat{\imath}(y)) \leq \varepsilon.$$

($\Leftarrow$) Apply Lemma 105 for all π_i. $\square$

Corollary 107. *The category* **GMet** *has all products, and they are computed like in* L**Spa**.[224]

[224] We showed that products in L**Spa** of objects in **GMet** also belong to **GMet**, it follows that this is also their products in **GMet** because the latter is a full subcategory of L**Spa**.

Unfortunately, this means that the notion of metric space originally defined in [Fré06], and incidentally what the majority of mathematicians calls a metric space, is not an instance of generalized metric space as we defined them. Since they only allow finite distances, some infinite products do not exist.[225] In general, if one wants to bound the distance above by some $B \in \mathsf{L}$, this can be done with the

[225] For instance let $\mathbf{A}_n$ be the metric space with two points $\{a, b\}$ at distance $n > 0 \in \mathbb{N}$ from each other. Then $\mathbf{A} = \prod_{n>0\in\mathbb{N}} \mathbf{A}_n$ exists in $[0,\infty]$**Spa** as we have just proven, but

$$d_{\mathbf{A}}(a^*, b^*) = \sup_{n>0\in\mathbb{N}} d_{\mathbf{A}_n}(a, b) = \sup_{n>0\in\mathbb{N}} n = \infty,$$

which means $\mathbf{A}$ is not a metric space in the sense of Definition 1.

equation $x, y \vdash x =_B y$, but the value B is still allowed as a distance. For instance $[0,1]\mathbf{Spa}$ is the full subcategory of $[0,\infty]\mathbf{Spa}$ defined by the equation $x, y \vdash x =_1 y$.

Arguably, this is only a superficially negative result since it is already common in parts of the literature [BvBR98, HST14] to allow infinite distances because the resulting category of metric spaces has better properties (like having infinite products and coproducts). However, there are some other conditions that one would like to impose on $[0,\infty]$-spaces which are not even preserved under finite products. We give two examples arising under the terminology partial metric.

Definition 108. A $[0,\infty]$-space (A, d) is called a **partial metric space** if it satisfies the following conditions [Mat94, Definition 3.1]:[226]

$$\forall a, b \in A, \quad a = b \iff d(a,a) = d(a,b) = d(b,b) \tag{93}$$

$$\forall a, b \in A, \quad d(a,a) \leq d(a,b) \tag{94}$$

$$\forall a, b \in A, \quad d(a,b) = d(b,a) \tag{95}$$

$$\forall a, b, c \in A, \quad d(a,c) \leq d(a,b) + d(b,c) - d(b,b) \tag{96}$$

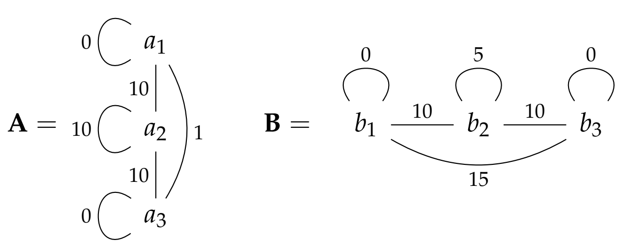
These conditions look similar to what we were able to translate into equations before, but the first and last are problematic.[227]

For (93), note that the forward implication is trivial, but for the converse, we would need to compare three distances at once inside the context, which seems impossible because the context only individually bounds distances by above. For (96), the problem comes from the minus operation on distances which will not interact well with upper bounds. Indeed, if we naively tried something like

$$x =_{\varepsilon_1} y, y =_{\varepsilon_2} z, y =_{\varepsilon_3} y \vdash x =_{\varepsilon_1 + \varepsilon_2 - \varepsilon_3} z,$$

we could always take ε_3 huge (even ∞) and make the distance between x and z as close to 0 as we would like (provided we can take ε_1 and ε_2 finite).

These are just informal arguments, but thanks to Corollary 107, we can prove formally that these conditions are not expressible as (classes of) quantitative equations. Let $\mathbf{A}$ and $\mathbf{B}$ be the $[0,\infty]$-spaces pictured below (the distances are symmetric).[228]



We can verify (by exhaustive checks) that $\mathbf{A}$ and $\mathbf{B}$ are partial metric spaces. If we take their product inside $[0,\infty]\mathbf{Spa}$, we find the following $[0,\infty]$-space (some distances are omitted) which does not satisfy (93) nor (96).[229]

[226] There is some ambiguity in what $+$ and $-$ means when dealing with ∞ (the original paper supposes distances are finite), but it is irrelevant for us.

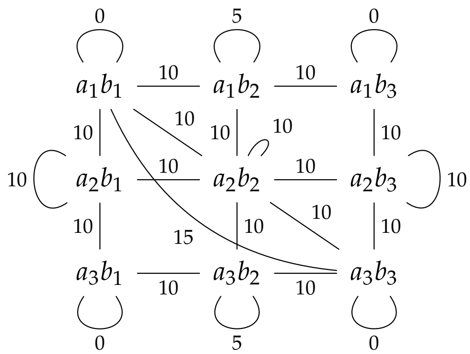
[227] We can translate (94) into $x =_\varepsilon y \vdash x =_\varepsilon x$, and (95) is just symmetry which we can translate into $y =_\varepsilon x \vdash x =_\varepsilon y$.

[228] The numbers on the lines indicate the distance between the ends of the line, e.g. $d_{\mathbf{A}}(a_1, a_1) = 0$, $d_{\mathbf{A}}(a_1, a_3) = 1$, and $d_{\mathbf{B}}(b_2, b_3) = 10$.

[229] For (93), the three points in the middle row $\{a_2b_1, a_2b_2, a_2b_3\}$ are all at distance 10 from each other and from themselves while not being equal. For (96), we have (on the diagonal)

$$d_{\mathbf{A}}(a_1b_1, a_3b_3) = 15, \text{ and}$$
$$d_{\mathbf{A}}(a_1b_1, a_2b_2) + d_{\mathbf{A}}(a_2b_2, a_3b_3) - d_{\mathbf{A}}(a_2b_2, a_2b_2) = 10,$$

but $15 > 10$.

$\mathbf{A} \times \mathbf{B} =$ 

We infer that there is no class $\hat{E}$ of quantitative equations such that $\mathbf{GMet}([0,\infty], \hat{E})$ is the full subcategory of $[0,\infty]\mathbf{Spa}$ containing all the partial metric spaces.[230]

[230] It is still possible that the category of partial metrics and nonexpansive maps is identified with some $\mathbf{GMet}(\mathsf{L}, \hat{E})$ for some cleverly picked L and $\hat{E}$. That would mean (infinite) products of partial metrics exist but they are not computed with supremums.

This result is a bit more damaging to our concept of generalized metric space (especially since partial metric spaces were motivated by some considerations in programming semantics [Mat94]), but we had to expect this would happen with how much time mathematicians had to use and abuse the name metric.

Here is another negative example.

Example 109 (ACC)**.** A binary relation $R \subseteq X \times X$ is said to have the **ascending chain condition** (ACC) if there is no infinite chain $x_0 \; R \; x_1 \; R \; x_2 \; R \cdots$. For example, $(\mathbb{N}, |^{\mathrm{op}})$ has the ACC, where $n \;|^{\mathrm{op}}\; m$ if and only if n is divisible by m and $n \neq m$.[231] Whenever R is reflexive (i.e. its corresponding B-relation satisfies (71)), R does not have the ACC because $x \; R \; x \; R \; x \cdots$ is an infinite chain.

[231] More famously, a ring is called Noetherian [MB99, §XI.1, p. 379] if its set of ideals ordered with strict inclusion has the ACC.

Similarly to Footnote 225, we can show that the infinite product of B-spaces does not preserve the ACC. Let $\mathbf{A} = \{0,1\}$ with $d_{\mathbf{A}}(0,1) = \bot$ and $d_{\mathbf{A}}(0,0) = d_{\mathbf{A}}(1,1) = d_{\mathbf{A}}(1,0) = \top$, i.e. the B-space corresponding to $\{0 < 1\}$. It has the ACC because there is only one chain $0 < 1$, while the infinite product $\prod_{n\in\mathbb{N}} \mathbf{A}$ does not:

$$(0,0,0,0,\dots) < (1,0,0,0,\dots) < (1,1,0,0,\dots) < (1,1,1,0,\dots) < \dots$$

Coproducts

The case of coproducts in $\mathbf{GMet}$ is more delicate. While $\mathsf{L}\mathbf{Spa}$ has coproducts, they do not always satisfy the equations satisfied by each of their components.

Proposition 110. *The category* $\mathbf{GMet}$ *has an initial object.*

Proof. The initial object $\varnothing$ in $\mathsf{L}\mathbf{Spa}$ is the empty set with the only possible L-relation $\varnothing \times \varnothing \to \mathsf{L}$ (the empty function). The empty function $f : \varnothing \to X$ is always nonexpansive from $\varnothing$ to $\mathbf{X}$ because (61) is vacuously satisfied.

Just as for the terminal object, since $\mathbf{GMet}$ is a full subcategory of $\mathsf{L}\mathbf{Spa}$, it suffices to show $\varnothing$ is in $\mathbf{GMet}$ to conclude it is initial in this subcategory. We do this by showing $\varnothing$ satisfies absolutely all quantitative equations, and in particular those of $\hat{E}$. This is easily done because when $\mathbf{X}$ is not empty,[232] there are no assignments $\mathbf{X} \to \varnothing$, so $\varnothing$ vacuously satisfies $\mathbf{X} \vdash x = y$ and $\mathbf{X} \vdash x =_\varepsilon y$. $\square$

[232] The context of a quantitative equation cannot be empty because the variables, say x and y, must belong to the context.

Proposition 111. *The category* $\mathsf{L}\mathbf{Spa}$ *has all coproducts.*

Proof. We just showed the empty coproduct (i.e. the initial object) exists. Let $\{\mathbf{A}_i = (A_i, d_i) \mid i \in I\}$ be a family of L-spaces indexed by a non-empty set I. We define the L-space $\mathbf{A} = (A, d)$ with carrier $A = \coprod_{i \in I} A_i$ (the disjoint union of the carriers) and L-relation $d : A \times A \to \mathsf{L}$ defined by:[233]

$$\forall a, b \in A, \quad d(a,b) = \begin{cases} d_i(a,b) & \exists i \in I, a, b \in A_i \\ \top & \text{otherwise} \end{cases}.$$

For each $i \in I$, we have the evident coprojection $\kappa_i : \mathbf{A}_i \to \mathbf{A}$ sending $a \in A_i$ to its copy in A, and it is nonexpansive because, by definition, for any $a, b \in A_i$, $d(a,b) = d_i(a,b)$.[234] We show $\mathbf{A}$ with these coprojections is the coproduct $\coprod_{i \in I} \mathbf{A}_i$.

Let $\mathbf{X}$ be some L-space and $f_i : \mathbf{A}_i \to \mathbf{X}$ be a family of nonexpansive maps. By the universal property of the coproduct in **Set**, there is a unique function $[f_i] : A \to X$ satisfying $[f_i] \circ \kappa_i = f_i$ for all $i \in I$. It remains to show $[f_i]$ is nonexpansive from $\mathbf{A}$ to $\mathbf{X}$. For any $a, b \in A$, suppose a belongs to A_i and b to A_j for some $i, j \in I$, then we have[235]

$$d_{\mathbf{X}}([f_i](a), [f_i](b)) = d_{\mathbf{X}}(f_i(a), f_j(b)) \leq \begin{cases} d_i(a,b) & i = j \\ \top & \text{otherwise} \end{cases} = d(a,b).$$

□

Because the distance between elements in different copies does not depend on the original spaces, it is easy to construct a quantitative equation that is not preserved by coproducts. For instance, even if all $\mathbf{A}_i$ satisfy $x, y \vdash x =_\varepsilon y$ for some fixed $\varepsilon \neq \top \in \mathsf{L}$,[236] the coproduct $\coprod_{i \in I} \mathbf{A}_i$ in L**Spa** does not satisfy it because some distances are $\top > \varepsilon$.

Still, **GMet** has coproducts as we will show in Corollary 177, but they are not that easy to define.[237]

[233] In words, $\mathbf{A}$ is the L-space with a copy of each $\mathbf{A}_i$ where the L-relation sends two points in different copies to $\top$ (intuitively, the copies are completely unrelated inside $\mathbf{A}$).

[234] Hence κ_i is even an isometric embedding.

[235] The first equation holds by definition of $[f_i]$ (it applies f_i to elements in the copy of A_i). The inequality holds by nonexpansiveness of f_i which is equal to f_j when $i = j$. The second equation is the definition of d.

[236] i.e. there is an upper bound smaller than $\top$ on all distances in all $\mathbf{A}_i$.

[237] Although in many cases like **Met** and **Poset**, they are computed like in L**Spa**.

Isometries

Since the forgetful functor $U : \mathsf{L}\mathbf{Spa} \to \mathbf{Set}$ preserves isomorphisms, we know that the underlying function of an isomorphism in L**Spa** is a bijection between the carriers. What is more, we show in Proposition 113 it must preserve distances on the nose, i.e. it is an isometry.

Definition 112 (Isometry)**.** A nonexpansive map $f : \mathbf{X} \to \mathbf{Y}$ is called an **isometry** if[238]

$$\forall x, x' \in X, \quad d_{\mathbf{Y}}(f(x), f(x')) = d_{\mathbf{X}}(x, x'). \tag{97}$$

If furthermore f is injective, we call it an **isometric embedding**.[239] If $f : \mathbf{X} \to \mathbf{Y}$ is an isometric embedding, we can identify $\mathbf{X}$ with the subspace of $\mathbf{Y}$ containing all the elements in the image of f. Conversely, the inclusion of a subspace of $\mathbf{Y}$ in $\mathbf{Y}$ is always an isometric embedding.

Proposition 113. *In* **GMet**, *isomorphisms are precisely the bijective isometries.*

[238] The inequality in (61) is replaced by an equation.

[239] This name is relatively rare because when dealing with metric spaces, the separation axiom implies that an isometry is automatically injective. This is also true for partial orders, where the name *order embedding* is common [DP02, Definition 1.34.(ii)].

Proof. We show a morphism $f : \mathbf{X} \to \mathbf{Y}$ has an inverse $f^{-1} : \mathbf{Y} \to \mathbf{X}$ if and only if it is a bijective isometry.

($\Rightarrow$) Since the underlying functions of f and f^{-1} are inverses, they must be bijections. Moreover, using (61) twice, we find that for any $x, x' \in X$,[240]

$$d_{\mathbf{X}}(x,x') = d_{\mathbf{X}}(f^{-1}f(x), f^{-1}f(x')) \leq d_{\mathbf{Y}}(f(x), f(x')) \leq d_{\mathbf{X}}(x,x'),$$

thus $d_{\mathbf{X}}(x,x') = d_{\mathbf{Y}}(f(x), f(x'))$, so f is an isometry.

($\Leftarrow$) Since f is bijective, it has an inverse $f^{-1} : Y \to X$ in **Set**, but we have to show f^{-1} is nonexpansive from $\mathbf{Y}$ to $\mathbf{X}$. For any $y, y' \in Y$, by surjectivity of f, there are $x, x' \in X$ such that $y = f(x)$ and $y' = f(x')$, then we have

$$d_{\mathbf{X}}(f^{-1}(y), f^{-1}(y')) = d_{\mathbf{X}}(f^{-1}f(x), f^{-1}f(x')) = d_{\mathbf{X}}(x,x') \stackrel{(97)}{=} d_{\mathbf{Y}}(f(x), f(x')) = d_{\mathbf{Y}}(y,y').$$

Hence f^{-1} is nonexpansive, it is even an isometry. $\square$

In particular, this means, as is expected, that isomorphisms preserve the satisfaction of quantitative equations. We can show a stronger statement: any isometric embedding reflects the satisfaction of quantitative equations.[241]

Proposition 114. *Let $f : \mathbf{Y} \to \mathbf{Z}$ be an isometric embedding between* L*-spaces and ϕ a quantitative equation, then*

$$\mathbf{Z} \vDash \phi \implies \mathbf{Y} \vDash \phi. \tag{98}$$

Proof. Let $\mathbf{X}$ be the context of ϕ. Any nonexpansive assignment $\hat{\iota} : \mathbf{X} \to \mathbf{Y}$ yields an assignment $f \circ \hat{\iota} : \mathbf{X} \to \mathbf{Z}$. By hypothesis, we know that $\mathbf{Z}$ satisfies ϕ for this particular assignment, namely,

$$\mathbf{Z} \vDash^{f \circ \hat{\iota}} \phi. \tag{99}$$

We can use this and the fact that f is an isometric embedding to show $\mathbf{Y} \vDash^{\hat{\iota}} \phi$. There are two very similar cases.

If $\phi = \mathbf{X} \vdash x = y$, then we have $\hat{\iota}(x) = \hat{\iota}(y)$ because we know $f\hat{\iota}(x) = f\hat{\iota}(x)$ by (99) and f is injective.

If $\phi = \mathbf{X} \vdash x =_\varepsilon y$, then we have $d_{\mathbf{Y}}(\hat{\iota}(x), \hat{\iota}(y)) = d_{\mathbf{Z}}(f\hat{\iota}(x), f\hat{\iota}(y)) \leq \varepsilon$, where the equation holds because f is an isometry and the inequality holds by (99). $\square$

Corollary 115. *Let $f : \mathbf{Y} \to \mathbf{Z}$ be an isometric embedding between* L*-spaces. If $\mathbf{Z}$ belongs to* **GMet***, then so does $\mathbf{Y}$. In particular, all the subspaces of a generalized metric space are also generalized metric spaces.*[242]

Examples 116. Corollary 115 can be useful to identify some properties of L-spaces that cannot be modelled with quantitative equations. Here are a few of examples.

1. A binary relation $R \subseteq X \times X$ is called **total** if for every $x \in X$, there exists $y \in X$ such that $(x, y) \in R$. Let **TotRel** be the full subcategory of B**Spa** containing only total relations. Is **TotRel** equal to some $\mathbf{GMet}(\mathsf{B}, \hat{E})$ for some $\hat{E}$? The existential quantification in the definition of total seems hard to simulate with a quantitative equation, but this is not a guarantee that maybe several equations cannot interact in such a counter-intuitive way.

[240] This is a general argument showing that any nonexpansive function with a right inverse is an isometry, it is also an isometric embedding because a right inverse in **Set** implies injectivity.

[241] This is stronger because we have just shown the inverse of an isomorphisms is an isometric embedding.

[242] Both parts are immediate. The first follows from applying (98) to all ϕ in $\hat{E}$, the class of quantitative equations defining **GMet**. The second follows from the inclusion of a subspace being an isometric embedding.

In order to prove that no class $\hat{E}$ defines total relations (i.e. $\mathbf{X} \vDash \hat{E}$ if and only if the relation corresponding to $d_{\mathbf{X}}$ is total), we can exhibit an example of a B-space that is total with a subspace that is not total. It follows that **TotRel** is not closed under taking subspaces, so it is not a category of generalized metric spaces by Corollary 115.[243]

Let $\mathbf{N}$ be the B-space with carrier $\mathbb{N}$ and B-relation $d_{\mathbf{N}}(n,m) = \bot \Leftrightarrow m = n+1$ (the corresponding relation is the graph of the successor function). This space satisfies totality, but the subspace obtained by removing 1 is not total because $d_{\mathbf{N}}(0,n) = \bot$ only when $n = 1$.

This same example works to show that surjectivity[244] cannot be defined via quantitative equations.

2. A very famous condition to impose on metric spaces is **completeness** (we do not need to define it here). Just as famous is the fact that $\mathbb{R}$ with the Euclidean metric from Example 85 is complete but the subspace $\mathbb{Q}$ is not. Thus, completeness cannot be defined via quantitative equations.[245]

With this characterization of isomorphisms, we can also show the forgetful functor $U : \mathbf{GMet} \to \mathbf{Set}$ is an isofibration which concretely means that if you have a bijection $f : X \to Y$ and a generalized metric $d_{\mathbf{Y}}$ on Y, then you can construct a generalized metric $d_{\mathbf{X}}$ on X such that $f : \mathbf{X} \to \mathbf{Y}$ is an isomorphism. Indeed, if you let $d_{\mathbf{X}}(x,x') = d_{\mathbf{Y}}(f(x), f(x'))$, then f is automatically a bijective isometry.[246]

Definition 117 (Isofibration)**.** A functor $P : \mathbf{C} \to \mathbf{D}$ is called an **isofibration**[247] if for any isomorphism $f : X \to PY$ in $\mathbf{D}$, there is an isomorphism $g : X' \to Y$ such that $Pg = f$, in particular $PX' = X$.

Proposition 118. *The forgetful functor $U : \mathbf{GMet} \to \mathbf{Set}$ is an isofibration.*

We wonder now how to complete the conceptual diagram below.

$$\begin{aligned} \text{isomorphism in } \mathbf{GMet} &\longleftrightarrow \text{bijective isometries} \\ \text{??? in } \mathbf{GMet} &\longleftrightarrow \text{isometric embeddings} \end{aligned}$$

Since isometric embeddings correspond to subspaces, one might think that they are the monomorphisms in **GMet**. Unfortunately, they are way more restrained.[248] Any nonexpansive map that is injective is a monomorphism. To prove this, we rely on the existence of a space $\mathbb{A}$ that informally *can pick elements*.

Proposition 119. *There is a generalized metric space $\mathbb{A}$ on the set $\{*\}$ such that for any other space $\mathbf{X}$, any function $f : \{*\} \to X$ is a nonexpansive map $\mathbb{A} \to \mathbf{X}$.*[249]

Proof. In L**Spa**, $\mathbb{A}$ is easy to find, its L-relation is defined by $d_{\mathbb{A}}(*,*) = \top$. Indeed, any function $f : \{*\} \to X$ is nonexpansive because $\top$ is the maximum value $d_{\mathbf{X}}$ can assign, so

$$d_{\mathbf{X}}(f(*), f(*)) \leq \top = d_{\mathbb{A}}(*,*).$$

Unfortunately, this L-space does not satisfy some quantitative equations (e.g. reflexivity $x \vdash x =_{\bot} x$), so we cannot guarantee it belongs to **GMet**.

[243] Actually, we have only proven that **TotRel** cannot be defined as a subcategory of B**Spa** with quantitative equations. There may still be some convoluted way that $\mathbf{TotRel} \cong \mathbf{GMet}(\mathsf{L}, \hat{E})$.

[244] This condition is symmetric to totality: $R \subseteq X \times X$ is **surjective** if for every $y \in X$, there exists $x \in X$ such that $(x,y) \in R$.

[245] Still with the caveat that the full subcategory of complete metric spaces might still be isomorphic to some $\mathbf{GMet}(\mathsf{L}, \hat{E})$.

[246] Clearly, it is the unique distance on X that works, and we know that $\mathbf{X}$ belongs to **GMet** thanks to Corollary 115.

[247] This term seems to have been coined by Lack and Paoli in [Lac07, §3.1] or [LP08, §6].

[248] They are the split monomorphisms, essentially by Footnote 240.

[249] In category theory speak, $\mathbb{A}$ is a representing object of the forgetful functor $U : \mathbf{GMet} \to \mathbf{Set}$.

Recall that $\mathbf{1}$ is a generalized metric space on the same set $\{*\}$, but with $d_{\mathbf{1}}(*,*) = \bot$. However, in many cases, $\mathbf{1}$ is not the right candidate either because if every function $f : \{*\} \to X$ is nonexpansive from $\mathbf{1}$ to $\mathbf{X}$, it means $d_{\mathbf{X}}(x,x) = \bot$ for all $x \in X$, which is not always the case.[250]

[250] It is equivalent to satisfying reflexivity.

We have two L-spaces at the extremes of a range of L-spaces $\{(\{*\}, d_\varepsilon)\}_{\varepsilon \in \mathsf{L}}$, where the L-relation d_ε sends $(*,*)$ to ε. At one extreme, we are guaranteed to be in **GMet**, but we are too restricted, and at the other extreme we might not belong to **GMet**. Getting inspiration from the intermediate value theorem, we can attempt to find a middle ground, namely, a value $\varepsilon \in \mathsf{L}$ such that setting $d_{\mathbb{A}}(*,*) = \varepsilon$ yields a space that lives in **GMet** but is not too restricted.

One natural thing to do is to take the biggest value (and hence the least restricted space that is in **GMet**). Formally, let

$$d_{\mathbb{A}}(*,*) = \sup \left\{ \varepsilon \in \mathsf{L} \mid (\{*\}, d_\varepsilon) \vDash \hat{E} \right\}.$$

It remains to check that any function $f : \{*\} \to X$ is nonexpansive from $\mathbb{A}$ to $\mathbf{X} \in \mathbf{GMet}$. Consider the image of f seen as a subspace of $\mathbf{X}$. By Corollary 115, it belongs to **GMet** and hence satisfies $\hat{E}$. Moreover, it is clearly isomorphic to the L-space $(\{*\}, d_\varepsilon)$ with $\varepsilon = d_{\mathbf{X}}(f(*), f(*))$, which means that L-space satisfies $\hat{E}$ as well (by Corollary 115 again). We conclude that $d_{\mathbf{X}}(f(*), f(*)) \leq d_{\mathbb{A}}(*,*)$.

As a bonus, one could check that for any $\varepsilon \in \mathsf{L}$ that is smaller than $d_{\mathbb{A}}(*,*)$, $(\{*\}, d_\varepsilon)$ also belongs to **GMet**.[251] □

[251] Use Lemma 105.

Proposition 120. *In* **GMet**, *monomorphisms are precisely the injective nonexpansive maps.*

Proof. We show a morphism $f : \mathbf{X} \to \mathbf{Y}$ is monic if and only if it is injective.

($\Rightarrow$) Let $x, x' \in X$ be such that $f(x) = f(x')$, and identify these elements with functions $x, x' : \{*\} \to X$ sending $*$ to x and x' respectively. By Proposition 119, we get two nonexpansive maps $x, x' : \mathbb{A} \to \mathbf{X}$. Post-composing by f, we find that $f \circ x = f \circ x'$ because they both send $*$ to $f(x) = f(x')$. By monicity of f, we find that $x = x'$ (as morphisms and hence as elements of X). We conclude f is injective.

($\Leftarrow$) Suppose that $f \circ g = f \circ h$ for some nonexpansive maps $g, h : \mathbf{Z} \to \mathbf{X}$. Applying the forgetful functor $U : \mathbf{GMet} \to \mathbf{Set}$, we find that $f \circ g = f \circ h$ also as functions. Since Uf is monic (i.e. injective), Ug and Uh must be equal, and since U is faithful, we obtain $g = h$. □

It remains to give a categorical characterization of isometric embeddings. This will rely on a well-known[252] abstract notion that we define here for completeness.

[252] While it is well-known, especially to those familiar with fibered category theory, it does not usually fit in a basic category theory course.

Definition 121 (Cartesian morphism)**.** Let $F : \mathbf{C} \to \mathbf{D}$ be a functor, and $f : A \to B$ be a morphism in $\mathbf{D}$. We say f is a **cartesian morphism** (with respect to F) if for every morphism $g : X \to B$ and factorization $Fg = Ff \circ u$, there exists a unique morphism $\widehat{u} : X \to A$ with $F\widehat{u} = u$ satisfying $x = f \circ \widehat{u}$. This can be summarized (without the quantifiers) in the diagram below.

$$\begin{tikzcd} X \arrow[d, dashed, "\widehat{u}"'] \arrow[dr, "g"] & & & FX \arrow[d, "u"'] \arrow[dr, "Fg"] & \\ A \arrow[r, "f"'] & B & \xrightarrow{F} & FA \arrow[r, "Ff"'] & FB \end{tikzcd}$$

Example 122 (in **GMet**)**.** Let us unroll this in the important case for us, when F is the forgetful functor $U : \mathbf{GMet} \to \mathbf{Set}$. A nonexpansive map $f : \mathbf{A} \to \mathbf{B}$ is a cartesian morphism if for any nonexpansive map $g : \mathbf{X} \to \mathbf{B}$, all functions $u : X \to A$ satisfying $g = f \circ u$ are nonexpansive maps $u : \mathbf{X} \to \mathbf{A}$.[253]

We can turn this around into an equivalent definition. The morphism $f : \mathbf{A} \to \mathbf{B}$ is cartesian if for all functions $u : X \to A$, $f \circ u$ being nonexpansive from $\mathbf{X}$ to $\mathbf{B}$ implies u is nonexpansive from $\mathbf{X}$ to $\mathbf{A}$.[254] In [AHS06, Definition 8.6], f is also called an *initial morphism*.

[253] We do not bother to write $\widehat{u}$ as it is automatically unique with underlying function u because U is faithful.

[254] If $f \circ u$ is nonexpansive from $\mathbf{X}$ to $\mathbf{B}$, then it is equal to g for some $g : \mathbf{X} \to \mathbf{B}$ which yields $u : \mathbf{X} \to \mathbf{A}$ being nonexpansive.

Proposition 123. *A morphism $f : \mathbf{A} \to \mathbf{B}$ in* **GMet** *is an isometric embedding if and only if it is monic and cartesian.*

Proof. By Proposition 120, being an isometric embedding is equivalent to being a monomorphism (i.e. being injective) and being an isometry. Therefore, it is enough to show that when f is injective, isometry $\iff$ cartesian.

($\Rightarrow$) Suppose f is an isometry, and let $u : X \to A$ be a function such that $f \circ u$ is nonexpansive from $\mathbf{X} \to \mathbf{B}$, we need to show u is nonexpansive from $\mathbf{X} \to \mathbf{A}$.[255] This is true because

$$\forall x, x' \in \mathbf{X}, \quad d_{\mathbf{A}}(u(x), u(x')) = d_{\mathbf{B}}(fu(x), fu(x')) \leq d_{\mathbf{X}}(x, x'),$$

where the equation follows from f being an isometry, and the inequality from nonexpansiveness of $f \circ u$.

[255] We use the second definition of cartesian in Example 122.

($\Leftarrow$) Suppose f is cartesian. For any $a, a' \in A$, we know that $d_{\mathbf{B}}(f(a), f(a')) \leq d_A(a, a')$, but we still need to show the converse inequality. Let $\mathbf{X}$ be the subspace of $\mathbf{B}$ containing only the image of a and a' (its carrier is $\{f(a), f(a')\}$), and $u : X \to A$ be the function sending $f(a)$ to a and $f(a')$ to a'.[256] Notice that $f \circ u$ is the inclusion of $\mathbf{X}$ in $\mathbf{B}$ which is nonexpansive. Because f is cartesian, u must then be nonexpansive from $\mathbf{X}$ to $\mathbf{A}$ which implies

$$d_{\mathbf{A}}(a, a') = d_{\mathbf{A}}(u(f(a)), u(f(a'))) \leq d_{\mathbf{X}}(f(a), f(a')) = d_{\mathbf{B}}(f(a), f(a')).$$

We conclude that f is an isometry. $\square$

[256] We use the injectivity of f here.

Corollary 124. *If the composition $\mathbf{A} \xrightarrow{f} \mathbf{B} \xrightarrow{g} \mathbf{C}$ is an isometric embedding, then f is an isometric embedding.*[257]

Proof. It is a standard result that if $g \circ f$ is monic then so is f. Even more standard for injectivity. Now, if $g \circ f$ is an isometry, we have for any $a, a' \in X$,[258]

$$d_{\mathbf{A}}(a, a') = d_{\mathbf{C}}(gf(a), gf(a')) \leq d_{\mathbf{B}}(f(a), f(a')) \leq d_{\mathbf{A}}(a, a'),$$

and we conclude that $d_{\mathbf{A}}(a, a') = d_{\mathbf{B}}(f(a), f(a'))$, hence f is an isometry. $\square$

[257] With the characterization of Proposition 123, this abstractly follows from [AHS06, Proposition 8.9]. We give the concrete proof anyways.

[258] The equation holds by hypothesis that $g \circ f$ is an isometry and the two inequalities hold by nonexpansiveness of g and f.

The question of concretely characterizing epimorphisms is harder to settle. We can do it for $\mathsf{L}\mathbf{Spa}$, but not for an arbitrary $\mathbf{GMet}$.

Proposition 125. *In* $\mathsf{L}\mathbf{Spa}$*, a morphism* $f : \mathbf{X} \to \mathbf{A}$ *is epic if and only if it is surjective.*

Proof. ($\Rightarrow$) Given any $a \in A$, we define the L-space $\mathbf{A}_a$ to be $\mathbf{A}$ with an additional copy of a with all the same distances. Namely, the carrier is $A + \{*_a\}$, for any $a' \in A$, $d_{\mathbf{A}_a}(*_a, a') = d_{\mathbf{A}}(a, a')$ and $d_{\mathbf{A}_a}(a', *_a) = d_{\mathbf{A}}(a', a)$, and all the other distances are as in $\mathbf{A}$.[259]

[259] This construction is already impossible to do in an arbitrary $\mathbf{GMet}$. For instance, if $\mathbf{A}$ satisfies $x =_0 y \vdash x = y$, then $\mathbf{A}_a$ does not because $d_{\mathbf{A}_a}(a, *_a) = 0$.

