THEORIES OF PRESHEAF TYPE

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Introduction

Let us say that a geometric theory T is of *presheaf type* if its classifying topos $\mathbb{B}[T]$ is (equivalent to) a presheaf topos. (We adhere to the convention that *geometric logic* allows arbitrary disjunctions, while *coherent logic* means geometric and finitary.) Write $\mathrm{Mod}(T)$ for the category of *Set*-models and homomorphisms of T. The next proposition is well known; see, for example, MacLane–Moerdijk [13], pp. 381-386, and the textbook of Adámek–Rosický [1] for additional information:

Proposition 0.1. *For a category $\mathcal{M}$, the following properties are equivalent:*

(i) *$\mathcal{M}$ is a finitely accessible category in the sense of Makkai–Paré* [14]*, i.e. it has filtered colimits and a small dense subcategory $\mathcal{C}$ of finitely presentable objects*
(ii) *$\mathcal{M}$ is equivalent to* $\mathrm{Pts}(Set^{\mathcal{C}})$*, the category of points of some presheaf topos*
(iii) *$\mathcal{M}$ is equivalent to the free filtered cocompletion (also known as Ind-$\mathcal{C}$) of a small category $\mathcal{C}$.*
(iv) *$\mathcal{M}$ is equivalent to* $\mathrm{Mod}(T)$ *for some geometric theory of presheaf type.*

Moreover, if these are satisfied for a given $\mathcal{M}$, then the $\mathcal{C}$ — in any of (i), (ii) and (iii) — can be taken to be the full subcategory of $\mathcal{M}$ consisting of finitely presentable objects. (There may be inequivalent choices of $\mathcal{C}$, as it is in general only determined up to idempotent completion; this will not concern us.)

This seems to completely solve the problem of identifying when T is of presheaf type: check whether $\mathrm{Mod}(T)$ is finitely accessible and if so, recover the presheaf topos as *Set*-functors on the full subcategory of finitely presentable models. There is a subtlety here, however, as pointed out (probably for the first time) by Johnstone [10]. It is exemplified by the word *some* in (iv) above. Namely, the presheaf topos one recovers this way (which indeed has $\mathcal{M}$ as its category of *Set*-models) need not coincide with the sought-for topos $\mathbb{B}[T]$. Take, for example, any axiomatization T_1 of the theory of fields by coherent sentences. (We take this merely to mean that $\mathrm{Mod}(T_1)$ is equivalent to the category of fields and homomorphisms.) That category is finitely accessible, so there are geometric theories T_2 of presheaf type such that $\mathrm{Mod}(T_2)$ is the category of fields. But T_1 is not one of them; there exists no coherent presheaf type axiomatization of fields. (See Cor. 2.2 below.) Such a T_1 and T_2 — while

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their categories of *Set*-models are equivalent[1] — can be thought of as distinct formulations of the notion of 'variable field'; and, of course, it was their desire to do commutative algebra in a topos that led Mulvey, Johnstone, Kock and others to investigate these notions in the 1970's.

In a different line of research, in the 1980's, finitely accessible categories were isolated by Lair and (independently) Makkai and Paré as the ones possessing the simplest structure, after locally presentable categories, among categories of models and homomorphisms of first-order theories. (From this point of view, from example, Hodges's [7] functorial approach to the Ehrenfeucht-Mostowski construction is made possible by the fact that the category of linear orders and strict monotone maps is finitely accessible, hence completely determined by its subcategory of finite linear orders.) The author's interest in theories of presheaf type is primarily through their link to classical model theory, a link that will be explored elsewhere. The present article serves mainly as a depository of many examples, some open questions and three theorems concerning theories of presheaf type. The first theorem, a gem of an argument due to Joyal and Wraith and appearing (slightly disguised) in [12] amounts to a recognition criterion for theories of presheaf type. The second one is a necessary and sufficient condition for a presheaf topos to be coherent; it yields a sufficient (but not necessary) condition for a finitely accessible category to be the category of models of a coherent theory. This has implications for the first-order axiomatizability of categories of ind-objects, which is our third theorem.

We note that a different recognition criterion appears in an unpublished note of Moerdijk [16] giving an example of a theory whose classifying topos is Connes' category of cyclic sets, and also that Johnstone [10] provides a close parallel of our discussion for the related case of disjunctive theories.

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1. Given a theory, decide if it is classified by a presheaf topos

One should begin by noting that it seems impossible to identify theories of presheaf type from any of the known direct constructions of the classifying topos: that of Makkai–Reyes [15], exhibiting $\mathbb{B}[T]$ as the formal completion (under coproducts and coequalizers of equivalence relations) of a coherent category $\mathrm{Def}[T]$ constructed out of T; that of sheaves on $\mathrm{Def}[T]$ for a certain (syntactic) Grothendieck topology (see e.g. MacLane–Moerdijk [13]); and the 'forcing' approach of Tierney [17], giving $\mathbb{B}[T]$ as the category of sheaves on an underlying site of finitely presentable models. So the following argument, hidden as a small part of the beautiful article of Joyal and Wraith [12], is especially welcome.

