0.1 A monad on a slice category arising from a monad in the underlying category

Let $\mathcal{C}$ be a category with Cartesian products, and (T, μ, η) a monad on $\mathcal{C}$. Recall that T is said to be strong with respect to the Cartesian product if there is given a natural transformation $\alpha_{X,Y} : T(X) \times Y \to T(X \times Y)$ satisfying some laws:

$$T(\pi_A) \circ \alpha_{A,1} = \pi_{TA} : TA \times 1 \to TA \tag{1}$$

$$\begin{CD} TA \times (B \times C) @>{\alpha_{A,B\times C}}>> @. T(A \times (B \times C)) \\ @V{\cong}VV @. @VV{\cong}V \\ (TA \times B) \times C @>{\alpha_{A,B} \times C}>> T(A \times B) \times C @>{\alpha_{A\times B,C}}>> T((A \times B) \times C) \end{CD} \tag{2}$$

$$\begin{array}{ccc} A \times B & & \\ \downarrow{\scriptstyle \eta_A \times B} & \searrow^{\eta_{A\times B}} & \\ TA \times B & \xrightarrow{\alpha_{A,B}} & T(A \times B) \end{array} \tag{3}$$

$$\begin{CD} TTA \times B @>{\alpha_{TA,B}}>> T(TA \times B) @>{T(\alpha_{A,B})}>> TT(A \times B) \\ @V{\mu \times B}VV @. @VV{\mu}V \\ TA \times B @>{\alpha_{A,B}}>> @. T(A \times B) \end{CD} \tag{4}$$

These equations are taken from nlab.

Strong monads were introduced by Kock. They have been used by Moggi to develop semantics of sequenced computations.

Fix a category $\mathcal{C}$ with finite limits, and let (T, μ, η) be a strong monad on $\mathcal{C}$.

Fix Y an object in $\mathcal{C}$. In this section we will construct a monad (R, μ^R, η^R) on $\mathcal{C}/Y$.

Lemma 1. *The following diagram commutes:*

$$\begin{array}{ccc} T(X) \times Y & \xrightarrow{\alpha_{X,Y}} & T(X \times Y) \\ & \searrow_{\pi_{TX}} & \downarrow{\scriptstyle T(\pi_X)} \\ & & TX \end{array} \tag{5}$$

Proof. Let $!_Y$ denote the unique morphism $Y \to 1$. Apply naturality of α to $(\mathrm{id}_X, !_Y)$ and use 1 with $A := X$. □

For $f : X \to Y$, we write $E(f)$ for $\operatorname{dom} R(f)$.

In what follows, for $f : X \to Y$, $\operatorname{gr} f$ denotes the *graph* of f, the map $(\mathrm{id}_X, f) : X \to X \times Y$.

We first define R on objects. We define $E(f)$ by the following pullback square:

$$\begin{array}{ccc} E(f) & \longrightarrow & T(X) \\ \downarrow & \lrcorner & \downarrow T \operatorname{gr} f \\ TX \times Y & \xrightarrow{\alpha_{X,Y}} & T(X \times Y) \end{array} \tag{6}$$

and $R(f)$ is defined to be the second component of the left map of 6, the composition

$$R(f) := E(f) \to TX \times Y \xrightarrow{\pi_Y} Y \tag{7}$$

If $f : X \to Y, f' : X \to Y$ are two objects in $\mathcal{C}/Y$, and $g : X \to X'$ a morphism in $\mathcal{C}/Y$, it is clear how to use the universal property of the pullback $E(f')$ to construct a map $R(g) : E(f) \to E(f')$, and immediate that it commutes with the maps $R(f)$ and $R(f')$. We omit routine verification of the identity and composition laws for R.

