Category Theory is known as a unifying language of mathematics [4]. In recent years, Applied Category Theory has begun to explore it as a language for all kinds of science [2]. I propose that category theory is the *language of thinking*, as follows.

The basic concepts of category theory

type and process, relation and transformation
identity and composition, adjunction and representation

are systematized in the language of a *bifibrant double category*, a concept presently known as "proarrow equipment" or "framed bicategory" [8]. Such a language can be understood simply as a *logic*, i.e. a system of *thoughts of a world*:

A world is a category of types of things, and processes between types.

A thought of the world is a relation of types (a judgement), and
a process of thinking is a transformation of relations (an inference).

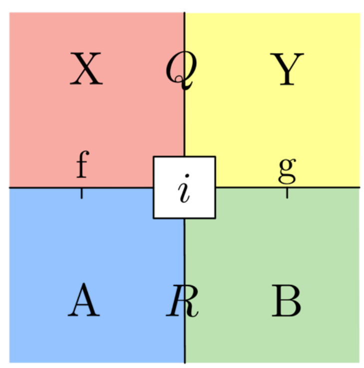
Relations and transformations form a category, and these "thoughts" form a bifibration from the world to the world, with operations of parallel composition and identity.

In this view, category theory is the realization that *thought is connection*, the dimension beyond the world in which type relates to type, and process transforms to process.

Yet a process is a special kind of connection, and so thought encompasses the world: each process forms a dual pair of relations. By composition, thoughts are pushed forward or pulled backward along processes; this is the "bifibrance" of a logic.

The language exists in two dual forms: *syntax* and *imagery*, a.k.a. string diagrams [6]: dual to object, arrow, square is color, string, bead. We distinguish processes from relations by a downward pointer, and their action on relations is drawn as bending.

bif. dbl. cat.	dim.	**logic**
object	0	type
tight morphism	V	process
loose morphism	H	relation
square	2	transformation



The simplest kind of logic is *binary logic*: sets and functions, relations and entailments; i.e. the predicate logic of sets. Type theory has realized that relations have content beyond truth values, and in a few decades we have made a multiverse of logics to explore.

So how do we make logics? This is summarized in the motto:

a category is a matrix with composition and identity.

A category is a type of objects, indexing a matrix of morphisms, with the structure of composition and identity. In [8], Shulman presented the two ways we construct logics:

1. A *bifibered monoidal category* $\mathcal{R} \to \mathbb{A}$ forms a logic, in which a relation $R : \mathrm{A} \,|\, \mathrm{B}$ is an object R over $\mathrm{A} \times \mathrm{B}$; this is a matrix, i.e. two-variable type $\mathrm{a} : \mathrm{A}, \mathrm{b} : \mathrm{B} \vdash R(\mathrm{a}, \mathrm{b}) : \mathbb{V}$.

2. *Monads* in a logic, self-relations with composition and identity, form a richer logic. A monad in a logic of matrices is a category, "enriched in" or "internal to" that logic.

The two constructions define the language of *co/ends* [5]: a bimodule of monads is a matrix with composition actions; these compose by *coend*, a coequalizer of a coproduct, and "divide" or transform by *end*, an equalizer of a product.

$$\begin{aligned} R \circ S &= \Sigma \mathrm{b} \quad R(-, \mathrm{b}) \otimes S(\mathrm{b}, -) \\ [P, Q] &= \Pi \mathrm{x}, \mathrm{y} \quad P(\mathrm{x}, \mathrm{y}) \to Q(\mathrm{x}, \mathrm{y}) \end{aligned}$$

Categories are self-relations, which *act* on relations of categories, defining "active logic": coend is the *bilinear* existential, and end is the *natural* universal. [3]

Category theory is presently seen as a network of concepts, without a central ground. While it is true that generality begets interdefinability, the "fundamentality" of concepts must be understood by how we *construct* universes of categories, and this leads directly to the language of coends and ends. In this way, category theory is generalized logic.