[1] Note that this coincidence of models over *Set*, in case both theories are finitary, is necessarily 'unnatural' in the sense that it is not induced by an interpretation of one theory in the other. Indeed, if $\mathcal{E} \xrightarrow{f} \mathcal{F}$ is a coherent morphism between coherent toposes that induces an equivalence of categories $\mathrm{Pts}(\mathcal{E}) \to \mathrm{Pts}(\mathcal{F})$ then, by the conceptual completeness theorem of Makkai and Reyes [15], f itself is an equivalence. This unnaturality need not hold if one of T_1 and T_2 is infinitary.

Theorem 1.1. *Let T be a finite limit theory and T^+ a geometric extension of T in the same language. Assume that T^+ has enough models in Set and that*

(F_∞) *Every T^+-model in Set, as an object of* $\mathrm{Mod}(T)$, *is a filtered colimit of T^+-models finitely presentable as objects of* $\mathrm{Mod}(T)$.

Then $\mathbb{B}[T^+]$ is the category of Set-functors on $\mathcal{C}_{T^+}$, the full subcategory of $\mathrm{Mod}(T)$ *consisting of finitely presentable objects that are models of T^+.*

Proof. Recall that $\mathrm{Mod}(T)$ is a locally finitely presentable category. Write $\mathcal{C}_T$ for its full subcategory of finitely presentable objects. Then $Set^{\mathcal{C}_T}$ is the classifying topos $\mathbb{B}[T]$, and $\mathbb{B}[T^+]$ is a subtopos of $\mathbb{B}[T]$; let v denote the inclusion. The inclusion functor $\mathcal{C}_{T^+} \hookrightarrow \mathcal{C}_T$ induces an essential morphism $Set^{\mathcal{C}_{T^+}} \xrightarrow{u} Set^{\mathcal{C}_T}$ that is also a topos inclusion. $Set^{\mathcal{C}_{T^+}}$ carries a tautologous T^+-model, given (as a diagram of T^+-models) by the identity functor on $\mathcal{C}_{T^+}$. Let $Set^{\mathcal{C}_{T^+}} \xrightarrow{w} \mathbb{B}[T^+]$ classify this. One then has a commutative diagram

$$\begin{array}{ccc} Set^{\mathcal{C}_{T^+}} & \xrightarrow{\quad w \quad} & \mathbb{B}[T^+] \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle v} & \\ & Set^{\mathcal{C}_T} & \end{array}$$

Since u, v are inclusions, so is w. Now take any point $Set \xrightarrow{p} \mathbb{B}[T^+]$, corresponding to the T^+-model P in Set, and express P as a colimit of some $\mathcal{D} \xrightarrow{d} \mathcal{C}_{T^+}$, $\mathcal{D}$ a filtered diagram. d, thought of as a filtered colimit of representables, gives a point $Set \xrightarrow{d} Set^{\mathcal{C}_{T^+}}$ and p^* is (naturally isomorphic to) d^*w^* since they both classify P.

Take now a morphism $f \in \mathbb{B}[T^+]$, and suppose $w^*(f)$ is an isomorphism. A fortiori $p^*(f)$ is an isomorphism for any point p of $\mathbb{B}[T^+]$. Since the latter was supposed to have enough points, f is iso. So w^* reflects isomorphisms. But as $Set^{\mathcal{C}_{T^+}}$ is a subtopos of $\mathbb{B}[T^+]$, w must be an equivalence. ☐

Remark 1.2. The argument works, unchanged, under the weaker assumption that T is of presheaf type; what it really does is to characterize certain extensions of presheaf theories as still being of presheaf type. One uses the case of finite limit theories and the Gabriel–Ulmer theorem (that the classifying toposes of finite limit theories are precisely the presheaf toposes $\mathrm{Pre}(\mathcal{C})$ where $\mathcal{C}$ has finite limits) as starting point.

Remark 1.3. Since the classifying topos of a theory is determined up to equivalence, whether or not a theory T is of presheaf type is well-defined, but the $\mathcal{C}$ s.t. $\mathbb{B}[T] = Set^{\mathcal{C}}$ is only defined up to idempotent completion. (Recall that $Set^{\mathcal{C}}$ is equivalent to $Set^{\mathcal{D}}$ if and only if $\mathcal{C}$ and $\mathcal{D}$ have equivalent idempotent completions.) The indexing category $\mathcal{C}_{T^+}$ produced by Thm. 1.1 will always be complete under retracts.

Remark 1.4. Let T, T^+ be as in Thm. 1.1. Then there is some cardinal κ such that T^+ satisfies (F_∞) iff it satisfies

(F_κ) Every T^+-model of size less than κ is a filtered colimit of T^+-models finitely presentable as objects of $\mathrm{Mod}(T)$.