If $f : \mathbf{X} \to \mathbf{A}$ is not surjective, then pick $a \in A$ that is not in the image of f, and define two functions $g_a, g_* : A \to A + \{*_a\}$ that act as identity on all A except a where $g_a(a) = a$ and $g_*(a) = *_a$. By construction, both g_a and g_* are nonexpansive from $\mathbf{A}$ to $\mathbf{A}_a$ and $g_a \circ f = g_* \circ f$. Since $g_a \neq g_*$, f cannot be epic, and we have proven the contrapositive of the forward implication.

($\Leftarrow$) Suppose that $g, g' : \mathbf{A} \to \mathbf{B}$ are morphisms in $\mathsf{L}\mathbf{Spa}$ such that $g \circ f = g' \circ f$. Apply the forgetful functor to get $Ug \circ Uf = Ug' \circ Uf$, and since U is epic in $\mathbf{Set}$, we know $Ug = Ug'$. Since U is faithful, we conclude that $g = g'$.[260] $\square$

[260] This direction works in an arbitrary $\mathbf{GMet}$, that is, surjections are epic in any $\mathbf{GMet}$.

The standard example to show that Proposition 125 does not generalize to an arbitrary $\mathbf{GMet}$ is the inclusion of $\mathbb{Q}$ into $\mathbb{R}$ with the Euclidean metric inside $\mathbf{Met}$. It is not surjective, but it is epic because any nonexpansive function from $\mathbb{R}$ is determined by its image on the rationals.[261]

[261] For any $r \in \mathbb{R}$, you can always find $q_n \in \mathbb{Q}$ such that $d(q_n, r) \leq \frac{1}{n}$, hence $d_{\mathbf{A}}(f(q_n), f(r)) \leq \frac{1}{n}$ for any nonexpansive $f : (\mathbb{R}, d) \to \mathbf{A}$. We infer that $f(r)$ is determined by the value of $f(q_n)$ for all n.

Proposition 126. *Let* $f : \mathbf{A} \to \mathbf{B}$ *be a split epimorphism between* L*-spaces and* ϕ *a quantitative equation, then*

$$\mathbf{A} \vDash \phi \implies \mathbf{B} \vDash \phi. \tag{100}$$

Proof. Let $g : \mathbf{B} \to \mathbf{A}$ be the right inverse of f (i.e. $f \circ g = \mathrm{id}_{\mathbf{B}}$) and $\mathbf{X}$ be the context of ϕ.[262] Any nonexpansive assignment $\hat{\iota} : \mathbf{X} \to \mathbf{B}$ yields an assignment $g \circ \hat{\iota} : \mathbf{X} \to \mathbf{A}$. By hypothesis, we know that $\mathbf{A}$ satisfies ϕ for this particular assignment, namely,

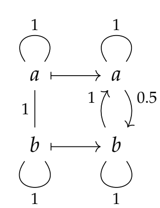
[262] Note that we already argued in Footnote 240 that the right inverse implies g is an isometric embedding. Then we could conclude by Corollary 115. The proof given here is essentially the same.

$$\mathbf{A} \vDash^{g \circ \hat{\iota}} \phi. \tag{101}$$

Now, we can apply Lemma 105 with $f : \mathbf{A} \to \mathbf{B}$ to obtain $\mathbf{B} \vDash^{f \circ g \circ \hat{\iota}} \phi$, and since $f \circ g = \mathrm{id}_{\mathbf{B}}$, we conclude $\mathbf{B} \vDash^{\hat{\iota}} \phi$. $\square$

Remark 127. It is not true in general that the image $f(A)$ of a nonexpansive function $f : \mathbf{A} \to \mathbf{B}$ (seen as a subspace of $\mathbf{B}$) satisfies the same equations as $\mathbf{A}$. For instance,[263] let $\mathbf{A}$ contain two points $\{a, b\}$ all at distance $1 \in [0, \infty]$ from each other (even from themselves). The $[0, \infty]$-relation is symmetric so it satisfies for all $\varepsilon \in [0, 1]$. $y =_\varepsilon x \vdash x =_\varepsilon y$. If we define $\mathbf{B}$ with the same points and distances except $d_{\mathbf{B}}(a, b) = 0.5$, then the identity function is nonexpansive from $\mathbf{A}$ to $\mathbf{B}$, but its image is $\mathbf{B}$ in which the distance is not symmetric.

[263] Here is a graphical depiction:



Proposition 126 is basically a dual of Proposition 114 because isometric embeddings are split monomorphisms, so we do not get additional examples of properties that cannot be expressed with quantitative equations.[264]

[264] In theory, duality may help in some settings, but I find isometric embeddings are easier to grasp.

Discrete Spaces

The forgetful functor $U : \mathbf{GMet} \to \mathbf{Set}$ has a left adjoint. Its concrete description is too involved, so we will prove this later in Corollary 175, but for the special case of $\mathsf{L}\mathbf{Spa}$, we can prove it now.

Proposition 128. *The forgetful functor $U : \mathsf{L}\mathbf{Spa} \to \mathbf{Set}$ has a left adjoint.*

Proof. For any set X, we define the **discrete space** $\mathbf{X}_\top$ to be the set X equipped with the L-relation $d_\top : X \times X \to \mathsf{L}$ sending any pair to $\top$.

For any L-space $\mathbf{A}$ and function $f : X \to A$, the function f is nonexpansive from $\mathbf{X}_\top$ to $\mathbf{A}$, thus $\mathbf{X}_\top$ is the free object on X (with respect to U). By categorical arguments, we obtain the left adjoint sending X to $\mathbf{X}_\top$. $\square$

3 Universal Quantitative Algebra

Saxophone concerto E minor OP 88

Deluxe

For a comprehensive introduction to the concepts and themes explored in this chapter, please refer to §0.3. Here, we only give a brief overview.

It is time to combine what we learned about universal algebra in Chapter 1 and about generalized metric spaces in Chapter 2 to develop universal quantitative algebra. This is the culminating point of several years of work with Matteo Mio and Valeria Vignudelli, during which we analyzed many choices and uncovered many subtleties in the existing accounts. The presentation we settled on highlights the fact that we are simply combining algebraic reasoning with the quantitative equations of Chapter 2. We give some examples (reusing those of the previous chapters) throughout this chapter.

Outline: In §3.1, we define quantitative algebras and quantitative equations over a signature, and we explain how to construct the free quantitative algebras. In §3.2, we give the rules for quantitative equational logic to derive quantitative equations from other quantitative equations, and we show it is sound and complete. In §3.3, we define presentations for monads on generalized metric spaces, and we give some examples.[265] In §3.4, we show that any monad lifting of a **Set** monad with an algebraic presentation to **GMet** can also be presented.

[265] Notice the parallel with the outline of Chapter 1.

In the sequel and unless otherwise stated, Σ is an arbitrary signature and **GMet** is an arbitrary category of generalized metric spaces defined by a class $\hat{E}_{\mathbf{GMet}}$ of quantitative equations.[266]

[266] Those defined in Definition 93.

3.1 Quantitative Algebras

Definition 129 (Quantitative algebra)**.** A **quantitative Σ-algebra** (or just quantitative algebra)[267] is a set A equipped with a Σ-algebra structure $(A, [\![-]\!]_A) \in \mathbf{Alg}(\Sigma)$ and a generalized metric space structure $(A, d_{\mathbf{A}}) \in \mathbf{GMet}$. We will switch between using the single symbol $\hat{\mathbb{A}}$ or the triple $(A, [\![-]\!]_A, d_{\mathbf{A}})$ when referring to a quantitative algebra, we will also write $\mathbb{A}$ for the **underlying** Σ-algebra, $\mathbf{A}$ for the underlying space, and A for the underlying set.

[267] We sometimes write simply algebra, with the `knowldege` link going to this definition.

A **homomorphism** from $\hat{\mathbb{A}}$ to $\hat{\mathbb{B}}$ is a function $h : A \to B$ between the underlying sets of $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ that is both a homomorphism $h : \mathbb{A} \to \mathbb{B}$ and a nonexpansive function $h : \mathbf{A} \to \mathbf{B}$. We sometimes emphasize and call h a nonexpansive homomorphism.[268] The identity maps $\mathrm{id}_A : A \to A$ and the composition of two homomorphisms are always homomorphisms, therefore we have a category whose objects are quantitative algebras and morphisms are nonexpansive homomorphisms. We denote it by $\mathbf{QAlg}(\Sigma)$.

[268] We will not distinguish between a nonexpansive homomorphism $h : \hat{\mathbb{A}} \to \hat{\mathbb{B}}$ and its underlying homomorphism or nonexpansive function or function. We may write Uh with U being the appropriate forgetful functor when necessary.

This category is concrete over $\mathbf{Set}$, $\mathbf{Alg}(\Sigma)$, $\mathbf{GMet}$ with forgetful functors:

- $U : \mathbf{QAlg}(\Sigma) \to \mathbf{Set}$ sends a quantitative algebra $\hat{\mathbb{A}}$ to its underlying set A and a nonexpansive homomorphism to the underlying function between carriers.
- $U : \mathbf{QAlg}(\Sigma) \to \mathbf{Alg}(\Sigma)$ sends $\hat{\mathbb{A}}$ to its underlying algebra $\mathbb{A}$ and a nonexpansive homomorphism to the underlying homomorphism.
- $U : \mathbf{QAlg}(\Sigma) \to \mathbf{GMet}$ sends $\hat{\mathbb{A}}$ to its underlying space $\mathbf{A}$ and a nonexpansive homomorphism to the underlying nonexpansive function.

One can quickly check that the following diagram commutes, and that it yields an alternative definition of $\mathbf{QAlg}(\Sigma)$ as a pullback of categories.[269] We have not found a technical use for this fact yet, but it starts making the case for universal quantitative algebra as a straightforward combination of universal algebra and generalized metric spaces.

[269] We can also mention there is another forgetful functor $U : \mathbf{QAlg}(\Sigma) \to \mathsf{L}\mathbf{Spa}$ obtained by composing $U : \mathbf{QAlg}(\Sigma) \to \mathbf{GMet}$ with the inclusion $\mathbf{GMet} \to \mathsf{L}\mathbf{Spa}$.

$$\begin{tikzcd}
\mathbf{QAlg}(\Sigma) \arrow[r, "U"] \arrow[d, "U"'] \arrow[dr, "U"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{GMet} \arrow[d, "U"] \\
\mathbf{Alg}(\Sigma) \arrow[r, "U"'] & \mathbf{Set}
\end{tikzcd}$$

Example 130. Since a quantitative algebra is just an algebra and a generalized metric space on the same set, we can find simple examples by combining pieces we have already seen.

1. In Example 5, we saw that an algebra for the signature $\Sigma = \{\mathsf{p} : 0\}$ is just a pair (X, x) comprising a set X with a distinguished point $x \in X$. In Example 85, we discussed the $\mathbb{N}_\infty$-space (H, d) where H is the set of humans and d is the collaboration distance. We can therefore consider the quantitative Σ-algebras $(H, \text{Paul Erdös}, d)$, which is the set of all humans with Paulo Erdös as a distinguished point and the collaboration distance.[270]
2. In Example 5, we saw the $\{\mathsf{f} : 1\}$-algebra $\mathbb{Z}$ where f is interpreted as adding 1. On top of that, we consider the B-relation corresponding to the partial order $\leq$ on $\mathbb{Z}$: $d_{\leq} : \mathbb{Z} \times \mathbb{Z} \to \mathsf{B}$ that sends (n, m) to $\bot$ if and only if $n \leq m$. We get a quantitative algebra $(\mathbb{Z}, - + 1, d_{\leq})$.[271]
3. In Example 85, we saw that $\mathbb{R}$ equipped with the Euclidean distance d is a metric space, i.e. an object of $\mathbf{GMet} = \mathbf{Met}$. The addition of real numbers is the most natural interpretation of $\Sigma = \{+ : 2\}$, thus we get a quantitative algebra $(\mathbb{R}, +, d)$.

[270] Note that $\mathbf{GMet}$ is instantiated as $\mathbb{N}_\infty\mathbf{Spa}$, i.e. $\mathsf{L} = \mathbb{N}_\infty$ and $\hat{E}_{\mathbf{GMet}} = \emptyset$.

[271] This time, $\mathbf{GMet}$ is instantiated as $\mathbf{Poset}$ with $\mathsf{L} = \mathsf{B}$ and $\hat{E}_{\mathbf{GMet}} = \hat{E}_{\mathbf{Poset}}$ as defined after Definition 101.

Remark 131. Already here, we covered three examples that are not possible with the original (and predominant in the literature) definition of quantitative algebras

[MPP16, Definition 3.1]. The first two are not possible because the base category is not **Met**. The third is not possible even if it deals with metric spaces.

Indeed, as already noted in [Adá22, Remark 3.1.(2)], the addition of real numbers is not a nonexpansive function $(\mathbb{R}, d) \times (\mathbb{R}, d) \to (\mathbb{R}, d)$, where $\times$ denotes the categorical product because,[272] recalling Corollary 107, we have

$$(d \times d)((1,1),(2,2)) = \sup\{d(1,2), d(1,2)\} = 1 < 2 = d(2,4) = d(1+1, 2+2).$$

Here are a two more compelling examples from the original paper [MPP16].

Example 132 (Hausdorff)**.** In Example 88, we defined the Hausdorff distance $d^{\uparrow}$ on $\mathcal{P}_{\mathrm{ne}}X$ that depends on an L-relation $d : X \times X \to \mathsf{L}$. In Example 67, we described a $\Sigma_{\mathbf{S}}$-algebra structure on $\mathcal{P}_{\mathrm{ne}}X$ (interpreting $\oplus$ as union). Combining these, we get a quantitative $\Sigma_{\mathbf{S}}$-algebra $(\mathcal{P}_{\mathrm{ne}}X, \cup, d^{\uparrow})$ for any L-space (X, d).

If we know that (X, d) satisfies some quantitative equations in $\hat{E}_{\mathbf{GMet}}$, we can sometimes prove that $(\mathcal{P}_{\mathrm{ne}}X, d^{\uparrow})$ does too. For instance, picking $\mathsf{L} = [0, 1]$ or $\mathsf{L} = [0, \infty]$, $\mathbf{GMet} = \mathbf{Met}$, and $\hat{E}_{\mathbf{GMet}} = \hat{E}_{\mathbf{Met}}$, one can show that if (X, d) belongs to **Met**, then so does $(\mathcal{P}_{\mathrm{ne}}X, d^{\uparrow})$, and we still get a quantitative $\Sigma_{\mathbf{S}}$-algebra $(\mathcal{P}_{\mathrm{ne}}X, \cup, d^{\uparrow})$, now over **Met**.[273]

Example 133 (Kantorovich)**.** Given a L-relation $d : X \times X \to [0, 1]$, we define the **Kantorovich distance** d_{K} on $\mathcal{D}X$ as follows:[274] for all $\varphi, \psi \in \mathcal{D}X$,

$$d_{\mathrm{K}}(\varphi, \psi) = \inf \left\{ \sum_{(x,x')} \tau(x, x') d(x, x') \mid \tau \in \mathcal{D}(X \times X), \mathcal{D}\pi_1(\tau) = \varphi, \mathcal{D}\pi_2(\tau) = \psi \right\}.$$

The distributions τ above range over **couplings** of φ and ψ, i.e. distributions over $X \times X$ whose marginals are φ and ψ. Thus, what d_{K} does, in words, is computing the average distance according to all couplings, and then taking the smallest one.

In Example 68, we gave a $\Sigma_{\mathbf{CA}}$-algebra structure on $\mathcal{D}X$ (interpreting $+_p$ as convex combination). Combining the algebra and the $[0, 1]$-space, we get a quantitative $\Sigma_{\mathbf{CA}}$-algebra $(\mathcal{D}X, [\![-]\!]_{\mathcal{D}X}, d_{\mathrm{K}})$. Once again, we can prove that if (X, d) is a metric space, then so is $(\mathcal{D}X, d_{\mathrm{K}})$, and we obtain a quantitative algebra $(\mathcal{D}X, [\![-]\!]_{\mathcal{D}X}, d_{\mathrm{K}})$ over **Met**.[275]

Unlike the first examples, the interpretations in $(\mathcal{P}_{\mathrm{ne}}X, \cup, d^{\uparrow})$ and $(\mathcal{D}X, [\![-]\!]_{\mathcal{D}X}, d_{\mathrm{K}})$ are nonexpansive with respect to the product distance. Concretely,

$$\forall S, S', T, T' \in \mathcal{P}_{\mathrm{ne}}X, \qquad d^{\uparrow}(S \cup S', T \cup T') \leq \max\left\{ d^{\uparrow}(S, T), d^{\uparrow}(S', T') \right\} \quad (102)$$

$$\forall \varphi, \varphi', \psi, \psi' \in \mathcal{D}X, \qquad d_{\mathrm{K}}(p\varphi + \overline{p}\varphi', p\psi + \overline{p}\psi') \leq \max\left\{ d_{\mathrm{K}}(\varphi, \psi), d_{\mathrm{K}}(\varphi', \psi') \right\}. \quad (103)$$

The initial motivation to remove this requirement and arrive at Definition 129[276] came from a variant of the Kantorovich distance called the **Łukaszyk–Karmowski** (ŁK for short) distance [Łuk04, Eq. (21)] which sends $\varphi, \psi \in \mathcal{D}X$ to

$$d_{\text{ŁK}}(\varphi, \psi) = \sum_{(x,x')} \varphi(x)\psi(x')d(x, x'). \quad (104)$$

[272] In [MPP16], the interpretation of an n-ary operation symbol is required to be a nonexpansive map from the n-wise product of the carrier to the carrier.

[273] This is the quantitative algebra denoted by $\Pi[M]$ in [MPP16, Theorem 9.2].

[274] This lifting of a distance on X to a distance on $\mathcal{D}X$ is well-known in optimal transport theory [Vil09]. You can find a well-written concise description of d_{K} in [BBKK18, §2.1] in the case $\mathsf{L} = [0, \infty]$ where it is denoted $d^{\downarrow\mathcal{D}}$. They also give a dual description as we did for the Hausdorff distance in Example 88, but the strong duality result ($d^{\downarrow\mathcal{D}} = d^{\uparrow\mathcal{D}}$) does not hold in general.

[275] This is the quantitative algebra denoted by $\Pi[M]$ in [MPP16, Theorem 10.4].

[276] Which imposes no further relation between the Σ-algebra and the L-space other than being on the same set.

In words, instead of looking at many different couplings to find the best one, we only look at the independent coupling $\tau(x,x') = \varphi(x)\psi(x')$.[277]

We showed in [MSV22, Lemma 5.3] that convex combination was not nonexpansive with respect to the product of the ŁK distance, namely, there exists a $[0,1]$-space (X,d) and distributions $\varphi, \varphi', \psi, \psi' \in \mathcal{D}X$ such that

$$d_{ŁK}(p\varphi + \overline{p}\varphi', p\psi + \overline{p}\psi') > \sup\left\{d_{ŁK}(\varphi,\psi), d_{ŁK}(\varphi',\psi')\right\}.$$

Therefore, $(\mathcal{D}X, [\![-]\!]_{\mathcal{D}X}, d_{ŁK})$ is always a quantitative algebra in the sense of Definition 129, but not always in the sense of [MPP16, Definition 3.1].[278]

[277] The ŁK distance is easier to compute than the Kantorovich distance because there is no optimization to do. It is the reason why it was considered in [CKPR21] for an application to reinforcement learning.

[278] In fact, even if d is a metric , $d_{ŁK}$ is not a metric (it does not satisfy $x \vdash x =_0 x$), so that is another reason why [MPP16] does not apply.

Quantitative Equations

Now, in order to get back the expressiveness of the original framework, we need a way to impose this property of nonexpansiveness with respect to the product distance, and we also need a way to impose other properties like the fact that $\oplus$ should be interpreted as a commutative operation. We achieve both things at once with the following definition.

Definition 134 (Quantitative Equation)**.** A **quantitative equation** (over Σ and L) is a tuple comprising an L-space $\mathbf{X}$ called the **context**,[279] two terms $s,t \in \mathcal{T}_\Sigma X$ and optionally a quantity $\varepsilon \in \mathsf{L}$. We write these as $\mathbf{X} \vdash s = t$ when no ε is given or $\mathbf{X} \vdash s =_\varepsilon t$ when it is given.

An quantitative algebra $\hat{\mathbb{A}}$ **satisfies** a quantitative equation[280]

- $\mathbf{X} \vdash s = t$ if for any nonexpansive assignment $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, $[\![s]\!]_A^{\hat{\imath}} = [\![t]\!]_A^{\hat{\imath}}$.
- $\mathbf{X} \vdash s =_\varepsilon t$ if for any nonexpansive assignment $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, $d_{\mathbf{A}}([\![s]\!]_A^{\hat{\imath}}, [\![t]\!]_A^{\hat{\imath}}) \leq \varepsilon$.

We use ϕ and ψ to refer to a quantitative equation, and we sometimes call them simply equations with the `knowldege` link going here. We write $\hat{\mathbb{A}} \vDash \phi$ when $\hat{\mathbb{A}}$ satisfies ϕ,[281] and we also write $\hat{\mathbb{A}} \vDash^{\hat{\imath}} \phi$ when the equality $[\![s]\!]_A^{\hat{\imath}} = [\![t]\!]_A^{\hat{\imath}}$ or the bound $d_{\mathbf{A}}([\![s]\!]_A^{\hat{\imath}}, [\![t]\!]_A^{\hat{\imath}}) \leq \varepsilon$ holds for a particular assignment $\hat{\imath} : \mathbf{X} \to \mathbf{A}$ (and not necessarily for all assignments).

[279] Note that even with algebras in **GMet**, the context is in L**Spa**.

[280] Formally, we would need to write $[\![-]\!]_A^{U\hat{\imath}}$ instead of $[\![-]\!]_A^{\hat{\imath}}$ because $U\hat{\imath} : X \to A$ is the assignment we use to interpret the terms.

[281] As usual, satisfaction generalizes to classes of quantitative equations, i.e. if $\hat{E}$ is a classes of quantitative equations, $\hat{\mathbb{A}} \vDash \hat{E}$ means $\hat{\mathbb{A}} \vDash \phi$ for all $\phi \in \hat{E}$.

Our overloading of the terminology *quantitative equation* (recall Definition 93) is practically harmless because a quantitative equation from Chapter 2 $\mathbf{X} \vdash x = y$ (or $\mathbf{X} \vdash x =_\varepsilon y$) can be seen as the new kind of quantitative equation by viewing x and y as terms via the embedding η_X^Σ. Formally, since $[\![\eta_X^\Sigma(x)]\!]_A^{\hat{\imath}} = \hat{\imath}(x)$ for any $x \in X$ and $\hat{\imath} : \mathbf{X} \to \mathbf{A}$,[282]

$$\begin{aligned} \mathbf{A} \vDash \mathbf{X} \vdash x = y &\iff \hat{\mathbb{A}} \vDash \mathbf{X} \vdash \eta_X^\Sigma(x) = \eta_X^\Sigma(y) \\ \mathbf{A} \vDash \mathbf{X} \vdash x =_\varepsilon y &\iff \hat{\mathbb{A}} \vDash \mathbf{X} \vdash \eta_X^\Sigma(x) =_\varepsilon \eta_X^\Sigma(y). \end{aligned} \tag{105}$$

In particular, since we assumed the underlying space of any $\hat{\mathbb{A}} \in \mathbf{QAlg}(\Sigma)$ to be a generalized metric space, we can say that $\hat{\mathbb{A}} \vDash \phi$ for any $\phi \in \hat{E}_{\mathbf{GMet}}$.[283] Another consequence is that over the empty signature $\Sigma = \varnothing$, the quantitative equations from Definition 93 and Definition 134 are the same.

[282] Later on, we will seldom distinguish between x and $\eta_X^\Sigma(x)$ and write the former for simplicity.

[283] We implicitly see the equations in $\hat{E}_{\mathbf{GMet}}$ as the new kind of equations from Definition 134.

Furthermore, the new quantitative equations also generalize the equations of universal algebra (Definition 12). Indeed, given an equation $X \vdash s = t$, we construct the quantitative equation $\mathbf{X}_\top \vdash s = t$ where the new context is the discrete space on the old context. We show that

$$\mathbb{A} \vDash X \vdash s = t \iff \hat{\mathbb{A}} \vDash \mathbf{X}_\top \vdash s = t. \tag{106}$$

By Proposition 128, any assignment $\iota : X \to A$ is nonexpansive from $\mathbf{X}_\top$ to $\mathbf{A}$. Any nonexpansive assignment $\hat{\iota} : \mathbf{X}_\top \to \mathbf{A}$ also yields an assignment $X \to A$ by applying the forgetful functor U since the carrier of $\mathbf{X}_\top$ is X. Therefore, the interpretations of s and t coincide under all assignments if and only if they coincide under all nonexpansive assignments.

Let us get to more interesting examples now.

Example 135 (Almost commutativity)**.** Let $+ : 2 \in \Sigma$ be a binary operation symbol. As shown above, to ensure $+$ is interpreted as a commutative operation in a quantitative algebra, we can use the quantitative equation $\mathbf{X}_\top \vdash x + y = y + x$ where $X = \{x, y\}$. In fact, using the same syntactic sugar as we did in Chapter 2 to avoid explicitly describing all the context, we can write $x, y \vdash x + y = y + x$.[284]

[284] Whenever we will write $x_1, \dots, x_n \vdash s = t$, we will mean $\mathbf{X}_\top \vdash s = t$ where $X = \{x_1, \dots, x_n\}$, and similarly for $=_\varepsilon$.

Since the context can be any L-space, we can now add some nuance to the commutativity property. For instance, we can guarantee that $+$ is commutative only between elements that are close to each other with $x =_\varepsilon y \vdash x + y = y + x$ where $\varepsilon \in \mathsf{L}$ is fixed.[285] Unrolling the syntactic sugar, the context is the L-space containing two points x and y with $d_{\mathbf{X}}(x, y) = \varepsilon$ and all other distances being $\top$. Therefore, a nonexpansive assignment $\hat{\iota} : \mathbf{X} \to \mathbf{A}$ is a choice of two elements $\hat{\iota}(x)$ and $\hat{\iota}(y)$ with $d_{\mathbf{A}}(\hat{\iota}(x), \hat{\iota}(y)) \leq \varepsilon$ and no other constraint. We conclude that $\hat{\mathbb{A}}$ satisfies $x =_\varepsilon y \vdash x + y = y + x$ if and only if $[\![+]\!]_A(a, b) = [\![+]\!]_A(b, a)$ whenever $d_{\mathbf{A}}(a, b) \leq \varepsilon$.

[285] This example comes from [Adá22, Example 2.8.(4)].

Another possible variant on commutativity is $x =_\perp x, y =_\perp y \vdash x + y = y + x$. This means $+$ is guaranteed to be commutative only on elements which have a self-distance of $\perp$. For instance, in distributions with the ŁK distance, $d_{\text{ŁK}}(\varphi, \varphi) = 0$ only when the elements in the support of φ are all at distance 0 from each other. In particular, when d is a metric, $d_{\text{ŁK}}(\varphi, \varphi) = 0$ if and only if φ is a Dirac distribution. So that quantitative equation would ensure commutativity only on Dirac distributions.

Example 136 (Nonexpansiveness)**.** We can translate (102) and (103) into the following (family of) quantitative equations.

$$\forall \varepsilon, \varepsilon' \in \mathsf{L}, \quad x =_\varepsilon y, x' =_{\varepsilon'} y' \vdash x \oplus x' =_{\max\{\varepsilon, \varepsilon'\}} y \oplus y' \tag{107}$$

$$\forall \varepsilon, \varepsilon' \in \mathsf{L}, \quad x =_\varepsilon y, x' =_{\varepsilon'} y' \vdash x +_p x' =_{\max\{\varepsilon, \varepsilon'\}} y +_p y' \tag{108}$$

The quantitative algebra from Example 132 satisfies (107), and the one from Example 133 satisfies (108), but the variant with the ŁK distance does not satisfy (108).

In general, if we want an n-ary operation symbol $\mathsf{op} \in \Sigma$ to be interpreted as a nonexpansive map $\mathbf{A}^n \to \mathbf{A}$, we can impose the equations[286]

[286] This is an axiom in the logic of [MPP16]. It is not in our formulation of quantitative equational logic.

$$\forall \{\varepsilon_i\}_{i \in I} \subseteq \mathsf{L}, \quad \{x_i =_{\varepsilon_i} y_i \mid 1 \leq i \leq n\} \vdash \mathsf{op}(x_1, \dots, x_n) =_{\max_i \varepsilon_i} \mathsf{op}(y_1, \dots, y_n). \tag{109}$$

Example 137 (L-nonexpansiveness)**.** In most papers on quantitative algebras this property is called "nonexpansiveness of the operations". In [MSV22], we remarked this can be ambiguous because one could consider a different distance on n-tuples of inputs than the product distance. We then presented quantitative algebras for *lifted signature* which can deal with more general operations.

In a lifted signature, each operation symbol $\mathsf{op} : n \in \Sigma$ comes with an assignment $(A, d) \mapsto (A^n, L_{\mathsf{op}}(d))$ (on generalized metric spaces) which specifies the distance on n-tuples that needs to be considered. We say that the interpretation $[\![\mathsf{op}]\!]_A$ is L_{op}-nonexpansive when it is a nonexpansive map $[\![\mathsf{op}]\!]_A : (A^n, L(d)) \to (A, d)$.[287] We can also express L_{op}-nonexpansiveness with a family of quantitative equations like we did in Example 136:[288]

$$\forall \mathbf{X} \in \mathbf{GMet}, \forall x, y \in X^n, \quad \mathbf{X} \vdash \mathsf{op}(x_1, \ldots, x_n) =_{L_{\mathsf{op}}(d_{\mathbf{X}})(x,y)} \mathsf{op}(y_1, \ldots, y_n). \tag{110}$$

If an algebra $\hat{\mathbb{A}}$ satisfies these equations, then in particular, for all $a, b \in A^n$, it satisfies $\mathbf{A} \vdash \mathsf{op}(a_1, \ldots, a_n) =_{L_{\mathsf{op}}(d_{\mathbf{A}})(a,b)} \mathsf{op}(b_1, \ldots, b_n)$ under the assignment $\mathrm{id}_A : \mathbf{A} \to \mathbf{A}$. This means

$$d_{\mathbf{A}}([\![\mathsf{op}]\!]_A(a_1, \ldots, a_n), [\![\mathsf{op}]\!]_A(b_1, \ldots, b_n)) \leq L_{\mathsf{op}}(d_{\mathbf{A}})(a, b),$$

so we conclude that $[\![\mathsf{op}]\!]_A : (A^n, L_{\mathsf{op}}(d_{\mathbf{A}})) \to \mathbf{A}$ is nonexpansive.

Now, we still have to show that L_{op}-nonexpansiveness is the only consequence of (110). This requires an assumption on L_{op} that morally says the distance between tuples x and y in $(X^n, L_{\mathsf{op}}(d_{\mathbf{X}}))$ depends only on the distances between the coordinates $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ in $\mathbf{X}$.[289] We refer to [MSV22] for more details, in particular Definitions 3.1 and 3.2 give the condition on L_{op}.[290]

As a particular case, one can take $L_{\mathsf{op}}(d)$ to be the product distance and recover the original nonexpansiveness of Example 136. Another interesting instance is taking $L_{\mathsf{op}}(d)$ to be the discrete distance (in case $\mathbf{GMet} = \mathsf{LSpa}$, $\forall x, y \in X^n, L_{\mathsf{op}}(x, y) = \top$), then (110) becomes trivial as we will see in Lemma 152. Intuitively, it is because any function from the discrete space on A^n to $\mathbf{A}$ is nonexpansive.

Example 138 (Convexity)**.** The quantitative algebra $(\mathcal{D}X, [\![-]\!]_{\mathcal{D}X}, d_{\mathrm{K}})$ satisfies another family of quantitative equations that is stronger than (108):[291]

$$\forall \varepsilon, \varepsilon' \in \mathsf{L}, \quad x =_\varepsilon y, x' =_{\varepsilon'} y' \vdash x +_p x' =_{p\varepsilon + \overline{p}\varepsilon'} y +_p y'. \tag{111}$$

This property of $[\![+_p]\!]_{\mathcal{D}X}$ is called convexity in, e.g., [MV20, Definition 30].