Universal constructions are systematically derived in the language: composition and transformation form a tensor-hom adjunction, giving formulae for lifts and extensions, and weighted limits and colimits are representations thereof.

$$\begin{array}{ccl} [R \circ S, T] & \equiv & \Pi \mathrm{a}, \mathrm{c} \; (\Sigma \mathrm{b} \; R(\mathrm{a}, \mathrm{b}) \otimes S(\mathrm{b}, \mathrm{c})) \to T(\mathrm{a}, \mathrm{c}) \\ \cong & & \qquad\qquad\qquad \cong \\ [R, S \to T] & \equiv & \Pi \mathrm{a}, \mathrm{b} \; R(\mathrm{a}, \mathrm{b}) \to (\Pi \mathrm{c} \; S(\mathrm{b}, \mathrm{c}) \to T(\mathrm{a}, \mathrm{c})) \end{array}$$

The coYoneda lemma is the fact that the hom of a category is its identity relation, and the Yoneda lemma is the curried form of this fact.

$$\begin{array}{rcl} R(\mathrm{a}, \mathrm{b}_1) & \cong & \Sigma \mathrm{b}_0 : \mathrm{B} \;\; R(\mathrm{a}, \mathrm{b}_0) \otimes \mathrm{B}(\mathrm{b}_0, \mathrm{b}_1) \\ \Pi \mathrm{b}_1 : \mathrm{B} \;\; \mathrm{B}(\mathrm{b}_0, \mathrm{b}_1) \to R(\mathrm{a}, \mathrm{b}_1) & \cong & R(\mathrm{a}, \mathrm{b}_0) \end{array}$$

Fundamental ideas are made simple and clear in the language. Presenting CT as logic provides not only a central ground of category theory, but also a systematic and direct exposition of the full power of the language of categories.

Moreover, string diagrams provide an intuitive and systematic presentation of these fundamental ideas. Because imagery is *dual* to syntax, no exclusionary choice is needed: the two combine to form the visual formal language of "color syntax".

A string diagram is the general form of a concept, and writing syntax in the diagram determines a specific instance, i.e. substitution into a dependent type. Reasoning can smoothly transition levels of generality, from an entire logic to a specific transformation. In color syntax, the language of categories is both intuitive and practical.

Based on these ideas, I will be developing an education program for category theory, called Logic in Color. To cultivate in the public mind a language for all kinds of thinking — I believe there is vast possibility. If you see it too, let's talk. [logic.in.color@gmail.com]

My dissertation is the metalanguage of bifibrant double categories, i.e. *metalogic*.

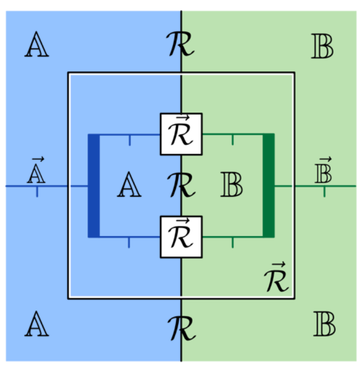
There is an analogy between categories and logics: just as a category is a matrix with composition and identity, a logic is a *matrix of categories* with composition and identity. So just as the language of categories is the co/end calculus, the language of logics is the *co/descent calculus*. We define a logic to be a "matrix category" pseudomonad, and thereby construct the three-dimensional category of logics.

Chapter 1: Spans of categories

A *span of categories* $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ is a equivalent to a matrix of categories $\mathcal{R}(\mathrm{A}, \mathrm{B})$ and a matrix of profunctors $\vec{\mathcal{R}}(\mathrm{a}, \mathrm{b})$, with sequential composition and identity; this is known as a *displayed category*, a.k.a. normal lax functor $\mathcal{R} : \mathbb{A} \times \mathbb{B} \to \mathbb{C}\mathrm{at}$ [9].

