# Extracting an $\mathbb{N}$ -graded differential modality from a differential modality

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#### Introduction

Differential modalities were introduced in [1] and graded differential modalities in [2]. We present a way of forming an  $\mathbb{N}$ graded differential modality from a differential modality, under mild conditions. The letters R, S will designate commutative semirings.

# Yet another definition

**DEF5** Let  $M = (!, m, u, \Delta, \epsilon, \partial)$  an R-graded differential modality, M' = $(!', m', u', \Delta', \epsilon', \partial')$  an S-graded differential modality and  $\rho : S \to R$  a semiring homomorphism. A  $\rho$ -morphism of graded differential modalities from M to M' is a family of natural transformations  $(\phi_s : !_{\rho(s)}A \rightarrow !'_sA)_{s \in S}$  such that the

#### An example

We can take  $\mathcal{L} = \mathbf{Vec}_k^{op}$  and !A = SAthe free symmetric algebra on A.

If char k = 0,  $S_{\leq n}(k) \simeq \{f(x) \in$  $k[x], \deg f \le n\}.$ 

But the story is different in positive characteristic. For instance, if  $k = \mathbb{Z}_2$ ,

### Acknowledgements

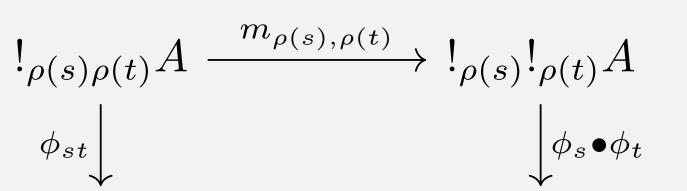
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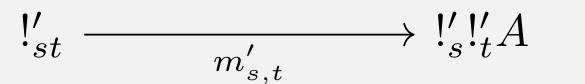
# The definitions

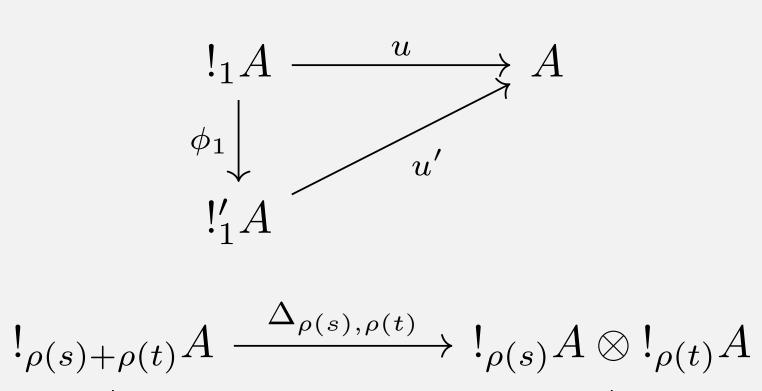
**DEF1** An additive symmetric monoidal category is a symmetric monoidal category such that every hom-set is a commutative monoid and for every morphism f we have that  $f \otimes -, - \otimes f$ , -; f and f; - preserve sums of maps and fzero maps.

**DEF2** An *R*-graded differential modality on an additive symmetric monoidal category  $\mathcal{L}$  is a tuple  $(!, m, u, \Delta, \epsilon, \partial)$  where

following diagrams commute (for every  $s, t \in S$ ):







then 
$$S_{\leq 0}(k) \simeq \{f(x^2), f \in k[x]\}$$
 and  
 $S_{\leq n}(k) = S(k) \simeq k[x]$  if  $n \ge 1$ .

# Useful identities

The following identities are essential to the proof of the theorem. They are satisfied by every differential modality.

(The notation  $\lambda \vdash n$  means that  $\lambda = (m_1(\lambda), ..., m_n(\lambda)) \in \mathbb{N}^n$  is such that  $1.m_1(\lambda) + ... + n.m_n(\lambda) = n$ , we define  $\Delta^n$  by  $\Delta^1 = id_{!A}$  and  $\Delta^{n+1} := \Delta; \Delta^n \otimes 1$ , and  $\tau_{\lambda}$  is an interleaving permutation associated to  $\lambda$ .)

