Companions and conjoints in a virtual equipment

Jana K. Nickel, 17 February 2026

In the following notes, we will introduce companions and conjoints in a virtual equipment and state and prove some related results in great detail.

The notes are mainly based on:

- Nathanael Arkor and Dylan McDermott: The formal theory of relative monads, 2025.
- G.S.H Cruttwell and Michael A. Shulman: A unified framework for generalized multicategories, 2010.

Yet, I will present the material in a slightly different style and with many additional details.

Construction.

Let $\mathbb{X}$ be a virtual equipment.

Let $f : A \longrightarrow B$ be any tight arrow in $\mathbb{X}$.

1) Then we have two niche configurations in $\mathbb{X}$

$$\begin{tikzcd} A \arrow[d, "f"'] & B \arrow[d, equal] \\ B \arrow[r, "e_B"', "\shortmid"{marking}] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[d, equal] & A \arrow[d, "f"] \\ B \arrow[r, "e_B"', "\shortmid"{marking}] & B. \end{tikzcd}$$

By assumption, they come along with cartesian 2-cells

$$\begin{tikzcd} A \arrow[r, "B(1{,}f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overline{B}(1{,}f)\ \mathrm{cart}"] & B \arrow[d, equal] \\ B \arrow[r, "e_B"', "\shortmid"{marking}] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r, "B(f{,}1)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\overline{B}(f{,}1)\ \mathrm{cart}"] & A \arrow[d, "f"] \\ B \arrow[r, "e_B"', "\shortmid"{marking}] & B, \end{tikzcd}$$

respectively.

We call the two loose arrows $B(1,f) : A \nrightarrow B$ and $B(f,1) : B \nrightarrow A$ in $\mathbb{X}$ also the base change loose arrows of the tight arrow $f : A \to B$.

2) Consider the identity 2-cell e_f of f in $\mathbb{X}$:

$$\begin{tikzcd} A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "e_f"] & A \arrow[d, "f"] \\ B \arrow[r, "e_B"', "\shortmid"{marking}] & B. \end{tikzcd}$$

This is the unique 2-cell in $\mathbb{X}$ of the form
$$\begin{tikzcd} A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "e_f"] & A \arrow[d, "f"] \\ B \arrow[r, "e_B"', "\shortmid"{marking}] & B \end{tikzcd}$$
satisfying $\underline{e}_B \fatsemi (f) = e_f \fatsemi \underline{e}_A$,

i.e.,

$$\begin{tikzcd}[column sep=small] & A \arrow[d, "f"'] & \\ & B \arrow[dl, equal, bend right] \arrow[dr, equal, bend left] \arrow[d, phantom, "\underline{e}_B\ \mathrm{opcart}"] & \\ B \arrow[rr, "e_B"', "\shortmid"{marking}] & & B \end{tikzcd} = \begin{tikzcd}[column sep=small] & A \arrow[dl, equal, bend right] \arrow[dr, equal, bend left] \arrow[d, phantom, "\underline{e}_A\ \mathrm{opcart}"] & \\ A \arrow[rr, "e_A"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[drr, phantom, "e_f"] & & A \arrow[d, "f"] \\ B \arrow[rr, "e_B"', "\shortmid"{marking}] & & B. \end{tikzcd}$$

2.1) We can write the 2-cell e_f as

$$
\begin{tikzcd}
A \arrow[r, "e_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "e_f"] & A \arrow[d, "f"] \\
B \arrow[r, "e_B"', "|" marking] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "e_A", "|" marking] \arrow[d, equal] & A \arrow[d, "f"] \\
A \arrow[d, "f"'] \arrow[r, phantom, "e_f"] & B \arrow[d, equal] \\
B \arrow[r, "e_B"', "|" marking] & B
\end{tikzcd},
$$

so by the universal property of the cartesian 2-cell
$$
\begin{tikzcd}
A \arrow[r, "B(1{,}f)", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\overline{B}(1{,}f) \ \mathrm{cart}"] & B \arrow[d, equal] \\
B \arrow[r, "e_B"', "|" marking] & B
\end{tikzcd}
$$
applied to e_f,