As a sanity check for our definitions, we can verify that homomorphisms preserve the satisfaction of quantitative equations.[292]

Lemma 139. *Let ϕ be an equation with context $\mathbf{X}$. If $h : \hat{\mathbb{A}} \to \hat{\mathbb{B}}$ is a homomorphism and $\hat{\mathbb{A}} \vDash^{\hat{\imath}} \phi$ for an assignment $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, then $\hat{\mathbb{B}} \vDash^{h \circ \hat{\imath}} \phi$.*

Proof. We have two very similar cases. Let ϕ be the equation $\mathbf{X} \vdash s = t$, we have

$$\hat{\mathbb{A}} \vDash^{\hat{\imath}} \phi \iff [\![s]\!]_A^{\hat{\imath}} = [\![t]\!]_A^{\hat{\imath}} \qquad \text{definition of } \vDash$$

[287] See [MSV22, Definitions 3.4 and 3.6].

[288] This is the L-NE rule of [MSV22, Definition 3.11], but it has been written more cleanly with quantitative equations with contexts.

[289] This is the case for nonexpansiveness with respect to the product distance. In fact, the only distances that matter there are the pairwise $d_{\mathbf{X}}(x_i, y_i)$ for all i. For L_{op}-nonexpansiveness, the other distances like $d_{\mathbf{X}}(x_1, x_1)$ or $d_{\mathbf{X}}(y_3, x_1)$ may be important, but never $d_{\mathbf{X}}(x, z)$ for some fresh z.

[290] Briefly, we need L_{op} to be a functor that preserves isometric embeddings.

[291] Instead of taking the maximum between ε and ε', we take their convex combination, and since the former is always larger than the latter, (111) is stronger than (108).

[292] Just like we did in Lemma 16 for $\mathbf{Set}$ and Lemma 105 for LSpa. In fact, the proofs are very similar.

$$\begin{aligned}
&\implies h(\llbracket s \rrbracket_A^{\hat{\iota}}) = h(\llbracket t \rrbracket_A^{\hat{\iota}}) \\
&\implies \llbracket s \rrbracket_B^{h\circ\hat{\iota}} = \llbracket t \rrbracket_B^{h\circ\hat{\iota}} && \text{by (10)} \\
&\iff \hat{\mathbb{B}} \vDash^{h\circ\hat{\iota}} \phi. && \text{definition of } \vDash
\end{aligned}$$

Let ϕ be the equation $\mathbf{X} \vdash s =_\varepsilon t$, we have

$$\begin{aligned}
\hat{\mathbb{A}} \vDash^{\hat{\iota}} \phi &\iff d_{\mathbf{A}}(\llbracket s \rrbracket_A^{\hat{\iota}}, \llbracket t \rrbracket_A^{\hat{\iota}}) \leq \varepsilon && \text{definition of } \vDash \\
&\implies d_{\mathbf{A}}(h(\llbracket s \rrbracket_A^{\hat{\iota}}), h(\llbracket t \rrbracket_A^{\hat{\iota}})) \leq \varepsilon \\
&\implies d_{\mathbf{A}}(\llbracket s \rrbracket_B^{h\circ\hat{\iota}}, \llbracket t \rrbracket_B^{h\circ\hat{\iota}}) \leq \varepsilon && \text{by (10)} \\
&\iff \hat{\mathbb{B}} \vDash^{h\circ\hat{\iota}} \phi. && \text{definition of } \vDash \qquad \square
\end{aligned}$$

Definition 140 (Quantitative variety)**.** Given a class $\hat{E}$ of quantitative equations, a $(\Sigma, \hat{E})$**-algebra** is a quantitative Σ-algebra that satisfies $\hat{E}$. We define $\mathbf{QAlg}(\Sigma, \hat{E})$, the category of $(\Sigma, \hat{E})$-algebras, to be the full subcategory of $\mathbf{QAlg}(\Sigma)$ containing only those algebras that satisfy $\hat{E}$. A **quantitative variety** is a category equal to $\mathbf{QAlg}(\Sigma, \hat{E})$ for some class of quantitative equations $\hat{E}$.[293]

There are many forgetful functors obtained by composing the forgetful functors from $\mathbf{QAlg}(\Sigma)$ with the inclusion functor $\mathbf{QAlg}(\Sigma, \hat{E}) \to \mathbf{QAlg}(\Sigma)$:

- $U : \mathbf{QAlg}(\Sigma, \hat{E}) \to \mathbf{Set} = \mathbf{QAlg}(\Sigma, \hat{E}) \to \mathbf{QAlg}(\Sigma) \xrightarrow{U} \mathbf{Set}$
- $U : \mathbf{QAlg}(\Sigma, \hat{E}) \to \mathbf{Alg}(\Sigma) = \mathbf{QAlg}(\Sigma, \hat{E}) \to \mathbf{QAlg}(\Sigma) \xrightarrow{U} \mathbf{Alg}(\Sigma)$
- $U : \mathbf{QAlg}(\Sigma, \hat{E}) \to \mathbf{GMet} = \mathbf{QAlg}(\Sigma, \hat{E}) \to \mathbf{QAlg}(\Sigma) \xrightarrow{U} \mathbf{GMet}$
- $U : \mathbf{QAlg}(\Sigma, \hat{E}) \to \mathsf{L}\mathbf{Spa} = \mathbf{QAlg}(\Sigma, \hat{E}) \to \mathbf{QAlg}(\Sigma) \xrightarrow{U} \mathsf{L}\mathbf{Spa}$

Examples 141. 1. With $\Sigma = \{\mathsf{p} : 0\}$, we now have a lot more varieties than we had in Example 21. Even restricting to a discrete context, we have the following quantitative equations where ε ranges over L:[294]

$$\begin{array}{lllll}
\vdash \mathsf{p} = \mathsf{p} & x \vdash x = x & x \vdash \mathsf{p} = x & & x, y \vdash x = y \\
\vdash \mathsf{p} =_\varepsilon \mathsf{p} & x \vdash x =_\varepsilon x & x \vdash \mathsf{p} =_\varepsilon x & x \vdash x =_\varepsilon \mathsf{p} & x, y \vdash x =_\varepsilon y
\end{array}$$

The meaning of the first row does not change from Example 21, and the meaning of the second row can be inferred by replacing equality between terms with distance between terms. For example, $\vdash \mathsf{p} =_\varepsilon \mathsf{p}$ says that the self-distance of the interpretation of the constant p is at most ε. Classifying the quantitative varieties for this signature would require a lot more work than for the classical varieties.[295]

2. When $\Sigma = \emptyset$, we mentioned that the quantitative equations are those of Chapter 2, so $\mathbf{QAlg}(\emptyset, \hat{E})$ is the subcategory of L-spaces that satisfy $\hat{E}$. In particular, the category $\mathbf{GMet}$ is a quantitative variety as it equals $\mathbf{QAlg}(\emptyset, \hat{E}_{\mathbf{GMet}})$.
3. If $\hat{E}$ contains the equations in $E_{\mathbf{CA}}$ and the equations in (111), then $\mathbf{QAlg}(\Sigma_{\mathbf{CA}}, \hat{E})$ is the category of convex algebras equipped with a convex metric [MV20, Definition 30] and nonexpansive homomorphisms.

[293] We will sometimes simply say variety with the `knowldege` link going to this definition.

[294] The first row comes from the classical case, and the second row replaces equality with equality up to ε ($=_\varepsilon$). The only difference being that $\mathsf{p} =_\varepsilon x$ and $x =_\varepsilon \mathsf{p}$ are not equivalent, so we need two distinct equations.

[295] Although I think it is feasible, tedious but feasible.

Definition 142 (Quantitative algebraic theory)**.** Given a class $\hat{E}$ of quantitative equations over Σ and L, the **quantitative algebraic theory** generated by $\hat{E}$, denoted by $\mathfrak{QTh}(\hat{E})$, is the class of quantitative equations that are satisfied in all $(\Sigma, \hat{E})$-algebras:[296]

$$\mathfrak{QTh}(\hat{E}) = \left\{\phi \mid \forall \hat{\mathbb{A}} \in \mathbf{QAlg}(\Sigma, \hat{E}), \hat{\mathbb{A}} \vDash \phi\right\}.$$

Equivalently, $\mathfrak{QTh}(\hat{E})$ contains the equations that are semantically entailed by $\hat{E}$,[297] namely $\phi \in \mathfrak{QTh}(\hat{E})$ if and only if

$$\forall \hat{\mathbb{A}} \in \mathbf{QAlg}(\Sigma), \quad \hat{\mathbb{A}} \vDash \hat{E} \implies \hat{\mathbb{A}} \vDash \phi. \tag{112}$$

We will see in §3.2 how to find which quantitative equations are entailed by others.

We call a class of quantitative equations a quantitative algebraic theory if it is generated by some class $\hat{E}$.

We will see twice[298] that the algebraic reasoning we are used to from Chapter 1 is embedded in quantitative algebraic reasoning. In particular, Example 23 which showed some equations which belong to the algebraic theory of commutative monoids can be read *unchanged* to find quantitative equations that belong to the quantitative algebraic theory of commutative monoids. These are only about equality ($=$), so let use give another example.

Example 143. We mentioned in Example 138 that the equations for convexity (111) are *stronger* than the equations for nonexpansiveness with respect to the product distance (108). Formally what this means is that if $\hat{E}$ contains (111), then the interpretation of $+_p$ in a $(\Sigma_{\mathbf{CA}}, \hat{E})$-algebra $\hat{\mathbb{A}}$ will be a nonexpansive map $\mathbf{A} \times \mathbf{A} \to \mathbf{A}$, hence $\hat{\mathbb{A}}$ will satisfy (108). Concisely, the equations of (108) belong to $\mathfrak{QTh}(\hat{E})$.

Free Quantitative Algebras

We turn to the construction of free algebras, and we start with a simple example.

Example 144 (Free metric)**.** We already have some intuitions about terms and equations from Example 24, thus we consider an empty signature in order to focus on the new contexts and quantities. For $\hat{E}$, let us take the set of equations defining a metric space (with $\mathsf{L} = [0,1]$),[299] so that $\mathbf{QAlg}(\varnothing, \hat{E}) = \mathbf{Met}$.

Now we wonder, given an L-space $\mathbf{X}$, what is the free metric space on it? Rehashing Definition 39, we want to find a metric space $F\mathbf{X}$ and a nonexpansive map $\eta : \mathbf{X} \to F\mathbf{X}$ such that any nonexpansive map from $\mathbf{X}$ to a metric space $\mathbf{A}$ factors through η uniquely. Of course, if $\mathbf{X}$ is already a metric space, then taking $F\mathbf{X} = \mathbf{X}$ and $\eta = \mathrm{id}_{\mathbf{X}}$ works. Otherwise, we can look at what prevents $d_{\mathbf{X}}$ from being a metric.

For instance, if $\mathbf{X}$ does not satisfy $\vdash x =_0 x$, it means there is some $x \in X$ such that $d_{\mathbf{X}}(x,x) > 0$. Inside $F\mathbf{X}$, we know that the distance between $\eta(x)$ and $\eta(x)$ must be 0. Note that if $\mathbf{A}$ is a metric space and $f : \mathbf{X} \to \mathbf{A}$ is nonexpansive, we know that $d_{\mathbf{A}}(f(x), f(x)) = 0$ too, so sending $\eta(x)$ to $f(x)$ will not be a problem.

For a second example, suppose $d_{\mathbf{X}}$ is not symmetric, i.e. $d_{\mathbf{X}}(x,y) < d_{\mathbf{X}}(y,x)$ for some fixed $x, y \in \mathbf{X}$. We know that $d_{F\mathbf{X}}(\eta(x), \eta(y)) = d_{F\mathbf{X}}(\eta(y), \eta(x))$, but what

[296] Again $\mathfrak{QTh}(\hat{E})$ is never a set (recall Definition 22).

[297] As in the non-quantitative case, $\mathfrak{QTh}(\hat{E})$ contains all of $\hat{E}$ but also many more equations like $x \vdash x = x$ or $x =_\varepsilon y \vdash x =_\varepsilon y$. Furthermore, $\mathfrak{QTh}(\hat{E})$ contains all the quantitative equations in $\hat{E}_{\mathbf{GMet}}$ because the underlying spaces of algebras in $\mathbf{QAlg}(\Sigma, \hat{E})$ belong to $\mathbf{GMet}$.

[298] In Examples 181 and 182.

[299] As a reminder, $\hat{E}$ contains

$$\begin{aligned} \forall \varepsilon \in [0,1], \quad y =_\varepsilon x &\vdash x =_\varepsilon y \\ &\vdash x =_0 x \\ x =_0 y &\vdash x = y \\ \forall \varepsilon, \delta \in [0,1], \quad x =_\varepsilon y, y =_\delta z &\vdash x =_{\varepsilon+\delta} z. \end{aligned}$$

value should it be? To ensure that η is nonexpansive, this value must be at most $d_{\mathbf{X}}(x,y)$, but why not smaller? If this lack of symmetry is the only thing preventing $d_{\mathbf{X}}$ from being a metric (i.e. defining d' everywhere like $d_{\mathbf{X}}$ except $d'(x,y) = d'(y,x)$ yields a metric) we cannot make $d_{F\mathbf{X}}(x,y)$ smaller, because the identity function id_X would be a nonexpansive map $\mathbf{X} \to (X,d')$ that does not factor through η (since $d'(x,y) > d_{F\mathbf{X}}(\eta(x),\eta(y))$). In fact, you can check that $F\mathbf{X} = (X,d')$ with $\eta = \mathrm{id}_X$ be the free metric space on $\mathbf{X}$ because our definition of d' fixed the only problem with $d_{\mathbf{X}}$.

In general, for any $x,y \in X$, we want $d_{F\mathbf{X}}(\eta(x),\eta(y))$ to be as large as possible while guaranteeing that $d_{F\mathbf{X}}$ is a metric and η is nonexpansive,[300] but it is not always that simple. The complexity comes from the possible interactions between different equations in $\hat{E}$. Say you have $d_{\mathbf{X}}(x,z) > d_{\mathbf{X}}(x,y) + d_{\mathbf{X}}(y,z)$ so the triangle inequality does not hold, hence you try to fix this by lowering $d_{F\mathbf{X}}(\eta x,\eta z)$ down exactly to $d_{F\mathbf{X}}(\eta x,\eta y) + d_{F\mathbf{X}}(\eta y,\eta z)$[301]. Then you need to lower $d_{F\mathbf{X}}(z,x)$ down to that same value, but after that you may need to lower $d_{F\mathbf{X}}(x,y)$ so that it is not bigger than the new value of $d_{F\mathbf{X}}(y,z) + d_{F\mathbf{X}}(z,x)$. In the end, you may end up back with $d_{F\mathbf{X}}(x,z) > d_{F\mathbf{X}}(x,y) + d_{F\mathbf{X}}(y,z)$, so you will have to do another round of fixes.

[300] You might think that we also want to guarantee that any $f : \mathbf{X} \to \mathbf{A}$ factors through η. It turns out that automatically holds when $d_{F\mathbf{X}}$ is a metric, but I do not have an explanation for this at the moment.

[301] Let us not write η each time for better readability, this is a bit informal as we will see.

Intuitively, $F\mathbf{X}$ is the space you obtain by iterating (possibly for infinitely many steps) and looking at the limit. We give a rigorous description below in the case of a more general signature, but we want to point out that this process does not deal only with distances, it can also force some equalities. For example, if $d_{\mathbf{X}}(x,y) = 0$ with $x \neq y$ at the start, you will end up with $\eta(x) = \eta(y)$ inside $F\mathbf{X}$.

Fix a class $\hat{E}$ of quantitative equations over Σ and L. For any generalized metric space $\mathbf{X}$, we can define a binary relation $\equiv_{\hat{E}}$ and an L-relation $d_{\hat{E}}$ on Σ-terms as follows:[302] for any $s,t \in \mathcal{T}_\Sigma X$,

$$s \equiv_{\hat{E}} t \iff \mathbf{X} \vdash s = t \in \mathfrak{QTh}(\hat{E}) \text{ and } d_{\hat{E}}(s,t) = \inf\{\varepsilon \mid \mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}(\hat{E})\}. \tag{113}$$

[302] The notation for $\equiv_{\hat{E}}$ and $d_{\hat{E}}$ should really depend on the space $\mathbf{X}$, but we prefer to omit this for better readability.

The definition of $\equiv_{\hat{E}}$ is completely analogous to what we did in the non-quantitative case (20). The definition of $d_{\hat{E}}$ is new but it also looks like how we defined an L-relation from an L-structure in Proposition 92. In fact, we can also prove a counterpart to (66), giving us an equivalent definition of $d_{\hat{E}}$: for any $s,t \in \mathcal{T}_\Sigma X$ and $\varepsilon \in \mathsf{L}$,[303]

$$d_{\hat{E}}(s,t) \leq \varepsilon \iff \mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}(\hat{E}). \tag{114}$$

[303] In words, $d_{\hat{E}}$ assigns a distance below ε to s and t if and only if their interpretations in each $(\Sigma,\hat{E})$-algebras are always at a distance below ε.

Proof of (114). ($\Leftarrow$) holds directly by definition of infimum. For ($\Rightarrow$), we need to show that any $(\Sigma,\hat{E})$-algebra satisfies $\mathbf{X} \vdash s =_\varepsilon t$. Let $\hat{\mathbb{A}} \in \mathbf{QAlg}(\Sigma,\hat{E})$ and $\hat{\imath} : \mathbf{X} \to \mathbf{A}$ be a nonexpansive assignment. We know that for every δ such that $\mathbf{X} \vdash s =_\delta t \in \mathfrak{QTh}(\hat{E})$, $d_{\mathbf{A}}([\![s]\!]_A^{\hat{\imath}}, [\![t]\!]_A^{\hat{\imath}}) \leq \delta$, thus

$$d_{\mathbf{A}}([\![s]\!]_A^{\hat{\imath}}, [\![t]\!]_A^{\hat{\imath}}) \leq \inf\{\delta \mid \mathbf{X} \vdash s =_\delta t \in \mathfrak{QTh}(\hat{E})\} = d_{\hat{E}}(s,t) \leq \varepsilon.$$

We conclude that $\hat{\mathbb{A}} \vDash^{\hat{\imath}} \mathbf{X} \vdash s =_\varepsilon t$, and we are done since $\hat{\mathbb{A}}$ and $\hat{\imath}$ were arbitrary. $\square$

Remark 145. In Example 144, we said that the distance should be made as large as possible while 1. ensuring the satisfaction of $\hat{E}$, and 2. ensuring some embedding

η is nonexpansive.[304] Let us see informally how to recover these two ideas in the definition of $d_{\hat{E}}$.

1.

Of course this is formally proven in what follows.

When we were not dealing with distances, we only had to prove that the relation $\equiv_E$ defined between terms was a congruence (Lemma 25), and then we were able to construct the term algebra by quotienting the set of terms and interpreting the operation symbols syntactically. Here we have to prove a bit more, namely that $d_{\hat{E}}$ is invariant under $\equiv_{\hat{E}}$ so the L-relation restricts to the quotient, and that the resulting L-space is a generalized metric space.

Let us decompose this in several small lemmas. We also collect here some more lemmas that look similar, many of which will be part of the proof of soundness when we introduce quantitative equational logic.[305] Let $\mathbf{X} \in \mathsf{L}\mathbf{Spa}$ and $\hat{\mathbb{A}} \in \mathbf{QAlg}(\Sigma)$ be universally quantified in all these lemmas.

First, Lemmas 146–149 say that $\equiv_{\hat{E}}$ is an equivalence relation and a congruence.[306]

[304] On a first read, there seems to be a conflict with defining $d_{\hat{E}}$ as an infimum and saying it should be as large as possible. Recall however that $\inf S$ is the *greatest* lower bound of S, and that is precisely what we need.

[305] We were less explicit back then, but that is what happened with Lemma 25 and soundness of equational logic.

[306] The proofs are exactly the same as for Lemma 25 because $\equiv_{\hat{E}}$ does not involve distances.

Lemma 146. *For any $t \in \mathcal{T}_\Sigma X$, $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash t = t$.*

Proof. Obviously, $[\![t]\!]_A^{\hat{\imath}} = [\![t]\!]_A^{\hat{\imath}}$ holds for all $\hat{\imath} : \mathbf{X} \to \mathbf{A}$. □

Lemma 147. *For any $s, t \in \mathcal{T}_\Sigma X$, if $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s = t$, then $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash t = s$.*

Proof. If $[\![s]\!]_A^{\hat{\imath}} = [\![t]\!]_A^{\hat{\imath}}$ holds for all $\hat{\imath}$, then $[\![t]\!]_A^{\hat{\imath}} = [\![s]\!]_A^{\hat{\imath}}$ holds too. □

Lemma 148. *For any $s, t, u \in \mathcal{T}_\Sigma X$, if $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s = t$ and $\mathbf{X} \vdash t = u$, then $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s = u$.*

Proof. If $[\![s]\!]_A^{\hat{\imath}} = [\![t]\!]_A^{\hat{\imath}}$ and $[\![t]\!]_A^{\hat{\imath}} = [\![u]\!]_A^{\hat{\imath}}$ holds for all $\hat{\imath}$, then $[\![s]\!]_A^{\hat{\imath}} = [\![u]\!]_A^{\hat{\imath}}$ holds too. □

Lemma 149. *For any $\mathsf{op} : n \in \Sigma$, $s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathcal{T}_\Sigma X$, if $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s_i = t_i$ for all $1 \leq i \leq n$, then $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash \mathsf{op}(s_1, \ldots, s_n) = \mathsf{op}(t_1, \ldots, t_n)$.*

Proof. For any assignment $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, we have $[\![s_i]\!]_A^{\hat{\imath}} = [\![t_i]\!]_A^{\hat{\imath}}$ for all i. Hence,

$$\begin{aligned}
[\![\mathsf{op}(s_1, \ldots, s_n)]\!]_A^{\hat{\imath}} &= [\![\mathsf{op}]\!]_A([\![s_1]\!]_A^{\hat{\imath}}, \ldots, [\![s_n]\!]_A^{\hat{\imath}}) && \text{by (7)} \\
&= [\![\mathsf{op}]\!]_A([\![t_1]\!]_A^{\hat{\imath}}, \ldots, [\![t_n]\!]_A^{\hat{\imath}}) && \forall i, [\![s_i]\!]_A^{\hat{\imath}} = [\![t_i]\!]_A^{\hat{\imath}} \\
&= [\![\mathsf{op}(s_1, \ldots, s_n)]\!]_A^{\hat{\imath}}. && \text{by (7)}
\end{aligned}$$
□

Lemmas 150 and 151 mean that $d_{\hat{E}}$ is well-defined on equivalence classes of $\equiv_{\hat{E}}$, namely, $d_{\hat{E}}(s,t) = d_{\hat{E}}(s',t')$ whenever $s \equiv_{\hat{E}} s'$ and $t \equiv_{\hat{E}} t'$.[307]

[307] By Lemmas 147 and 150, if $t \equiv_{\hat{E}} t'$, then

$$\mathbf{X} \vdash s =_\varepsilon t \iff \mathbf{X} \vdash s =_\varepsilon t.$$

By Lemmas 147 and 151, if $s \equiv_{\hat{E}} s'$, then

$$\mathbf{X} \vdash s =_\varepsilon t' \iff \mathbf{X} \vdash s' =_\varepsilon t'.$$

Combining these with (114), we get

$$d_{\hat{E}}(s,t) \leq \varepsilon \iff d_{\hat{E}}(s',t') \leq \varepsilon,$$

for all $\varepsilon \in \mathsf{L}$, and we conclude $d_{\hat{E}}(s,t) = d_{\hat{E}}(s',t')$.

Lemma 150. *For any $s, t, t' \in \mathcal{T}_\Sigma X$ and $\varepsilon \in \mathsf{L}$, if $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s =_\varepsilon t$ and $\mathbf{X} \vdash t = t'$, then $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s =_\varepsilon t'$.*

Proof. For any $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, we have $d_{\mathbf{A}}([\![s]\!]_A^{\hat{\imath}}, [\![t]\!]_A^{\hat{\imath}}) \leq \varepsilon$ and $[\![t]\!]_A^{\hat{\imath}} = [\![t']\!]_A^{\hat{\imath}}$, thus

$$d_{\mathbf{A}}([\![s]\!]_A^{\hat{\imath}}, [\![t']\!]_A^{\hat{\imath}}) = d_{\mathbf{A}}([\![s]\!]_A^{\hat{\imath}}, [\![t]\!]_A^{\hat{\imath}}) \leq \varepsilon.$$
□

Lemma 151. *For any $s, s', t \in \mathcal{T}_\Sigma X$ and $\varepsilon \in \mathsf{L}$, if $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s =_\varepsilon t$ and $\mathbf{X} \vdash s = s'$, then $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s' =_\varepsilon t$.*

Proof. Symmetric argument to the previous proof. □

Lemmas 152–155 will correspond to other rules in quantitative equational logic, and they will be explained in more details in §3.2.

Lemma 152. *For any $s, t \in \mathcal{T}_\Sigma X$, $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s =_\top t$.*

Proof. By definition of $\top$ (the supremum of all L), for any $\hat{\imath}$, $d_{\mathbf{A}}(\llbracket s \rrbracket_A^{\hat{\imath}}, \llbracket t \rrbracket_A^{\hat{\imath}}) \leq \top$. □

Lemma 153. *For any $x, x' \in X$, if $d_{\mathbf{X}}(x, x') = \varepsilon$, then $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash x =_\varepsilon x'$.*

Proof. For any nonexpansive $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, we have[308]

$$d_{\mathbf{A}}(\llbracket x \rrbracket_A^{\hat{\imath}}, \llbracket x' \rrbracket_A^{\hat{\imath}}) = d_{\mathbf{A}}(\hat{\imath}(x), \hat{\imath}(x')) \leq d_{\mathbf{X}}(x, x') = \varepsilon. \qquad \square$$

[308] The equation holds by definition of $\llbracket - \rrbracket_A^{\hat{\imath}}$ on variables, and the inequality holds by definition of nonexpansiveness.

Lemma 154. *For any $s, t \in \mathcal{T}_\Sigma X$ and $\varepsilon, \varepsilon' \in \mathsf{L}$, if $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s =_\varepsilon t$ and $\varepsilon \leq \varepsilon'$, then $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s =_{\varepsilon'} t$.*[309]

[309] In words, if the interpretations of s and t are at distance at most ε, then they are also at distance at most ε' when $\varepsilon \leq \varepsilon'$.

Proof. For any $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, we have $d_{\mathbf{A}}(\llbracket s \rrbracket_A^{\hat{\imath}}, \llbracket t \rrbracket_A^{\hat{\imath}}) \leq \varepsilon \leq \varepsilon'$. □

Lemma 155. *For any $s, t \in \mathcal{T}_\Sigma X$ and $\{\varepsilon_i\}_{i \in I} \subseteq \mathsf{L}$, if $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s =_{\varepsilon_i} t$ for all $i \in I$, then $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s =_\varepsilon t$ with $\varepsilon = \inf_{i \in I} \varepsilon_i$.*

Proof. For any $\hat{\imath}$ and for all $i \in I$, we have $d_{\mathbf{A}}(\llbracket s \rrbracket_A^{\hat{\imath}}, \llbracket t \rrbracket_A^{\hat{\imath}}) \leq \varepsilon_i$ by hypothesis. By definition of infimum, this means $d_{\mathbf{A}}(\llbracket s \rrbracket_A^{\hat{\imath}}, \llbracket t \rrbracket_A^{\hat{\imath}}) \leq \inf_{i \in I} \varepsilon_i = \varepsilon$. □

This shall take care of all except two rules in quantitative equational logic which we will get to in no time. The following result is a generalization of Lemma 99, and it morally says that $\mathcal{T}_\Sigma f$ is well-defined and nonexpansive when f is nonexpansive.

Lemma 156. *Let $f : \mathbf{X} \to \mathbf{Y}$ be a nonexpansive map. If $\mathbf{A}$ satisfies $\mathbf{X} \vdash s = t$ (resp. $\mathbf{X} \vdash s =_\varepsilon t$), then $\mathbf{A}$ satisfies $\mathbf{Y} \vdash \mathcal{T}_\Sigma f(s) = \mathcal{T}_\Sigma f(t)$ (resp. $\mathbf{Y} \vdash \mathcal{T}_\Sigma f(s) =_\varepsilon \mathcal{T}_\Sigma f(t)$).*[310]

[310] Note that when s and t are variables, we get back Lemma 99.

Proof. Any nonexpansive assignment $\hat{\imath} : \mathbf{Y} \to \mathbf{A}$, yields a nonexpansive assignment $\hat{\imath} \circ f : \mathbf{X} \to \mathbf{A}$. Moreover, by functoriality of $\mathcal{T}_\Sigma$, we have

$$\llbracket - \rrbracket_A^{\hat{\imath} \circ f} \overset{(8)}{=} \llbracket - \rrbracket_A \circ \mathcal{T}_\Sigma(\hat{\imath} \circ f) = \llbracket - \rrbracket_A \circ \mathcal{T}_\Sigma \hat{\imath} \circ \mathcal{T}_\Sigma f = \llbracket \mathcal{T}_\Sigma f(-) \rrbracket_A^{\hat{\imath}}.$$

By hypothesis, we have

$$\mathbf{A} \vDash^{\hat{\imath} \circ f} \mathbf{X} \vdash s = t \qquad (\text{resp. } \mathbf{A} \vDash^{\hat{\imath} \circ f} \mathbf{X} \vdash s =_\varepsilon t),$$

which means

$$\begin{aligned} \llbracket \mathcal{T}_\Sigma f(s) \rrbracket_A^{\hat{\imath}} &= \llbracket s \rrbracket_A^{\hat{\imath} \circ f} = \llbracket t \rrbracket_A^{\hat{\imath} \circ f} = \llbracket \mathcal{T}_\Sigma f(t) \rrbracket_A^{\hat{\imath}} \\ \text{resp. } d_{\mathbf{A}}(\llbracket \mathcal{T}_\Sigma f(s) \rrbracket_A^{\hat{\imath}}, \llbracket \mathcal{T}_\Sigma f(t) \rrbracket_A^{\hat{\imath}}) &= d_{\mathbf{A}}(\llbracket s \rrbracket_A^{\hat{\imath} \circ f}, \llbracket t \rrbracket_A^{\hat{\imath} \circ f}) \leq \varepsilon. \end{aligned}$$

Thus, we conclude

$$\mathbf{A} \vDash^{\hat{\imath}} \mathbf{Y} \vdash \mathcal{T}_\Sigma f(s) = \mathcal{T}_\Sigma f(t) \qquad (\text{resp. } \mathbf{A} \vDash^{\hat{\imath}} \mathbf{Y} \vdash \mathcal{T}_\Sigma f(s) =_\varepsilon \mathcal{T}_\Sigma f(t)). \qquad \square$$

Let us end our list of small results with Lemmas 157–159 which are for later.

Lemma 157. *For any $s, t \in \mathcal{T}_\Sigma X$ if $\hat{\mathbb{A}}$ satisfies $\mathbf{X}_\top \vdash s = t$, then $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s = t$, and for any $\varepsilon \in \mathsf{L}$, if $\hat{\mathbb{A}}$ satisfies $\mathbf{X}_\top \vdash s =_\varepsilon t$, then $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s =_\varepsilon t$.*[311]

Proof. For any nonexpansive assignment $\hat{\iota} : \mathbf{X} \to \mathbf{A}$, you can pre-compose it with $\mathrm{id}_X : \mathbf{X}_\top \to \mathbf{X}$ (which is nonexpansive) without changing the interpretation of terms: $[\![s]\!]_A^{\hat{\iota}} = [\![s]\!]_A^{\hat{\iota}\circ\mathrm{id}_X}$. By hypothesis, we know that $\hat{\mathbb{A}}$ satisfies $s = t$ (resp. $s =_\varepsilon t$) under the nonexpansive assignment $\hat{\iota} \circ \mathrm{id}_X : \mathbf{X}_\top \to \mathbf{A}$, and we conclude $\hat{\mathbb{A}}$ also satisfies $s = t$ (resp. $s =_\varepsilon t$) under the assignment $\hat{\iota}$. □

Lemma 158. *For any $s, t \in \mathcal{T}_\Sigma X$, if $\mathbb{A}$ satisfies $X \vdash s = t$, then $\hat{\mathbb{A}}$ satisfies $\mathbf{X} \vdash s = t$.*[312]

Proof. Any nonexpansive assignment $\hat{\iota} : \mathbf{X} \to \mathbf{A}$ is in particular an assignment $\hat{\iota} : X \to A$, thus $[\![s]\!]_A^{\hat{\iota}} = [\![t]\!]_A^{\hat{\iota}}$ hold by hypothesis that $\mathbb{A}$ satisfies $X \vdash s = t$. □

Lemma 159. *For any $s, t \in \mathcal{T}_\Sigma X$, if $\hat{\mathbb{A}}$ satisfies $\mathbf{X}_\top \vdash s = t$, then $\mathbb{A}$ satisfies $X \vdash s = t$.*[313]

Proof. This follows by definition of the discrete space $\mathbf{X}_\top$. Indeed, any assignment $\iota : X \to A$ is the underlying function of a nonexpansive assignment $\hat{\iota} : \mathbf{X} \to \mathbf{A}$, and since $\hat{\mathbb{A}}$ satisfies $s = t$ under $\hat{\iota}$ by hypothesis, $\mathbb{A}$ satisfies $s = t$ under ι. □

We can now get back to the equality $\equiv_{\hat{E}}$ and distance $d_{\hat{E}}$ between terms, and define the underlying space of the quantitative term algebra.