This is due to the downward Löwenheim-Skolem theorem, i.e. that for some κ, every T^+-model is the directed union of its elementary submodels of size less than κ. The least such cardinal κ depends on the size of the language, the size of the disjunctions and of the quantifications occurring in the theory (in the case of infinitary logics). In the case that concerns us the most, a geometric theory in a countable language (possibly with countable disjunctions) $\kappa = \aleph_1$ will work.

One cannot change the *filtered colimit* of condition (F_κ) to *directed union* without losing generality; that is to say, for Thm. 1.1 to apply, it need not be the case that every T^+-model is a directed union of elementary T^+-submodels that are finitely presentable as models of T. See Remark 3.7.

Remark 1.5. As discussed above, whether or not T^+ is of presheaf type is not determined by the abstract category $\mathrm{Mod}(T^+)$ alone; so any categorical recognition method must employ some auxiliary device. In the Joyal–Wraith theorem, this is the full inclusion $\mathrm{Mod}(T^+) \hookrightarrow \mathrm{Mod}(T)$.

Question 1.6. Does Thm. 1.1 present a necessary condition in the following sense: if T^+ is a theory of presheaf type, then there exists a finite limit theory T in the same language to which the assumptions apply?

A natural guess for such a T is what Coste [4] calls the *lim-part* of T^+, i.e. the set of finite limit sentences implied by T^+.

To be sure, Thm. 1.1 gives a necessary condition in the following sense: for any finitely accessible category $\mathcal{M}$, there exists a pair (T, T^+) to which the theorem applies such that $\mathcal{M} = \mathrm{Mod}(T^+)$. Such a T can be chosen to be the theory of presheaves, and T^+ that of flat presheaves on an appropriate subcategory of $\mathcal{M}$; see below.

Remark 1.7. Wraith [18] surmises that a theory is of presheaf type iff any of its models in any topos $\mathcal{E}$ is an $\mathcal{E}$-filtered $\mathcal{E}$-colimit of constant models that are finitely presentable (in *Set*). As stated, this is hardly (meant to be) a practical recognition theorem, since it involves quantification over *all* Grothendieck toposes. Note that there is no obvious intersection with Thm. 1.1, which assumes (in addition to enough *Set*-models, which is certainly necessary) only that every *Set*-model is a filtered colimit of 'finitely presentable' models — but finitely presentable in a weaker theory.

2. Given a presheaf topos, find a theory it classifies

Any site $(\mathcal{C}, J)$ of definition for a topos $\mathcal{E}$ allows one to specify a theory whose classifying topos $\mathcal{E}$ is; in particular, for any presheaf topos $\mathrm{Pre}(\mathcal{C})$, one can write down a canonical geometric theory T, that of flat functors on $\mathcal{C}$, such that $\mathbb{B}[T] = \mathrm{Pre}(\mathcal{C})$. This theory employs infinite disjunctions in general. However, one often wishes to know whether there is a finitary theory classified by $\mathrm{Pre}(\mathcal{C})$. Thm. 2.1 below gives a necessary and sufficient condition for that in terms of the combinatorics of $\mathcal{C}$.

Flat functors and coherence. Let $\mathcal{C}$ be a category. The theory of flat functors on $\mathcal{C}$ has a sort for every object of $\mathcal{C}$, and a function symbol for every arrow; the axioms express that the category of elements of the functor is filtered. (See e.g. p.386 of MacLane–Moerdijk [13].) For readability, we use the symbol "$x \in X$" to express that x is a variable of sort X, and omit parentheses around some function arguments.

$$\forall x \in X(\mathbf{id}_X(x) = x) \tag{2.1}$$

$$\forall x \in X\big((\mathbf{gf})x = \mathbf{g}(\mathbf{f}x)\big) \tag{2.2}$$

$$\bigvee_{X \in \mathrm{ob}\,\mathcal{C}} \exists x \in X \tag{2.3}$$

$$\forall x \in X\, \forall y \in Y \left(\bigvee_{X \xleftarrow{\mathbf{f}} Z \xrightarrow{\mathbf{g}} Y} \exists z \in Z \left(\mathbf{f}z = x \wedge \mathbf{g}z = y\right)\right) \tag{2.4}$$

$$\forall x \in X\left(\mathbf{f}x = \mathbf{g}x \implies \bigvee_{Z \xrightarrow{\mathbf{h}} X \underset{\mathbf{g}}{\overset{\mathbf{f}}{\rightrightarrows}} Y} \exists z \in Z \left(\mathbf{h}z = x\right)\right) \tag{2.5}$$

Some elaboration: in (2.2), there is one sentence for each composable pair $X \xrightarrow{\mathbf{f}} Y \xrightarrow{\mathbf{g}} Z$ of arrows. Thus the first two sets of sentences state that one has a functor $\mathcal{C} \to \mathit{Set}$. In the axiom scheme (2.4), there is one sentence for each pair X, Y of objects; the disjunction is over all diagrams $X \leftarrow \bullet \to Y$ that exist in $\mathcal{C}$. The last axiom scheme contains one sentence for each pair of parallel arrows $X \underset{\mathbf{g}}{\overset{\mathbf{f}}{\rightrightarrows}} Y$ in $\mathcal{C}$, and the disjunction is over all $\mathbf{h}$ that equalize them.