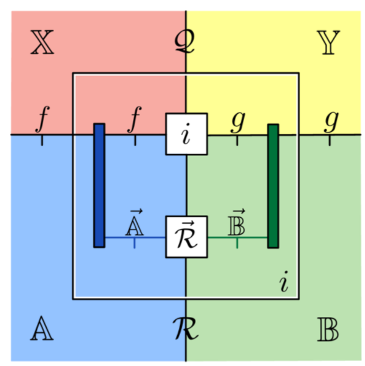
Yet also, a *span of profunctors* $i \leftarrow f \rightarrow g$ is equivalent to a matrix of profunctors $i(\mathrm{f}, \mathrm{g}) : \mathcal{Q}(\mathrm{X}, \mathrm{Y}) \,|\, \mathcal{R}(\mathrm{A}, \mathrm{B})$ with sequential composition and identity. The new concept of *displayed profunctor* $i : f \times g \to \mathrm{Prof}$ is a bimodule of displayed categories. [7]

We introduce three-dimensional string diagrams: horizontal, vertical, and *transversal*, i.e. "inner to outer". For $\mathrm{Span}\mathbb{C}\mathrm{at}$, the dimensions are spans of categories, profunctors, and functors, respectively — the latter is drawn as a closed loop or "bead within a bead".



span category

$\vec{\mathcal{R}}(\mathrm{a}_1, \mathrm{b}_1) \circ \vec{\mathcal{R}}(\mathrm{a}_2, \mathrm{b}_2) \Rightarrow \vec{\mathcal{R}}(\mathrm{a}_1\mathrm{a}_2, \mathrm{b}_1\mathrm{b}_2)$



span profunctor

$i(\mathrm{f}, \mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a}, \mathrm{b}) \Rightarrow i(\mathrm{fa}, \mathrm{gb})$

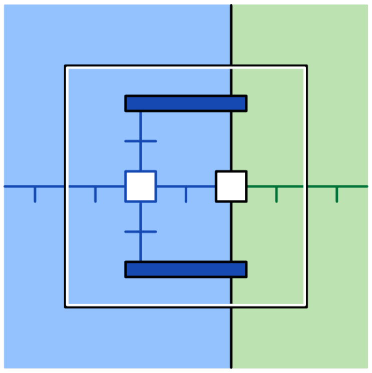
We show the double category of span categories $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ to be equivalent to that of displayed categories $\mathcal{R} : \mathbb{A} \times \mathbb{B} \to \mathbb{C}\mathrm{at}$. These matrices of categories of relations $\mathcal{R}(\mathrm{A}, \mathrm{B})$ are the basic data of metalogic, i.e. the co/descent calculus.

Chapter 2: Matrix categories

A logic is a span of categories $\underline{\mathbb{A}} \leftarrow \mathbb{A} \rightarrow \underline{\mathbb{A}}$, with *actions* of morphisms in $\underline{\mathbb{A}}$ (processes) on objects in $\mathbb{A}$ (relations). The concepts of fibered and opfibered category are unified in the concept of a *two-sided fibration*, which is a bimodule of arrow double categories [11]. Yet these latter are not *logics*, because they only have companions and not conjoints.

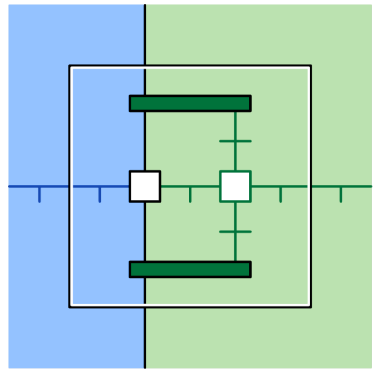
This limits the reasoning of arrow double categories, and leads to an obstruction to composition of "fibered profunctors". So, we determine how a category *does* form a logic: the *weave double category* $\langle \mathbb{A} \rangle$ is the union of the arrow double category and its opposite $\overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$; this is generated by squares in $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$, plus isomorphisms of their identities.

