Extracting an $\mathbb{N}$-graded differential modality from a differential modality

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Introduction

Differential modalities were introduced in [1] and graded differential modalities in [2]. We present a way of forming an $\mathbb{N}$-graded differential modality from a differential modality, under mild conditions. The letters R, S will designate commutative semirings.

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The definitions

DEF1 An additive symmetric monoidal category is a symmetric monoidal category such that every hom-set is a commutative monoid and for every morphism f we have that $f \otimes -$, $- \otimes f$, $-; f$ and $f; -$ preserve sums of maps and zero maps.

DEF2 An R-graded differential modality on an additive symmetric monoidal category $\mathcal{L}$ is a tuple $(!, m, u, \Delta, \epsilon, \partial)$ where $! = (!_r : \mathcal{L} \to \mathcal{L})_{r \in R}$ is a family of endofunctors; $m = (m_{r,s} : !_{rs}A \to !_r!_sA)_{r,s \in R}$ is a family of natural transformations and $u : !_1A \to A$ is a natural transformation such that $(!, m, u)$ is an R-graded comonad; $\Delta = (\Delta_{r,s} : !_{r+s}A \to !_rA \otimes !_sA)_{r,s \in R}$ is a family of natural transformations and $\epsilon : !_0A \to I$ is a natural transformation such that $(!A, \Delta, \epsilon)$ is an R-graded cocommutative comonoid; m is an R-graded comonoid morphism and the following equalities are verified (for every $r, s \in R$):

(i) Linear rule: $\partial_0; u = \epsilon \otimes 1$

(ii) Product rule: $\partial_{r+s+1}; \Delta_{r+1,s+1} = (\Delta_{r+1,s} \otimes 1); (1 \otimes \partial_r) + (\Delta_{r,s+1} \otimes 1); (1 \otimes \gamma); (\partial_r \otimes 1)$

(iii) Chain rule: $\partial_{rs+r+s}; m_{r+1,s+1} = (\Delta_{rs+r,s} \otimes 1); (m_{r,s+1} \otimes \partial_s); \partial_r$

(iv) Symmetry rule: $(1 \otimes \sigma); (\partial_r \otimes 1); \partial_{r+1} = (\partial_r \otimes 1); \partial_{r+1}$

DEF3 A differential modality is a 0-graded differential modality.

DEF4 Given a differential modality, we define $\partial^n : !A \otimes A^{\otimes n} \to !A$ by $\partial^0 = 1_{!A}$ and $\partial^{n+1} = \partial^n \otimes 1; \partial$.

Yet another definition

DEF5 Let $M = (!, m, u, \Delta, \epsilon, \partial)$ an R-graded differential modality, $M' = (!', m', u', \Delta', \epsilon', \partial')$ an S-graded differential modality and $\rho : S \to R$ a semiring homomorphism. A ρ-morphism of graded differential modalities from M to M' is a family of natural transformations $(\phi_s : !_{\rho(s)}A \to !'_sA)_{s \in S}$ such that the following diagrams commute (for every $s, t \in S$):

$$\begin{array}{ccc} !_{\rho(s)\rho(t)}A & \xrightarrow{m_{\rho(s),\rho(t)}} & !_{\rho(s)}!_{\rho(t)}A \\ \downarrow{\scriptstyle \phi_{st}} & & \downarrow{\scriptstyle \phi_s \bullet \phi_t} \\ !'_{st} & \xrightarrow[m'_{s,t}]{} & !'_s!'_tA \end{array}$$

$$\begin{array}{ccc} !_1A & \xrightarrow{u} & A \\ \downarrow{\scriptstyle \phi_1} & \nearrow{\scriptstyle u'} & \\ !'_1A & & \end{array}$$

$$\begin{array}{ccc} !_{\rho(s)+\rho(t)}A & \xrightarrow{\Delta_{\rho(s),\rho(t)}} & !_{\rho(s)}A \otimes !_{\rho(t)}A \\ \downarrow{\scriptstyle \phi_{s+t}} & & \downarrow{\scriptstyle \phi_s \otimes \phi_t} \\ !'_{s+t}A & \xrightarrow[\Delta'_{s,t}]{} & !'_sA \otimes !'_tA \end{array}$$

$$\begin{array}{ccc} !_0A & \xrightarrow{\phi_0} & !'_0A \\ & \searrow{\scriptstyle \epsilon} & \downarrow{\scriptstyle \epsilon'} \\ & & I \end{array}$$

$$\begin{array}{ccc} !_{\rho(s)}A \otimes A & \xrightarrow{\phi_s \otimes 1} & !'_sA \otimes A \\ \downarrow{\scriptstyle \partial_s} & & \downarrow{\scriptstyle \partial'_s} \\ !_{\rho(s)+1}A & \xrightarrow[\phi_{r+1}]{} & !'_{s+1}A \end{array}$$