Thus the theory employs infinitary disjunctions for infinite $\mathcal{C}$; but if $\mathrm{Pre}(\mathcal{C})$ happens to be a coherent topos, there must exist a coherent theory classified by $\mathrm{Pre}(\mathcal{C})$. There is a practical recognition criterion for when a presheaf topos is coherent; minus the logical aspect, it is stated as Exercise 2.17 of SGA4, Tome 2, Expose VI. First, some terminology. A category $\mathcal{C}$ is said to have *fc* (for '*finite cone*') terminals if it possesses a finite set $\mathbb{T}$ of objects s.t. any $X \in \mathrm{ob}\,\mathcal{C}$ permits some map $X \to T$ to some $T \in \mathbb{T}$. (Neither the map nor T is assumed to be unique.) The empty category (tautologously) has *fc* terminals. Let $\mathcal{D} \xrightarrow{F} \mathcal{C}$ be a functor; F is said to have an *fc limit* if the category of cones in $\mathcal{C}$ over F possesses *fc* terminals. The concept of *fc colimit* is defined dually.

Terminological aside. The nomenclature for various weakenings of the notion of (co)limit is not completely standard. A *weakly terminal* object is one that receives a map (not necessarily unique) from any object. *Multi-terminals* for a category mean a set of objects such that any object permits a map to a unique one of them, and that map is unique. The term *cone-terminal*, when both senses of uniqueness are dropped (but a restriction remains on the cardinality of the terminating set of objects) was suggested by Jiří Rosický.

Theorem 2.1. *Let $\mathcal{C}$ be a small category. The following are equivalent:*

(i) $\mathrm{Pre}(\mathcal{C})$ *is a coherent topos.*
(ii) $\mathcal{C}$ *has all* fc *finite limits. (That is, any functor* $\mathcal{D} \xrightarrow{F} \mathcal{C}$ *with* $\mathcal{D}$ *a finite diagram has an* fc *limit in* $\mathcal{C}$*.)*
(iii) $\mathcal{C}$ *has* fc *terminal objects,* fc *products, and* fc *equalizers.*

Proof. (i) $\Longrightarrow$ (ii): let G be the set of (representatives of isomorphism classes of) coherent objects of $\mathrm{Pre}(\mathcal{C})$. By the assumption that $\mathrm{Pre}(\mathcal{C})$ is coherent, G is a set of generators, and is closed under finite limits in $\mathrm{Pre}(\mathcal{C})$. Let X be a representable object of $\mathrm{Pre}(\mathcal{C})$. The set of all maps from coherent objects to X is collectively epi, by the assumption that G generates. X is finitely presentable, a fortiori quasi-compact,[2] so a finite subset of these maps already covers it. A finite coproduct of coherent objects is coherent, so we have found a coherent Y and an epi $Y \twoheadrightarrow X$. But X is projective, so a retract of Y, so itself coherent. The upshot we need is that a finite limit of representables in $\mathrm{Pre}(\mathcal{C})$ is quasi-compact (coherent, in fact).

Let now $\mathcal{D} \xrightarrow{F} \mathcal{C}$ be a functor with $\mathcal{D}$ finite, and L the limit of the composite $\mathcal{D} \xrightarrow{F} \mathcal{C} \xrightarrow{y} \mathrm{Pre}(\mathcal{C})$ with the Yoneda embedding. For a cone $\mathcal{K}$ on F in $\mathcal{C}$, let $\mathsf{V}_{\mathcal{K}}$ denote its vertex. Since L is the presheaf that sends $X \in \mathrm{ob}\,\mathcal{C}$ to the set of cones on F with vertex X, precomposition of $\mathcal{K}$ with arrows $X \to V$ gives a natural map $y(\mathsf{V}_{\mathcal{K}}) \to L$. As $\mathcal{K}$ ranges over all cones on F, this set of maps is collectively epi. Since L is quasi-compact, a finite subset $y(\mathsf{V}_{\mathcal{K}_i}) \to L$ already covers L. But this means precisely that the $\mathcal{K}_i$ provide *fc* limits for F.
(ii) $\Longrightarrow$ (iii): take $\mathcal{D}$ to be the empty diagram resp. $\bullet\bullet$ resp. $\bullet \rightrightarrows \bullet$.
(iii) $\Longrightarrow$ (i): under these conditions, the axioms (2.3)-(2.4)-(2.5) can be replaced by (intuitionistically) equivalent finitary ones, by replacing the set of cones indexing the disjunctions by their respective *fc* terminal subsets. Let us call the theory thus obtained that of "coherent flat functors". (It is not defined canonically, unless some systematic choice of the terminating cones can be made.) □

Corollary 2.2. *There exists no coherent axiomatization of fields that is of presheaf type.*