A *matrix category* or *two-sided bifibration* $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ is a span of categories, which forms a bimodule from $\langle\mathbb{A}\rangle$ to $\langle\mathbb{B}\rangle$. Hence $\mathcal{R}$ has actions by both arrows and "op-arrows" of $\mathbb{A}$ and $\mathbb{B}$; these are drawn as follows.



matrix category

$\odot_{\mathbb{A}} : \langle\mathbb{A}\rangle(\mathrm{A}_0, \mathrm{A}_1) \times \mathcal{R}(\mathrm{A}_1, \mathrm{B}) \to \mathcal{R}(\mathrm{A}_0, \mathrm{B})$

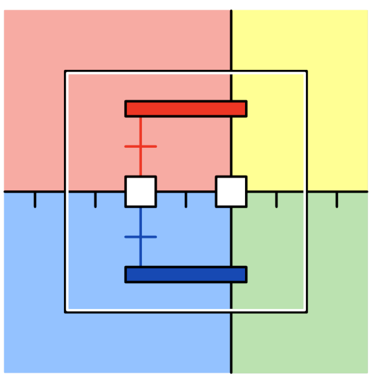


matrix category

$\odot_{\mathbb{B}} : \mathcal{R}(\mathrm{A}, \mathrm{B}_0) \times \langle\mathbb{B}\rangle(\mathrm{B}_0, \mathrm{B}_1) \to \mathcal{R}(\mathrm{A}, \mathrm{B}_1)$

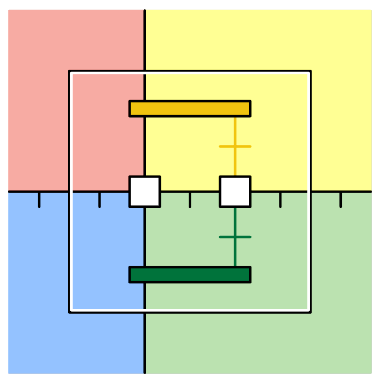
This generalizes from categories to profunctors: the *arrow profunctor* $\vec{f} : \vec{\mathbb{X}} \,|\, \vec{\mathbb{A}}$ consists of commutative squares $\mathrm{f}_0 \cdot \mathrm{a} = \mathrm{x} \cdot \mathrm{f}_1$, and horizontal composition defines a monad structure. The *weave vertical profunctor* $\langle f\rangle : \langle\mathbb{A}\rangle \,|\, \langle\mathbb{B}\rangle$ is the union of $\vec{f}$ and its opposite.

A *matrix profunctor* $i(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \,|\, \mathcal{R}(\mathbb{A},\mathbb{B})$ is a span of profunctors $f \leftarrow i \rightarrow g$, which forms a bimodule from $\langle f\rangle$ to $\langle g\rangle$.



matrix profunctor

$\odot_f : \langle f\rangle(\mathrm{f}_0, \mathrm{f}_1) \times i(\mathrm{f}_1, \mathrm{g}) \Rightarrow i(\mathrm{f}_0, \mathrm{g})$



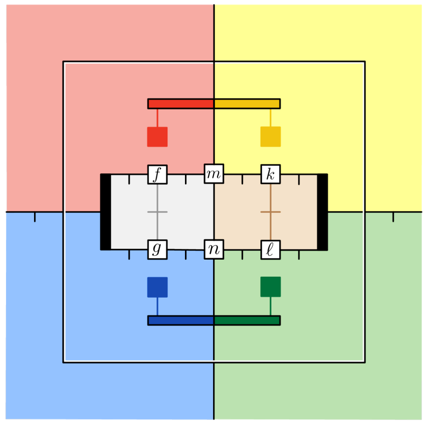
matrix profunctor

$\odot_g : i(\mathrm{f}, \mathrm{g}_0) \times \langle g\rangle(\mathrm{g}_0, \mathrm{g}_1) \Rightarrow i(\mathrm{f}, \mathrm{g}_1)$