$\exists!$ 2-cell $\nu(1,f)$ in $\mathbb{X}$ of the form

$$
\begin{tikzcd}
A \arrow[r, "e_A", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\nu(1{,}f)"] & A \arrow[d, "f"] \\
A \arrow[r, "B(1{,}f)"', "|" marking] & B
\end{tikzcd}
$$

with $e_f = \overline{B}(1,f) \circ \nu(1,f)$, i.e.,

$$
\begin{tikzcd}
A \arrow[r, "e_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "e_f"] & A \arrow[d, "f"] \\
B \arrow[r, "e_B"', "|" marking] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "e_A", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\nu(1{,}f)"] & A \arrow[d, "f"] \\
A \arrow[r, "B(1{,}f)"', "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\overline{B}(1{,}f) \ \mathrm{cart}"] & B \arrow[d, equal] \\
B \arrow[r, "e_B"', "|" marking] & B
\end{tikzcd}.
$$

2.2) Similarly, we can write the 2-cell e_f as

$$
\begin{tikzcd}
A \arrow[r, "e_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "e_f"] & A \arrow[d, "f"] \\
B \arrow[r, "e_B"', "|" marking] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "e_A", "|" marking] \arrow[d, "f"'] & A \arrow[d, equal] \\
B \arrow[d, equal] \arrow[r, phantom, "e_f"] & A \arrow[d, "f"] \\
B \arrow[r, "e_B"', "|" marking] & B
\end{tikzcd},
$$

so by the universal property of the cartesian 2-cell
$$
\begin{tikzcd}
B \arrow[r, "B(f{,}1)", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\overline{B}(f{,}1) \ \mathrm{cart}"] & A \arrow[d, "f"] \\
B \arrow[r, "e_B"', "|" marking] & B
\end{tikzcd}
$$
applied to e_f,

$\exists!$ 2-cell $\nu(f,1)$ in $\mathbb{X}$ of the form

$$
\begin{tikzcd}
A \arrow[r, "e_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\nu(f{,}1)"] & A \arrow[d, equal] \\
B \arrow[r, "B(f{,}1)"', "|" marking] & A
\end{tikzcd}
$$

with $e_f = \overline{B}(f,1) \circ \nu(f,1)$, i.e.,

$$
\begin{tikzcd}
A \arrow[r, "e_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "e_f"] & A \arrow[d, "f"] \\
B \arrow[r, "e_B"', "|" marking] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "e_A", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\nu(f{,}1)"] & A \arrow[d, equal] \\
B \arrow[r, "B(f{,}1)"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\overline{B}(f{,}1) \ \mathrm{cart}"] & A \arrow[d, "f"] \\
B \arrow[r, "e_B"', "|" marking] & B
\end{tikzcd}.
$$

Remark.

Let again $\mathbb{X}$ be a virtual equipment and $f\colon A \to B$ any tight arrow in $\mathbb{X}$.

1) Then we have the following equality of 2-cells in $\mathbb{X}$:

$$
\begin{tikzcd}
A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "v(1{,}f)"] & A \arrow[r, "B(1{,}f)", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\overline{B}(1{,}f)\ \mathrm{cart}"] & B \arrow[d, equal] \\
A \arrow[r, "B(1{,}f)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\lambda_{B(1{,}f)}\ \mathrm{opcart}"] & B \arrow[r, "e_B"', "\shortmid"{marking}] & B \arrow[d, equal] \\
A \arrow[rr, "B(1{,}f)"', "\shortmid"{marking}] & & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\rho_{B(1{,}f)}\ \mathrm{opcart}"] & A \arrow[r, "B(1{,}f)", "\shortmid"{marking}] & B \arrow[d, equal] \\
A \arrow[rr, "B(1{,}f)"', "\shortmid"{marking}] & & B
\end{tikzcd},
$$

i.e., $\lambda_{B(1,f)} \circ (v(1,f), \overline{B}(1,f)) = \rho_{B(1,f)}$.

Proof.

By the universal property of the cartesian 2-cell
$$
\begin{tikzcd}
A \arrow[r, "B(1{,}f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overline{B}(1{,}f)\ \mathrm{cart}"] & B \arrow[d, equal] \\
B \arrow[r, "e_B"', "\shortmid"{marking}] & B
\end{tikzcd},
$$

it suffices to show that the post-compositions of the two above 2-cells with $\overline{B}(1,f)$ agree.