If there were, its classifying topos would have to be $\mathit{Set}^{\mathbb{F}}$, where $\mathbb{F}$ is spanned by fields that are finitely presentable as objects of the category of fields and homomorphisms. But $\mathit{Set}^{\mathbb{F}}$ is not a coherent topos, since $\mathbb{F}$ has infinitely many components — to wit, each of the prime fields $\mathbb{F}_p$ as well as the rationals $\mathbb{Q}$ are finitely presentable as objects of fields(!), and no two of them are connected by a zig-zag of homomorphisms — whereas for a coherent $\mathit{Set}^{\mathbb{F}}$, $\mathbb{F}$ has to have *fc* initial objects, a fortiori a finite number of connected components.
Remark 2.3. Note that Thm. 1.1 — taking T to be the theory of rings and T^+ that of fields — does not apply since e.g. $\mathbb{Q}$ is not a filtered colimit of fields that are finitely presentable *even as rings.*

There do exist (infinitary) geometric axiomatizations of presheaf type for fields, in enriched languages of rings. See Johnstone [10] who, in a slightly different set-up, also observes Cor. 2.2.

[2] We use the terminology of SGA4; Johnstone [9] Def. 7.31 employs *compact* for the same concept.

Remark 2.4. The argument given for the implication (iii) $\Longrightarrow$ (i) in Thm. 2.1 is brief to state, but conceptually perhaps indirect. It amounts to the fact that the finite limit completion of $\mathcal{C}$ in $\mathrm{Pre}(\mathcal{C})$ under the Yoneda embedding consists of quasi-compact objects, hence (via Giraud's theorem) provides a coherent site of definition for $\mathrm{Pre}(\mathcal{C})$.

Remark 2.5. Note that even under the assumption that $\mathrm{Pre}(\mathcal{C})$ is coherent, the language employed in (iii) has the size of $\mathcal{C}$. Finding a finite theory classified by a given $\mathrm{Pre}(\mathcal{C})$, with $\mathcal{C}$ countable and $\mathrm{Pre}(\mathcal{C})$ coherent, is more of an art than a science. One typically tries to decode a candidate from the explicit combinatorial definition of $\mathcal{C}$.

As an immediate consequence of Prop. 0.1 and Thm. 2.1:

Corollary 2.6. *For a category $\mathcal{M}$, the following properties are equivalent:*

(i) *$\mathcal{M}$ is a finitely accessible category some (equivalently, all) of whose dense subcategories consisting of finitely presentable objects possess* fc *finite colimits*
(ii) *$\mathcal{M}$ is equivalent to $\mathrm{Pts}(Set^{\mathcal{C}})$, i.e. the free filtered cocompletion of a small category $\mathcal{C}$ that has all* fc *finite colimits*
(iii) *$\mathcal{M}$ is equivalent to $\mathrm{Mod}(T)$ for some coherent theory of presheaf type.*

Moreover, if these are satisfied for a given $\mathcal{M}$, then the $\mathcal{C}$ in (ii) can be taken to be the full subcategory of $\mathcal{M}$ consisting of finitely presentable objects, and the theory T of (iii) can be taken to be that of coherent flat functors.

Remark 2.7. One, then, has intrinsic characterizations of those categories that are equivalent to $\mathrm{Mod}(T)$ for some geometric T of presheaf type, resp. some coherent T of presheaf type. No such characterization is known if T is allowed to run through the entire class of geometric resp. coherent theories. One should mention that for a coherent theory T (even with just one axiom) $\mathrm{Mod}(T)$ may fail to be finitely accessible (see Remark 2.59 of Adámek–Rosický [1]), though it will always be $\aleph_1$-accessible. More generally, for a geometric T, little is known of $\mathrm{Mod}(T)$ besides that it is accessible and has filtered colimits, and that these two properties do not characterize the categories thus arising.

Axiomatizing ind-objects. The same argument, from a slightly different point of view, can be used to produce (not necessarily finitary, but first-order) axioms for certain categories of ind-objects. This describes, at the same time, the 'generic situation' underlying Thm. 1.1. Recall that ind-objects are formal filtered diagrams of objects; such devices are often useful in commutative algebra or algebraic geometry as replacements for infinitary constructions that would lead one outside of one's category of interest. In the theorem below, the reader may as well keep simple examples such as flat modules (see 3.5) in mind.

Theorem 2.8. *Let $\mathcal{K}$ be a locally finitely presentable category, and $\mathcal{G}$ a full subcategory of $\mathcal{K}$ such that objects of $\mathcal{G}$ are finitely presentable in $\mathcal{K}$. Let $\mathcal{A}$ be the closure of $\mathcal{G}$ in $\mathcal{K}$ under filtered colimits, i.e. the full subcategory of $\mathcal{K}$ with all objects that can be written as colimits of a filtered diagram in $\mathcal{G}$. Then $\mathcal{A}$ is a finitely accessible category, equivalent to Ind-$\mathcal{G}$.*

Assume moreover $\mathcal{K} = \mathrm{Mod}(T)$, where T is a finite limit theory. Then there exists, in the language of T, a geometric theory T^+ of presheaf type such that $\mathcal{A} = \mathrm{Mod}(T^+)$. If $\mathcal{G}$ has fc *finite colimits, then T^+ can be chosen coherent.*

Proof. $\mathcal{A}$ inherits filtered colimits from $\mathcal{K}$ and $\mathcal{G}$ becomes a small dense subcategory of finitely presentable objects for $\mathcal{A}$; so $\mathcal{A}$ is finitely accessible. Cf. Prop. 0.1.