Morphisms of the above are *matrix functors* and *matrix transformations*; these form a double category $\mathrm{Mat}\mathbb{C}\mathrm{at}$ over $\mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$. Sequential composition of matrix profunctors over that of profunctors is defined by a coequalizer, which nullifies the action of zig-zags that reassociate equivalent pairs; given $m(f,k) : \mathcal{R}(\mathbb{X},\mathbb{A}) \,|\, \mathcal{S}(\mathbb{Y},\mathbb{B})$ and $n(g,\ell) : \mathcal{S}(\mathbb{Y},\mathbb{B}) \,|\, \mathcal{T}(\mathbb{Z},\mathbb{C})$, the composite $(m \diamond n)(f \circ g, k \circ \ell) : \mathcal{R}(\mathbb{X},\mathbb{A}) \,|\, \mathcal{T}(\mathbb{Z},\mathbb{C})$ consists of pairs (m,n) so that for all

$$\begin{aligned} \mathrm{f} \cdot \mathrm{g} : \quad & \langle f \circ g\rangle([(\mathrm{f}_0, \mathrm{g}_0)], [(\mathrm{f}_1, \mathrm{g}_1)])(\mathrm{id.X}, \mathrm{id.Z}) \\ \mathrm{k} \cdot \mathrm{l} : \quad & \langle k \circ \ell\rangle([(\mathrm{k}_0, \mathrm{l}_0)], [(\mathrm{k}_1, \mathrm{l}_1)])(\mathrm{id.A}, \mathrm{id.C}) \end{aligned}$$

the pair (m,n) is equated with $(\mathrm{f} \odot m \odot \mathrm{k}, \mathrm{g} \odot n \odot \mathrm{l})$.

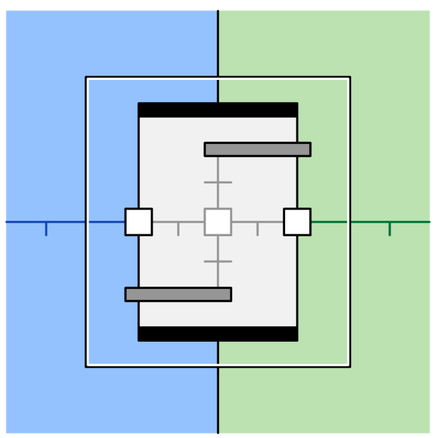


sequential composition

$$(m,n) \;\equiv\; (\mathrm{f}\odot m\odot \mathrm{k}, \mathrm{g}\odot n\odot \mathrm{k}) \;:\; m \diamond n$$

Moreover, $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is a logic, and $\mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathbb{C}\mathrm{at}\times\mathbb{C}\mathrm{at}$ is a *fibered double category* [1]: sequential composition of matrix profunctors preserves substitution of transformations. Hence we call the structure $\mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathbb{C}\mathrm{at}\times\mathbb{C}\mathrm{at}$ a *fibered logic*.

We then define *parallel composition* of matrix categories. While profunctors compose by quotient, matrix categories compose by *codescent object* [11], which adjoins an associator isomorphism for the action by arrows and oparrows of the middle category.



parallel composition

$$\alpha : (R, \bar{\mathrm{b}}\odot S) \cong (R\odot \bar{\mathrm{b}}, S)$$

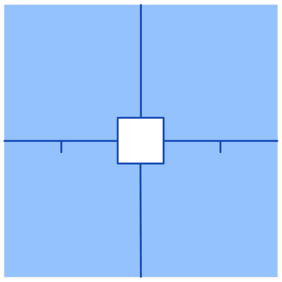
Dually, a category of matrix functors is formed by a *descent object* [10]. So, composition and transformation form a tensor-hom adjunction, just as in the co/end calculus.