Moreover, by the universal property of the opcartesian 2-cell
$$
\begin{tikzcd}
& A \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "e_A\ \mathrm{opcart}"] & \\
A \arrow[rr, "e_A"', "\shortmid"{marking}] & {} & A
\end{tikzcd},
$$

it suffices to show that their pre-compositions with $(e_A, 1_{B(1,f)})$ agree, meaning the following two composite 2-cells:

$$
\phi :=
\begin{tikzcd}
& A \arrow[r, "B(1{,}f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
A \arrow[ur, equal, bend left, "e_A\ \mathrm{opcart}"] \arrow[r, "e_A"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "v(1{,}f)"] & A \arrow[r, "B(1{,}f)"', "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\overline{B}(1{,}f)"] & B \arrow[d, equal] \\
A \arrow[r, "B(1{,}f)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\lambda_{B(1{,}f)}"] & B \arrow[r, "e_B"', "\shortmid"{marking}] & B \arrow[d, equal] \\
A \arrow[rr, "B(1{,}f)"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[drr, phantom, "\overline{B}(1{,}f)\ \mathrm{cart}"] & & B \arrow[d, equal] \\
B \arrow[rr, "e_B"', "\shortmid"{marking}] & & B
\end{tikzcd}
\quad \text{and} \quad
\psi :=
\begin{tikzcd}
& A \arrow[r, "B(1{,}f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
A \arrow[ur, equal, bend left, "e_A\ \mathrm{opcart}"] \arrow[r, "e_A"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\rho_{B(1{,}f)}"] & A \arrow[r, "B(1{,}f)"', "\shortmid"{marking}] & B \arrow[d, equal] \\
A \arrow[rr, "B(1{,}f)"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[drr, phantom, "\overline{B}(1{,}f)\ \mathrm{cart}"] & & B \arrow[d, equal] \\
B \arrow[rr, "e_B"', "\shortmid"{marking}] & & B
\end{tikzcd}.
$$

In the following, we will frequently use the associativity and axiom and the identity axiom of the virtual double category $\mathbb{X}$ without explicit mention.

We have

$$\begin{aligned}\psi &= (\overline{\mathbb{B}}(1,f) ⨾ \rho_{\mathbb{B}(1,f)}) ⨾ (e_A, 1_{\mathbb{B}(1,f)}) \\ &= \overline{\mathbb{B}}(1,f) ⨾ (\rho_{\mathbb{B}(1,f)} ⨾ (e_A, 1_{\mathbb{B}(1,f)})) \\ &= \overline{\mathbb{B}}(1,f) ⨾ 1_{\mathbb{B}(1,f)}\end{aligned}$$

by construction of $\rho_{\mathbb{B}(1,f)}$

$$= \overline{\mathbb{B}}(1,f).$$

We need to show that $\phi = \overline{\mathbb{B}}(1,f)$.

By the naturality property of λ wrt $\overline{\mathbb{B}}(1,f)$, we have the following equality of composite 2-cells in $\mathbb{X}$:

$$\begin{tikzcd}
A \arrow[r, "\mathbb{B}(1,f)"] \arrow[d, equal] & B \arrow[r, "e_B"] \arrow[d, phantom, "\lambda_{\mathbb{B}(1,f)}"] & B \arrow[d, equal] \\
A \arrow[rr, "\mathbb{B}(1,f)"'] \arrow[d, "f"'] & {} \arrow[d, phantom, "\overline{\mathbb{B}}(1,f)"] & B \arrow[d, equal] \\
B \arrow[rr, "e_B"'] & {} & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "\mathbb{B}(1,f)"] \arrow[d, "f"'] \arrow[dr, phantom, "\overline{\mathbb{B}}(1,f)"] & B \arrow[r, "e_B"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
B \arrow[r, "e_B"'] \arrow[d, equal] & B \arrow[r, "e_B"'] \arrow[d, phantom, "\lambda_{e_B}"] & B \arrow[d, equal] \\
B \arrow[rr, "e_B"'] & {} & B
\end{tikzcd},$$

i.e., $\overline{\mathbb{B}}(1,f) ⨾ \lambda_{\mathbb{B}(1,f)} = \lambda_{e_B} ⨾ (\overline{\mathbb{B}}(1,f), 1_{e_B})$.