Let now $\mathcal{K} = \mathrm{Mod}(T)$ as assumed. Let G be any finitely presentable object, and X any object of $\mathcal{K}$. Choose any finite presentation $\langle \bar{c}; \Phi \rangle$ for G. Recall that this will be a finite set $\bar{c}$ of new constants (in the sorts of G) and a finite set Φ of atomic formulas in the language of T (which we always assume has equality for all its sorts) without free variables, but using the constants $\bar{c}$. T-homomorphisms $G \to X$ then biject with those interpretations of the constants $\bar{c}$ in the structure underlying X that satisfy $X \models \phi$ for all $\phi \in \Phi$.

An object $X \in Mod(T)$ belongs to $\mathcal{A}$ if and only if the category of elements of the presheaf represented by X on the category $\mathcal{G}$ is filtered, meaning if and only if the comma category $\mathcal{G} \downarrow X$ is filtered. That means

- there exists $G \in \mathcal{G}$ that allows a map $G \to X$
- for each pair $G_1, G_2 \in \mathcal{G}$ and each pair of maps $G_i \xrightarrow{x_i} X$, there exists $G_3 \in \mathcal{G}$ with maps $G_i \xrightarrow{g_i} G_3$ and map $G_3 \xrightarrow{x_3} X$ such that $x_3 g_i = x_i$ ($i = 1, 2$ throughout)
- for each pair $g_1, g_2 : G_1 \rightrightarrows G_2$ of parallel arrows between objects of $\mathcal{G}$ and each $G_2 \xrightarrow{x} X$ such that $x g_1 = x g_2$, there exists $G_3 \in \mathcal{G}$ with maps $G_2 \xrightarrow{g} G_3$ and $G_3 \xrightarrow{z} X$ such that $g g_1 = g g_2$ and $z g = x$.

By the above remark on presentations, each of these axioms can be phrased as a geometric sentence in the language of T. (For the second axiom, for each pair G_1, G_2, fix finite presentations $\langle \bar{c}_1; \Phi_1 \rangle$ resp. $\langle \bar{c}_2; \Phi_2 \rangle$. If G_3 is such that $G_i \xrightarrow{g_i} G_3$ exist, then without loss of generality we may assume that G_3 has a presentation $\langle \bar{c}_3; \Phi_3 \rangle$ where $\bar{c}_3$ contains both the constant symbols $\bar{c}_1$ and the $\bar{c}_2$, and g_i is induced by the inclusion of generators $\bar{c}_i \hookrightarrow \bar{c}_3$. Quantification over maps from explicitly finitely presented models into X is then first-order. A similar trick works with the third axiom. Cf. formulas (2.3)-(2.5).)

Thus the theory of flat $\mathcal{G}$-functors becomes a geometric theory T^+ in the language of T such that the pair $\mathcal{A} \subseteq \mathcal{K}$ gets identified with $\mathrm{Mod}(T^+) \subseteq \mathrm{Mod}(T)$. Thm. 1.1 applies by construction. The size of the disjunctions needed in T^+ can be bounded by the cardinality of any cone-initial set of diagrams of the requisite shape in $\mathcal{G}$. If $\mathcal{G}$ has *fc* finite colimits, then finite cone-initial sets exist. □

3. A bestiary

Most of our examples are organized into infinite (logical or algebraic) families. Some of the infinite logical families are of a 'relative' nature: if some theory is of presheaf type, so is a modification of it. Of course, one can always start with finite limit theories; the modifications produced will not be such.

Example 3.1. Negated positive sentences
Let T be any theory of presheaf type and Φ a set of sentences, each of which is a positive-existential combination of atomic formulas in the language of T. The validity of such sentences is preserved by homomorphisms of structures. Set T^+ to be T together with the negations of these sentences: $\phi \implies \bot$ for each $\phi \in \Phi$.

Any model X of T can be written as the colimit of a filtered diagram $\mathcal{D}$ of finitely presentable T-models. If X happens to be a model of T^+ then, to be sure, all of the models employed in $\mathcal{D}$ must be models of T^+ as well (since they map to X, they cannot satisfy ϕ!). But that means precisely that the condition of Thm. 1.1 is satisfied. (See also Remark 1.2.)

With this device, one can sometimes 'exclude' certain models (and, necessarily, what they map to) from a theory; for example, by demanding $\mathbf{0} \neq \mathbf{1}$ in the theory of rings (where $\mathbf{0}$ stands for the additive and $\mathbf{1}$ for the multiplicative unit). Or one can deny solutions to equations. Let T be any axiomatization of presheaf type for the theory of fields in a language containing that of rings. Add the axioms

$$\exists x_1, x_2, \dots, x_n \, (x_1^2 + x_2^2 + \cdots + x_n^2 = -1) \implies \bot$$

for each n. This is the theory of formally real fields, which then must be of presheaf type as well.