$$\begin{aligned} \mathcal{R}\otimes\mathcal{S} &= \vec{\Sigma}\mathrm{B}.\quad \mathcal{R}(-,\mathrm{B})\times\mathcal{S}(\mathrm{B},-) \\ [\mathcal{P},\mathcal{Q}] &= \vec{\Pi}\mathrm{X},\mathrm{Y}.\quad \mathcal{P}(\mathrm{X},\mathrm{Y})\to\mathcal{Q}(\mathrm{X},\mathrm{Y}) \end{aligned}$$

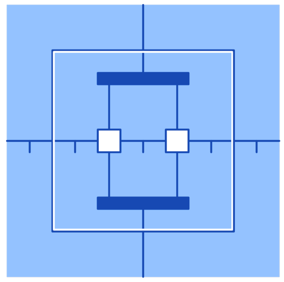
Parallel composition does not preserve sequential composition of matrix profunctors: because both dimensions are bimodules, both compositions involve colimits which the other cannot represent. So $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is "a triple category without interchange": we define a *metalogic* to be a fibered logic $\mathbb{M}\to\mathbb{C}\times\mathbb{C}$, which forms a tricategory internal to $\mathrm{Span}\mathbb{C}\mathrm{at}$.

Chapter 3: The metalogic of logics

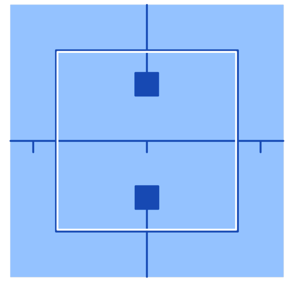
A bifibrant double category, i.e. a logic, is a pseudomonad in $\mathrm{Mat}\mathbb{C}\mathrm{at}$.



logic

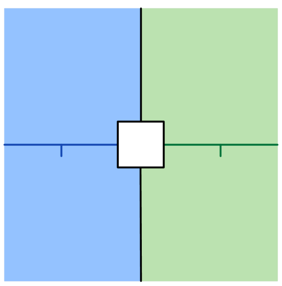


composition

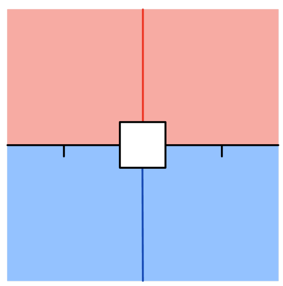


unit

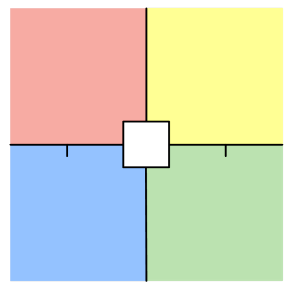
Because a logic is two-dimensional, there are *two* kinds of relations between logics: a *vertical profunctor* consists of processes between logics, and a *horizontal profunctor* consists of relations between logics. Two pairs are connected by a *double profunctor*, which consists of inferences between relations, along processes.



meta relation
[horiz. profunctor]

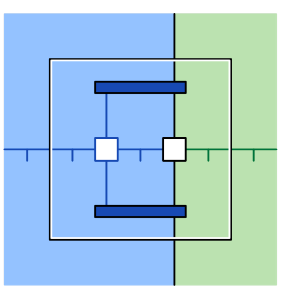


meta process
[vert. profunctor]

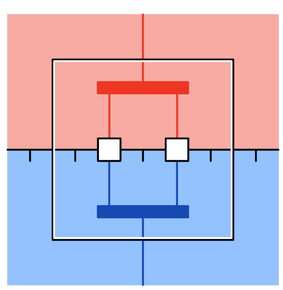


meta inference
[dbl. profunctor]

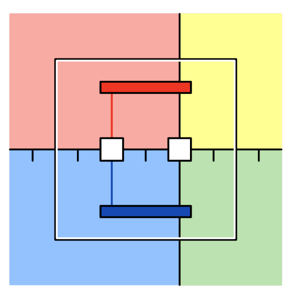
For a horizontal profunctor, parallel composition is a familiar *bimodule* structure. Yet because vertical profunctors are orthogonal, parallel composition is a *monad* structure, and then double profunctors are bimodules thereof.