The theorem

Given a differential modality on $\mathcal{L}$, suppose that we have such a coequalizer diagram for every object A and every $n \in \mathbb{N}$:

$$!A \otimes A^{\otimes n} \underset{0}{\overset{\partial^{n+1}}{\rightrightarrows}} !A \xrightarrow{s_n} !_{\leq n}A$$

Suppose also that we keep a coequalizer diagram by applying $- \otimes A$ to this diagram.

Then, there exists a unique $\mathbb{N}$-graded differential modality $(!_{\leq n}, \epsilon^{\leq}, u^{\leq}, \Delta^{\leq}_{n,p}, m^{\leq}_{n,p}, \partial^{\leq}_n)$ on $\mathcal{L}$ such that:

1) the endofunctor $!_{\leq n} : \mathcal{L} \to \mathcal{L}$ gives $!_{\leq n}A$ when applied to an object A,
2) $(s_n)_{n \in \mathbb{N}}$ is a $(0 : \mathbb{N} \to 0)$-morphism of graded differential modalities.

An example

We can take $\mathcal{L} = \mathbf{Vec}_k^{op}$ and $!A = SA$ the free symmetric algebra on A.

If $\mathrm{char}\, k = 0$, $S_{\leq n}(k) \simeq \{f(x) \in k[x],\ \deg f \leq n\}$.

But the story is different in positive characteristic. For instance, if $k = \mathbb{Z}_2$, then $S_{\leq 0}(k) \simeq \{f(x^2),\ f \in k[x]\}$ and $S_{\leq n}(k) = S(k) \simeq k[x]$ if $n \geq 1$.

Useful identities

The following identities are essential to the proof of the theorem. They are satisfied by every differential modality.

(The notation $\lambda \vdash n$ means that $\lambda = (m_1(\lambda), ..., m_n(\lambda)) \in \mathbb{N}^n$ is such that $1.m_1(\lambda) + ... + n.m_n(\lambda) = n$, we define Δ^n by $\Delta^1 = id_{!A}$ and $\Delta^{n+1} := \Delta; \Delta^n \otimes 1$, and τ_λ is an interleaving permutation associated to λ.)

(v) Constant rule:

$$\partial; \epsilon = 0$$

(vi) Linear bis rule:

$$\partial^2; u = 0$$

(vii) Higher-order product rule:

$$\partial^n; \Delta = \sum_{0 \leq k \leq n} \binom{n}{k} \Delta \otimes 1; \quad 1 \otimes \gamma_{!A, A^{\otimes k}} \otimes 1; \quad \partial^k \otimes \partial^{n-k}$$

(viii) Faà di Bruno rule:

$$\partial^n; m = \sum_{\lambda \vdash n} \frac{n!}{\prod_{j=1}^{n} m_j(\lambda)!(j!)^{m_j(\lambda)}} \left[\Delta^{1+m_1(\lambda)+...+m_n(\lambda)} \otimes 1_{A^{\otimes n}}\right]; \tau_\lambda; \left[m \otimes (\partial^1)^{\otimes m_1(\lambda)} \otimes ... \otimes (\partial^n)^{\otimes m_n(\lambda)}\right]; \partial_{!A}^{m_1(\lambda)+...+m_n(\lambda)}$$

(ix) Higher-order symmetry rule: (for every $\tau \in \mathfrak{S}_n$)

$$1_{!A} \otimes \tau; \partial^n = \partial^n$$

References

[1] Blute R.F., Cockett J.R.B. and Seely R.A.G.: *Differential categories*, Math. Struct. in Comp. Science (2006)

[2] Lemay J.-S. P. and Vienney J.-B.: *Graded differential categories and graded differential linear logic*, Proceedings of MFPS XXXIX (2023)