Example 3.2. Non-empty domains
Let T be any theory of presheaf type, and let T^+ contain in addition, for a finite collection of sorts, that their exists an element in that domain (much as classical model theory demands). Writing, as usual, any model X of T^+ as a filtered colimit of finitely presentable T-models, each of the chosen sorts must have an element in it eventually; since finitely many sorts were chosen, one can still write X as a filtered colimit of T^+-models finitely presentable as T-models.

Example 3.3. Universal relational theories
Consider a language with only relation symbols and constants; we take T to be the theory of this signature (no axioms). Any first order theory T^+ axiomatized in such a language by sentences of the form $\forall \overline{x} \, \phi(\overline{x})$, ϕ quantifier-free, has the property that any substructure of a model of T^+ is still a model. A fortiori, any model of T^+ is a directed union of its submodels that contain finitely many elements besides the constants, hence are finitely presentable as models of T, and (when T^+ is geometric) Thm. 1.1 applies.

Theories of (partially) ordered structures are often of this type:

- Cosimplicial sets

Let T^+ state that $\prec$ is a linear order on a non-empty domain (cf. Example 3.2). The category of totally ordered, non-empty finite sets and order-preserving maps is, by definition, (equivalent to) the cosimplicial indexing category Δ; so $\mathbb{B}[T^+] = Set^{\Delta}$.

- Simplicial sets

Add two constants $\mathbf{b}$, $\mathbf{t}$ to the theory of the previous example, and let T^+ say that $\prec$ is a linear order with distinct bottom element $\mathbf{b}$ and top element $\mathbf{t}$. Write Order for the category

$\mathcal{C}_{T^+}$; its objects are linearly ordered finite sets of cardinality at least 2, with morphisms maps that preserve the order and the maximal and minimal elements. There are contravariant equivalences

$$\mathsf{Order} \xrightarrow{\hom_{\mathsf{Order}}(-,\{\mathbf{b},\mathbf{t}\})} \Delta \tag{3.1}$$

$$\mathsf{Order} \xleftarrow{\hom_{\Delta}(-,\Delta[1])} \Delta \tag{3.2}$$

where $0 \prec 1$ is (in the guise of $\{\mathbf{b}, \mathbf{t}\}$) the initial object of Order and (in the guise of $\Delta[1]$) an object of Δ. Thus simplicial sets classify linear orders with distinct, named endpoints.

Remark 3.4. The above equivalences are of "Isbell type", with $\{\mathbf{b}, \mathbf{t}\} = \Delta[1]$ as "schizophrenic object". (See MacLane–Moerdijk [13] p.480 for a brief, and Johnstone [11] VI.4 for a thorough discussion of similar dualities.) The case of (3.1) and (3.2) is but the first level of the order-disk duality of Joyal, leading to his definition of "bundle of orders" and the category Θ.

- Preserving and reflecting formulas

Let T be any theory in (infinitary) first-order logic, and ϕ any formula in its language. Add a new predicate Φ (of the same arity as ϕ) to the language and the axiom

$$\forall \overline{x}\big(\phi \iff \Phi\big)$$

Call the resulting theory T_Φ. Or add a new predicate $\overline{\Phi}$ and the axioms

$$\forall \overline{x}\big(\phi \vee \overline{\Phi}\big)$$

$$\forall \overline{x}\big(\phi \wedge \overline{\Phi} \implies \bot\big)$$

Call the resulting theory $T_{\overline{\Phi}}$. Models of these extended theories will be exactly the same as T-models, but homomorphisms $X \to Y$ will be those T-homomorphisms that satisfy: if $X \models \phi(\overline{a})$ for some tuple of elements then $Y \models \phi(\overline{a})$ (in the case of T_Φ) resp. if $Y \models \phi(\overline{a})$ then $X \models \phi(\overline{a})$ (in the case of $T_{\overline{\Phi}}$). If T and ϕ are geometric and universal relational, T_Φ and $T_{\overline{\Phi}}$ will be of presheaf type. This applies e.g. to posets and monotone maps, posets and injective monotone maps, posets and embeddings of posets, to the analogues of these notions for linear and cyclic orders. . . .

Example 3.5. Flat modules

Consider a (left, say) R-module M over a ring R. There are several characterizations of when M is flat; the equivalence of the following three is highly non-trivial:

(i) The functor $- \otimes_R M$ preserves exactness.
(ii) Let R be a row vector of length n with entries in R and X a column vector of size n with entries in M such that $\mathsf{RX} = [\mathbf{0}]$. Then there exist some positive integer m, column vector B of size m with entries in M and matrix C of size $n \times m$ with entries in R such that $\mathsf{CB} = \mathsf{X}$ and $\mathsf{RC} = [\mathbf{0}, \mathbf{0}, \dots, \mathbf{0}]$ (this being a row vector of length n).
(iii) M can be written as a filtered colimit of free modules.