H-prof. composition



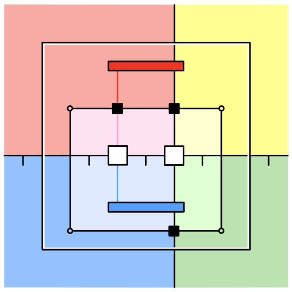
V-prof. composition



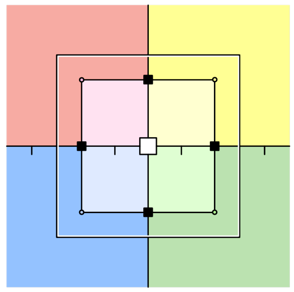
D-prof. composition

So logics have two kinds of "relations", and one kind of "function": a *double functor* $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ maps squares of $\mathbb{A}_0$ to squares of $\mathbb{A}_1$, preserving relation composition and unit up to coherent isomorphism.

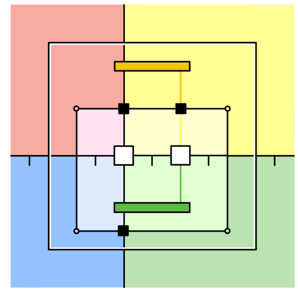
This generalizes to transformations of vertical, horizontal, and double profunctors; all four are defined by mapping squares in a way that coheres with parallel composition.



left comp. coherence



double transformation



right comp. coherence

Logics form a metalogic: morphisms are functors, profunctors, and matrix categories; squares are vertical transformations, horizontal transformations, and double profunctors; and cubes are double transformations.

Below, the outline: we first construct the metalogic of matrix categories, and then apply the "pseudomonad construction" to form the metalogic of bifibrant double categories; and we give a metalogical interpretation of this structure.

	$\mathrm{Mat}\mathbb{C}\mathrm{at}$	$\mathrm{H.PsMnd}(-)$	$bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$	Logic
0	category	(H)-pseudomonad	bifibrant double category	logic
V	profunctor	(H)-vertical monad	vertical profunctor	meta process
H	matrix category	(H)-pseudobimodule	horizontal profunctor	meta relation
VH	matrix profunctor	(H)-vertical bimodule	double profunctor	meta inference
T	functor	ps. mnd. morphism	double functor	flow type
TV	transformation	v. mnd. morphism	vertical transformation	flow process
TH	matrix functor	ps. bim. morphism	horizontal transformation	flow relation
TVH	matrix transformation	v. bim. morphism	double transformation	flow inference

The language is vast and powerful. There are just three basic limitations of metalogic.

1. *No interchange.* Parallel (horizontal) composition is neither lax nor colax with respect to sequential (vertical) composition of double profunctors.

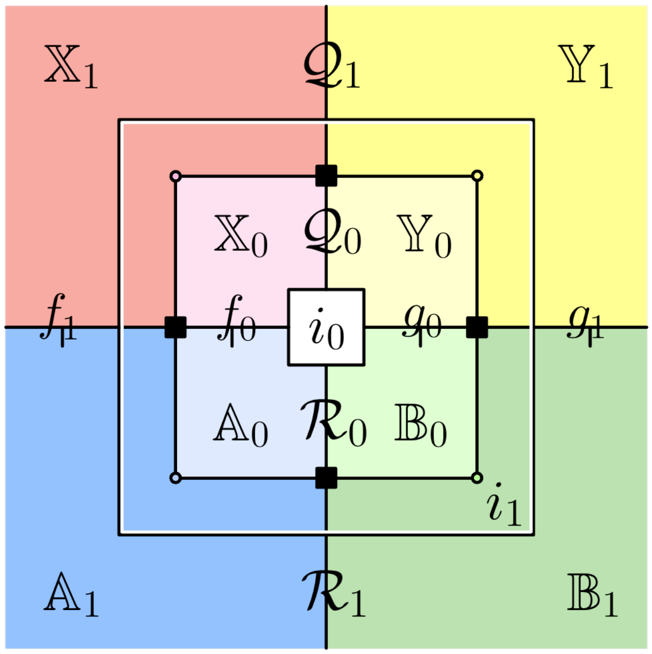
2. *No metaprocess collage.* In general there is no collage of a vertical profunctor, because its elements do not act on the relations of the bifibrant double categories.