Since $\equiv_{\hat{E}}$ is an equivalence relation for any $\mathbf{X}$, we can consider the set $\mathcal{T}_\Sigma X/\equiv_{\hat{E}}$ of **terms modulo** $\hat{E}$.[314] We denote with $[-]_{\hat{E}} : \mathcal{T}_\Sigma X \to \mathcal{T}_\Sigma X/\equiv_{\hat{E}}$ the canonical quotient map, and by Lemmas 150 and 151, we can define an L-relation on terms modulo $\hat{E}$ by factoring $d_{\hat{E}}$ through $[-]_{\hat{E}}$. We obtain the L-relation $d_{\hat{E}}$ as the unique function making the triangle below commute.[315]

$$\begin{tikzcd} \mathcal{T}_\Sigma X \times \mathcal{T}_\Sigma X \arrow[r, "d_{\hat{E}}"] \arrow[d, "{[-]_{\hat{E}} \times [-]_{\hat{E}}}"'] & \mathsf{L} \\ \mathcal{T}_\Sigma X/\equiv_{\hat{E}} \times \mathcal{T}_\Sigma X/\equiv_{\hat{E}} \arrow[ur, dashed, "d_{\hat{E}}"'] & \end{tikzcd} \tag{116}$$

We write $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ for the resulting L-space $(\mathcal{T}_\Sigma X/\equiv_{\hat{E}}, d_{\hat{E}})$. We still have an alternative definition analog to (114) for the new L-relation d_E.[316]

$$d_{\hat{E}}([s]_{\hat{E}}, [t]_{\hat{E}}) \leq \varepsilon \iff \mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}(\hat{E}). \tag{117}$$

This will be the carrier of the term algebra on $\mathbf{X}$, so we need to prove that $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ belongs to **GMet**. We rely on the following generalization of Lemma 37. It essentially states that satisfaction of quantitative equations is preserved by substitutions that are nonexpansive. This result will also take care of the last two rules of quantitative equational logic.

Lemma 160. *Let $\mathbf{Y}$ be an L-space and $\sigma : Y \to \mathcal{T}_\Sigma X$ be an assignment such that*[317]

$$\forall y, y' \in Y, \quad \mathbf{X} \vdash \sigma(y) =_{d_{\mathbf{Y}}(y,y')} \sigma(y') \in \mathfrak{QTh}(\hat{E}), \tag{118}$$

and $\hat{\mathbb{A}}$ a $(\Sigma, \hat{E})$-algebra. If $\hat{\mathbb{A}}$ satisfies $\mathbf{Y} \vdash s = t$ (resp. $\mathbf{Y} \vdash s =_\varepsilon t$), then it also satisfies $\mathbf{X} \vdash \sigma^*(s) = \sigma^*(t)$ (resp. $\mathbf{X} \vdash \sigma^*(s) =_\varepsilon \sigma^*(t)$).

[311] In words, if $\hat{\mathbb{A}}$ satisfies an equation where the context is the discrete space on X, then $\hat{\mathbb{A}}$ satisfies that same equation with the context replaced by any other L-space on X. This is also a special case of Lemma 156 where $f : \mathbf{X}_\top \to \mathbf{X}$ is the identity map.

[312] In words, if the underlying (not quantitative) algebra satisfies an equation, then so does the quantitative algebra where the context can be endowed with any L-relation.

[313] Combining Lemmas 158 and 159, we find

$$\mathbb{A} \vDash X \vdash s = t \iff \hat{\mathbb{A}} \vDash \mathbf{X}_\top \vdash s = t. \tag{115}$$

This can be useful when comparing equational logic and quantitative equational logic in Example 182.

[314] Keep in mind that for different L-relations on X, we may get different equivalence relations on $\mathcal{T}_\Sigma X$.

[315] We used the same symbol, because the first $d_{\hat{E}}$ was only used to define this new $d_{\hat{E}}$.

[316] In particular, the quotient map is nonexpansive:

$$[-]_{\hat{E}} : (\mathcal{T}_\Sigma X, d_{\hat{E}}) \to \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}.$$

[317] By combining (118) with (114) we find that σ is a nonexpansive map $\mathbf{Y} \to (\mathcal{T}_\Sigma X, d_{\hat{E}})$, and any such nonexpansive map satisfies (118). We explicitly write (118) to better emulate the corresponding rules in quantitative equational logic.

Proof. Let $\hat{\imath} : \mathbf{X} \to \mathbf{A}$ be a nonexpansive assignment, we need to show $[\![\sigma^*(s)]\!]_A^{\hat{\imath}} = [\![\sigma^*(t)]\!]_A^{\hat{\imath}}$ (resp. $d_{\mathbf{A}}([\![\sigma^*(s)]\!]_A^{\hat{\imath}}, [\![\sigma^*(t)]\!]_A^{\hat{\imath}}) \leq \varepsilon$). Just like in Lemma 37, we define the assignment $\hat{\imath}_\sigma : Y \to A$ that sends $y \in Y$ to $[\![\sigma(y)]\!]_A^{\hat{\imath}}$, and we had already proven $[\![-]\!]_A^{\hat{\imath}_\sigma} = [\![\sigma^*(-)]\!]_A^{\hat{\imath}}$. Now, it is enough to show $\hat{\imath}_\sigma$ is nonexpansive $\mathbf{Y} \to \mathbf{A}$[318] and the lemma will follow because by hypothesis, $[\![s]\!]_A^{\hat{\imath}_\sigma} = [\![t]\!]_A^{\hat{\imath}_\sigma}$ (reps. $d_{\mathbf{A}}([\![s]\!]_A^{\hat{\imath}_\sigma}, [\![t]\!]_A^{\hat{\imath}_\sigma}) \leq \varepsilon$).

[318] Something we did not have to do in the non-quantitative case.

For any $y, y' \in Y$, we have

$$d_{\mathbf{A}}(\hat{\imath}_\sigma(y), \hat{\imath}_\sigma(y')) = d_{\mathbf{A}}([\![\sigma(y)]\!]_A^{\hat{\imath}}, [\![\sigma(y')]\!]_A^{\hat{\imath}}) \leq d_{\mathbf{Y}}(y, y'),$$

where the equation holds by definition of $\hat{\imath}_\sigma$, and the inequality holds because $\hat{\mathbb{A}}$ belongs to $\mathbf{QAlg}(\Sigma, \hat{E})$ and hence satisfies $\mathbf{X} \vdash \sigma(y) =_{d_{\mathbf{Y}}(y,y')} \sigma(y') \in \mathfrak{QTh}(\hat{E})$ (in particular under the nonexpansive assignment $\hat{\imath}$). Hence $\hat{\imath}_\sigma$ is nonexpansive. $\square$

Lemma 161. *For any* L*-space* $\mathbf{X}$ *and any quantitative equation* $\phi \in \hat{E}_{\mathbf{GMet}}$, $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \vDash \phi$.

Proof. We mentioned in Footnote 297 that $\phi \in \mathfrak{QTh}(\hat{E})$ because the carriers of $(\Sigma, \hat{E})$-algebras are generalized metric spaces, so any $(\Sigma, \hat{E})$-algebra $\hat{\mathbb{A}}$ satisfies it.

Let $\hat{\imath} : \mathbf{Y} \to \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ is a nonexpansive assignment. By the axiom of choice,[319] there is a function $\sigma : Y \to \mathcal{T}_\Sigma X$ satisfying $[\sigma(y)]_{\hat{E}} = \hat{\imath}(y)$ for all $y \in Y$. This assignment satisfies (118) because for all $y, y' \in Y$, (117) yields

[319] Choice implies the quotient map $[-]_{\hat{E}}$ has a right inverse $r : \mathcal{T}_\Sigma X/\equiv_{\hat{E}} \to \mathcal{T}_\Sigma X$, and we set $\sigma = r \circ \hat{\imath}$.

$$d_{\hat{E}}([\sigma(y)]_{\hat{E}}, [\sigma(y')]_{\hat{E}}) \leq d_{\mathbf{Y}}(y, y') \overset{(117)}{\iff} \mathbf{X} \vdash \sigma(y) =_{d_{\mathbf{Y}}(y,y')} \sigma(y') \in \mathfrak{QTh}(\hat{E}),$$

and the $\mathtt{L.H.S.}$ holds because $\hat{\imath}$ is nonexpansive.

Therefore, if ϕ has the shape $\mathbf{Y} \vdash y = y'$ (resp. $\mathbf{Y} \vdash y =_\varepsilon y'$), by Lemma 160, all $(\Sigma, \hat{E})$-algebras satisfy $\mathbf{X} \vdash \sigma(y) = \sigma(y')$ (resp. $\mathbf{X} \vdash \sigma(y) =_\varepsilon \sigma(y')$). By definition of $\equiv_{\hat{E}}$ (resp. by definition of $d_{\hat{E}}$ (117)), we have

$$\hat{\imath}(y) = [\sigma(y)]_{\hat{E}} = [\sigma(y')]_{\hat{E}} = \hat{\imath}(y') \quad (\text{resp. } d_{\hat{E}}(\hat{\imath}(y), \hat{\imath}(y')) = d_{\hat{E}}([\sigma(y)]_{\hat{E}}, [\sigma(y')]_{\hat{E}}) \leq \varepsilon\ ),$$

which means $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ satisfies ϕ under $\hat{\imath}$. Since $\hat{\imath}$ and ϕ were arbitrary, we conclude $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ satisfies all of $\hat{E}_{\mathbf{GMet}}$, i.e. it is a generalized metric space. $\square$

As for **Set**, we obtain a functor $\widehat{\mathcal{T}}_{\Sigma,\hat{E}} : \mathbf{GMet} \to \mathbf{GMet}$[320] by setting $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f$ equal to the unique function making (119) commute. Concretely, we have $\widehat{\mathcal{T}}_{\Sigma,E}f([t]_{\hat{E}}) = [\mathcal{T}_\Sigma f(t)]_{\hat{E}}$ which is well-defined by one part of Lemma 156.

[320] In fact, we defined a functor $\mathsf{L}\mathbf{Spa} \to \mathbf{GMet}$, but we are interested in its restriction to **GMet**.

$$\begin{tikzcd} \mathcal{T}_\Sigma X \arrow[r, "{[-]_{\hat{E}}}"] \arrow[d, "\mathcal{T}_\Sigma f"'] & \mathcal{T}_\Sigma X/\equiv_{\hat{E}} \arrow[d, dashed, "\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f"] \\ \mathcal{T}_\Sigma Y \arrow[r, "{[-]_{\hat{E}}}"'] & \mathcal{T}_\Sigma Y/\equiv_{\hat{E}} \end{tikzcd} \tag{119}$$

Although we do have to check that $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f$ is nonexpansive whenever f is, and we use the other part of Lemma 156.

Lemma 162. *If* $f : \mathbf{X} \to \mathbf{Y}$ *is nonexpansive, then so is* $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f : \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \to \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{Y}$.

Proof. For any $s, t \in \mathcal{T}_\Sigma X$, we have

$$\begin{aligned} d_{\hat{E}}([s]_{\hat{E}}, [t]_{\hat{E}}) \leq \varepsilon &\iff \mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}(\hat{E}) && \text{by (117)} \\ &\implies \mathbf{X} \vdash \mathcal{T}_\Sigma f(s) =_\varepsilon \mathcal{T}_\Sigma f(t) \in \mathfrak{QTh}(\hat{E}) && \text{Lemma 156} \\ &\iff d_{\hat{E}}([\mathcal{T}_\Sigma f(s)]_{\hat{E}}, [\mathcal{T}_\Sigma f(t)]_{\hat{E}}) \leq \varepsilon && \text{by (117)} \\ &\iff d_{\hat{E}}(\widehat{\mathcal{T}}_{\Sigma,\hat{E}} f[s]_{\hat{E}}, \widehat{\mathcal{T}}_{\Sigma,\hat{E}} f[t]_{\hat{E}}) \leq \varepsilon. && \text{by (119)} \end{aligned}$$

Therefore, $d_{\hat{E}}(\widehat{\mathcal{T}}_{\Sigma,\hat{E}} f[s]_{\hat{E}}, \widehat{\mathcal{T}}_{\Sigma,\hat{E}} f[t]_{\hat{E}}) \leq d_{\hat{E}}([s]_{\hat{E}}, [t]_{\hat{E}})$. $\square$

We may now define the interpretation of operation symbols syntactically to obtain the quantitative term algebra.

Definition 163 (Quantitative term algebra, semantically)**.** The **quantitative term algebra** for $(\Sigma, \hat{E})$ on $\mathbf{X}$ is the quantitative Σ-algebra whose underlying space is $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ and whose interpretation of $\mathsf{op} : n \in \Sigma$ is defined by[321]

$$[\![\mathsf{op}]\!]_{\widehat{\mathbb{T}}\mathbf{X}}([t_1]_{\hat{E}}, \dots, [t_n]_{\hat{E}}) = [\mathsf{op}(t_1, \dots, t_n)]_{\hat{E}}. \tag{120}$$

We denote this algebra by $\widehat{\mathbb{T}}_{\Sigma,\hat{E}}\mathbf{X}$ or simply $\widehat{\mathbb{T}}\mathbf{X}$.

This should feel very familiar to what we had done in Definition 26.[322] In particular, we still have that $[-]_{\hat{E}}$ is a homomorphism from $\mathcal{T}_\Sigma X$ to the underlying algebra of $\widehat{\mathbb{T}}\mathbf{X}$,[323] namely, (121) commutes (recall Footnote 20).

$$\begin{tikzcd} \mathcal{T}_\Sigma \mathcal{T}_\Sigma X \arrow[r, "\mathcal{T}_\Sigma[-]_{\hat{E}}"] \arrow[d, "\mu^\Sigma_X"'] & \mathcal{T}_\Sigma \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[d, "[\![-]\!]_{\widehat{\mathbb{T}}\mathbf{X}}"] \\ \mathcal{T}_\Sigma X \arrow[r, "{[-]_{\hat{E}}}"'] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \end{tikzcd} \tag{121}$$

While (121) is a diagram in **Set**, we write $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ instead of the underlying set $\mathcal{T}_\Sigma X/{\equiv_{\hat{E}}}$ for better readability. We will keep this habit.

Your intuition for $[\![-]\!]_{\widehat{\mathbb{T}}\mathbf{X}}$ (the interpretation of arbitrary terms) should be exactly the same as the one for $[\![-]\!]_{\mathbb{T}X}$ in *classical* universal algebra: it takes a term in $\mathcal{T}_\Sigma \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$, replaces the leaves with a representative term, and gives back the equivalence class of the resulting term. We can also use it to define an analog to flattening.[324] For any space $\mathbf{X}$, let $\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{X}}$ be the unique function making (122) commute.

$$\begin{tikzcd} \mathcal{T}_\Sigma \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[rr, "[\![-]\!]_{\widehat{\mathbb{T}}\mathbf{X}}"] \arrow[dr, "{[-]_{\hat{E}}}"'] & & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \\ & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[ur, dashed, "\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{X}}"'] & \end{tikzcd} \tag{122}$$

Let us show that $\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{X}}$ is nonexpansive and natural.

Lemma 164. *For any space $\mathbf{X}$, $\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{X}}$ is a nonexpansive map $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \to \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$.*

[321] This is well-defined by Lemma 149.

[322] In fact, we can make the connection more precise, $\mathbb{T}X$ is constructed by quotienting $\mathcal{T}_\Sigma X$ by the congruence $\equiv_E$ and (the underlying algebra of) $\widehat{\mathbb{T}}\mathbf{X}$ by quotienting $\mathcal{T}_\Sigma X$ by the congruence $\equiv_{\hat{E}}$ (see Remark 27).

[323] Put $h = [-]_{\hat{E}}$ in (1) to get (120)

[324] Just as we did in (26).

Proof. Let $[s]_{\hat{E}}, [t]_{\hat{E}} \in \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ be such that $d_{\hat{E}}([s]_{\hat{E}}, [t]_{\hat{E}}) \leq \varepsilon$. By (117), this means

$$\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}(\hat{E}), \tag{123}$$

namely, the distance between interpretations of s and t is bounded above by ε in all $(\Sigma, \hat{E})$-algebras. We need to show $d_{\hat{E}}(\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{X}}([s]_{\hat{E}}), \widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{X}}([t]_{\hat{E}})) \leq \varepsilon$, or using (122),

$$d_{\hat{E}}(\llbracket s \rrbracket_{\widehat{\mathbb{T}}\mathbf{X}}, \llbracket t \rrbracket_{\widehat{\mathbb{T}}\mathbf{X}}) \leq \varepsilon. \tag{124}$$

We want to use (117) again to reduce that inequality to a bound on distances between interpretations, but that requires choosing representatives for $\llbracket s \rrbracket_{\widehat{\mathbb{T}}\mathbf{X}}, \llbracket t \rrbracket_{\widehat{\mathbb{T}}\mathbf{X}} \in \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$.

Instead of choosing them naively, let $s', t' \in \mathcal{T}_\Sigma\mathcal{T}_\Sigma X$ be such that $\mathcal{T}_\Sigma[-]_{\hat{E}}(s') = s$ and $\mathcal{T}_\Sigma[-]_{\hat{E}}(t') = t$. In words, s' and t' are the same as s and t where equivalence classes at the leaves are replaced representative terms.[325] Commutativity of (121) implies $[\mu^\Sigma_X(s')]_{\hat{E}} = \llbracket s \rrbracket_{\widehat{\mathbb{T}}\mathbf{X}}$ and similarly for t. We can now use (117) to infer that proving (124) is equivalent to proving

[325] Since s and t have finitely many leaves, we are only doing finitely many choices of representatives.

$$\mathbf{X} \vdash \mu^\Sigma_X(s') =_\varepsilon \mu^\Sigma_X(t') \in \mathfrak{QTh}(\hat{E}). \tag{125}$$

This means we need to show that, for all $\hat{\mathbb{A}} \in \mathbf{QAlg}(\Sigma, \hat{E})$ and $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, $d_{\mathbf{A}}(\llbracket \mu^\Sigma_X(s') \rrbracket^{\hat{\imath}}_A, \llbracket \mu^\Sigma_X(t') \rrbracket^{\hat{\imath}}_A) \leq \varepsilon$.

We already know by (123) that for all $\hat{\sigma} : \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \to \mathbf{A}$, $d_{\mathbf{A}}(\llbracket s \rrbracket^{\hat{\sigma}}_A, \llbracket t \rrbracket^{\hat{\sigma}}_A) \leq \varepsilon$, so it suffices to find, for each $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, a nonexpansive assignment $\hat{\sigma}_{\hat{\imath}} : \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \to \mathbf{A}$ such that

$$\llbracket \mu^\Sigma_X(s') \rrbracket^{\hat{\imath}}_A = \llbracket s \rrbracket^{\hat{\sigma}_{\hat{\imath}}}_A \text{ and } \llbracket \mu^\Sigma_X(t') \rrbracket^{\hat{\imath}}_A = \llbracket t \rrbracket^{\hat{\sigma}_{\hat{\imath}}}_A. \tag{126}$$

We define $\hat{\sigma}_{\hat{\imath}} : \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \to \mathbf{A}$ to be the unique function making (127) commute.[326]

[326] It exists because $\hat{\mathbb{A}}$ satisfies all the equations in $\mathfrak{QTh}(\hat{E})$ so if $s \equiv_{\hat{E}} t$ then

$$\llbracket \mathcal{T}_\Sigma \hat{\imath}(s) \rrbracket_A \overset{(8)}{=} \llbracket s \rrbracket^{\hat{\imath}}_A = \llbracket t \rrbracket^{\hat{\imath}}_A \overset{(8)}{=} \llbracket \mathcal{T}_\Sigma \hat{\imath}(t) \rrbracket_A.$$

$$\begin{tikzcd}
\mathcal{T}_\Sigma X \arrow[r, "\mathcal{T}_\Sigma \hat{\imath}"] \arrow[d, "{[-]_{\hat{E}}}"'] & \mathcal{T}_\Sigma A \arrow[d, "{\llbracket - \rrbracket_A}"] \\
\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[r, dashed, "\hat{\sigma}_{\hat{\imath}}"'] & A
\end{tikzcd} \tag{127}$$

First, $\hat{\sigma}_{\hat{\imath}}$ is a nonexpansive map $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \to \mathbf{A}$ because for any $[u]_{\hat{E}}, [v]_{\hat{E}} \in \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$,

$$d_{\mathbf{A}}(\hat{\sigma}_{\hat{\imath}}[u]_{\hat{E}}, \hat{\sigma}_{\hat{\imath}}[v]_{\hat{E}}) \overset{(127)}{=} d_{\mathbf{A}}(\llbracket \mathcal{T}_\Sigma \hat{\imath}(u) \rrbracket_A, \llbracket \mathcal{T}_\Sigma \hat{\imath}(v) \rrbracket_A) \overset{(8)}{=} d_{\mathbf{A}}(\llbracket u \rrbracket^{\hat{\imath}}_A, \llbracket v \rrbracket^{\hat{\imath}}_A) \leq d_{\hat{E}}([u]_{\hat{E}}, [v]_{\hat{E}}),$$

where the inequality holds by definition of $d_{\hat{E}}$ and because $\hat{\mathbb{A}}$ satisfies all the equations in $\mathfrak{QTh}(\hat{E})$.

Second, we can prove that

$$\llbracket - \rrbracket^{\hat{\imath}}_A \circ \mu^\Sigma_X = \llbracket - \rrbracket^{\hat{\sigma}_{\hat{\imath}}}_A \circ \mathcal{T}_\Sigma[-]_{\hat{E}}, \tag{128}$$

which implies (126) holds (by applying both sides of (128) to s' and t'). We pave the following diagram.

Showing (129) commutes:

(a) Apply $\mathcal{T}_\Sigma$ to (127).

(b) By (13).

(c) By (8).

$$\begin{tikzcd}
\mathcal{T}_\Sigma\mathcal{T}_\Sigma X \arrow[rrr, "\mathcal{T}_\Sigma[-]_{\hat{E}}"] \arrow[dr, "\mathcal{T}_\Sigma\mathcal{T}_\Sigma\hat{\imath}"] \arrow[dd, "\mu^\Sigma_X"'] & \arrow[r, phantom, "\text{(a)}"] & & \mathcal{T}_\Sigma\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[dl, "\mathcal{T}_\Sigma\hat{\sigma}_{\hat{\imath}}"'] \arrow[dd, "{[\![-]\!]}^{\hat{\sigma}_{\hat{\imath}}}_A"] \\
& \mathcal{T}_\Sigma X \arrow[r, "\mathcal{T}_\Sigma[\![-]\!]_A"'] \arrow[d, phantom, "\text{(b)}"] & \mathcal{T}_\Sigma A \arrow[dr, "{[\![-]\!]}_A"'] \arrow[r, phantom, "\text{(c)}"] & \\
\mathcal{T}_\Sigma X \arrow[rrr, "{[\![-]\!]}^{\hat{\imath}}_A"'] & & & A
\end{tikzcd} \tag{129}$$

□

Lemma 165. *The family of maps* $\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{X}} : \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \to \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ *is natural in* $\mathbf{X}$.

Proof. We will (for posterity) reproduce the proof we did for Proposition 30, but it is important to note that nothing changes except the notation which now has lots of little hats.

We need to prove that for any function $f : \mathbf{X} \to \mathbf{Y}$, the square below commutes.

$$\begin{tikzcd}
\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[r, "\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f"] \arrow[d, "\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{X}}"'] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{Y} \arrow[d, "\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{Y}}"] \\
\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[r, "\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f"'] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{Y}
\end{tikzcd} \tag{130}$$

We can pave the following diagram.

$$\begin{tikzcd}
\mathcal{T}_\Sigma\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[r, "{[-]}_{\hat{E}}"] \arrow[dr, "\mathcal{T}_\Sigma\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f"] \arrow[ddr, "{[\![-]\!]}_{\widehat{\mathbb{T}}\mathbf{X}}"] \arrow[dd, "{[-]}_{\hat{E}}"'] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[r, "\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f"] \arrow[d, phantom, "\text{(a)}"] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{Y} \arrow[dd, "\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{Y}}"] \\
\arrow[r, phantom, "\text{(b)}\quad\text{(c)}"] & \mathcal{T}_\Sigma\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{Y} \arrow[ur, "{[-]}_{\hat{E}}"'] \arrow[dr, "{[\![-]\!]}_{\widehat{\mathbb{T}}\mathbf{Y}}"] \arrow[r, phantom, "\text{(d)}"] & \\
\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[r, "\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{X}}"'] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[r, "\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f"'] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{Y}
\end{tikzcd}$$

All of (a), (b) and (d) commute by definition. In more details, (a) is an instance of (119) with $\mathbf{X}$ replaced by $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$, $\mathbf{Y}$ by $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{Y}$ and f by $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f$, and both (b) and (d) are instances of (122). To show (c) commutes, we draw another diagram that looks like a cube and where (c) is the front face. We can show all the other faces commute, and then use the fact that $\mathcal{T}_\Sigma[-]_{\hat{E}}$ is surjective (i.e. epic) to conclude that the front face must also commute.[327]

[327] In more details, the left and right faces commute by (121), the bottom and top faces commute by (119), and the back face commutes by (6).

The function $\mathcal{T}_\Sigma[-]_{\hat{E}}$ is surjective (i.e. epic) because $[-]_{\hat{E}}$ is (it is a canonical quotient map) and functors on **Set** preserve epimorphisms (if we assume the axiom of choice). Thus, it suffices to show that $\mathcal{T}_\Sigma[-]_{\hat{E}}$ pre-composed with the bottom path or the top path of the front face gives the same result.

Now it is just a matter of going around the cube using the commutativity of the other faces. Here is the complete derivation (we write which face was

$$\begin{tikzcd}
\mathcal{T}_\Sigma\mathcal{T}_\Sigma X \arrow[rr, "\mathcal{T}_\Sigma\mathcal{T}_\Sigma f"] \arrow[dr, "\mathcal{T}_\Sigma[-]_{\hat{E}}"'] \arrow[dd, "\mu^\Sigma_X"'] & & \mathcal{T}_\Sigma\mathcal{T}_\Sigma Y \arrow[dr, "\mathcal{T}_\Sigma[-]_{\hat{E}}"] \arrow[dd, "\mu^\Sigma_Y"' near start] & \\
& \mathcal{T}_\Sigma\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[rr, "\mathcal{T}_\Sigma\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f" near start] \arrow[dd, "\llbracket - \rrbracket_{\widehat{\mathbb{T}}\mathbf{X}}"' near start] & & \mathcal{T}_\Sigma\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{Y} \arrow[dd, "\llbracket - \rrbracket_{\widehat{\mathbb{T}}\mathbf{Y}}"] \\
\mathcal{T}_\Sigma X \arrow[rr, "\mathcal{T}_\Sigma f" near end] \arrow[dr, "{[-]_{\hat{E}}}"'] & & \mathcal{T}_\Sigma Y \arrow[dr, "{[-]_{\hat{E}}}"] & \\
& \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[rr, "\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f"'] & & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{Y}
\end{tikzcd}$$

The first diagram we paved implies (27) commutes because $[-]_{\hat{E}}$ is surjective. $\square$

From the front face of the cube above, we find that for any $f : \mathbf{X} \to \mathbf{Y}$, $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f$ is a homomorphism between the underlying algebras of $\widehat{\mathbb{T}}\mathbf{X}$ and $\widehat{\mathbb{T}}\mathbf{Y}$. We already showed $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f$ is nonexpansive in Lemma 162, thus it is a homomorphism between the quantitative algebras $\widehat{\mathbb{T}}\mathbf{X}$ and $\widehat{\mathbb{T}}\mathbf{Y}$.

We now prove generalizations of results from Chapter 1[328] in order to show that $\widehat{\mathbb{T}}\mathbf{X}$ is not just a quantitative Σ-algebra but a $(\Sigma, \hat{E})$-algebra.

We can prove, analogously to Lemma 31, that for any $\hat{\mathbb{A}} \in \mathbf{QAlg}(\Sigma, \hat{E})$, $\llbracket - \rrbracket_A$ is a homomorphism between $\widehat{\mathbb{T}}\mathbf{A}$ and $\hat{\mathbb{A}}$.

[328] Contrary to what we did for Lemma 165, we will not reproduce the arguments that can be reused, you can trust me that it would go as smoothly for the other reults.

Lemma 166. *For any $(\Sigma, \hat{E})$-algebra $\hat{\mathbb{A}}$, the square (131) commutes, and $\llbracket - \rrbracket_A$ is a nonexpansive map $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A} \to \mathbf{A}$.*[329]

$$\begin{tikzcd}
\mathcal{T}_\Sigma\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A} \arrow[r, "\mathcal{T}_\Sigma\llbracket - \rrbracket_A"] \arrow[d, "\llbracket - \rrbracket_{\widehat{\mathbb{T}}\mathbf{A}}"'] & \mathcal{T}_\Sigma A \arrow[d, "\llbracket - \rrbracket_A"] \\
\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A} \arrow[r, "\llbracket - \rrbracket_A"'] & A
\end{tikzcd} \tag{131}$$

[329] We use the same convention as in (30) and write $\llbracket - \rrbracket_A$ for both maps $\mathcal{T}_\Sigma A \to A$ and $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A} \to A$. Recall the latter is well-defined because whenever $[s]_{\hat{E}} = [t]_{\hat{E}}$, $\hat{\mathbb{A}}$ must satisfy $\mathbf{A} \vdash s = t$, and in particular under the assignment $\mathrm{id}_A : \mathbf{A} \to \mathbf{A}$, this yields $\llbracket s \rrbracket_A = \llbracket t \rrbracket_A$.

Proof. For the commutative square, we can reuse the proof of Lemma 31. For nonexpansiveness, if $d_{\hat{E}}([s]_{\hat{E}}, [t]_{\hat{E}}) \leq \varepsilon$, then by (117) $\mathbf{A} \vdash s =_\varepsilon t$ belongs to $\mathfrak{QTh}(\hat{E})$ which means $\hat{\mathbb{A}}$ must satisfy that equation, and in particular under the assignment $\mathrm{id}_A : \mathbf{A} \to \mathbf{A}$, this yields $d_{\mathbf{A}}(\llbracket s \rrbracket_A, \llbracket t \rrbracket_A) \leq \varepsilon$. $\square$

We can prove, analogously to Lemma 32, that for any $\mathbf{X}$, $\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{X}}$ is a homomorphism from $\widehat{\mathbb{T}}\widehat{\mathbb{T}}\mathbf{X}$ to $\widehat{\mathbb{T}}\mathbf{X}$.

Lemma 167. *For any generalized metric space $\mathbf{X}$, the following square commutes, and $\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{X}}$ is a nonexpansive map $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \to \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$.*

$$\begin{tikzcd}
\mathcal{T}_\Sigma\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[r, "\mathcal{T}_\Sigma\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{X}}"] \arrow[d, "\llbracket - \rrbracket_{\widehat{\mathbb{T}}\widehat{\mathbb{T}}\mathbf{X}}"'] & \mathcal{T}_\Sigma\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[d, "\llbracket - \rrbracket_{\widehat{\mathbb{T}}\mathbf{X}}"] \\
\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[r, "\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{X}}"'] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}
\end{tikzcd} \tag{132}$$

Proof. For the commutative square, we can reuse the proof of Lemma 32. For non-expansiveness, we have already shown this in Lemma 164. □

Of course, paired with the flattening we also have a map $\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}$ which sends elements $x \in X$ to the equivalence lass containing x seen as a trivial term, namely,

$$\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}} = \mathbf{X} \xrightarrow{\eta^{\Sigma}_{X}} \mathcal{T}_{\Sigma}X \xrightarrow{[-]_{\hat{E}}} \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}. \tag{133}$$

We need to show $\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}$ is nonexpansive and natural in $\mathbf{X}$.

Lemma 168. *For any space $\mathbf{X}$, $\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}$ is a nonexpansive map $\mathbf{X} \to \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$.*

Proof. This is a direct consequence of Lemma 153. For any $x, x' \in X$ and $\varepsilon \in \mathsf{L}$,

$$\begin{aligned} d_{\mathbf{X}}(x,x') \leq \varepsilon &\implies \mathbf{X} \vdash x =_{\varepsilon} x' \in \mathfrak{QTh}(\hat{E}) && \text{by Lemma 153} \\ &\iff d_{\hat{E}}([x]_{\hat{E}}, [x']_{\hat{E}}) \leq \varepsilon. && \text{by (117)} \end{aligned}$$

Therefore, $d_{\hat{E}}([x]_{\hat{E}}, [x']_{\hat{E}}) \leq d_{\mathbf{X}}(x,x')$. □

Lemma 169. *For any nonexpansive map $f : \mathbf{X} \to \mathbf{Y}$, the following square commutes.*[330]

$$\begin{tikzcd} \mathbf{X} \arrow[r, "\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}"] \arrow[d, "f"'] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[d, "\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f"] \\ \mathbf{Y} \arrow[r, "\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{Y}}"'] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{Y} \end{tikzcd} \tag{134}$$

[330] Naturality of $\eta^{\Sigma,E}$ was easier in **Set** because it is the vertical composition of two natural transformations, η^{Σ} and $[-]_E$, which do not have counterparts in **GMet**.