It is amusing to observe that (i), the textbook definition, is the key to the usefulness of flatness in homological algebra; property (ii) — which is, morally, that "any linear dependence among the variables X is due to linear dependence among their coefficients C, when expressed with respect to some basis B" — shows that this notion is definable in the language of modules, via geometric sentences with countable disjunctions; and Lazard's theorem, the equivalence of (iii) with (ii), shows that Thm. 1.1 applies, by taking T to be the theory of R-modules. (Any free module is a filtered union of finitely generated free, hence finitely presented modules.)

If one knows that (iii) implies (ii), there is an obvious way to prove it: given R, X as above, all elements of X must exist together at some stage of the colimit, and it suffices to prove the desired statement for free modules, where it is easy. What is surprising a priori is that the highly transfinitary characterization, (iii), should be equivalent to *any* set of axioms in first-order logic at all. One explanation for this is Thm. 2.8 — it applies whenever one closes, within all modules, or indeed within a fixed variety of universal algebras, any subcategory consisting of finitely presentable objects (for example those that are free on a finite set).

Specialize Example 3.5 to $R = \mathbb{Z}$. A flat $\mathbb{Z}$-module is precisely the same as a torsion-free abelian group. Now the axiomatization of torsion-free abelian groups as a theory of presheaf type, in the spirit of 3.5(ii), would employ a unary function symbol $\mathbf{n}(-)$ ("multiplication by n") for each natural number n, and countable disjunctions. But the obvious axiomatization of torsion-free by the sentences (one for each iterated sum)

$$x + x + \cdots + x = 0 \implies x = 0$$

in the language of abelian groups is also of presheaf type and *coherent*. The explanation for this is again Thm. 2.8; in fact, flat R-modules possess a first-order axiomatization (in the language of R-modules) if and only if they possess a coherent axiomatization if and only if R is a coherent ring. Details of this theorem will appear elsewhere.

The next family comes from commutative algebra too; the reason is that the property exploited, 'finitely generated implies finitely presented', is quite rare.

Example 3.6. Hereditary module properties over noetherian rings
Let P be any property of modules that is inherited by submodules and preserved by filtered colimits. (Typical examples of such properties are being torsion and being torsion-free. 'Property' here is to be understood in its everyday meaning, independent of any notion of logical syntax.) Fix a noetherian ring R, and take T to be the theory of R-modules. Any module (with property P) is a directed union of its finitely generated submodules (with property P, by assumption of heredity), which are finitely presented as R-modules, by the noetherian assumption. By Thm. 2.8, there exists a geometric theory T^+ of R-modules with property P, and Thm. 1.1 applies.

Remark 3.7. Hereditary noetherian module conditions and universal relational theories share the following property: any finitely generated submodel of a T^+-model is finitely presentable even as a T-model; therefore condition (F_∞) is satisfied through a directed union of submodels. In Thm. 1.1, however, one cannot change 'filtered colimit' to 'directed union' without losing generality, as the example of coherent modules shows. By a theorem of Jensen [8], for any cardinal κ there exists a ring R and a flat R-module M such that M is not the union of its κ-generated flat submodules — a fortiori, there exists a flat module that is not the union of its finitely *presented* flat submodules.

In closing, we should mention what is probably the most surprising example of Thm. 2.8 (coming, as it does, from combinatorial homotopy, whose interaction with model theory proper has been quite limited). It is due to Joyal and Wraith [12], and is the key part of their proof of Wraith's conjecture on the cohomology of Eilenberg–MacLane toposes. (It is fair to say that one of the author's chief motivations for the present article was the desire to understand that proof.) The category $\mathcal{K}$ underlying that example is simplicial sets. (See e.g. Goerss–Jardine [6] for a detailed introduction to simplicial homotopy theory.) Take the distinguished set $\mathcal{G}$ of finitely presentable objects to be those simplicial sets that have finitely many non-degenerate simplices (this means precisely that they are finitely presentable as objects of *SSet*) and whose geometric realization is a contractible topological space.

From results of Gabriel–Zisman [5] it follows that the completion of $\mathcal{G}$ under filtered colimits in *SSet* is precisely the class of simplicial sets whose geometric realization is a contractible topological space. This property can therefore be phrased in first-order terms in the signature of simplicial sets, which is quite surprising. (There is another known proof, which involves analyzing the syntax of the statement "all homotopy groups of $\mathsf{Ex}^\infty(X)$ are trivial", where $SSet \xrightarrow{\mathsf{Ex}^\infty} SSet$ is Kan's simplicial fibrant replacement functor, using Kan's definition of the n^{th} homotopy group of a fibrant simplicial set as a certain subquotient of its collection of n-simplices.)

It would be valuable to perform such analyses of other situations in homotopical algebra, for example of the cofibrant-weakly contractible objects in the homotopy theories of Bousfield and Kan [3] on the functor category $SSet^{\mathcal{D}}$ (or more generally on simplicial objects in a Grothendieck topos) or of acyclic groups (Baumslag–Dyer–Heller [2]).

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