We compute:

$$\phi =
\begin{tikzcd}
 & A \arrow[r, "\mathbb{B}(1,f)"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
A \arrow[ur, bend left, equal] \arrow[r, "e_A"'] \arrow[d, equal] \arrow[dr, phantom, "\nu(1,f)"] \arrow[u, phantom, "e_A"] & A \arrow[r, "\mathbb{B}(1,f)"'] \arrow[d, "f"] \arrow[dr, phantom, "\overline{\mathbb{B}}(1,f)"] & B \arrow[d, equal] \\
A \arrow[r, "\mathbb{B}(1,f)"'] \arrow[d, equal] & B \arrow[r, "e_B"] \arrow[d, phantom, "\lambda_{\mathbb{B}(1,f)}"] & B \arrow[d, equal] \\
A \arrow[rr, "\mathbb{B}(1,f)"] \arrow[d, "f"'] & {} \arrow[d, phantom, "\overline{\mathbb{B}}(1,f)"] & B \arrow[d, equal] \\
B \arrow[rr, "e_B"'] & {} & B
\end{tikzcd}
\underset{\text{naturality of } \lambda \text{ wrt } \overline{\mathbb{B}}(1,f)}{=}
\begin{tikzcd}
 & A \arrow[r, "\mathbb{B}(1,f)"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
A \arrow[ur, bend left, equal] \arrow[r, "e_A"'] \arrow[d, equal] \arrow[dr, phantom, "\nu(1,f)"] \arrow[u, phantom, "e_A"] & A \arrow[r, "\mathbb{B}(1,f)"'] \arrow[d, "f"] \arrow[dr, phantom, "\overline{\mathbb{B}}(1,f)"] & B \arrow[d, equal] \\
A \arrow[r, "\mathbb{B}(1,f)"'] \arrow[d, "f"'] \arrow[dr, phantom, "\overline{\mathbb{B}}(1,f)"] & B \arrow[r, "e_B"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
B \arrow[r, "e_B"'] \arrow[d, equal] & B \arrow[r, "e_B"'] \arrow[d, phantom, "\lambda_{e_B}"] & B \arrow[d, equal] \\
B \arrow[rr, "e_B"'] & {} & B
\end{tikzcd}
\underset{\text{construction of } \nu(1,f)}{=}
\begin{tikzcd}
 & A \arrow[r, "\mathbb{B}(1,f)"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
A \arrow[ur, bend left, equal] \arrow[r, "e_A"'] \arrow[d, "f"'] \arrow[dr, phantom, "e_f"] \arrow[u, phantom, "e_A"] & A \arrow[r, "\mathbb{B}(1,f)"'] \arrow[d, "f"] \arrow[dr, phantom, "\overline{\mathbb{B}}(1,f)"] & B \arrow[d, equal] \\
B \arrow[r, "e_B"'] \arrow[d, equal] & B \arrow[r, "e_B"'] \arrow[d, phantom, "\lambda_{e_B}"] & B \arrow[d, equal] \\
B \arrow[rr, "e_B"'] & {} & B
\end{tikzcd}$$

$$\underset{\text{construction of } e_f}{=}
\begin{tikzcd}
 & A \arrow[r, "\mathbb{B}(1,f)"] \arrow[d, "f"'] \arrow[dr, phantom, "\overline{\mathbb{B}}(1,f)"] & B \arrow[d, equal] \\
 & B \arrow[r, "e_B"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
B \arrow[ur, bend left, equal] \arrow[r, "e_B"'] \arrow[d, equal] \arrow[u, phantom, "e_B"] & B \arrow[r, "e_B"'] \arrow[d, phantom, "\lambda_{e_B}"] & B \arrow[d, equal] \\
B \arrow[rr, "e_B"'] & {} & B
\end{tikzcd}
\underset{\text{construction of } \lambda_{e_B}}{=}
\begin{tikzcd}
A \arrow[r, "\mathbb{B}(1,f)"] \arrow[d, "f"'] \arrow[dr, phantom, "\overline{\mathbb{B}}(1,f)"] & B \arrow[d, equal] \\
B \arrow[r, "e_B"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\
B \arrow[r, "e_B"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "\mathbb{B}(1,f)"] \arrow[d, "f"'] \arrow[dr, phantom, "\overline{\mathbb{B}}(1,f)"] & B \arrow[d, equal] \\
B \arrow[r, "e_B"'] & B
\end{tikzcd}.$$

2) Similarly, we have the following equality of 2-cells in $\mathbb{X}$:

$$
\begin{tikzcd}
B \arrow[r, "B(f,1)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\overline{B}(f,1)\ \mathrm{cart}"] & A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\nu(f,1)"] & A \arrow[d, equal] \\
B \arrow[r, "e_B"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\rho_{B(f,1)}\ \mathrm{opcart}"] & B \arrow[r, "B(f,1)"', "\shortmid"{marking}] & A \arrow[d, equal] \\
B \arrow[rr, "B(f,1)"', "\shortmid"{marking}] & & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
B \arrow[r, "B(f,1)", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\lambda_{B(f,1)}\ \mathrm{opcart}"] & A \arrow[r, "e_A", "\shortmid"{marking}] & A \arrow[d, equal] \\
B \arrow[rr, "B(f,1)"', "\shortmid"{marking}] & & A,
\end{tikzcd}
$$

i.e., $\rho_{B(f,1)} \circ (\overline{B}(f,1), \nu(f,1)) = \lambda_{B(f,1)}$.