Lemma 2. *Let $t(f) : E(f) \to T(X)$ be the top map in 6 and $s(f) : E(f) \to T(X) \times Y$ the left map in 6. Then $t(f) = \pi_{TX} \circ s(f)$, or equivalently $s(f) = (t(f), R(f))$.*

Proof. By 1, $\pi_{TX} = T(\pi_X) \circ \alpha_{X,Y}$, so

$$\begin{aligned} \pi_{TX} \circ s(f) &= T(\pi_X) \circ \alpha_{X,Y} \circ s(f) \\ &= T(\pi_X) \circ T(\operatorname{gr}(f)) \circ t(f) \\ &= T(\pi_X \circ \operatorname{gr}(f)) \circ t(f) \\ &= t(f) \end{aligned}$$

□

We will use the notation $t(f)$ for the canonical map $E(f) \to T(X)$ again in what follows.

We describe the unit $\eta^R_f : f \to R(f)$. To construct a map $X \to E(f)$, by the universal property of the pullback it is necessary to give maps $a_0 : X \to TX, a_1 : X \to Y, a_2 : X \to TX$, subject to the requirement that $\alpha_{X,Y} \circ (a_0, a_1) = T(\operatorname{gr} f) \circ a_2$. We will take $a_0 = a_2 = \eta_X$ and $a_1 = f$.

Let us verify that the necessary coherence condition is satisfied:

$$\begin{aligned} &T(\operatorname{gr} f) \circ \eta_X \\ &\qquad = \eta_{X \times Y} \circ \operatorname{gr} f \\ &\qquad = \alpha_{X,Y} \circ (\eta_X \times \mathrm{id}_Y) \circ \operatorname{gr} f \\ &\qquad = \alpha_{X,Y} \circ (\eta_X, f) \end{aligned}$$

as desired.

It is clear by the choice of $a_1 := f$ that η^R_f is a morphism in the slice category $\mathcal{C}/Y$.

Let us turn to the multiplication μ^R.

In the following, $T(f)^*\alpha_{1,Y} : E(f) \to T(X)$ denotes the leg of the pullback cone defining $E(f)$ over $T(f)$ in 6.

To give a map $E(R(f)) \to E(f)$, it suffices to give maps $b_0 : ERf \to TX$ and $b_1 : ERf \to TX \times Y$ with $\alpha_{X\times Y} \circ b_1 = T(\mathrm{gr}\, f) \circ b_0$. We will take b_0 to be

$$b_0 := ERf \xrightarrow{t(Rf)} T(Ef) \xrightarrow{T(t(f))} T^2X \xrightarrow{\mu} TX \tag{8}$$

and we will take $b_1 = (b_0, R^2(f))$.

Let us prove that the necessary coherence condition is satisfied.

$$\begin{aligned}
&T(\mathrm{gr}\, f) \circ b_0 \\
&= T(\mathrm{gr}\, f) \circ \mu_X \circ T(t(f)) \circ t(Rf) \\
&= \mu_{X\times Y} \circ T^2(\mathrm{gr}\, f) \circ T(t(f)) \circ t(Rf) \\
&= \mu_{X\times Y} \circ T(\alpha_{X,Y}) \circ T((t(f), R(f))) \circ t(Rf) \\
&= \mu_{X\times Y} \circ T(\alpha_{X,Y}) \circ T(t(f) \times 1_Y) \circ T(\mathrm{gr}(Rf)) \circ t(Rf) \\
&= \mu_{X\times Y} \circ T(\alpha_{X,Y}) \circ T(t(f) \times 1_Y) \circ \alpha_{Ef,Y} \circ (t(Rf), R^2 f) \\
&= \mu_{X\times Y} \circ T(\alpha_{X,Y}) \circ \alpha_{TX,Y} \circ (T(tf) \times 1_Y) \circ (t(Rf), R^2 f) \\
&= \alpha_{X,Y} \circ (\mu_X \times 1_Y) \circ (T(tf) \times 1_Y) \circ (t(Rf), R^2 f) \\
&= \alpha_{X,Y} \circ (b_0, R^2 f)
\end{aligned}$$