Proof. We pave the following diagram (in **Set**, but that is enough since $U : \mathbf{GMet} \to \mathbf{Set}$ is faithful).

$$\begin{tikzcd} \mathbf{X} \arrow[rr, "\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}"] \arrow[dr, "\eta^{\Sigma}_{X}"] \arrow[ddd, "f"'] & \text{(a)} & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[ddd, "\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f"] \\ & \mathcal{T}_{\Sigma}X \arrow[ur, "{[-]_{\hat{E}}}"'] \arrow[d, "\mathcal{T}_{\Sigma}f"] & \\ \text{(b)} & \mathcal{T}_{\Sigma}Y \arrow[dr, "{[-]_{\hat{E}}}"] & \text{(c)} \\ \mathbf{Y} \arrow[ur, "\eta^{\Sigma}_{Y}"] \arrow[rr, "\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{Y}}"'] & \text{(d)} & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{Y} \end{tikzcd} \tag{135}$$

Showing (135) commutes:
(a) Definition of $\widehat{\eta}^{\Sigma,\hat{E}}$ (133).
(b) Naturality of η^{Σ} (4).
(c) Definition of $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f$ (119).
(d) Definition of $\widehat{\eta}^{\Sigma,\hat{E}}$ (133).

□

We also have the following technical lemma and its corollary analogous to Lemma 33 and Lemma 34.

Lemma 170. *For any generalized metric space $\mathbf{X}$, $[\![-]\!]^{\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}}_{\widehat{\mathbb{T}}\mathbf{X}} = [-]_{\hat{E}}$.*[331]

[331] We can reuse the proof for Lemma 33.

We get that for any quantitative equation ϕ with context $\mathbf{X}$, ϕ belongs to $\mathfrak{QTh}(\hat{E})$ if and only if the algebra $\widehat{\mathbb{T}}_{\Sigma,\hat{E}}\mathbf{X}$ satisfies it under the assignment $\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}$.

Lemma 171. *Let ϕ be an equation with context $\mathbf{X}$, $\phi \in \mathfrak{QTh}(\hat{E})$ if and only if $\widehat{\mathbb{T}}\mathbf{X} \vDash^{\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}} \phi$.*

Proof. We have two cases to show.

- $\mathbf{X} \vdash s = t \in \mathfrak{QTh}(\hat{E})$ if and only if $\widehat{\mathbb{T}}\mathbf{X} \vDash^{\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}} \mathbf{X} \vdash s = t$, and
- $\mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}(\hat{E})$ if and only if $\widehat{\mathbb{T}}\mathbf{X} \vDash^{\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}} \mathbf{X} \vdash s =_\varepsilon t$.

By Lemma 170,

$$[\![s]\!]^{\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}}_{\widehat{\mathbb{T}}X} = [s]_{\hat{E}} \text{ and } [\![t]\!]^{\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}}_{\widehat{\mathbb{T}}X} = [t]_{\hat{E}}, \tag{136}$$

then by using definitions, we have (as desired)

$$\begin{aligned}
\mathbf{X} \vdash s = t \in \mathfrak{QTh}(\hat{E}) \quad &\overset{(113)}{\iff} \quad [s]_{\hat{E}} = [t]_{\hat{E}} \quad \overset{(136)}{\iff} \quad [\![s]\!]^{\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}}_{\widehat{\mathbb{T}}X} = [\![t]\!]^{\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}}_{\widehat{\mathbb{T}}X} \\
\mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}(\hat{E}) \quad &\overset{(117)}{\iff} \quad d_{\hat{E}}([s]_{\hat{E}}, [t]_{\hat{E}}) \leq \varepsilon \quad \overset{(136)}{\iff} \quad d_{\hat{E}}([\![s]\!]^{\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}}_{\widehat{\mathbb{T}}X}, [\![t]\!]^{\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}}_{\widehat{\mathbb{T}}X}) \leq \varepsilon. \qquad \square
\end{aligned}$$

The next result, analogous to Lemma 35, tells us that $\widehat{\eta}^{\Sigma,\hat{E}}$ and $\widehat{\mu}^{\Sigma,\hat{E}}$ interact together like the unit and multiplication of a monad.

Lemma 172. *The following diagram commutes.*[332]

$$\begin{tikzcd}
\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[r, "\widehat{\eta}^{\Sigma,\hat{E}}_{\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}}"] \arrow[dr, "\mathrm{id}_{\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}}"'] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[d, "\widehat{\mu}^{\Sigma,\hat{E}}\mathbf{X}"] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[l, "\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}"'] \arrow[dl, "\mathrm{id}_{\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}}"] \\
& \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} &
\end{tikzcd}$$

[332] We can reuse the proof of Lemma 35, although when using naturality of $[-]_{\hat{E}}$ in **Set**, we replace it by (119) which is not formally a naturality property (because $\mathcal{T}_\Sigma$ is not a functor on **GMet**).

Finally, we can show that $\widehat{\mathbb{T}}_{\Sigma,\hat{E}}\mathbf{X}$ is $(\Sigma, \hat{E})$-algebra (analogous to Proposition 38).

Proposition 173. *For any space $\mathbf{A}$, the term algebra $\widehat{\mathbb{T}}_{\Sigma,\hat{E}}\mathbf{A}$ satisfies all the equations in $\hat{E}$.*

Proof. Let $\phi \in \hat{E}$ be an equation with context $\mathbf{X}$ and $\hat{\imath} : \mathbf{X} \to \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A}$ be a nonexpansive assignment. We factor $\hat{\imath}$ into[333]

$$\hat{\imath} = X \xrightarrow{\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}} \widehat{\mathcal{T}}_{\Sigma,\hat{E}}X \xrightarrow{\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\hat{\imath}} \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A} \xrightarrow{\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{A}}} \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A}.$$

Now, Lemma 171 says that ϕ is satisfied in $\widehat{\mathbb{T}}\mathbf{X}$ under the assignment $\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}$. We also know by Lemma 139 that homomorphisms preserve satisfaction, so we can apply it twice using the facts that $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\hat{\imath}$ and $\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{A}}$ are homomorphisms (the former was shown after Lemma 165 and the latter in Lemma 167) to conclude that $\widehat{\mathbb{T}}\mathbf{A}$ satisfies ϕ under $\widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{A}} \circ \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\hat{\imath} \circ \widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}} = \hat{\imath}$. $\square$

[333] This factoring is correct because

$$\begin{aligned}
\hat{\imath} &= \mathrm{id}_{\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A}} \circ \hat{\imath} \\
&= \widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{A}} \circ \widehat{\eta}^{\Sigma,\hat{E}}_{\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A}} \circ \hat{\imath} && \text{Lemma 172} \\
&= \widehat{\mu}^{\Sigma,\hat{E}}_{\mathbf{A}} \circ \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\hat{\imath} \circ \widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}}. && \text{naturality of } \widehat{\eta}^{\Sigma,\hat{E}}
\end{aligned}$$

We end this section just like we ended §1.1 by showing that $\widehat{\mathbb{T}}\mathbf{X}$ is the free $(\Sigma, \hat{E})$-algebra.[334]

[334] In both [MSV22] and [MSV23], we constructed the free algebra using quantitative equational logic.

Theorem 174. *For any space $\mathbf{X}$, the term algebra $\widehat{\mathbb{T}}X$ is the free $(\Sigma, \hat{E})$-algebra on $\mathbf{X}$.*

Proof. Note that the morphism witnessing freeness of $\widehat{\mathbb{T}}\mathbf{X}$ is $\widehat{\eta}^{\Sigma,\hat{E}}_{\mathbf{X}} : \mathbf{X} \to \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$. As expected, the proof goes exactly like for Proposition 41 except, we have to show that when $f : \mathbf{X} \to \mathbf{A}$ is nonexpansive, so is $f^* : \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \to \mathbf{A}$. This follows by the following derivation.[335]

[335] We implicitly use nonexpansiveness of f in the second step, where f is used as a nonexpansive assignment.

$$\begin{aligned}
d_{\hat{E}}([s]_{\hat{E}}, [t]_{\hat{E}}) \leq \varepsilon &\iff \mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}(\hat{E}) && \text{by (117)} \\
&\implies d_{\mathbf{A}}([\![s]\!]^f_A, [\![t]\!]^f_A) \leq \varepsilon && \hat{\mathbb{A}} \in \mathbf{QAlg}(\Sigma, \hat{E}) \\
&\iff d_{\mathbf{A}}([\![\mathcal{T}_\Sigma f(s)]\!]_A, [\![\mathcal{T}_\Sigma f(t)_A]\!]) && \text{by (8)} \\
&\iff d_{\mathbf{A}}([\![[\mathcal{T}_\Sigma f(s)]_{\hat{E}}]\!]_A, [\![[\mathcal{T}_\Sigma f(t)]_{\hat{E}}]\!]_A) && \text{Footnote 329} \\
&\iff d_{\mathbf{A}}([\![\widehat{\mathcal{T}}_{\Sigma,\hat{E}} f[s]_{\hat{E}}]\!]_A, [\![\widehat{\mathcal{T}}_{\Sigma,\hat{E}} f[t]_{\hat{E}}]\!]_A) && \text{by (119)} \\
&\iff d_{\mathbf{A}}(f^*[s]_{\hat{E}}, f^*[t]_{\hat{E}}) && \text{definition of } f^*
\end{aligned}$$
□

Since we have a free $(\Sigma, \hat{E})$-algebra $\widehat{\mathbb{T}}\mathbf{X}$ for every generalized metric space $\mathbf{X}$, we get a left adjoint to $U : \mathbf{QAlg}(\Sigma, \hat{E}) \to \mathbf{GMet}$. This automatically yields a monad structure on $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}$ that we will study after developing quantitative equational logic. Before that, we make use of a special case of the adjunction above.

Corollary 175. *The forgetful functor* $U : \mathbf{GMet} \to \mathbf{Set}$ *has a left adjoint.*

Proof. The following adjoints compose to yield a left adjoint to $U : \mathbf{GMet} \to \mathbf{Set}$.[336]

[336] The adjunction between $\mathsf{L}\mathbf{Spa}$ and $\mathbf{Set}$ was described in Proposition 128. The adjunction between $\mathbf{GMet}$ and $\mathsf{L}\mathbf{Spa}$ is the one we just obtained via Theorem 174 that we instantiate with $\mathbf{GMet} = \mathbf{QAlg}(\varnothing, \hat{E}_{\mathbf{GMet}})$ (recall Example 141).

$$\begin{tikzcd}
\mathbf{GMet} \arrow[r, shift left] \arrow[r, phantom, "\top"] \arrow[rr, bend left, "U"] & \mathsf{L}\mathbf{Spa} \arrow[l, shift left] \arrow[r, shift left, "U"] \arrow[r, phantom, "\top"] & \mathbf{Set} \arrow[l, shift left]
\end{tikzcd}$$
□

Example 176 (Discrete metric)**.** To make this more concrete, one can wonder what is the free metric space on a set X (with $\mathsf{L} = [0,1]$). According to the diagram above, we first need to construct the discrete space $\mathbf{X}_\top$ on X, then construct the free metric space on $\mathbf{X}_\top$. We know how to do the first step (Proposition 128), and the second step is also fairly easy to do.[337] The only thing that prevents $\mathbf{X}_\top$ from being a metric is reflexivity, i.e. $d_\top(x,x) = 1 \neq 0$. If we define $d_{\mathbf{X}}$ just like $d_\top$ except with $d_{\mathbf{X}}(x,x) = 0$, then it is a metric,[338] and $(X, d_{\mathbf{X}})$ is the free metric space over X.

[337] Even though we said in Example 144 that the free metric space on an arbitrary $\mathbf{X}$ is harder to describe.

[338] Identity of indiscernibles and symmetry hold because $d_{\mathbf{X}}(x,y) = d_{\mathbf{X}}(y,x) = 1$ when $x \neq y$. The triangle inequality holds because
$$d_{\mathbf{X}}(x,z) = 1 \leq 1 + 1 = d_{\mathbf{X}}(x,y) + d_{\mathbf{X}}(y,z).$$

With the help of quantitative algebraic theories and free algebras, we can now define coproducts inside $\mathbf{GMet}$.

Corollary 177. *The category* $\mathbf{GMet}$ *has coproducts.*

Proof. We will only do the case of binary coproducts for exposition's sake, but the proof can be adapted to arbitrary families. For any generalized metric space $\mathbf{A}$, the quantitative algebraic theory of $\mathbf{A}$ is generated by the signature $\Sigma_{\mathbf{A}} = \{a : 0 \mid a \in A\}$ and the quantitative equations[339]

[339] Note that a and a' are seen as constants, not variables, so the context of these equations is the empty L-space.

$$\hat{E}_{\mathbf{A}} = \left\{ \vdash a =_{d_{\mathbf{A}}(a,a')} a' \mid a, a' \in A \right\}.$$

A $(\Sigma_{\mathbf{A}}, \hat{E}_{\mathbf{A}})$-algebra $\hat{\mathbb{B}}$ is a generalized metric space $\mathbf{B}$ equipped with an interpretation $[\![a]\!]_B$ for every $a \in A$ such that $d_{\mathbf{B}}([\![a]\!]_B, [\![a']\!]_B) \leq d_{\mathbf{A}}(a,a')$ for every $a, a' \in A$.

Equivalently, all the interpretations can be seen as a single nonexpansive map $[\![-]\!]_B : \mathbf{A} \to \mathbf{B}$. Therefore, $\mathbf{QAlg}(\Sigma_{\mathbf{A}}, \hat{E}_{\mathbf{A}})$ is the coslice category $\mathbf{A}/\mathbf{GMet}$.

Given another space $\mathbf{A}'$, if we combine the theories of $\mathbf{A}$ and $\mathbf{A}'$ with no additional equations, we get the category $\mathbf{QAlg}(\Sigma_{\mathbf{A}} + \Sigma_{\mathbf{A}'}, \hat{E}_{\mathbf{A}} + \hat{E}_{\mathbf{A}'})$ of spaces $\mathbf{B}$ equipped with two nonexpansive maps $[\![-]\!]_B : \mathbf{A} \to \mathbf{B}$ and $[\![-]\!]'_B : \mathbf{A}' \to B$. This category has an initial object, the free algebra on the initial generalized metric space from Proposition 110. Moreover, this category can be equivalently described as the comma category $[\mathbf{A}, \mathbf{A}'] \downarrow \mathrm{id}_{\mathbf{GMet}}$ where $[\mathbf{A}, \mathbf{A}'] : \mathbf{1} + \mathbf{1} \to \mathbf{GMet}$ is the constant functor sending the two objects in the domain to $\mathbf{A}$ and $\mathbf{A}'$ respectively.[340] The initial object of this category (we just showed it exists) is the coproduct $\mathbf{A} + \mathbf{A}'$ (by definition of coproducts and comma categories). $\square$

[340] The category $\mathbf{1} + \mathbf{1}$ has two objects, their identity morphisms and that is it.

3.2 Quantitative Equational Logic

It is now time to introduce quantitative equational logic (QEL), which you can think of as both a generalization and an extension of equational logic. It is a generalization because it is parametrized by a complete lattice L, and when instantiating $\mathsf{L} = 1$, we get back equational logic as explained in Example 181. It is an extension because all the rules of equational logic are valid in QEL when replacing the contexts with discrete spaces as explained in Example 182. Figure 3.1 displays the inference rules of **quantitative equational logic**. The notion of **derivation** is straightforwardly adapted from Definition 42, the crucial difference is that proof trees can now be infinite.[341]

[341] This is necessary due to the rules SUB, SUBQ, and CONT.

Given any class of quantitative equations $\hat{E}$, we denote by $\mathfrak{QTh}'(\hat{E})$ the class of equations that can be proven from $\hat{E}$ in quantitative equational logic, in other words, $\phi \in \mathfrak{QTh}'(\hat{E})$ if and only if there is a derivation of ϕ in QEL with axioms $\hat{E}$.

Our goal now is to prove that $\mathfrak{QTh}'(\hat{E}) = \mathfrak{QTh}(\hat{E})$. We say that QEL is sound and complete for $(\Sigma, \hat{E})$-algebras. Less concisely, soundness means that whenever QEL proves an equation ϕ with axioms $\hat{E}$, ϕ is satisfied by all $(\Sigma, \hat{E})$-algebras, and completeness says that whenever an equation ϕ is satisfied by all $(\Sigma, \hat{E})$-algebras, there is a derivation of ϕ in QEL with axioms $\hat{E}$.

Just like for equational logic, all the rules in Figure 3.1 are sound for any fixed quantitative algebra meaning that if $\hat{\mathbb{A}}$ satisfies the equations on top of a rule, it must satisfy the conclusion of that rule. Let us explain the rules as we prove soundness.

The first four rules say that equality is an equivalence relation that is preserved by the operations, we showed they were sound in Lemmas 146–149. More formally, we can define (for any $\mathbf{X}$) a binary relation $\equiv'_{\hat{E}}$ on Σ-terms[342] that contains the pair (s, t) whenever $\mathbf{X} \vdash s = t$ can be proven in QEL (c.f. (113)): for any $s, t \in \mathcal{T}_\Sigma X$,

[342] Again, we omit the L-space $\mathbf{X}$ from the notation.

$$s \equiv'_{\hat{E}} t \iff \mathbf{X} \vdash s = t \in \mathfrak{QTh}'(\hat{E}). \tag{137}$$

Then, REFL, SYMM, TRANS, and CONG make $\equiv'_{\hat{E}}$ a congruence relation.

Lemma 178. *For any L-space $\mathbf{X}$, the relation $\equiv'_{\hat{E}}$ is reflexive, symmetric, transitive, and for any $\mathsf{op} : n \in \Sigma$ and $s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathcal{T}_\Sigma X$,*[343]

[343] i.e. $\equiv'_{\hat{E}}$ is a congruence on the Σ-algebra $\mathcal{T}_\Sigma X$ defined in Remark 18.

$$\frac{}{\mathbf{X} \vdash t = t}\ \textsc{Refl} \qquad \frac{\mathbf{X} \vdash s = t}{\mathbf{X} \vdash t = s}\ \textsc{Symm} \qquad \frac{\mathbf{X} \vdash s = t \qquad \mathbf{X} \vdash t = u}{\mathbf{X} \vdash s = u}\ \textsc{Trans}$$

$$\frac{\mathsf{op} : n \in \Sigma \qquad \forall 1 \leq i \leq n,\ \mathbf{X} \vdash s_i = t_i}{\mathbf{X} \vdash \mathsf{op}(s_1, \dots, s_n) = \mathsf{op}(t_1, \dots, t_n)}\ \textsc{Cong}$$

$$\frac{\sigma : Y \to \mathcal{T}_\Sigma X \qquad \mathbf{Y} \vdash s = t \qquad \forall y, y' \in Y,\ \mathbf{X} \vdash \sigma(y) =_{d_{\mathbf{Y}}(y,y')} \sigma(y')}{\mathbf{X} \vdash \sigma^*(s) = \sigma^*(t)}\ \textsc{Sub}$$

$$\frac{}{\mathbf{X} \vdash s =_\top t}\ \textsc{Top} \qquad \frac{d_{\mathbf{X}}(x, x') = \varepsilon}{\mathbf{X} \vdash x =_\varepsilon x'}\ \textsc{Vars} \qquad \frac{\mathbf{X} \vdash s =_\varepsilon t \qquad \varepsilon \leq \varepsilon'}{\mathbf{X} \vdash s =_{\varepsilon'} t}\ \textsc{Max}$$

$$\frac{\forall i, \mathbf{X} \vdash s =_{\varepsilon_i} t \qquad \varepsilon = \inf_i \varepsilon_i}{\mathbf{X} \vdash s =_\varepsilon t}\ \textsc{Cont} \qquad \frac{\phi \in \hat{E}_{\mathbf{GMet}}}{\phi}\ \textsc{GMet}$$

$$\frac{\mathbf{X} \vdash s = t \qquad \mathbf{X} \vdash s =_\varepsilon u}{\mathbf{X} \vdash t =_\varepsilon u}\ \textsc{CompL} \qquad \frac{\mathbf{X} \vdash s = t \qquad \mathbf{X} \vdash u =_\varepsilon s}{\mathbf{X} \vdash u =_\varepsilon t}\ \textsc{CompR}$$

$$\frac{\sigma : Y \to \mathcal{T}_\Sigma X \qquad \mathbf{Y} \vdash s =_\varepsilon t \qquad \forall y, y' \in Y, \mathbf{X} \vdash \sigma(y) =_{d_{\mathbf{Y}}(y,y')} \sigma(y')}{\mathbf{X} \vdash \sigma^*(s) =_\varepsilon \sigma^*(t)}\ \textsc{SubQ}$$

Figure 3.1: Rules of quantitative equational logic over the signature Σ and the complete lattice L, where $\mathbf{X}$ and $\mathbf{Y}$ can be any L-space, s, t, u, s_i and t_i can be any term in $\mathcal{T}_\Sigma X$, and ε, ε' and ε_i range over L. As indicated in the premises of the rules Cong, Sub and SubQ, they can be instantiated for any n-ary operation symbol and for any function σ respectively.

$$\forall 1 \leq i \leq n, s_i \equiv'_{\hat{E}} t_i \implies \mathsf{op}(s_1, \dots, s_n) \equiv'_{\hat{E}} \mathsf{op}(t_1, \dots, t_n). \tag{138}$$

We denote with $\wr - \int_{\hat{E}}$ the canonical quotient map $\mathcal{T}_\Sigma X \to \mathcal{T}_\Sigma X / \equiv'_{\hat{E}}$.

Skipping Sub for now, the Top rule says that $\top$ is an upper bound for all distances since it is the maximum element of L. We showed it is sound in Lemma 152.

The Vars rule is, in a sense, the quantitative version of Refl. It reflects the fact that assignments of variables are nonexpansive with respect to the distance in the context. Indeed, $\hat{\iota} : \mathbf{X} \to \mathbf{A}$ is nonexpansive precisely when, for all $x, x' \in X$,

$$d_{\mathbf{A}}(\hat{\iota}(x), \hat{\iota}(x')) = d_{\mathbf{A}}([\![x]\!]_A^{\hat{\iota}}, [\![x']\!]_A^{\hat{\iota}}) \leq d_{\mathbf{X}}(x, x').$$

How is this related to Refl? Letting $t = x \in X$, Refl says that for any assignment $\hat{\iota} : \mathbf{X} \to \mathbf{A}$, $\hat{\iota}(x) = \hat{\iota}(x)$. This seems trivial, but it hides a deeper fact that the assignment must be deterministic (a functional relation), as it cannot assign two different values to the same input.[344] So just like Refl imposes the constraint of determinism on assignments, Vars imposes nonexpansiveness. We showed Vars is sound in Lemma 153.

The rules Max and Cont should remind you of the definition of L-structure (Definition 90). Very briefly, they ensure that equipping the set of terms over X with the relations $R_\varepsilon^{\mathbf{X}} \subseteq \mathcal{T}_\Sigma X \times \mathcal{T}_\Sigma X$ defined by

$$s\ R_\varepsilon^{\mathbf{X}}\ t \iff \mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}'(\hat{E}), \tag{139}$$

yields an L-structure.[345] We showed they are sound in Lemmas 154 and 155. Note

[344] A similar thing happens for Cong which says that the interpretations of operation are deterministic (both in equational logic and QEL). In [MPP16], the logic has a rule NExp which morally says that the interpretations of operations are nonexpansive too, i.e. NExp is to Cong what Vars is to Refl. We said more on our choice to omit NExp in §0.3.

[345] Monotonicity and continuity hold by Max and Cont respectively. This is where the name Cont comes from, and this is why I prefer it over the other names in the literature.

that Top is an instance of Cont with the empty index set (recall that $\top = \inf \emptyset$).

The soundness of GMet is a consequence of (105) and the definition of quantitative algebra which requires the underlying space to satisfy all the equations in $\hat{E}_{\mathbf{GMet}}$.

CompL and CompR guarantee that the L-structure we just defined factors through the quotient $\mathcal{T}_\Sigma X/\equiv'_{\hat{E}}$.[346] We showed they are sound in Lemmas 150 and 151. In the presence of a symmetry axiom, only one of them would be sufficient.

Finally, we get to the substitutions Sub and SubQ, they are the same except for replacing $=$ with $=_\varepsilon$. Recall that the substitution rule in equational logic is

$$\frac{\sigma : Y \to \mathcal{T}_\Sigma X \qquad Y \vdash s = t}{X \vdash \sigma^*(s) = \sigma^*(t)},$$

which morally means that variables in the context Y are universally quantified. In Sub and SubQ, there is an additional condition on σ which arises because the variables in Y are *not* universally quantified, an assignment $Y \to A$ is considered in the definition of satisfaction only if it is nonexpansive from $\mathbf{Y}$ to $\mathbf{A}$.[347]

We proved Sub and SubQ are sound in Lemma 160, and we can compare with the proof of soundness of Sub in equational logic (Lemma 37) to find the same key argument: the interpretation of $\sigma^*(t)$ under some assignment $\hat{\imath}$ is equal to the interpretation of t under the assignment $\hat{\imath}_\sigma$ sending y to the interpretation of $\sigma(y)$ under $\hat{\imath}$. Since satisfaction for quantitative algebras only deals with nonexpansive assignments, we needed to check that $\hat{\imath}_\sigma$ is nonexpansive whenever $\hat{\imath}$ is, and this was true thanks to the conditions on σ. Let us give an illustrative example of why the extra conditions are necessary.

Example 179. We work over $\mathsf{L} = [0,1]$, $\mathbf{GMet} = \mathbf{Met}$, $\Sigma = \emptyset$, and $\hat{E} = \emptyset$. Let $\mathbf{Y} = \{y_0, y_1\}$ with $d_{\mathbf{Y}}(y_0, y_1) = d_{\mathbf{Y}}(y_1, y_0) = \frac{1}{2}$ and $\mathbf{X} = \{x_0, x_1\}$ with $d_{\mathbf{X}}(x_0, x_1) = d_{\mathbf{X}}(x_1, x_0) = 1$.[348] We consider the algebra $\hat{\mathbb{A}}$ whose underlying space is $\mathbf{A} = \mathbf{X}$ (since Σ is empty that is the only data required to define an algebra). It satisfies the equation $\mathbf{Y} \vdash y_0 = y_1$ because any nonexpansive assignment of $\mathbf{Y}$ into $\mathbf{A}$ must identify y_0 and y_1 (there are no distinct points with distance less than $\frac{1}{2}$).

Take the substitution $\sigma : Y \to \mathcal{T}_\Sigma X$ defined by $y_0 \mapsto x_0$ and $y_1 \mapsto x_1$, we can check $\hat{\mathbb{A}}$ does not satisfy $\mathbf{X} \vdash \sigma^*(y_0) = \sigma^*(y_1)$.[349] This means that σ cannot satisfy the extra conditions in Sub. Indeed, $\hat{\mathbb{A}}$ does not satisfy $\mathbf{X} \vdash \sigma(y_0) =_{\frac{1}{2}} \sigma(y_1)$ (take the assignment $\mathrm{id}_{\mathbf{X}}$ again).

By proving each rule is sound, we have shown that QEL is sound.

Theorem 180 (Soundness)**.** *If $\phi \in \mathfrak{QTh}'(\hat{E})$, then $\phi \in \mathfrak{QTh}(\hat{E})$.*

Let us explain how to recover equational logic from quantitative equational logic in two different ways.

Example 181 (Recovering equational logic I)**.** In Example 91, we saw that $1\mathbf{Spa}$ is the category $\mathbf{Set}$. Here we show that QEL over the complete lattice 1 with $\hat{E}_{\mathbf{GMet}} = \emptyset$ is the same thing as equational logic. First, what is a quantitative equation ϕ over 1? Since the context is a 1-space, it is just a set,[350] and furthermore, since 1 contains

[346] i.e. the following relation is well-defined:

$$\wr s \int_{\hat{E}} R^{\mathbf{X}}_\varepsilon \wr t \int_{\hat{E}} \iff \mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}'(\hat{E}), \quad (140)$$

[347] Put differently, the variables are universally quantified subject to certain constraints on their distances relative to the context $\mathbf{Y}$.

[348] We can see both $\mathbf{Y}$ and $\mathbf{X}$ as subspaces of $[0,1]$ with the Euclidean metric, where e.g. y_0 is embedded as 0 and y_1 as $\frac{1}{2}$, and x_0 is embedded as 0 and x_1 as 1.

[349] That equation is $\mathbf{X} \vdash x_0 = x_1$ and with the assignment $\mathrm{id}_{\mathbf{X}} : \mathbf{X} \to \mathbf{X} = \mathbf{A}$, we have

$$[\![x_0]\!]_A^{\mathrm{id}_{\mathbf{X}}} = x_0 \neq x_1 = [\![x_1]\!]_A^{\mathrm{id}_{\mathbf{X}}}.$$

[350] In other words, X and $\mathbf{X}$ are the same thing.

a single element (which we call $\top$ here, but it is equal to $\bot$) ϕ is either

$$X \vdash s = t \quad \text{or} \quad X \vdash s =_\top t.$$

Now, the second equation always belongs to $\mathfrak{QTh}'(\hat{E})$ for any $\hat{E}$ by Top. Therefore, the rules whose conclusions have an equation with a quantity (all but the first five) can be replaced by Top. The remaining rules are exactly those of equational logic except the substitution rule which has some additional constraints. The latter require proving only equations with quantities which we can always do with Top.

Thus, we can infer that for any $\hat{E}$, the equations without quantities in $\mathfrak{QTh}'(\hat{E})$ are exactly the equations in $\mathfrak{Th}'(E)$, where E contains the quantitative equations without quantities of $\hat{E}$ seen as equations.[351]

[351] i.e. $E = \{X \vdash s = t \mid \mathbf{X} \vdash s = t \in \hat{E}\}$

Example 182 (Recovering equational logic II)**.** There is a less trivial way to see that equational reasoning faithfully embeds into quantitative equational reasoning.

We are back to the general case of L being an arbitrary complete lattice and $\hat{E}_{\mathbf{GMet}}$ being possibly non-empty. Let E be a class of non-quantitative equations, and let $\hat{E}$ contain every equation in E seen as a quantitative equation with its context being the discrete space, i.e.

$$\hat{E} = \{\mathbf{X}_\top \vdash s = t \mid X \vdash s = t \in E\}. \tag{141}$$

Claim. If $X \vdash s = t \in \mathfrak{Th}'(E)$, then $\mathbf{X}_\top \vdash s = t \in \mathfrak{QTh}'(\hat{E})$.[352]

[352] Depending on the equations inside $\hat{E}_{\mathbf{GMet}}$, it is possible that $\mathfrak{QTh}'(\hat{E})$ contains more equations without quantities than $\mathfrak{Th}'(E)$. Nevertheless, we show that everything you can prove in equational logic can also be proven in QEL.

Proof 1. You can show by induction that a derivation of $X \vdash s = t$ in equational logic with axioms E can be transformed into a derivation of $\mathbf{X}_\top \vdash s = t$ in QEL with axioms $\hat{E}$. The base cases are handled by the definition of $\hat{E}$ and the rule Refl in QEL instantiated with the discrete spaces which perfectly emulates the rule Refl in equational logic.

For the inductive step, the rules Symm, Trans, and Cong in equational logic all have perfect counterparts in QEL. The substitution rule needs a bit more work. If the last rule in the derivation in equational logic is

$$\frac{\sigma : Y \to \mathcal{T}_\Sigma X \qquad Y \vdash s = t}{X \vdash \sigma^*(s) = \sigma^*(t)}\ \textsc{Sub},$$

then by induction hypothesis, there is a derivation of $\mathbf{Y}_\top \vdash s = t$ in QEL. We obtain the following derivation noting that for all $y, y' \in Y$, $d_\top(y, y') = \top$.

$$\dfrac{\sigma : Y \to \mathcal{T}_\Sigma X \qquad \dfrac{\text{I.H.}}{\mathbf{Y}_\top \vdash s = t} \qquad \dfrac{}{\forall y, y' \in Y,\ \mathbf{X}_\top \vdash \sigma(y) =_{d_\top(y,y')} \sigma(y')}\ \textsc{Top}}{\mathbf{X}_\top \vdash \sigma^*(s) = \sigma^*(t)}\ \textsc{Sub}$$

□

Proof 2. The proof above reasoning on derivations is useful to get familiar with QEL, but there is a faster *semantic* proof that relies on completeness. By soundness and completeness,[353] it is enough to prove that if $X \vdash s = t \in \mathfrak{Th}(E)$, then $\mathbf{X}_\top \vdash s = t \in \mathfrak{QTh}(\hat{E})$. This follows from the equivalence (115) (which was easy to prove):

[353] Of both equational logic (**??** 44**??** 49) and QEL (**??** 180**??** 187).

$$\hat{\mathbb{A}} \vDash \hat{E} \overset{(115)}{\iff} \mathbb{A} \vDash E \overset{(17)}{\implies} \mathbb{A} \vDash X \vdash s = t \overset{(115)}{\iff} \hat{\mathbb{A}} \vDash \mathbf{X}_\top \vdash s = t.$$ □

This second proof also points to a stronger version of the claim that we state as a lemma for future use.