Constant rule:  $(\mathbf{V})$ 

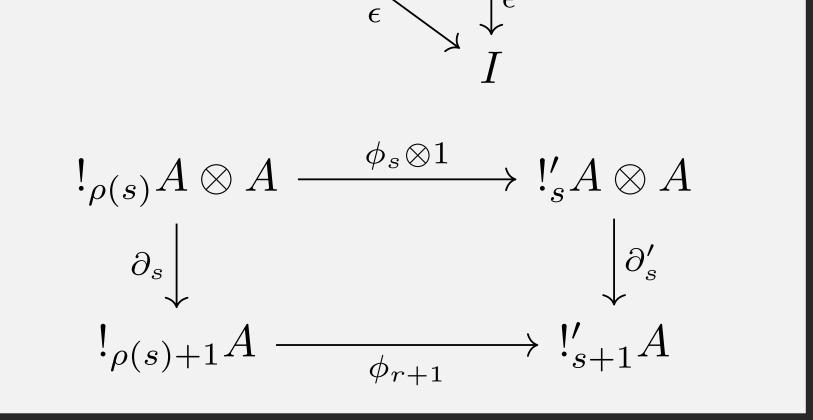
 $\partial; \epsilon = 0$ 

(vi) Linear bis rule:

 $! = (!_r : \mathcal{L} \to \mathcal{L})_{r \in R}$  is a family of endofunctors;  $m = (m_{r,s} : !_{rs}A \rightarrow !_r !_s A)_{r,s \in R}$ is a family of natural transformations and  $u : !_1 A \rightarrow A$  is a natural transformation such that (!, m, u) is an *R*-graded comonad;  $\Delta = (\Delta_{r,s} : !_{r+s}A \rightarrow !_rA \otimes$  $!_s A)_{r,s\in R}$  is a family of natural transformations and  $\epsilon : !_0 A \to I$  is a natural transformation such that  $(!A, \Delta, \epsilon)$  is an R-graded cocommutative comonoid; m is an *R*-graded comonoid morphism and the following equalities are verified (for every  $r, s \in R$ ):

(i) Linear rule:  $\partial_0; u = \epsilon \otimes 1$ 

- (ii) Product rule:  $\partial_{r+s+1}; \Delta_{r+1,s+1} =$  $(\Delta_{r+1,s}\otimes 1); (1\otimes \partial_r) + (\Delta_{r,s+1}\otimes 1); (1\otimes$  $\gamma$ );  $(\partial_r \otimes 1)$
- (iii) Chain rule:  $\partial_{rs+r+s}; m_{r+1,s+1}$  $(\Delta_{rs+r,s} \otimes 1); (m_{r,s+1} \otimes \partial_s); \partial_r$
- (iv) Symmetry rule:  $(1 \otimes \sigma); (\partial_r \otimes 1); \partial_{r+1} =$  $(\partial_r \otimes 1); \partial_{r+1}$



# The theorem

Given a differential modality on  $\mathcal{L}$ , suppose that we have such a coequalizer diagram for every object A and every  $n \in \mathbb{N}$ :

Suppose also that we keep a coequalizer diagram by applying  $- \otimes A$  to this diagram.

unique Then, exists there

#### $\partial^2$ ; u = 0

(vii) Higher-order product rule:

$$\partial^{n}; \Delta = \sum_{0 \leq k \leq n} {n \choose k} \Delta \otimes 1;$$
  
 $1 \otimes \gamma_{!A,A \otimes k} \otimes 1;$   
 $\partial^{k} \otimes \partial^{n-k}$ 

(viii) Faà di Bruno rule:

 $\partial^n$ 

$$T; m = \sum_{\lambda \vdash n} \frac{n!}{\prod_{j=1}^{n} m_j(\lambda)! (j!)^{m_j(\lambda)}} \\ \left[ \Delta^{1+m_1(\lambda)+\ldots+m_n(\lambda)} \otimes 1_{A \otimes n} \right]; \tau_{\lambda}; \\ \left[ m \otimes (\partial^1)^{\otimes m_1(\lambda)} \otimes \ldots \otimes (\partial^n)^{\otimes m_n(\lambda)} \right] \\ \partial^{m_1(\lambda)+\ldots+m_n(\lambda)}_{!A}$$

Higher-order symmetry rule: (ix)(for every  $\tau \in \mathfrak{S}_n$ )

 $1_{!A} \otimes \tau; \partial^n = \partial^n$ 

**DEF3** A differential modality is a 0-graded differential modality.

**DEF4** Given a differential modality, we define  $\partial^n : !A \otimes A^{\otimes n} \to !A$  by  $\partial^0 = 1_{!A}$ and  $\partial^{n+1} = \partial^n \otimes 1; \partial$ .

 $\mathbb{N}$ -graded differential modality  $(!_{\leq n}, \epsilon^{\leq}, u^{\leq}, \Delta_{n,p}^{\leq}, m_{n,p}^{\leq}, \partial_n^{\leq})$  $\mathcal{L}$ on such that:

1) the endofunctor  $!_{\leq n} : \mathcal{L} \to \mathcal{L}$  gives  $!_{< n}A$  when applied to an object A,

2)  $(s_n)_{n \in \mathbb{N}}$  is a  $(0 : \mathbb{N} \to 0)$ -morphism of graded differential modalities.

#### References

[1] Blute R.F., Cockett J.R.B. and Seely R.A.G.: Differential categories, Math. Struct. in Comp. Science (2006)

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