3. *No base closure.* The base logic — bifibrant double categories and functors, vertical profunctors and transformations — is not closed, and neither is the total logic.

Yet $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$ is horizontally closed: lifts and extensions are derived in the same way as in the co/end calculus, and this gives formulae for weighted double co/limits. This provides a unified method of construction in category theory.

As a double profunctor consists of inferences between logics, a double transformation is a "flow" of *meta*-reasoning, a way to transform one system of reasoning into another.

In this sense, the language of $bf.\mathbb{D}\mathrm{bl}\mathbb{C}\mathrm{at}$ is the language of metalogic.



(Yes, this framework extends to virtual equipments, and moreover their polycategorical generalization. The key is the notion of matrix profunctor; one can specify any kind of "shape" of 2-cell, equipped with multi- or poly- composition. As of now, I do not know of an aspect of category theory which is beyond the scope of this metalanguage.

The pseudomonad construction generalizes either to lax or colax double functors; but this replaces the iso-inserter of the descent formula with an inserter, which complicates the co/descent calculus. It is likely best to use pseudo double functors, and encode co/laxity.)

Conclusion

When the notion of elementary topos was defined, humanity took a great leap forward by generalizing logic; this began an extensive program of unifying mathematics. Yet a topos forms a *locally thin* logic, as its relations are valued in propositions.

The notions of *stacks* and *descent*, by which Grothendieck et. al. have revolutionized algebraic geometry and beyond, are immanent within the co/descent calculus of logics.

There is an immense program here. I am just one person, who hardly knows much mathematics beyond category theory. My productivity alone is inadequate to this endeavor, and the sooner the collaboration the better. So, this is an invitation to everyone who sees the potential of this language.

I plan to organize a seminar. Although the current draft of my dissertation is rough, it is enough to communicate the definitions and theorems. If you're interested, let me know. Let's develop and explore the three-dimensional language of metalogic.

References

[1] Geoffrey Cruttwell, Michael Lambert, Dorette Pronk, and Martin Szyld. Double fibrations, 2022.

[2] Brendan Fong and David I Spivak. Seven sketches in compositionality: An invitation to applied category theory, 2018.

[3] F. William Lawvere. Metric spaces, generalized logic, and closed categories. *Theory and Applications of Categories*, 1:1–37, 2002.

[4] F. William Lawvere. Taking categories seriously. *Theory and Applications of Categories*, 8:1–24, 2005.

[5] Fosco Loregian. *(Co)end Calculus*. Cambridge University Press, 2021.

[6] David Jaz Myers. String diagrams for double categories and equipments, 2018.

[7] Robert Paré. Composition of modules for lax functors. *Theory and Applications of Categories*, 27:393–444, 2013.

[8] Michael Shulman. Framed bicategories and monoidal fibrations. *Theory and Applications of Categories*, 20:650–738, 2008.

[9] Ross Street. Powerful functors, 2001.

[10] Ross Street. Categorical and combinatorial aspects of descent theory. *Applied Categorical Structures*, 12(5/6):537–576, 2004.

[11] Tamara von Glehn. Polynomials, fibrations, and distributive laws. *Theory and Applications of Categories*, 33:1111–1144, 2018.