Lemma 183. *Let E be a class of non-quantitative equations and $\hat{E}$ be defined as in* (141)*. If $X \vdash s = t \in \mathfrak{Th}'(E)$, then $\mathbf{X} \vdash s = t \in \mathfrak{QTh}'(\hat{E})$.*[354]

[354] Follow the second proof above but instead of the second use of (115), use Lemma 158. (This requires assuming $\mathfrak{QTh}(\hat{E}) = \mathfrak{QTh}'(\hat{E})$ which we prove soon.)

Let us get back to our goal of showing QEL is complete. We follow the proof sketch of completeness for equational logic.[355] We define a quantitative algebra exactly like $\widehat{\mathbb{T}}\mathbf{X}$ but using the equality relation and L-relation induced by $\mathfrak{QTh}'(\hat{E})$ instead of $\mathfrak{QTh}(\hat{E})$, and then we show it satisfies $\hat{E}$ which, by construction, will imply $\mathfrak{QTh}(\hat{E}) \subseteq \mathfrak{QTh}'(\hat{E})$.

[355] Our proof of completeness for the logic in [MSV22] seems more complex (in my opinion), but it morally follows the same sketch. It is obfuscated however by the fact that [MSV22] did not deal with contexts, instead we were using what we now call syntactic sugar to describe quantitative equations.

Definition 184 (Quantitative term algebra, syntactically)**.** The *new* quantitative term algebra for $(\Sigma, \hat{E})$ on $\mathbf{X}$ is the quantitative Σ-algebra whose underlying space is $\mathcal{T}_\Sigma X/{\equiv'}_{\hat{E}}$ equipped with the L-relation corresponding to the L-structure defined in (140),[356] and whose interpretation of $\mathsf{op} : n \in \Sigma$ is defined by[357]

$$[\![\mathsf{op}]\!]_{\widehat{\mathbb{T}}'\mathbf{X}}(\wr t_1 \int_{\hat{E}}, \dots, \wr t_n \int_{\hat{E}}) = \wr \mathsf{op}(t_1, \dots, t_n) \int_{\hat{E}}. \tag{143}$$

We denote this algebra by $\widehat{\mathbb{T}}'_{\Sigma,\hat{E}}\mathbf{X}$ or simply $\widehat{\mathbb{T}}'\mathbf{X}$.

[356] Explicitly, it is the L-relation $'d'_{\hat{E}}$ that satisfies

$$d'_{\hat{E}}(\wr s \int_{\hat{E}}, \wr t \int_{\hat{E}}) \leq \varepsilon \iff \mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}'(\hat{E}). \tag{142}$$

[357] This is well-defined (i.e. invariant under change of representative) by (138).

We will prove this alternative definition of the term algebra coincides with $\widehat{\mathbb{T}}\mathbf{X}$. First, we have to show that $\widehat{\mathbb{T}}'\mathbf{X}$ belongs to $\mathbf{QAlg}(\Sigma, \hat{E})$ like we did for $\widehat{\mathbb{T}}X$ in Proposition 173, and we state a technical lemma before that.

Lemma 185. *Let $\iota : Y \to \mathcal{T}_\Sigma X/{\equiv'}_E$ be any assignment. For any function $\sigma : Y \to \mathcal{T}_\Sigma X$ satisfying $\wr \sigma(y) \int_{\hat{E}} = \iota(y)$ for all $y \in Y$, we have $[\![-]\!]^\iota_{\widehat{\mathbb{T}}'X} = \wr \sigma^*(-) \int_{\hat{E}}$.*[358]

[358] The proof goes as in the classical case (Lemma 47). We do not even need to ask ι to be nonexpansive, but we will use the result with a nonexpansive assignment.

Proposition 186. *For any space $\mathbf{X}$, $\widehat{\mathbb{T}}'\mathbf{X}$ satisfies all the equations in $\hat{E}$.*

Proof. Let $\mathbf{Y} \vdash s = t$ (resp. $\mathbf{Y} \vdash s =_\varepsilon t$) belong to $\hat{E}$ and $\hat{\iota} : \mathbf{Y} \to (\mathcal{T}_\Sigma X/{\equiv'}_{\hat{E}}, d'_{\hat{E}})$ be a nonexpansive assignment. By the axiom of choice,[359] there is a function $\sigma : Y \to \mathcal{T}_\Sigma X$ satisfying $\wr \sigma(y) \int_{\hat{E}} = \hat{\iota}(y)$ for all $y \in Y$. Thanks to Lemma 185, it is enough to show $\wr \sigma^*(s) \int_{\hat{E}} = \wr \sigma^*(t) \int_{\hat{E}}$ (resp. $d'_{\hat{E}}(\wr \sigma^*(s) \int_{\hat{E}}, \wr \sigma^*(t) \int_{\hat{E}}) \leq \varepsilon$).[360]

[359] Choice implies the quotient map $\wr - \int_{\hat{E}}$ has a right inverse $r : \mathcal{T}_\Sigma X/{\equiv'}_{\hat{E}} \to \mathcal{T}_\Sigma X$, and we set $\sigma = r \circ \hat{\iota}$.

[360] By Lemma 185, it implies

$$[\![s]\!]^{\hat{\iota}}_{\widehat{\mathbb{T}}'\mathbf{X}} = \wr \sigma^*(s) \int_{\hat{E}} = \wr \sigma^*(t) \int_{\hat{E}} = [\![t]\!]^{\iota}_{\widehat{\mathbb{T}}'X},$$

resp. $d'_{\hat{E}}([\![s]\!]^{\hat{\iota}}_{\widehat{\mathbb{T}}'\mathbf{X}}, [\![t]\!]^{\hat{\iota}}_{\widehat{\mathbb{T}}'\mathbf{X}}) = d'_{\hat{E}}(\wr \sigma^*(s) \int_{\hat{E}}, \wr \sigma^*(t) \int_{\hat{E}}) \leq \varepsilon$

and since $\hat{\iota}$ was arbitrary, we conclude that $\widehat{\mathbb{T}}'\mathbf{X}$ satisfies $\mathbf{Y} \vdash s = t$ (resp. $\mathbf{Y} \vdash s =_\varepsilon t$).

Equivalently, by definition of $\wr - \int_{\hat{E}}$ and $\mathfrak{QTh}'(\hat{E})$, we can just exhibit a derivation of $\mathbf{X} \vdash \sigma^*(s) = \sigma^*(t)$ (resp. $\mathbf{X} \vdash \sigma^*(s) =_\varepsilon \sigma^*(t)$) in QEL with axioms $\hat{E}$. That equation can be proven with the SUB (resp. SUBQ) rule instantiated with $\sigma : Y \to \mathcal{T}_\Sigma X$ and the equation $\mathbf{Y} \vdash s = t$ (resp. $\mathbf{Y} \vdash s =_\varepsilon$) which is an axiom, but we need derivations showing σ satisfies the side conditions of the substitution rules. This follows from nonexpansiveness of $\hat{\iota}$ because for any $y, y' \in Y$, we know that

$$d_{\hat{E}}(\wr \sigma(y) \int_{\hat{E}}, \wr \sigma(y) \int_{\hat{E}}) = d_{\hat{E}}(\hat{\iota}(y), \hat{\iota}(y')) \leq d_{\mathbf{Y}}(y, y'),$$

which means by (142) that $\mathbf{X} \vdash \sigma(y) =_{d_{\mathbf{Y}}(y,y')} \sigma(y)$ belongs to $\mathfrak{QTh}'(\hat{E})$. $\square$

Completeness of quantitative equational logic readily follows.

Theorem 187 (Completeness)**.** *If $\phi \in \mathfrak{QTh}(\hat{E})$, then $\phi \in \mathfrak{QTh}'(\hat{E})$.*

Proof. Let $\phi \in \mathfrak{QTh}(\hat{E})$ and $\mathbf{X}$ be its context. By Proposition 186 and definition of $\mathfrak{QTh}(\hat{E})$, we know that $\widehat{\mathbb{T}}'\mathbf{X} \vDash \phi$. In particular, $\widehat{\mathbb{T}}'\mathbf{X}$ satisfies ϕ under the assignment

$$\hat{\imath} = \mathbf{X} \xrightarrow{\eta^{\Sigma}_{X}} \mathcal{T}_{\Sigma}X \xrightarrow{\wr - \int_{\hat{E}}} \mathcal{T}_{\Sigma}X/{\equiv'_{\hat{E}}},$$

which is nonexpansive by Vars.[361]

[361] Explicitly, Vars means $\mathbf{X} \vdash x =_{d_{\mathbf{X}}(x,x')} x'$ belongs to $\mathfrak{QTh}'(\hat{E})$, hence, (142) implies

$$d'_{\hat{E}}(\wr x \int_{\hat{E}}, \wr x' \int_{\hat{E}}) \leq d_{\mathbf{X}}(x,x').$$

Moreover with $\sigma = \eta^{\Sigma}_{X}$, we can show σ satisfies the hypothesis of Lemma 185 and $\sigma^* = \mathrm{id}_{\mathcal{T}_{\Sigma}X}$,[362] thus we conclude

[362] We defined $\hat{\imath}$ precisely to have $\wr \eta^{\Sigma}_{X}(x) \int_{\hat{E}} = \hat{\imath}(x)$. To show $\sigma^* = \eta^{\Sigma *}_{X}$ is the identity, use (34) and the fact that $\mu^{\Sigma} \cdot \eta^{\Sigma}\mathcal{T}_{\Sigma} = \mathbb{1}_{\mathcal{T}_{\Sigma}}$ (it holds by definition (5)).

- if $\phi = \mathbf{X} \vdash s = t$: $\wr s \int_{\hat{E}} = [\![s]\!]^{\hat{\imath}}_{\widehat{\mathbb{T}}'\mathbf{X}} = [\![t]\!]^{\hat{\imath}}_{\widehat{\mathbb{T}}'\mathbf{X}} = \wr t \int_{\hat{E}}$, and
- if $\phi = \mathbf{X} \vdash s =_{\varepsilon} t$: $d'_{\hat{E}}(\wr s \int_{\hat{E}}, \wr t \int_{\hat{E}}) = d'_{\hat{E}}([\![s]\!]^{\hat{\imath}}_{\widehat{\mathbb{T}}'\mathbf{X}}, [\![t]\!]^{\hat{\imath}}_{\widehat{\mathbb{T}}'\mathbf{X}}) \leq \varepsilon$.

By definition of $\equiv'_{\hat{E}}$ (137) and $d'_{\hat{E}}$ (142), this implies $\mathbf{X} \vdash s = t$ (resp. $\mathbf{X} \vdash s =_{\varepsilon} t$) belongs to $\mathfrak{QTh}'(\hat{E})$. $\square$

Note that because $\widehat{\mathbb{T}}\mathbf{X}$ and $\widehat{\mathbb{T}}'\mathbf{X}$ were defined in the same way in terms of $\mathfrak{QTh}(\hat{E})$ and $\mathfrak{QTh}'(\hat{E})$ respectively, and since we have proven the latter to be equal, we obtain that $\widehat{\mathbb{T}}\mathbf{X}$ and $\widehat{\mathbb{T}}'\mathbf{X}$ are the same quantitative algebra. In the sequel, we will work with $\widehat{\mathbb{T}}\mathbf{X}$ mostly but we may use the facts that $s \equiv_{\hat{E}} t$ (resp. $d_{\hat{E}}(s,t) \leq \varepsilon$) if and only if there is a derivation of $\mathbf{X} \vdash s = t$ (resp. $\mathbf{X} \vdash s =_{\varepsilon} t$) in QEL.[363]

[363] i.e. when proving that an equation holds in some theory $\mathfrak{QTh}(\hat{E})$, we can either use the rules of QEL or the several lemmas from §3.1 which are morally the semantic counterparts to the inference rules.

Remark 188. Mirroring Remark 50, we would like to say that the axiom of choice was not necessary in the proofs above. Unfortunately, this situation is more delicate, and I do not know for sure that we can avoid using choice (although I expect we can).

At first, you might think that since terms are still finite, we can still restrict the context to the free variables which is finite. Unfortunately, even if $x \in \mathrm{FV}\{s,t\}$ and $y \notin \mathrm{FV}\{s,t\}$, it is possible that the distance between x and y in the context is necessary to state the right property. Here is an example that we carry with $\mathbf{GMet} = [0,1]\mathbf{Spa}$, $\Sigma = \varnothing$, and $\hat{E}$ defining discrete metrics:[364]

[364] When $d_{\mathbf{A}}(a,b)$ is not 1, it must be that $a = b$ by the first set of equations, by the second set, it must be that $d_{\mathbf{A}}(a,b) = 0$. Under such constraints $\mathbf{A}$ must be the discrete metric on A that we described in Example 176, so $\mathbf{QAlg}(\varnothing, \hat{E})$ is the category of discrete metrics.

$$\hat{E} = \{x =_{\varepsilon} y \vdash x = y \mid 1 \neq \varepsilon \in \mathsf{L}\} \cup \{x = y \vdash x =_0 y\}.$$

Let $\mathbf{X} = \{x,z\}$ and $\mathbf{Y} = \{x,y,z\}$ with the following distances ($\mathbf{X}$ is a subspace of $\mathbf{Y}$):

$$\begin{tikzcd} x \arrow[loop above, no head, "0"] \arrow[r, no head, "\frac{1}{2}"] & y \arrow[loop above, no head, "0"] \arrow[r, no head, "\frac{1}{2}"] & z \arrow[loop above, no head, "0"] \end{tikzcd}$$

The equation $\mathbf{Y} \vdash x = z$ belongs to $\mathfrak{QTh}(\hat{E})$. Indeed, if $\mathbf{A} \vDash \hat{E}$, then $d_{\mathbf{A}}(a,b) \leq \frac{1}{2}$ implies $a = b$, so any nonexpansive assignment $\hat{\imath} : \mathbf{Y} \to \mathbf{A}$ must identify x and y, and y and z, hence $\hat{\imath}(x) = \hat{\imath}(z)$. However, the equation $\mathbf{X} \vdash x = z$ is not in $\mathfrak{QTh}(\hat{E})$ because you can have $d_{\mathbf{A}}(\hat{\imath}(x), \hat{\imath}(z)) \leq 1$ without $\hat{\imath}(x) = \hat{\imath}(z)$.

This shows that some variables in the context which are not used in the terms of the equation (in this instance y) might still be important. One may still wonder whether it is possible to restrict the contexts to be finite or countable.[365] I do not know if that is true, but I expect that countable contexts are enough and that finite contexts are not.

[365] i.e. for any equation ϕ, is there an equation ψ with finite (or countable) context such that

$$\hat{\mathbb{A}} \vDash \phi \iff \hat{\mathbb{A}} \vDash \psi.$$

In summary, while there can be an analog to the derivable Add rule in equational logic, the obvious counterpart to the Del rule is not even sound.

Let us highlight one last feature of quantitative equational logic: the rule GMet defining what kind of generalized metric spaces are considered is independent of all the other rules.[366] As a consequence, and we give more details in [MSV23, §8], you can choose to work over $\mathsf{L}\mathbf{Spa}$ all the time and add the equations in $\hat{E}_{\mathbf{GMet}}$ as axioms in $\hat{E}$ anytime you wish to restrict to algebras whose carriers are generalized metric spaces. Written a bit ambiguously,[367]

$$\mathbf{QAlg}(\Sigma, \hat{E}) = \mathbf{QAlg}(\Sigma, \hat{E} \cup \hat{E}_{\mathbf{GMet}}) \quad \text{and} \quad \mathfrak{QTh}(\hat{E}) = \mathfrak{QTh}(\hat{E} \cup \hat{E}_{\mathbf{GMet}}). \quad (144)$$

[366] Although it was less explicit because only **Met** was considered, this was already a feature of the logic in [MPP16].

[367] What we really mean is that on the left, **QAlg** and $\mathfrak{QTh}$ are the operators we described with the parameter **GMet** built in, and on the right, they are the same operators instantiated with $\mathsf{L}\mathbf{Spa}$ instead.

3.3 Quantitative Algebraic Presentations

In order to obtain a more categorical understanding of quantitative algebras, a first step is to show that the functor $\widehat{\mathcal{T}}_{\Sigma,\hat{E}} : \mathbf{GMet} \to \mathbf{GMet}$ we constructed is a monad.

Proposition 189. *The functor $\widehat{\mathcal{T}}_{\Sigma,\hat{E}} : \mathbf{GMet} \to \mathbf{GMet}$ defines a monad on $\mathbf{GMet}$ with unit $\widehat{\eta}^{\Sigma,\hat{E}}$ and multiplication $\widehat{\mu}^{\Sigma,\hat{E}}$. We call it the **term monad** for $(\Sigma, \hat{E})$.*

Proof. A first proof uses a standard result of category theory. Since we showed that $\widehat{\mathbb{T}}_{\Sigma,\hat{E}}\mathbf{A}$ is the free $(\Sigma, \hat{E})$-algebra on $\mathbf{A}$ for every space $\mathbf{A}$ (Theorem 174), we obtain a monad sending $\mathbf{A}$ to the underlying space of $\widehat{\mathbb{T}}_{\Sigma,\hat{E}}\mathbf{A}$, i.e. $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A}$.[368]

One could also follow the proof we gave for **Set** and explicitly show that $\widehat{\eta}^{\Sigma,\hat{E}}$ and $\widehat{\mu}^{\Sigma,\hat{E}}$ obey the laws for the unit and multiplication (most of the work having been done earlier in this chapter). □

[368] The unit is automatically $\widehat{\eta}^{\Sigma,\hat{E}}$, but some computations are needed to show the multiplication is $\widehat{\mu}^{\Sigma,\hat{E}}$.

What is arguably more important is that quantitative $(\Sigma, \hat{E})$-algebras on a space $\mathbf{A}$ correspond to $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}$-algebras on $\mathbf{A}$.[369] We construct an isomorphism between $\mathbf{QAlg}(\Sigma, \hat{E})$ and $\mathbf{EM}(\widehat{\mathcal{T}}_{\Sigma,\hat{E}})$ using the isomorphism $P : \mathbf{Alg}(\Sigma) \cong \mathbf{EM}(\mathcal{T}_\Sigma) : P^{-1}$ that we defined in Proposition 59,[370] the forgetful functor $U : \mathbf{QAlg}(\Sigma, \hat{E}) \to \mathbf{Alg}(\Sigma)$ that sends $\hat{\mathbb{A}}$ to the underlying algebra $\mathbb{A}$, and the functor $\mathbf{EM}(\widehat{\mathcal{T}}_{\Sigma,\hat{E}}) \to \mathbf{EM}(\mathcal{T}_\Sigma)$ we define below.

[369] i.e. $U : \mathbf{QAlg}(\Sigma, \hat{E}) \to \mathbf{GMet}$ is monadic.

[370] Take the statement of Proposition 59 with $E = \emptyset$.

Lemma 190. *For any $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}$-algebra (A, α), the map $U\alpha \circ [-]_{\hat{E}} : \mathcal{T}_\Sigma A \to A$ is a $\mathcal{T}_\Sigma$-algebra. Furthermore, this defines a functor $U^{[-]_{\hat{E}}} : \mathbf{EM}(\widehat{\mathcal{T}}_{\Sigma,\hat{E}}) \to \mathbf{EM}(\mathcal{T}_\Sigma)$.*

Proof. Apply Proposition 71 after checking that $(U, [-]_{\hat{E}})$ is monad functor from $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}$ to $\mathcal{T}_\Sigma$.[371] □

[371] The appropriate diagrams (55) and (56) commute by (133) and a combination of (121) and (122).

Theorem 191. *There is an isomorphism $\mathbf{QAlg}(\Sigma, \hat{E}) \cong \mathbf{EM}(\widehat{\mathcal{T}}_{\Sigma,\hat{E}})$.*[372]

[372] We follow [MSV22] which does not rely on monadicity theorems (recall Remark 60). For a proof that does, see [MSV23, Theorems 6.3 and 8.10] where monadicity for L-spaces is proved first, then monadicity for generalized metric spaces is proven using (144).

Proof. In the diagram below, we already have the functors drawn with solid arrows, and we want to construct $\widehat{P}$ and $\widehat{P}^{-1}$ drawn with dashed arrows before proving they are inverses to each other.

$$\begin{tikzcd}
\mathbf{QAlg}(\Sigma, \hat{E}) \arrow[r, dashed, shift left, "\widehat{P}"] \arrow[d, "U"'] & \mathbf{EM}(\widehat{\mathcal{T}}_{\Sigma,\hat{E}}) \arrow[l, dashed, shift left, "\widehat{P}^{-1}"] \arrow[d, "U^{[-]_{\hat{E}}}"] \\
\mathbf{Alg}(\Sigma) \arrow[r, shift left, "P"] & \mathbf{EM}(\mathcal{T}_\Sigma) \arrow[l, shift left, "P^{-1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{QAlg}(\Sigma, \hat{E}) \arrow[rr, dashed, shift left, "\widehat{P}"] \arrow[dr, "U"'] & & \mathbf{EM}(\widehat{\mathcal{T}}_{\Sigma,\hat{E}}) \arrow[ll, dashed, shift left, "\widehat{P}^{-1}"] \arrow[dl, "U^{\widehat{\mathcal{T}}_{\Sigma,\hat{E}}}"] \\
& \mathbf{GMet} &
\end{tikzcd}$$

A (meaningful) sidequest for us is to make the diagrams above commute, namely, the underlying $\mathcal{T}_\Sigma$-algebra of $\widehat{P}\hat{\mathbb{A}}$ should be $P\mathbb{A}$ and the underlying space of $\widehat{P}\hat{\mathbb{A}}$ should be the underlying space of $\hat{\mathbb{A}}$, and similarly for $\widehat{P}^{-1}$. It turns out this completely determines our functors, up to some quick checks. We will move between spaces and their underlying sets without indicating it by $U : \mathbf{GMet} \to \mathbf{Set}$.

Given $\hat{\mathbb{A}} \in \mathbf{QAlg}(\Sigma, \hat{E})$, we look at the underlying Σ-algebra $\mathbb{A}$, apply P to it to get $\alpha_{\mathbb{A}} : \mathcal{T}_\Sigma A \to A$ which sends a term t to its interpretation $\llbracket t \rrbracket_A$, and we need to check that it factors through $[-]_{\hat{E}}$ and a nonexpansive map $\widehat{\alpha}_{\hat{\mathbb{A}}}$ as in (145).

$$\begin{tikzcd} \mathcal{T}_\Sigma A \arrow[rr, "\alpha_{\mathbb{A}}"] \arrow[dr, "{[-]_{\hat{E}}}"'] & & \mathbf{A} \\ & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A} \arrow[ur, "\widehat{\alpha}_{\hat{\mathbb{A}}}"'] & \end{tikzcd} \tag{145}$$

First, $\alpha_{\mathbb{A}}$ is well-defined on terms modulo $\hat{E}$ because if $s \equiv_{\hat{E}} t$, then $\hat{\mathbb{A}}$ satisfies $\mathbf{A} \vdash s = t \in \mathfrak{Q}\mathfrak{T}\mathfrak{h}(\hat{E})$, and this in turn means (taking the assignment $\mathrm{id}_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}$):

$$\alpha_{\mathbb{A}}(s) = \llbracket s \rrbracket_A = \llbracket s \rrbracket_A^{\mathrm{id}_{\mathbf{A}}} = \llbracket t \rrbracket_A^{\mathrm{id}_{\mathbf{A}}} = \llbracket t \rrbracket_A = \alpha_{\mathbb{A}}(t).$$

Next, the factor we obtain $\widehat{\alpha}_{\hat{\mathbb{A}}} : \mathcal{T}_\Sigma A/{\equiv_{\hat{E}}} \to A$ is nonexpansive from $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A}$ to $\mathbf{A}$. Indeed, if $d_{\hat{E}}([s]_{\hat{E}}, [t]_{\hat{E}}) \leq \varepsilon$, then $\hat{\mathbb{A}}$ satisfies $\mathbf{A} \vdash s =_\varepsilon t \in \mathfrak{Q}\mathfrak{T}\mathfrak{h}(\hat{E})$, and this means:

$$d_{\mathbf{A}}(\widehat{\alpha}_{\hat{\mathbb{A}}}[s]_{\hat{E}}, \widehat{\alpha}_{\hat{\mathbb{A}}}[t]_{\hat{E}}) = d_{\mathbf{A}}(\alpha_{\mathbb{A}}(s), \alpha_{\mathbb{A}}(t)) = d_{\mathbf{A}}(\llbracket s \rrbracket_A, \llbracket t \rrbracket_A) = d_{\mathbf{A}}(\llbracket s \rrbracket_A^{\mathrm{id}_{\mathbf{A}}}, \llbracket t \rrbracket_A^{\mathrm{id}_{\mathbf{A}}}) \leq \varepsilon.$$

Finally, if $h : \hat{\mathbb{A}} \to \hat{\mathbb{B}}$ is a homomorphism, then by definition it is nonexpansive $\mathbf{A} \to \mathbf{B}$ and it commutes with $\llbracket - \rrbracket_A$ and $\llbracket - \rrbracket_B$. The latter means it commutes with $\alpha_{\mathbb{A}}$ and $\alpha_{\mathbb{B}}$, which in turn means it commutes with $\widehat{\alpha}_{\hat{\mathbb{A}}}$ and $\widehat{\alpha}_{\hat{\mathbb{B}}}$ because $[-]_{\hat{E}}$ is epic (see (146)). We obtain our functor $\widehat{P} : \mathbf{QAlg}(\Sigma, \hat{E}) \to \mathbf{EM}(\widehat{\mathcal{T}}_{\Sigma,\hat{E}})$.

$$\begin{tikzcd} \mathcal{T}_\Sigma A \arrow[rr, "\mathcal{T}_\Sigma h"] \arrow[drr, "\alpha_{\mathbb{A}}"] \arrow[ddr, "{[-]_{\hat{E}}}"'] & & \mathcal{T}_\Sigma B \arrow[drr, "\alpha_{\mathbb{B}}"] \arrow[ddr, "{[-]_{\hat{E}}}"'] & & \\ & & \mathbf{A} \arrow[rr, "h"'] & & \mathbf{B} \\ & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A} \arrow[ur, "\widehat{\alpha}_{\hat{\mathbb{A}}}"'] \arrow[rr, "\widehat{\mathcal{T}}_{\Sigma,\hat{E}}h"'] & & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{B} \arrow[ur, "\widehat{\alpha}_{\hat{\mathbb{B}}}"'] & \end{tikzcd} \tag{146}$$

The top face of the prism in (146) commutes because h is a homomorphism, the back face commutes by (119), and the side faces commute by (145). Thus, the bottom face commutes because $[-]_{\hat{E}}$ is epic.

Given a $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}$-algebra $\widehat{\alpha} : \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{A} \to \mathbf{A}$, we look at the $\mathcal{T}_\Sigma$-algebra

$$U^{[-]_{\hat{E}}}\widehat{\alpha} = U\widehat{\alpha} \circ [-]_{\hat{E}} : \mathcal{T}_\Sigma A \to A$$

obtained via Lemma 190, then we apply P^{-1} to get the Σ-algebra $(A, \llbracket - \rrbracket_{U^{[-]_{\hat{E}}}\widehat{\alpha}})$. Since $\mathbf{A} = (A, d_{\mathbf{A}})$ is a generalized metric space (because $\widehat{\alpha}$ belongs to $\mathbf{EM}(\widehat{\mathcal{T}}_{\Sigma,\hat{E}})$), we obtain a quantitative algebra $\hat{\mathbb{A}}_{\widehat{\alpha}} = (A, \llbracket - \rrbracket_{U^{[-]_{\hat{E}}}\widehat{\alpha}}, d_{\mathbf{A}})$, and we need to check it satisfies the equations in $\hat{E}$.

Recall from the proof of Proposition 59 that interpreting terms in $\hat{\mathbb{A}}_{\widehat{\alpha}}$ is the same thing as applying $U^{[-]_{\hat{E}}}\widehat{\alpha} = U\widehat{\alpha} \circ [-]_{\hat{E}}$. Therefore, given any L-space $\mathbf{X}$, nonexpansive assignment $\hat{\iota} : \mathbf{X} \to \mathbf{A}$, and $t \in \mathcal{T}_\Sigma X$, we have

$$\llbracket t \rrbracket^{\hat{\iota}}_{U^{[-]_{\hat{E}}}} \overset{(8)}{=} \llbracket \mathcal{T}_\Sigma \hat{\iota}(t) \rrbracket_{U^{[-]_{\hat{E}}}} = \widehat{\alpha}[\mathcal{T}_\Sigma \hat{\iota}(t)]_{\hat{E}}.$$

Now, if $\mathbf{X} \vdash s = t \in \hat{E}$, we also have $\mathbf{A} \vdash \mathcal{T}_\Sigma \hat{\iota}(s) = \mathcal{T}_\Sigma \hat{\iota}(t) \in \mathfrak{Q}\mathfrak{T}\mathfrak{h}(\hat{E})$ by Lemma 156, which means

$$\llbracket s \rrbracket^{\hat{\iota}}_{U^{[-]_{\hat{E}}}} = \widehat{\alpha}[\mathcal{T}_\Sigma \hat{\iota}(s)]_{\hat{E}} = \widehat{\alpha}[\mathcal{T}_\Sigma \hat{\iota}(t)]_{\hat{E}} = \llbracket t \rrbracket^{\hat{\iota}}_{U^{[-]_{\hat{E}}}}.$$

Similarly for $\mathbf{X} \vdash s =_\varepsilon t \in \hat{E}$, Lemma 156 means $\mathbf{A} \vdash \mathcal{T}_\Sigma \hat{\iota}(s) =_\varepsilon \mathcal{T}_\Sigma \hat{\iota}(t) \in \mathfrak{Q}\mathfrak{T}\mathfrak{h}(\hat{E})$, so[373]

$$d_{\mathbf{A}}(\llbracket s \rrbracket^{\hat{\iota}}_{U^{[-]_{\hat{E}}}}, \llbracket t \rrbracket^{\hat{\iota}}_{U^{[-]_{\hat{E}}}}) = d_{\mathbf{A}}(\widehat{\alpha}[\mathcal{T}_\Sigma \hat{\iota}(s)]_{\hat{E}}, \widehat{\alpha}[\mathcal{T}_\Sigma \hat{\iota}(t)]_{\hat{E}}) \leq d_{\hat{E}}([\mathcal{T}_\Sigma \hat{\iota}(s)]_{\hat{E}}, [\mathcal{T}_\Sigma \hat{\iota}(t)]_{\hat{E}}) \leq \varepsilon.$$

[373] The first inequality holds by nonexpansiveness of $\widehat{\alpha}$ and the second by definition of $d_{\hat{E}}$ (117).

Finally, if $h : (\mathbf{A}, \widehat{\alpha}) \to (\mathbf{B}, \widehat{\beta})$ is $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}$-homomorphism, then by definition, it is nonexpansive $\mathbf{A} \to \mathbf{B}$, and by Lemma 190 it commutes with $U^{[-]_{\hat{E}}}\widehat{\alpha}$ and $U^{[-]_{\hat{E}}}\widehat{\beta}$ which means it is a homomorphism of the underlying algebras of $\hat{\mathbb{A}}_{\widehat{\alpha}}$ and $\hat{\mathbb{B}}_{\widehat{\beta}}$. We conclude it is also a homomorphism between the quantitative algebras $\hat{\mathbb{A}}_{\widehat{\alpha}}$ and $\hat{\mathbb{B}}_{\widehat{\beta}}$.[374] We obtain our functor $\widehat{P}^{-1} : \mathbf{EM}(\widehat{\mathcal{T}}_{\Sigma,\hat{E}}) \to \mathbf{QAlg}(\Sigma, \hat{E})$.

[374] Recall that homomorphisms between quantitative algebras are just nonexpansive homomorphisms.

The diagrams at the start of the proof commute by construction, and P and P^{-1} are inverses by Proposition 59. That is enough to conclude that $\widehat{P}$ and $\widehat{P}^{-1}$ are also inverses. Indeed, by commutativity of the triangle, $\widehat{P}$ and $\widehat{P}^{-1}$ preserve the underlying spaces, and if we fix a space $\mathbf{A}$, the forgetful functors U and $U^{[-]_{\hat{E}}}$ are injective.[375] Then, still with a fixed space $\mathbf{A}$, by commutativity of the square, we have

[375] For U, it is clear because it only forgets the L-relation. For $U^{[-]_{\hat{E}}}$, it is also not too hard to see, and it is because $U : \mathbf{GMet} \to \mathbf{Set}$ is faithful and $[-]_{\hat{E}}$ is epic.

$$U\widehat{P}^{-1}\widehat{P}\hat{\mathbb{A}} = P^{-1}U^{[-]_{\hat{E}}}\widehat{P}\hat{\mathbb{A}} = P^{-1}PU\hat{\mathbb{A}} = U\hat{\mathbb{A}}, \text{ and}$$
$$U^{[-]_{\hat{E}}}\widehat{P}\widehat{P}^{-1}\widehat{\alpha} = PU\widehat{P}^{-1}\widehat{\alpha} = PP^{-1}U^{[-]_{\hat{E}}}\widehat{\alpha} = U^{[-]_{\hat{E}}}\widehat{\alpha},$$

with which we can conclude by injectivity of U and $U^{[-]_{\hat{E}}}$. □

This motivates the following definition.

Definition 192 (**GMet** presentation)**.** Let M be a monad on **GMet**, a **quantitative algebraic presentation** of M is signature Σ and a class of quantitative equations $\hat{E}$ along with a monad isomorphism $\rho : \widehat{\mathcal{T}}_{\Sigma,\hat{E}} \cong M$. We also say M is presented by $(\Sigma, \hat{E})$. By Proposition 65 and Theorem 191, this is equivalent to having an isomorphism $\mathbf{EM}(\widehat{\mathcal{T}}_{\Sigma,\hat{E}}) \cong \mathbf{QAlg}(\Sigma, \hat{E})$ that commutes with the forgetful functors.

Example 193 (Hausdorff)**.** We saw in Example 67 that the monad $\mathcal{P}_{\text{ne}}$ on **Set** is presented by the theory of semilattices. In this example,[376] we define the theory of quantitative semilattices and show it presents a monad which sends (X, d) to $\mathcal{P}_{\text{ne}}X$ equipped with the Hausdorff distance $d^{\uparrow}$.

[376] We adapted it from [MPP16, §9.1].

A **quantitative semilattice** is a semilattice (i.e. a $(\Sigma_{\mathbf{S}}, E_{\mathbf{S}})$-algebra) equipped with an L-relation such that the interpretation of the semilattice operation is nonexpansive with respect to the product distance. Equivalently, it is a quantitative $\Sigma_{\mathbf{S}}$-algebra that satisfies $\hat{E}_{\mathbf{S}}$ which contains:[377]

[377] The first three equations are those of $E_{\mathbf{S}}$ seen with the discrete context as in Example 182. The last row is (107) which enforces the nonexpansiveness property of $[\![\oplus]\!]$.

$$x \vdash x = x \oplus x$$
$$x, y \vdash x \oplus y = y \oplus x$$
$$x, y, z \vdash x \oplus (y \oplus z) = (x \oplus y) \oplus z$$
$$\forall \varepsilon, \varepsilon' \in \mathsf{L}, \quad x =_{\varepsilon} y, x' =_{\varepsilon'} y' \vdash x \oplus x' =_{\max\{\varepsilon,\varepsilon'\}} y \oplus y'$$

We can give an alternative description of the free quantitative semilattice.

Lemma 194. *The free quantitative semilattice on (X, d) is $\hat{\mathbb{P}}_{(X,d)} = (\mathcal{P}_{\text{ne}}X, \cup, d^{\uparrow})$.*[378]

[378] This corresponds to [MPP16, Theorem 9.3].

Proof. We know from Example 67 that $(\mathcal{P}_{\text{ne}}X, \cup)$ is the free semilattice and hence satisfies $E_{\mathbf{S}}$, thus by Lemma 158, $\hat{\mathbb{P}}_{(X,d)}$ satisfies the first three equations above. We already mentioned that $\hat{\mathbb{P}}_{(X,d)}$ satisfies (107) because it satisfies (102).[379] Thus, $\hat{\mathbb{P}}_{(X,d)}$ is a quantitative semilattice.

[379] We did not give a proof for (102).

Let $\hat{\mathbb{A}}$ be a quantitative semilattice and $f : (X, d) \to \mathbf{A}$ be a nonexpansive map. By Lemma 159, $\mathbb{A}$ is a semilattice, hence the universal property of the free semilattice gives a unique homomorphism of $(\Sigma_{\mathbf{S}}, E_{\mathbf{S}})$-algebras $f^* : (\mathcal{P}_{\text{ne}}X, \cup) \to \mathbb{A}$ such that $f^*(\{x\}) = f(x)$ for all $x \in X$. It remains to show that f^* is a nonexpansive map $(\mathcal{P}_{\mathrm{ne}}X, d^{\uparrow}) \to \mathbf{A}$.[380]

Let $S, T \in \mathcal{P}_{\mathrm{ne}}X$, $C \in \mathcal{P}_{\mathrm{ne}}(X \times X)$ be a coupling for S and T, and suppose C is ordered with $C = \{c_1, \dots, c_n\}$. In particular, we have $S = \pi_1(c_1) \cup \cdots \cup \pi_1(c_n)$ and $T = \pi_2(c_1) \cup \cdots \cup \pi_2(c_n)$. Since f^* is a homomorphism of semilattices, this implies

$$\begin{aligned} f^*(S) &= f(\pi_1(c_1)) [\![\oplus]\!]_A \cdots [\![\oplus]\!]_A f(\pi_1(c_n)), \text{and} \\ f^*(T) &= f(\pi_2(c_1)) [\![\oplus]\!]_A \cdots [\![\oplus]\!]_A f(\pi_2(c_n)). \end{aligned}$$

Now, we can use the fact that $\hat{\mathbb{A}}$ satisfies the equations in (107) n times in the first step of the following derivation.

$$\begin{aligned} d_{\mathbf{A}}(f^*(S), f^*(T)) &\leq \max_{1 \leq i \leq n} d_{\mathbf{A}}(f(\pi_1(c_i)), f(\pi_2(c_i))) && \text{by (107)} \\ &\leq \max_{1 \leq i \leq n} d(\pi_1(c_i), \pi_2(c_i)) && f \text{ nonexpansive} \\ &\leq d^{\downarrow}(S, T) && \text{definition of } d^{\downarrow} \\ &= d^{\uparrow}(S, T) && \text{Lemma 89} \end{aligned}$$

We conclude that f^* is a homomorphism between the quantitative algebras $\hat{\mathbb{P}}_{(X,d)}$ and $\hat{\mathbb{A}}$. The uniqueness follows from it being unique as a homomorphism of semilattices and the faithfulness of $U : \mathbf{QAlg}(\Sigma_{\mathbf{S}}, \hat{E}_{\mathbf{S}}) \to \mathbf{Alg}(\Sigma_{\mathbf{S}})$. □

Since $\widehat{\mathbb{T}}(X, d)$ is also the free quantitative semilattice on (X, d) by Theorem 174 and free objects are unique by Proposition 40, there is an isomorphism of quantitative algebras $\rho_{(X,d)} : \widehat{\mathbb{T}}(X, d) \cong \hat{\mathbb{P}}_{(X,d)}$. After some abstract categorical arguments we do not reproduce, one finds that ρ is a monad isomorphism $\widehat{\mathcal{T}_{\Sigma_{\mathbf{S}}, \hat{E}_{\mathbf{S}}}} \cong \mathcal{P}_{\mathrm{ne}}^{\uparrow}$, where $\mathcal{P}_{\mathrm{ne}}^{\uparrow} : \mathbf{GMet} \to \mathbf{GMet}$ sends (X, d) to $(\mathcal{P}_{\mathrm{ne}}X, d^{\uparrow})$ and its unit and multiplication act just like those of $\mathcal{P}_{\mathrm{ne}}$.[381]

The second example of presentation is from [MPP16, §10.1].

Example 195 (Kantorovich)**.** We saw in Example 68 that the monad $\mathcal{D}$ on **Set** is presented by the theory of convex algebras. Let $\mathsf{L} = [0, \infty]$ and $\mathbf{GMet} = \mathbf{Met}$. The theory of **quantitative convex algebras** is generated by $\hat{E}_{\mathbf{CA}}$ which contains the equations of $E_{\mathbf{CA}}$ seen as quantitative equations (as explained in Example 182) and the quantitative equations for convexity (111).[382]

Let $(\mathcal{D}X, [\![-]\!]_{\mathcal{D}X})$ be the free convex algebra, where $+_p$ is interpreted as convex combination of distributions (54). Thanks to Lemma 158, we know that for any metric d on X, we can equip $\mathcal{D}X$ with the Kantorovich distance d_{K} and obtain a quantitative algebra $(\mathcal{D}X, [\![-]\!]_{\mathcal{D}X}, d_{\mathrm{K}})$ that satisfies the equations of convex algebras (seen with a discrete context). Moreover, with Example 138 we can infer that $(\mathcal{D}X, [\![-]\!]_{\mathcal{D}X}, d_{\mathrm{K}})$ is a quantitative convex algebra (i.e. it also satisfies (111)). In [MPP16, Theorem 10.5], the authors show that, along with the map $\eta_X^{\mathcal{D}} : (X, d) \to (\mathcal{D}X, d_{\mathrm{K}})$ sending x to the Dirac distribution on x, it is the free quantitative convex algebra on (X, d).

[380] Actually, you also have to prove that $\eta : (X, d) \to (\mathcal{P}_{\mathrm{ne}}X, d^{\uparrow})$ sending x to $\{x\}$ is nonexpansive. This is easy to check.

[381] This monad is famous independently of quantitative algebras, variations of it were studied in, e.g., [ACT10, §4], [Tho12, §4], [BBKK18, Example 8.3], and [DFM23, §6].

[382] As a reminder, $\hat{E}_{\mathbf{CA}}$ contains

$$\begin{aligned} x &\vdash x = x +_p x \\ x, y &\vdash x +_p y = y +_{1-p} x \\ x, y, z &\vdash (x +_p y) +_q z = x +_{pq} + (y +_{\frac{p(1-q)}{1-pq}} z) \\ x =_{\varepsilon} y, x' =_{\varepsilon'} y' &\vdash x +_p x' =_{p\varepsilon + \overline{p}\varepsilon'} y +_p y' \end{aligned}$$

We can conclude that $(\Sigma_{\mathbf{CA}}, \hat{E}_{\mathbf{CA}})$ presents a monad $\mathcal{D}_{\mathrm{K}} : \mathbf{Met} \to \mathbf{Met}$ which sends (X, d) to $(\mathcal{D}X, d_{\mathrm{K}})$ and whose unit and multiplication act just like those of the **Set** monad $\mathcal{D}$.[383]

Here is one last example.

Example 196 (Maybe)**.** We saw in Example 63 that the maybe monad on **Set** is presented by the theory of $\Sigma = \{\mathsf{p} : 0\}$ with no equations. Let us generalize this to the maybe monad on **GMet**.[384] We saw in Corollary 177 that $\mathbf{QAlg}(\Sigma, \hat{E}_{\mathbf{1}}) \cong \mathbf{1}/\mathbf{GMet}$, where $\hat{E}_{\mathbf{1}}$ contains the single equation $\vdash \mathsf{p} =_{\varepsilon} \mathsf{p}$ with ε being the self-distance of the unique element in $\mathbf{1}$, are the same thing as objects in the coslice. This isomorphism commutes with the forgetful functors to **GMet**,[385] and we get that the monad $\widehat{\mathcal{T}}_{\Sigma,\hat{E}_{\mathbf{1}}}$ obtained via the existence of free algebras is isomorphic to the monad $- + \mathbf{1}$ which is obtained via the existence of free objects in $\mathbf{1}/\mathbf{GMet}$.[386]

[383] This monad is famous independently of quantitative algebras, variations of it were studied in, e.g., [vB05, §5], [MMM12], [BBKK18, Example 8.4], and [FP19].

[384] It exists because **GMet** has a terminal object (Proposition 103) and coproducts (Corollary 177).

[385] The functor $U : \mathbf{1}/\mathbf{GMet} \to \mathbf{GMet}$ sends the pair $(\mathbf{X}, f : \mathbf{1} \to \mathbf{X})$ to $\mathbf{X}$.

[386] You need to check that $\mathbf{X} + \mathbf{1}$ is indeed the free object on $\mathbf{X}$ in this coslice.

3.4 Lifting Presentations

Most examples of **GMet** presentations in the literature [MPP16, MV20, MSV21, MSV22] (including Examples 193, 195 and 196) are built on top of a **Set** presentation. In summary, there is a monad M on **Set** with a known algebraic presentation (Σ, E) (e.g. $\mathcal{P}_{\mathrm{ne}}$ and semilattices or $\mathcal{D}$ and convex algebras) and a lifting of every space (X, d) to a space $(MX, \widehat{d})$. Then, a quantitative algebraic theory $(\Sigma, \hat{E})$ over the same signature is generated by counterparts to the equations in E as well as new quantitative equations to model the liftings. Finally, it is shown how the theory axiomatizes the lifting, namely, the **GMet** monad induced by the theory is isomorphic to a monad whose action on objects is the assignment $(X, d) \mapsto (MX, \widehat{d})$.

In this section, we prove Theorem 207 which makes this process more automatic and gives necessary and sufficient conditions for when it can actually be done. Throughout, we fix a monad (M, η, μ) on **Set** and an algebraic theory (Σ, E) presenting M via an isomorphism $\rho : \mathcal{T}_{\Sigma,E} \cong M$. We first give multiple definitions to make precise what we mean by *lifting*.

Definition 197 (Liftings)**.** We have three different notions of lifting that we introduce from weakest to strongest.

- A **mere lifting** of M to **GMet** is an assignment $(X, d_{\mathbf{X}}) \mapsto (MX, \widehat{d_{\mathbf{X}}})$ defining a generalized metric on MX for every generalized metric on X.[387]
- A **functor lifting** of M to **GMet** is a functor $\widehat{M} : \mathbf{GMet} \to \mathbf{GMet}$ that makes the square below commute.

$$\begin{array}{ccc} \mathbf{GMet} & \xrightarrow{\widehat{M}} & \mathbf{GMet} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathbf{Set} & \xrightarrow[M]{} & \mathbf{Set} \end{array} \tag{147}$$

  Note in particular that for every space $\mathbf{X}$, the carrier of $\widehat{M}\mathbf{X}$ is MX, so we obtain a mere lifting $\mathbf{X} \mapsto \widehat{M}\mathbf{X}$. Furthermore, given a nonexpansive map $f : \mathbf{X} \to \mathbf{Y}$, the underlying function of $\widehat{M}f$ is Mf, i.e. $Mf : \widehat{M}\mathbf{X} \to \widehat{M}\mathbf{Y}$ is nonexpansive.

[387] The name *lifting* more commonly refers to what we call functor lifting or monad lifting which require more conditions than a mere lifting, hence the name *mere lifting*.

In fact, if we have a mere lifting $(X, d_{\mathbf{X}}) \mapsto (MX, \widehat{d_{\mathbf{X}}})$ such that for every nonexpansive map $f : \mathbf{X} \to \mathbf{Y}$, $Mf : (MX, \widehat{d_{\mathbf{X}}}) \to (MY, \widehat{d_{\mathbf{Y}}})$ is nonexpansive, we automatically get a functor lifting $\widehat{M}$ whose action on objects is given by the mere lifting.[388] We conclude that functor liftings are just mere liftings with that additional condition.

[388] The action on morphisms is prescribed by (147), namely, the underlying function of $\widehat{M}f$ is Mf which is nonexpansive by hypothesis, and since U is faithful, that determines $\widehat{M}f$.

- A **monad lifting** of M to **GMet** is a monad $(\widehat{M}, \widehat{\eta}, \widehat{\mu})$ on **GMet** such that $\widehat{M}$ is a functor lifting of M and furthermore $U\widehat{\eta} = \eta U$ and $U\widehat{\mu} = \mu U$. These two equations mean that the underlying functions of the unit and multiplication $\widehat{\eta}_{\mathbf{X}}$ and $\widehat{\mu}_{\mathbf{X}}$ are η_X and μ_X for any space $\mathbf{X}$.[389] In particular, the maps

$$\eta_X : \mathbf{X} \to \widehat{M}\mathbf{X} \quad \text{and} \quad \mu_X : \widehat{M}\widehat{M}\mathbf{X} \to \widehat{M}\mathbf{X}$$

are nonexpansive for every $\mathbf{X}$. In fact, since U is faithful, that completely determines $\widehat{\eta}_{\mathbf{X}}$ and $\widehat{\mu}_{\mathbf{X}}$, and we conclude as before that a monad lifting is just a mere lifting with three additional conditions:

[389] In summary, the description of a monad M and its monad lifting $\widehat{M}$ are exactly the same after forgetting about distances. In particular, the action of $\widehat{M}$ on morphisms does not depend on the distances at the source or the target, and similarly, the unit and multiplication maps do not depend on the distance of the space.

1. $Mf : (MX, \widehat{d_{\mathbf{X}}}) \to (MY, \widehat{d_{\mathbf{Y}}})$ is nonexpansive if $f : \mathbf{X} \to \mathbf{Y}$ is nonexpansive,
2. $\eta_X : (X, d_{\mathbf{X}}) \to (MX, \widehat{d_{\mathbf{X}}})$ is nonexpansive for every $\mathbf{X}$, and
3. $\mu_X : (MMX, \widehat{\widehat{d_{\mathbf{X}}}}) \to (MX, \widehat{d_{\mathbf{X}}})$ is nonexpansive for every $\mathbf{X}$.

In practice, when defining a monad lifting, we will define a mere lifting and check Items 1–3. Let us give an example.

Example 198. Given an L-space (X, d), we define an L-relation $\widehat{d}$ on $\mathcal{P}_{\mathrm{ne}}X$ as follows: for any non-empty finite $S, S' \subseteq X$,

$$\widehat{d}(S, S') = \begin{cases} \bot & S = S' \\ d(x, y) & S = \{x\} \text{ and } S' = \{y\} \\ \top & \text{otherwise} \end{cases}. \tag{148}$$

Instantiating **GMet** with the category of L-spaces that satisfy reflexivity ($x \vdash x =_\bot x$), (148) defines a mere lifting of $\mathcal{P}_{\mathrm{ne}}$ to **GMet** given by $(X, d) \mapsto (\mathcal{P}_{\mathrm{ne}}X, \widehat{d})$.[390] Viewing $\mathcal{P}_{\mathrm{ne}}$ as modelling nondeterminism, this lifting could model a system where nondeterministic processes cannot be meaningfully compared (they are put at maximum distance) unless the sets of possible outcomes are the same (distance is minimal) or both processes are deterministic (distance is inherited from the distance between the only possible outcomes).

[390] We need reflexivity to ensure the first and second cases do not clash. You can also check that whenever d is a metric space, $\widehat{d}$ is as well, so we get a mere lifting of $\mathcal{P}_{\mathrm{ne}}$ to **Met**.

We show this is a monad lifting of $(\mathcal{P}_{\mathrm{ne}}, \eta, \mu)$,[391] with Lemmas 199–201.

[391] The unit and multiplication of $\mathcal{P}_{\mathrm{ne}}$ were defined in Example 53.

Lemma 199. *If $f : (X, d) \to (Y, \Delta)$ is nonexpansive, then so is the direct image function $\mathcal{P}_{\mathrm{ne}}f : (\mathcal{P}_{\mathrm{ne}}X, \widehat{d}) \to (\mathcal{P}_{\mathrm{ne}}Y, \widehat{\Delta})$.*[392]

[392] We write $f(S)$ instead of $\mathcal{P}_{\mathrm{ne}}f(S)$ for better readability.

Proof. Let $S, S' \in \mathcal{P}_{\mathrm{ne}}X$. If $S = S'$, then $f(S) = f(S')$, so

$$\widehat{\Delta}(f(S), f(S')) = \bot \leq \bot = \widehat{d}(S, S').$$

If $S = \{x\}$ and $S' = \{y\}$, then $f(S) = \{f(x)\}$ and $f(S') = \{f(y)\}$, so[393]

[393] The inequality holds because f is nonexpansive.

$$\widehat{\Delta}(f(S), f(S')) = \Delta(f(x), f(y)) \leq d(x,y) = \widehat{d}(S,S').$$

Otherwise, $\widehat{d}(S,S') = \top$ and $\widehat{\Delta}(f(S), f(S'))$ is always less or equal to $\top$. □

Lemma 200. *For any (X,d), the map $\eta_X : (X,d) \to (\mathcal{P}_{\mathrm{ne}}X, \widehat{d})$ is nonexpansive.*

Proof. Recall that $\eta_X(x) = \{x\}$. For any $x, y \in X$, $\widehat{d}(\{x\},\{y\}) = d(x,y)$, so η_X is even an isometry. □

Lemma 201. *For any (X,d), the map $\mu_X : (\mathcal{P}_{\mathrm{ne}}\mathcal{P}_{\mathrm{ne}}X, \widehat{\widehat{d}}) \to (\mathcal{P}_{\mathrm{ne}}X, \widehat{d})$ is nonexpansive.*

Proof. Recall that $\mu_X(F) = \cup F$ and let $F, F' \in \mathcal{P}_{\mathrm{ne}}\mathcal{P}_{\mathrm{ne}}X$. The case $F = F'$ is dealt with like in Lemma 199, it implies $\cup F = \cup F'$, hence the distances on both sides are $\perp$. If $F = \{S\}$ and $F' = \{S'\}$, $\cup F = S$ and $\cup F' = S'$, then

$$\widehat{d}(\mu_X(F), \mu_X(F')) = \widehat{d}(S,S') = \widehat{\widehat{d}}(\{S\},\{S'\}).$$

Otherwise, $\widehat{\widehat{d}}(F,F') = \top$, so the inequality holds because $\widehat{d}(\mu_X(F), \mu_X(F'))$ is always less or equal to $\top$. □

Many monads of interest on different **GMet** categories are monad liftings of **Set** monads which have an algebraic presentation. We already mentioned the Hausdorff and Kantorovich monad liftings in Examples 193 and 195, but there is also a combination of the two: the Hausdorff–Kantorovich monad lifting of the convex sets of distributions monad [MV20] to **Met**. In [MSV21], we further combined these with the maybe monad on **Met**. Another example is the formal ball monad on quasi-metric spaces [GL19] which is a monad lifting of a writer monad on **Set**. All of these happen to have a quantitative algebraic presentation,[394] and we will show that this is not a coincidence.

Given a monad lifting $\widehat{M}$, we know that it acts on sets just like M does, and that can be described algebraically through the presentation $\rho : \mathcal{T}_{\Sigma,E} \cong M$. This can help to understand how $\widehat{M}$ acts on distances. For any space $\mathbf{X}$, we see the distance $\widehat{d_{\mathbf{X}}}$ on MX as a distance $\widehat{d}$ on terms modulo E via the bijection ρ_X:[395]

$$\widehat{d}([s]_E, [t]_E) = \widehat{d_{\mathbf{X}}}(\rho_X[s]_E, \rho_X[t]_E).$$

Can we find some quantitative equations $\hat{E}$ that axiomatize $\widehat{d}$, i.e. such that $d_{\hat{E}}$ and $\widehat{d}$ are isomorphic (uniformly for all $\mathbf{X}$)?

First of all, for the distances to be isomorphic, they need to be on the same set, namely, we need to have $\mathcal{T}_\Sigma X/{\equiv_E} \cong \mathcal{T}_\Sigma X/{\equiv_{\hat{E}}}$, or equivalently, $s \equiv_E t \iff s \equiv_{\hat{E}} t$. At once, this removes some options for which equations to add in $\hat{E}$. For instance, we cannot add $\mathbf{X} \vdash s = t$ if $X \vdash s = t$ does not already belong to $\mathfrak{Th}(E)$. Conversely, if $X \vdash s = t \in \mathfrak{Th}(E)$, we need to ensure $\mathbf{X} \vdash s = t$ belongs to $\mathfrak{QTh}(\hat{E})$. We can do this by adding $\mathbf{X}_\top \vdash s = t$ to $\hat{E}$ thanks to Example 182.

After that, we will have to add quantitative equations with quantities to axiomatize $\widehat{d}$, but we have to be careful not to break the equivalence we just obtained between $\equiv_E$ and $\equiv_{\hat{E}}$. For instance, if $\mathbf{GMet} = \mathbf{Met}$, $\mathsf{f} : 1 \in \Sigma$ and $E = \emptyset$, then we

[394] Goubault-Larrecq does not talk about quantitative algebras in [GL19], but the quantitative writer monad of [BMPP21, §4.3.2] has a presentation which can easily be adapted to present the monad of [GL19].

[395] Recall Proposition 118.

cannot have $x =_{\frac{1}{2}} y \vdash \mathsf{f}x =_0 \mathsf{f}y \in \hat{E}$, because using the equation $x =_0 y \vdash x = y$ that defines **Met**, we could conclude that $x =_{\frac{1}{2}} y \vdash \mathsf{f}x = \mathsf{f}y$ belongs to $\mathfrak{QTh}(\hat{E})$, which means $\mathsf{f}x \equiv_{\hat{E}} \mathsf{f}y$ whenever $d_{\mathbf{X}}(x,y) \leq \frac{1}{2}$ while $\mathsf{f}x \not\equiv_E \mathsf{f}y$.

The relation between $\hat{E}$ and E seems to mimic our intuition about mere liftings. We say that $\hat{E}$ extends E.

Definition 202 (Extension)**.** Given a class E of equations over Σ and a class $\hat{E}$ of quantitative equations over Σ, we say that $\hat{E}$ is an **extension** of E if for all $\mathbf{X} \in \mathbf{GMet}$ and $s, t \in \mathcal{T}_\Sigma X$,

$$X \vdash s = t \in \mathfrak{Th}(E) \iff \mathbf{X} \vdash s = t \in \mathfrak{QTh}(\hat{E}). \tag{149}$$

Remark 203. Let us make two delicate points on the quantification of $\mathbf{X}$ in (149).

First, it happens *before* the equivalence. This means that equalities[396] that hold in $\mathcal{T}_{\Sigma,E}X$ coincide with the equalities that hold in $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ for each $\mathbf{X}$ individually. In particular, if $\mathbf{X}$ and $\mathbf{X}'$ are spaces on the same set X, then the equalities that hold in $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ and $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}'$ coincide. This intuitively corresponds to the fact that the action of $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}$ does not depend on distances.

[396] This is not a formal term: by *equalities that hold*, we mean which Σ-terms are in the same equivalence class.

If instead of (149) we had the following equivalence with the quantification inside,

$$X \vdash s = t \in \mathfrak{Th}(E) \iff \forall \mathbf{X} \in \mathbf{GMet}, \mathbf{X} \vdash s = t \in \mathfrak{QTh}(\hat{E}),$$

then the equalities in $\mathcal{T}_{\Sigma,E}X$ would be those that hold in all $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ (for all spaces $\mathbf{X}$ with carrier X). In particular, $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ and $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}'$ could have different equivalence classes. That is not desirable when defining a mere lifting.

Second, even though the context of a quantitative equation can be any L-space, $\mathbf{X}$ is only quantified over generalized metric spaces here. This implies that the equivalence classes of $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ and $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}'$ may be different if $d_{\mathbf{X}}$ and $d'_{\mathbf{X}}$ are two different L-relations on X. This does not contradict our intuition about liftings because we only care about the action of $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}$ on L-spaces that belong to **GMet**.

For instance, let $\Sigma = \{\mathsf{f} : 1\}$, $E = \emptyset$, $\hat{E} = \emptyset$, and **GMet** be defined by the equation $x =_\perp y \vdash x = x$. If $X = \{x, y\}$ and $d_{\mathbf{X}}(x,y) = \perp$, then $\mathbf{X} \vdash \mathsf{f}x = \mathsf{f}y$ belongs to $\mathfrak{QTh}(\hat{E})$ while $\mathsf{f}x \not\equiv_E \mathsf{f}y$.[397] Still, it makes sense that $\hat{E}$ extend E since both have no equations.

[397] Here is the derivation (the application of **GMet** implicitly uses the fact that $x =_\perp y \vdash x = x$ is syntactic sugar for $\mathbf{X} \vdash x =_\perp y$):

$$\cfrac{\cfrac{}{\mathbf{X} \vdash x = y}\ \textsc{GMet}}{\mathbf{X} \vdash \mathsf{f}x = \mathsf{f}y}\ \textsc{Cong}$$

It turns out that extensions are stronger than mere liftings because we can show the monad we constructed via terms modulo $\hat{E}$ is a monad lifting of $\mathcal{T}_{\Sigma,E}$.

Proposition 204. *If $\hat{E}$ is an extension of E, then $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}$ is a monad lifting of $\mathcal{T}_{\Sigma,E}$.*

Proof. We need to check the following three equations where $U : \mathbf{GMet} \to \mathbf{Set}$ is the forgetful functor:

$$U\widehat{\mathcal{T}}_{\Sigma,\hat{E}} = \mathcal{T}_{\Sigma,E}U \qquad U\widehat{\eta}^{\Sigma,\hat{E}} = \eta^{\Sigma,E}U \qquad U\widehat{\mu}^{\Sigma,\hat{E}} = \mu^{\Sigma,E}U.$$

First, we have to show that for any space $\mathbf{X}$, $U\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} = \mathcal{T}_{\Sigma,E}U\mathbf{X}$. By definitions, the L.H.S. is $\mathcal{T}_\Sigma X/{\equiv_{\hat{E}}}$ and the R.H.S. is $\mathcal{T}_\Sigma X/{\equiv_E}$, so it boils down to showing that for all $s, t \in \mathcal{T}_\Sigma X$, $s \equiv_{\hat{E}} t \iff s \equiv_E t$. This readily follows from the definitions of $\equiv_{\hat{E}}$ and $\equiv_E$, and from (149):[398]

[398] Note again the importance of being able to do this for each $\mathbf{X}$ individually.

$$s \equiv_{\hat{E}} t \overset{(113)}{\iff} \mathbf{X} \vdash s = t \in \mathfrak{QTh}(\hat{E}) \overset{(149)}{\iff} X \vdash s = t \in \mathfrak{Th}(E) \overset{(20)}{\iff} s \equiv_E t.$$

Next, we have to show that $U\widehat{\mathcal{T}}_{\Sigma,\hat{E}}f = \mathcal{T}_{\Sigma,E}f$ for any $f : \mathbf{X} \to \mathbf{Y}$. This is done rather quickly by comparing their definitions, they make the same squares (22) and (119) commute now that we know $\equiv_{\hat{E}}$ and $\equiv_E$ coincide.

This takes care of the first equation, and the other two are done very similarly, we compare the definitions of $\widehat{\eta}^{\Sigma,\hat{E}}$ and $\eta^{\Sigma,E}$ (resp. $\widehat{\mu}^{\Sigma,\hat{E}}$ and $\mu^{\Sigma,E}$) and conclude they are the same when $\equiv_{\hat{E}}$ and $\equiv_E$ coincide.[399] $\square$

[399] We defined $\widehat{\eta}^{\Sigma,\hat{E}}$ in (133), $\eta^{\Sigma,E}$ in Footnote 63, $\widehat{\mu}^{\Sigma,\hat{E}}$ in (122), and $\mu^{\Sigma,E}$ in (31).

So if we are able to construct an extension $\hat{E}$ of E, we can obtain a monad lifting of M by passing through the isomorphism $\rho : \mathcal{T}_{\Sigma,E} \cong M$.

Corollary 205. *If M is presented by (Σ, E), and $\hat{E}$ is an extension of E, then $\hat{E}$ presents a monad lifting of M.*

Proof. We first construct a monad lifting of (M, η, μ). For any space $\mathbf{X}$, we have an isomorphism $\rho_X^{-1} : MX \to \mathcal{T}_{\Sigma,E}X$, and a generalized metric $d_{\hat{E}}$ on $\mathcal{T}_{\Sigma,E}$ (since the underlying set of $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}$ is $\mathcal{T}_{\Sigma,E}$ by Proposition 204). We can define a generalized metric $\widehat{d_{\mathbf{X}}}$ on MX as we have done for Proposition 118 to guarantee that $\rho_X^{-1} : (MX, \widehat{d_{\mathbf{X}}}) \to \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X}$ is an isomorphism:[400]

$$\widehat{d_{\mathbf{X}}}(m, m') = d_{\hat{E}}(\rho_X^{-1}(m), \rho_X^{-1}(m')). \tag{150}$$

[400] In words, the distance between m and m' in MX is computed by viewing them as (equivalence classes of) terms in $\mathcal{T}_\Sigma X$, then using the distance between them given by $d_{\hat{E}}$.

This yields a mere lifting $(X, d_{\mathbf{X}}) \mapsto (MX, \widehat{d_{\mathbf{X}}})$.

In order to show this is a monad lifting, we use the following diagrams (quantified for all $\mathbf{X} \in \mathbf{GMet}$ and nonexpansive $f : \mathbf{X} \to \mathbf{Y}$) which commute because ρ is a monad isomorphism with inverse ρ^{-1}.[401]

$$\begin{tikzcd}
(MX, \widehat{d_{\mathbf{X}}}) \arrow[r, "\rho_X^{-1}"] \arrow[d, "Mf"'] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[d, "\mathcal{T}_{\Sigma,E}f"] \\
(MY, \widehat{d_{\mathbf{Y}}}) & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{Y} \arrow[l, "\rho_Y"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{X} \arrow[r, "\eta_X^{\Sigma,E}"] \arrow[dr, "\eta_X"'] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[d, "\rho_X"] \\
& (MX, \widehat{d_{\mathbf{X}}})
\end{tikzcd}$$

$$\begin{tikzcd}
(MMX, \widehat{\widehat{d_{\mathbf{X}}}}) \arrow[r, "\rho_{MX}^{-1}"] \arrow[d, "\mu_X"'] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}(X, \widehat{d_{\mathbf{X}}}) \arrow[r, "\mathcal{T}_{\Sigma,E}\rho_X^{-1}"] & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[d, "\mu_X^{\Sigma,E}"] \\
(MX, \widehat{d_{\mathbf{X}}}) & & \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \arrow[ll, "\rho_X"]
\end{tikzcd}$$

[401] The first holds by naturality, the second by (48), and the third by (49). Moreover, all the functions in these diagrams are nonexpansive (with the sources and targets as drawn) by previous results:

- We just showed the components of ρ are isometries.
- We showed $\mathcal{T}_{\Sigma,E}f$ is the underlying function of $\widehat{\mathcal{T}}_{\Sigma,E}f$ because $\widehat{\mathcal{T}}_{\Sigma,E}$ is a monad lifting of $\mathcal{T}_{\Sigma,E}$ (Proposition 204), so $\mathcal{T}_\Sigma Ef$ is nonexpansive when f is nonexpansive.
- By the previous two points, $\mathcal{T}_{\Sigma,E}\rho_X^{-1}$ is nonexpansive.
- Again since $\widehat{\mathcal{T}}_{\Sigma,\hat{E}}$ is a monad lifting of $\mathcal{T}_{\Sigma,E}$, $\eta_X^{\Sigma,E}$ and $\mu_X^{\Sigma,E}$ are nonexpansive.

These show (detailed in the footnote) that Mf, η_X and μ_X are compositions of nonexpansive maps, and hence are nonexpansive. We obtain a monad lifting $\widehat{M}$ of M to $\mathbf{GMet}$ which sends $(X, d_{\mathbf{X}})$ to $(MX, \widehat{d_{\mathbf{X}}})$.

It remains to show that $\widehat{M}$ is presented by $(\Sigma, \hat{E})$. By construction, we have the isomorphism $\widehat{\rho}_{\mathbf{X}} : \widehat{\mathcal{T}}_{\Sigma,\hat{E}}\mathbf{X} \to \widehat{M}\mathbf{X}$ whose underlying function is ρ_X for every $\mathbf{X}$. The fact that $\widehat{\rho}$ is a monad morphism follows from the facts that ρ is a monad morphism, and that $U : \mathbf{GMet} \to \mathbf{Set}$ is faithful so it reflects commutativity of diagrams.[402] $\square$

[402] Let us detail the argument for naturality, the others would follow the same pattern. We need to show that $\widehat{\rho}_{\mathbf{Y}} \circ \widehat{M}f = \widehat{M}f \circ \widehat{\rho}_{\mathbf{X}}$. Applying U, we get $\rho_Y \circ Mf = Mf \circ \rho_X$ which is true because ρ is natural, hence $U(\widehat{\rho}_{\mathbf{Y}} \circ \widehat{M}f) = U(\widehat{M}f \circ \widehat{\rho}_{\mathbf{X}})$. Since U is faithful, and the desired equation holds.

Now, we would like to have a converse result. Namely, if $(X, d_{\mathbf{X}}) \mapsto (MX, \widehat{d_{\mathbf{X}}})$ is given by a monad lifting $\widehat{M}$ of M to **GMet**, our goal is to construct an extension $\hat{E}$ of E such that the monad lifting corresponding to $\hat{E}$ (given in Corollary 205) is $\widehat{M}$. There is no obvious reason this is even possible, maybe $\widehat{M}$ is a monad lifting that has no quantitative algebraic presentation.[403] Our next theorem shows that such an $\hat{E}$ always exists. In fact, it is constructed very naively.

As discussed in Example 182, when $\hat{E}$ contains all the quantitative equations in

$$\hat{E}_1 = \{\mathbf{X}_\top \vdash s = t \mid X \vdash s = t \in E\}, \tag{151}$$

then we have at least one direction of (149), namely, that $X \vdash s = t \in \mathfrak{Th}(E)$ implies $\mathbf{X} \vdash s = t \in \mathfrak{QTh}(\hat{E})$ for all $\mathbf{X}$ and $s, t \in \mathcal{T}_\Sigma X$.[404] Next, we include in $\hat{E}$ all the possible equations $\mathbf{X} \vdash s =_\varepsilon t$ where ε is the distance between s and t when viewed inside $\widehat{M}\mathbf{X}$ (via ρ_X),[405] namely, $\hat{E}_2 \subseteq \hat{E}$ where

$$\hat{E}_2 = \left\{\mathbf{X} \vdash s =_\varepsilon t \mid \mathbf{X} \in \mathbf{GMet}, s, t \in \mathcal{T}_\Sigma X, \varepsilon = \widehat{d_{\mathbf{X}}}(\rho_X[s]_E, \rho_X[t]_E)\right\}. \tag{152}$$

This is a very large bunch of equations (it is not even a set), but it leaves no stone unturned, meaning that the distance computed by $\hat{E}$ will always be smaller than the distance in $\widehat{M}\mathbf{X}$. Indeed, for any $m, m' \in MX$, letting $s, t \in \mathcal{T}_\Sigma X$ be such that $\rho_X[s]_E = m$ and $\rho_X[t]_E = m'$ (by surjectivity of ρ_X), we have[406]

$$\begin{aligned}\widehat{d_{\mathbf{X}}}(m, m') \leq \varepsilon \implies & \mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}(\hat{E}) \\ \iff & d_{\hat{E}}([s]_E, [t]_E) \leq \varepsilon \\ \iff & d_{\hat{E}}(\rho_X^{-1}(m), \rho_X^{-1}(m')) \leq \varepsilon.\end{aligned}$$

In order to conclude that $\hat{E} = \hat{E}_1 \cup \hat{E}_2$ presents $\widehat{M}$, we need to show that $\hat{E}$ is an extension of E, i.e. the other direction of (149), and that the monad lifting defined in Corollary 205 coincides with $\widehat{M}$, i.e. the converse implication of the previous derivation holds. We will prove these by constructing a (family of) special algebras in $\mathbf{QAlg}(\Sigma, \hat{E})$.[407]

For any generalized metric space $\mathbf{A}$, we denote by $\mathbb{M}\mathbf{A}$ the quantitative Σ-algebra $(MA, [\![-]\!]_{\mu_A}, \widehat{d_{\mathbf{A}}})$, where

- $(MA, \widehat{d_{\mathbf{A}}})$ is the space obtained by applying $\widehat{M}$ to $\mathbf{A}$, and
- $(MA, [\![-]\!]_{\mu_A})$ is the Σ-algebra obtained by applying the isomorphism $\mathbf{Alg}(\Sigma, E) \cong \mathbf{EM}(M)$ (from the presentation) to the M-algebra (MA, μ_A) (from Example 58).

We can show that $\mathbb{M}\mathbf{A}$ belongs to $\mathbf{QAlg}(\Sigma, \hat{E}_1 \cup \hat{E}_2)$.

Lemma 206. *For all $\phi \in \hat{E}_1 \cup \hat{E}_2$, $\mathbb{M}\mathbf{A} \vDash \phi$.*

Proof. If $\phi = \mathbf{X}_\top \vdash s = t \in \hat{E}_1$, then by construction $(MA, [\![-]\!]_{\mu_A})$ satisfies $X \vdash s = t \in E$. So $\mathbb{M}\mathbf{A}$ satisfies ϕ by Lemma 158.

Suppose now that $\phi = \mathbf{X} \vdash s =_\varepsilon t \in \hat{E}_2$ with $\varepsilon = \widehat{d_{\mathbf{X}}}(\rho_X[s]_E, \rho_X[t]_E)$. A bit of unrolling[408] shows that for an assignment $\iota : X \to MA$, the interpretation $[\![-]\!]^\iota_{\mu_A}$ is

[403] Or maybe $\widehat{M}$ has a presentation that is not an extension of E, but our informal discussion leading to the definition of extensions indicates that is less probable.

[404] We use Lemma 183.

[405] We are essentially doing the opposite of (150).

[406] The implication follows because by definition, $\hat{E}$ will contain $\mathbf{X} \vdash s =_{d_{\mathbf{X}}(m,m')} t$, hence by the Max rule, we will have $\mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}(\hat{E})$. The first equivalence is (117), and the second holds because ρ_X^{-1} is the inverse of ρ_X.

[407] In turns out (after the rest of the proof) we are constructing the free algebra over $\mathbf{A}$, but we feel it is not necessary to make that explicit.

[408] Look at the definition of P^{-1} in Proposition 59, in particular what we proved in Footnote 114, and the definition of $-\rho$ in (53).

the composite

$$\mathcal{T}_\Sigma X \xrightarrow{\mathcal{T}_\Sigma \iota} \mathcal{T}_\Sigma MA \xrightarrow{[-]_E} \mathcal{T}_{\Sigma,E} MA \xrightarrow{\rho_{MA}} MMA \xrightarrow{\mu_A} MA.$$

For later use, we apply the naturality of $[-]_E$ (22) and ρ to rewrite the composite as

$$[\![-]\!]^\iota_{\mu_A} = \mathcal{T}_\Sigma X \xrightarrow{[-]_E} \mathcal{T}_{\Sigma,E} X \xrightarrow{\rho_X} MX \xrightarrow{M\iota} MMA \xrightarrow{\mu_A} MA. \tag{153}$$

We conclude that $\mathbb{M}\mathbf{A} \vDash \phi$ with the following derivation which holds for all nonexpansive $\hat{\iota} : \mathbf{X} \to \widehat{M}\mathbf{A}$.[409]

$$\begin{aligned}
\widehat{d_{\mathbf{A}}}([\![s]\!]^{\hat\iota}_{\mu_A}, [\![t]\!]^{\hat\iota}_{\mu_A}) &= \widehat{d_{\mathbf{A}}}\left(\mu_A(M\hat\iota(\rho_X[s]_E)), \mu_A(M\hat\iota(\rho_X[t]_E))\right) && \text{by (153)}\\
&\le \widehat{\widehat{d_{\mathbf{A}}}}\left(M\hat\iota(\rho_X[s]_E), M\hat\iota(\rho_X[t]_E)\right) && \mu_A \text{ is nonexpansive}\\
&\le \widehat{d_{\mathbf{X}}}\left(\rho_X[s]_E, \rho_X[t]_E\right) && M\hat\iota \text{ is nonexpansive}\\
&= \varepsilon && \square
\end{aligned}$$

[409] Our hypothesis that $\widehat{M}$ is a monad lifting yields nonexpansiveness of μ_A and $M\hat\iota$.

Theorem 207. *Let $\widehat{M}$ be a monad lifting of M to* **GMet**, *and $\hat{E} = \hat{E}_1 \cup \hat{E}_2$. Then, $\hat{E}$ is an extension of E and it presents $\widehat{M}$.*

Proof. We already showed the forward implication of (149) when we defined $\hat{E}_1$ (151). For the converse, suppose that $\mathbf{X} \vdash s = t \in \mathfrak{QTh}(\hat{E})$, we saw in Lemma 206 that $\mathbb{M}\mathbf{X}$ satisfies $\mathbf{X} \vdash s = t$. Taking the assignment $\eta_X : \mathbf{X} \to \widehat{M}\mathbf{X}$ which is nonexpansive because $\widehat{M}$ is a monad lifting, we have $[\![s]\!]^{\eta_X}_{\mu_X} = [\![t]\!]^{\eta_X}_{\mu_X}$. Using (153) and the monad law $\mu_X \circ M\eta_X = \mathrm{id}_{MX}$ (left triangle in (39)), we find

$$\rho_X[s]_E = \mu_X(M\eta_X(\rho_X[s]_E)) = [\![s]\!]^{\eta_X}_{\mu_X} = [\![t]\!]^{\eta_X}_{\mu_X} = \mu_X(M\eta_X(\rho_X[t]_E)) = \rho_X[t]_E.$$

Finally, since ρ_X is a bijection, we have $[s]_E = [t]_E$, i.e. $X \vdash s = t \in \mathfrak{Th}(E)$.

We already showed that $\widehat{d_{\mathbf{X}}}(m, m') \ge d_{\hat{E}}(\rho_X^{-1}(m), \rho_X^{-1}(m'))$ when defining $\hat{E}_2$. For the converse, let $m = \rho_X[s]_E$ and $m' = \rho_X[t]_E$ for some $s, t \in \mathcal{T}_\Sigma X$ and suppose that $d_{\hat{E}}([s]_E, [t]_E) \le \varepsilon$, or equivalently by (117), that $\mathbf{X} \vdash s =_\varepsilon t \in \mathfrak{QTh}(\hat{E})$. As above, Lemma 206 says that $\mathbb{M}\mathbf{X}$ satisfies that equation. Taking the assignment $\eta_X : \mathbf{X} \to \widehat{M}\mathbf{X}$ which is nonexpansive because $\widehat{M}$ is a monad lifting, we have[410]

$$\widehat{d_{\mathbf{X}}}(m, m') = \widehat{d_{\mathbf{X}}}\left(\rho_X[s]_E, \rho_X[t]_E\right) = \widehat{d_{\mathbf{X}}}\left([\![s]\!]^{\eta_X}_{\mu_X}, [\![t]\!]^{\eta_X}_{\mu_X}\right) \le \varepsilon.$$

[410] The second inequality holds again by (153) and (39).

Comparing with (150), we conclude that $\widehat{M}$ is exactly the monad lifting from Corollary 205. In particular, $\hat{E}$ presents $\widehat{M}$ via $\widehat{\rho}$ whose component at $\mathbf{X}$ is ρ_X. $\square$

Remark 208. A deeper result hides behind the last line. It follows from our constructions that if you start from an extension $\hat{E}$, build a monad lifting $\widehat{M}$ from $\hat{E}$ with Corollary 205, then build an extension $\hat{E}'$ from $\widehat{M}$ with Theorem 207, you obtain two *equivalent* classes of equations, i.e. $\mathfrak{QTh}(\hat{E}) = \mathfrak{QTh}(\hat{E}')$. Similarly, if you start with a monad lifting $\widehat{M}$, then build an extension $\hat{E}$, then build a monad lifting $\widehat{M}'$, then $\widehat{M} = \widehat{M}'$.[411]

This does not yield a bijection but almost. If you restrict extensions of E to those that are quantitative algebraic theories,[412] then you get a bijection with monad

[411] We have equality on the nose because monad liftings are defined with equality on the nose. One can probably relax these to be isomorphisms.

[412] i.e. they are *saturated*, you cannot add more quantitative equations without changing the algebras

liftings of M.

I believe it is a simple exercise in categorical logic to make this remark into an (dual) equivalence of categories. A more challenging task would be to allow M and E to vary.

When constructing the extension $\hat{E} = \hat{E}_1 \cup \hat{E}_2$, $\hat{E}_1$ can be fairly small since it has the size of E, but $\hat{E}_2$ as defined is always huge (not even a set). In theory, some results in the literature could allow us to restrict the size of contexts to be of a moderate size only with mild size conditions on L and $\hat{E}_{\mathbf{GMet}}$.[413] In practice, we can sometimes find some simple set of quantitative equations which will be equivalent to $\hat{E}_2$ (when $\hat{E}_1$ is present), and we give a couple of examples below. They require some *clever* arguments that depend on the application, but there may be room for optimization in the definition of $\hat{E}_2$.

Example 209 (Trivial Lifting of $\mathcal{P}_{\mathrm{ne}}$)**.** Recall the monad lifting of $\mathcal{P}_{\mathrm{ne}}$ to $\mathbf{GMet} = \mathbf{QAlg}(\emptyset, \{x \vdash x =_\perp X\})$ from Example 198. Let us denote it by $\widehat{\mathcal{P}}$, and its action on objects by $(X, d) \mapsto (\mathcal{P}_{\mathrm{ne}}X, \widehat{d_{\mathbf{X}}})$.[414] We also denote with ρ the monad isomorphism witnessing that $\mathcal{P}_{\mathrm{ne}}$ is presented by the theory of semilattices $(\Sigma_{\mathbf{S}}, E_{\mathbf{S}})$ (recall Example 67). By Theorem 207, there is a quantitative algebraic presentation for $\widehat{\mathcal{P}}$ given by[415]

$$\hat{E}_1 = \{\mathbf{X}_\top \vdash s = t \mid X \vdash s = t \in E_{\mathbf{S}}\} \text{ and } \hat{E}_2 = \left\{\mathbf{X} \vdash s =_\varepsilon t \mid \varepsilon = \widehat{d_{\mathbf{X}}}\left(\rho_X[s]_{E_{\mathbf{S}}}, \rho_X[t]_{E_{\mathbf{S}}}\right)\right\}.$$

We claim that the equations in $\hat{E}_1$ are enough, namely, $\mathfrak{QTh}(\hat{E}_1 \cup \hat{E}_2) = \mathfrak{QTh}(\hat{E}_1)$.

First, since $\hat{E}_1 \subseteq \hat{E}_1 \cup \hat{E}_2$, we infer that $\mathfrak{QTh}(\hat{E}_1) \subseteq \mathfrak{QTh}(\hat{E}_1 \cup \hat{E}_2)$.[416]

Second, recall from Lemma 183 that with the equations in $\hat{E}_1$, we can already prove all the equations in the theory of semilattices. This means that for any $\mathbf{X} \vdash s =_\varepsilon t \in \hat{E}_2$ with $\varepsilon = \widehat{d_{\mathbf{X}}}\left(\rho_X[s]_{E_{\mathbf{S}}}, \rho_X[t]_{E_{\mathbf{S}}}\right)$, we have the three following cases.

- If $[s]_{E_{\mathbf{S}}} = [t]_{E_{\mathbf{S}}}$ and $\varepsilon = \perp$, i.e. s and t represent the same subset of X, then the equation $X \vdash s = t$ is in $\mathfrak{Th}(E_{\mathbf{S}})$ which implies $\mathbf{X} \vdash s = t$ is in $\mathfrak{QTh}(\hat{E}_1)$. It follows by the following derivation that $\mathbf{X} \vdash s =_0 t \in \mathfrak{QTh}(\hat{E}_1)$ as desired.[417]

$$\dfrac{\mathbf{X} \vdash s = t \qquad \dfrac{\sigma = x \mapsto s \qquad \dfrac{}{x \vdash x =_\perp x}\,\textsc{GMet} \qquad \dfrac{}{\mathbf{X} \vdash s =_\top s}\,\textsc{Top}}{\mathbf{X} \vdash s =_\perp s}\,\textsc{SubQ}}{\mathbf{X} \vdash s =_\perp t}\,\textsc{CompR}$$

- If $[s]_{E_{\mathbf{S}}} = [x]_{E_{\mathbf{S}}}$ and $[t]_{E_{\mathbf{S}}} = [y]_{E_{\mathbf{S}}}$ for some $x, y \in X$ and $\varepsilon = d_{\mathbf{X}}(x, y)$, then the equations $X \vdash s = x$ and $X \vdash y = t$ are in $\mathfrak{Th}(E_{\mathbf{S}})$ which implies $\mathbf{X} \vdash s = x$ and $\mathbf{X} \vdash y = t$ are in $\mathfrak{QTh}(\hat{E}_1)$. Furthermore, Lemma 153 implies $\mathbf{X} \vdash x =_\varepsilon y \in \mathfrak{QTh}(\hat{E}_1)$, and finally by Lemmas 150 and 151, $\mathbf{X} \vdash s =_\varepsilon t$ also belongs to $\mathfrak{QTh}(\hat{E}_1)$ as desired.

- Otherwise, $\varepsilon = \top$, so $\mathbf{X} \vdash s =_\varepsilon t$ belongs to $\mathfrak{QTh}(\hat{E}_1)$ by Lemma 152.

We have shown that $\hat{E}_2 \subseteq \mathfrak{QTh}(\hat{E}_1)$, and it follows that $\mathfrak{QTh}(\hat{E}_1 \cup \hat{E}_2) \subseteq \mathfrak{QTh}(\hat{E}_1)$.[418]

In conclusion, we found that $\widehat{\mathcal{P}}$ is presented by the equations in $\hat{E}_1$ which we rewrite below:

$$x \vdash x = x \oplus x \qquad x, y \vdash x \oplus y = y \oplus x \qquad x, y, z \vdash x \oplus (y \oplus z) = (x \oplus y) \oplus z.$$

[413] I will not write the proofs because I am not confident enough with the literature on accessible and presentable categories, but I believe [FMS21, Propositions 3.8 and 3.9] make it possible to adapt the arguments of Remark 50 replacing $\aleph_0$ with a different cardinal (we implicitly used $\aleph_0$ because $\lambda < \aleph_0 \Leftrightarrow \lambda$ finite).

[414] The distance $\widehat{d_{\mathbf{X}}}$ was defined in (148).

[415] We are a bit concise in the quantifications for $\hat{E}_2$.

[416] There are two ways to understand this. Semantically, the equations that are satisfied by all algebras in $\mathbf{QAlg}(\Sigma, \hat{E}_1)$ are also satisfied by all algebras in $\mathbf{QAlg}(\Sigma, \hat{E}_1 \cup \hat{E}_2)$ because the second category is contained in the first. Syntactically, if you have more axioms, you can prove more things.

[417] Recall that the context of $x \vdash x =_\perp x$, after unrolling the syntactic sugar, is the L-space with x at distance $\top$ from itself, so we only need to prove $\sigma(x)$ is also at distance $\top$ from itself (we do it with Top).

[418] Again, there are two different ways to understand this. Semantically, if all algebras in $\mathbf{QAlg}(\Sigma, \hat{E}_1)$ satisfy $\hat{E}_2$, then $\mathbf{QAlg}(\Sigma, \hat{E}_1)$ and $\mathbf{QAlg}(\Sigma, \hat{E}_1 \cup \hat{E}_2)$ are the same categories. Syntactically, in any derivation with axioms $\hat{E}_1 \cup \hat{E}_2$, you can replace each axiom in $\hat{E}_2$ by a derivation using only axioms in $\hat{E}_1$.

Compared to the presentation of $\mathcal{P}_{\text{ne}}^{\uparrow}$, we simply removed (107).

In a sense, $\widehat{\mathcal{P}}$ can be seen as a *trivial* monad lifting of $\mathcal{P}_{\text{ne}}$ because we simply viewed the equations presenting $\mathcal{P}_{\text{ne}}$ as quantitative equations as we did in (141), and we added nothing else. After this example, you may want to conjecture that whenever $\hat{E}$ is constructed from E like that, then $\hat{E}$ presents a monad lifting of the $\mathcal{T}_{\Sigma,E}$, or equivalently thanks to Corollary 205 and Theorem 207, $\hat{E}$ is an extension of E. That is not true. We showed in [MSV21, Theorem 44] that $\hat{E}$ can sometimes prove more equations than E.

We end this chapter with a final example, the one that motivated a lot of ideas in this manuscript.

Example 210 (ŁK)**.** The ŁK distance on probability distributions defined in (104) defines a mere lifting $(X, d) \mapsto (\mathcal{D}X, d_{\text{ŁK}})$ of $\mathcal{D}$ to $\mathbf{GMet} = [0, 1]\mathbf{Spa}$.[419] We show this is a monad lifting of $(\mathcal{D}, \eta, \mu)$ (as defined in Example 54) with Lemmas 211–213.

[419] Of course, you can take $[0, \infty]\mathbf{Spa}$ as well. You can also show that this mere lifting preserves the satisfaction of all the equations defining metric spaces except reflexivity ($x \vdash x =_0 x$). Indeed, we have $d_{\text{ŁK}}(\varphi, \varphi) = 0$ if and only if $d(x, y) = 0$ for all $x, y \in \text{supp}(\varphi)$ (if d is reflexive, this forces $\varphi = \delta_x$). For instance, you can take $\mathbf{GMet}$ to be the category of diffuse metric spaces as we did in [MSV22, §5.3].

Lemma 211. *If $f : (X, d) \to (Y, \Delta)$ is nonexpansive, then so is $\mathcal{D}f : (\mathcal{D}X, d_{\text{ŁK}}) \to (\mathcal{D}Y, \Delta_{\text{ŁK}})$.*

Proof. Let $\varphi, \psi \in \mathcal{D}X$, we have

$$
\begin{aligned}
& d_{\text{ŁK}}(\mathcal{D}f(\varphi), \mathcal{D}f(\psi)) \\
&= \sum_{(y,y')} \mathcal{D}f(\varphi)(y)\mathcal{D}f(\psi)(y')\Delta(y, y') \\
&= \sum_{(y,y')} \left(\sum_{x \in f^{-1}(y)} \varphi(x) \right) \left(\sum_{x' \in f^{-1}(y')} \psi(x') \right) \Delta(y, y') && \text{definition of } \mathcal{D}f \\
&= \sum_{(y,y')} \sum_{x \in f^{-1}(y)} \sum_{x' \in f^{-1}(y')} \varphi(x)\psi(x')\Delta(y, y') \\
&= \sum_{(x,x')} \varphi(x)\psi(x')\Delta(f(x), f(x')) \\
&\leq \sum_{(x,x')} \varphi(x)\psi(x')d(f(x), f(x')) && f \text{ is nonexpansive} \\
&= d_{\text{ŁK}}(\varphi, \psi). && \text{definition of } d_{\text{ŁK}} \quad \square
\end{aligned}
$$

Lemma 212. *For any (X, d), the map $\eta_X : (X, d) \to (\mathcal{D}X, d_{\text{ŁK}})$ is nonexpansive.*

Proof. For any $a, a' \in X$, we have[420]

[420] Notice that η_X is even an isometric embedding.

$$
d_{\text{ŁK}}(\delta_a, \delta_{a'}) \overset{(104)}{=} \sum_{(x,x')} \delta_a(x)\delta_{a'}(x')d(x, x') = \delta_a(a)\delta_{a'}(a')d(a, a') = d(a, a'). \quad \square
$$

Lemma 213. *For any (X, d), the map $\mu_X : (\mathcal{D}\mathcal{D}X, d_{\text{ŁKŁK}}) \to (\mathcal{D}X, d_{\text{ŁK}})$ is nonexpansive.*

Proof. $\square$

Let us denote this monad lifting by $\mathcal{D}_{\text{ŁK}}$. In [MSV22, §5.3], we gave a relatively simple quantitative algebraic presentation for $\mathcal{D}_{\text{ŁK}}$, but Theorem 207 will help us find a simpler one. Since, by Example 68, the theory of convex algebras generated by $(\Sigma_{\mathbf{CA}}, E_{\mathbf{CA}})$ presents $\mathcal{D}$ (via a monad isomorphism that we write ρ), the theorem gives us a theory presenting $\mathcal{D}_{ŁK}$ generated by $\hat{E}_1 \cup \hat{E}_2$ where

$$\hat{E}_1 = \{\mathbf{X}_\top \vdash s = t \mid X \vdash s = t \in E_{\mathbf{CA}}\} \text{ and}$$
$$\hat{E}_2 = \left\{(X,d) \vdash s =_\varepsilon t \mid \varepsilon = d_{ŁK}\left(\rho_X[s]_{E_{\mathbf{CA}}}, \rho_X[t]_{E_{\mathbf{CA}}}\right)\right\}.$$

In order to simplify $\hat{E}_2$, we rely on two property that $d_{ŁK}$ has (one symmetric to the other) : for any $\varphi, \varphi', \psi \in \mathcal{D}X$ and $p \in [0,1]$,

$$d_{ŁK}(p\varphi + \overline{p}\varphi', \psi) = pd_{ŁK}(\varphi, \psi) + \overline{p}d_{ŁK}(\varphi', \psi) \text{ and} \tag{154}$$
$$d_{ŁK}(\varphi, p\varphi + \overline{p}\varphi') = pd_{ŁK}(\psi, \varphi) + \overline{p}d_{ŁK}(\psi, \varphi'). \tag{155}$$

Intuitively, this means that we can compute the distance between s and t by decomposing the terms into their variables, computing simple distances, then combining them to get back to s and t.[421] Formally, we only need to keep the quantitative equations in $\hat{E}_2$ that belong to[422]

$$\begin{aligned}\hat{E}_2' =& \{x =_{\varepsilon_1} y, x =_{\varepsilon_2} z \vdash x =_{p\varepsilon_1 + \overline{p}\varepsilon_2} y +_p z \mid \varepsilon_1, \varepsilon_2 \in [0,1], p \in (0,1)\} \\ &\cup \{y =_{\varepsilon_1} x, z =_{\varepsilon_2} x \vdash y +_p z =_{p\varepsilon_1 + \overline{p}\varepsilon_2} x \mid \varepsilon_1, \varepsilon_2 \in [0,1], p \in (0,1)\}.\end{aligned}$$

We will prove that for any $\hat{\mathbb{A}} \in \mathbf{QAlg}(\Sigma_{\mathbf{CA}})$, $\hat{\mathbb{A}} \vDash \hat{E}_1 \cup \hat{E}_2'$ implies $\hat{\mathbb{A}} \vDash \hat{E}_1 \cup \hat{E}_2$.[423] Suppose $\hat{\mathbb{A}} \vDash \hat{E}_1 \cup \hat{E}_2'$, we proceed by induction on the structure of s and t to show that $\hat{\mathbb{A}}$ satisfies $(X,d) \vdash s =_\varepsilon t$, where $\varepsilon = d_{ŁK}\left(\rho_X[s]_{E_{\mathbf{CA}}}, \rho_X[t]_{E_{\mathbf{CA}}}\right)$.

If s and t are variables, then $\rho_X[s]_{E_{\mathbf{CA}}} = \delta_x$ and $\rho_X[t]_{E_{\mathbf{CA}}} = \delta_y$ for some $x, y \in X$, thus $\varepsilon = d(x,y)$ and $(X,d) \vdash x =_{d(x,y)} y$ is satisfied by $\hat{\mathbb{A}}$ (by 153).

Otherwise, without loss of generality,[424] we write $t = t_1 +_p t_2$, and let $\varepsilon_i = d_{ŁK}\left(\rho_X[s]_{E_{\mathbf{CA}}}, \rho_X[t_i]\right)$. By the induction hypothesis, $\hat{\mathbb{A}} \vDash (X,d) \vdash s =_{\varepsilon_i} t_i$ for $i = 1,2$. Then, we define a substitution map $\sigma : \{x,y,z\} \to \mathcal{T}_\Sigma X$ with $x \mapsto s$, $y \mapsto t_1$ and $z \mapsto t_2$, and since $\hat{\mathbb{A}}$ satisfies $x =_{\varepsilon_1} y, x =_{\varepsilon_2} z \vdash x =_{p\varepsilon_1 + \overline{p}\varepsilon_2} y +_p z \in \hat{E}_2'$, we can apply Lemma 160 to conclude $\hat{\mathbb{A}}$ satisfies $(X,d) \vdash s =_{\varepsilon'} t$ with

$$\begin{aligned}\varepsilon' &= pd_{ŁK}\left(\rho_X[s]_{E_{\mathbf{CA}}}, \rho_X[t_1]\right) + \overline{p}d_{ŁK}\left(\rho_X[s]_{E_{\mathbf{CA}}}, \rho_X[t_2]\right) \\ &= d_{ŁK}\left(\rho_X[s]_{E_{\mathbf{CA}}}, p\rho_X[t_1] + \overline{p}\rho_X[t_2]\right) && \text{by (154)} \\ &= d_{ŁK}\left(\rho_X[s]_{E_{\mathbf{CA}}}, \rho_X[t_1 +_p t_2]\right) \\ &= d_{ŁK}\left(\rho_X[s]_{E_{\mathbf{CA}}}, \rho_X[t]_{E_{\mathbf{CA}}}\right) = \varepsilon.\end{aligned}$$

We conclude that $\hat{E}_1 \cup \hat{E}_2'$ presents $\mathcal{D}_{ŁK}$.

[421] This is very similar to what happens for the Kantorovich distance and (111).

[422] If you have symmetry ($x =_\varepsilon y \vdash y =_\varepsilon x$) as an axiom in **GMet** already, you can keep only one of these sets.

[423] It follows that $\mathfrak{QTh}(\hat{E}_1 \cup \hat{E}_2') = \mathfrak{QTh}(\hat{E}_1 \cup \hat{E}_2)$ because we already have the $\supseteq$ inclusion as explained in Footnote 418.

[424] If s is a term of depth > 0 but t is a variable, you decompose s instead, and then you have to use a symmetric argument.

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