Proof.

This can be shown by an analogous reasoning as in 1).

Definition.

Let $\mathbb{X}$ be a virtual double category with nullary opcartesian 2-cells.

A companion tuple in $\mathbb{X}$ is a quadruple (f, p, α, β) consisting of

- a tight arrow $f: A \to B$ in $\mathbb{X}$,
- a loose arrow $p: A \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} B$ in $\mathbb{X}$,
- a 2-cell in $\mathbb{X}$ of the form
$$
\begin{tikzcd}
A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "f"] \\
A \arrow[r, "p"', "\shortmid"{marking}] & B
\end{tikzcd},
$$
- a 2-cell in $\mathbb{X}$ of the form
$$
\begin{tikzcd}
A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
B \arrow[r, "e_B"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

satisfying the following equalities of 2-cells in $\mathbb{X}$:

(i)
$$
\begin{tikzcd}
A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "f"] \\
A \arrow[r, "p"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
B \arrow[r, "e_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "e_f"] & A \arrow[d, "f"] \\
B \arrow[r, "e_B"', "\shortmid"{marking}] & B
\end{tikzcd},
$$

i.e., $\beta \circ \alpha = e_f$,

(ii)

$$
\begin{tikzcd}
A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\beta"] & B \arrow[d, equal] \\
A \arrow[r, "p"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\lambda_p \ \mathrm{opcart}"] & B \arrow[r, "e_B"', "\shortmid"{marking}] & B \arrow[d, equal] \\
A \arrow[rr, "p"', "\shortmid"{marking}] & & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\rho_p \ \mathrm{opcart}"] & A \arrow[r, "p", "\shortmid"{marking}] & B \arrow[d, equal] \\
A \arrow[rr, "p"', "\shortmid"{marking}] & & B,
\end{tikzcd}
$$

i.e., $\lambda_p \,\circ\, (\alpha, \beta) = \rho_p$.

In this case, we also call (f, p) a companion pair in $\mathbb{X}$ and say that $p: A \nrightarrow B$ is a loose companion of $f: A \to B$ in $\mathbb{X}$ and that $f: A \to B$ is a tight companion of $p: A \nrightarrow B$ in $\mathbb{X}$. We also say that the tight arrow $f: A \to B$ and the loose arrow $p: A \nrightarrow B$ are companions of each other in $\mathbb{X}$.

Definition.

Let $\mathbb{X}$ be a virtual double category with nullary opcartesian 2-cells.

A conjunction in $\mathbb{X}$ is a quadruple (f, p, α, β) consisting of

- a tight arrow $f: A \to B$ in $\mathbb{X}$,
- a loose arrow $p: B \nrightarrow A$ in $\mathbb{X}$, N.B. the order of B and A!
- a 2-cell in $\mathbb{X}$ of the form

$$
\begin{tikzcd}
A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, equal] \\
B \arrow[r, "p"', "\shortmid"{marking}] & A
\end{tikzcd},
$$

- a 2-cell in $\mathbb{X}$ of the form

$$
\begin{tikzcd}
B \arrow[r, "p", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\beta"] & A \arrow[d, "f"] \\
B \arrow[r, "e_B"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

satisfying the following equalities of 2-cells in $\mathbb{X}$:

(i)

$$
\begin{tikzcd}
A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, equal] \\
B \arrow[r, "p"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\beta"] & A \arrow[d, "f"] \\
B \arrow[r, "e_B"', "\shortmid"{marking}] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "e_f"] & A \arrow[d, "f"] \\
B \arrow[r, "e_B"', "\shortmid"{marking}] & B
\end{tikzcd},
$$

i.e., $\beta \circ \alpha = e_f$,

(ii)

$$
\begin{tikzcd}
B \arrow[r, "p", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\beta"] & A \arrow[r, "e_A", "\shortmid" marking] \arrow[d, "f"] \arrow[dr, phantom, "\alpha"] & A \arrow[d, equal] \\
B \arrow[r, "e_B"', "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\rho_p \ \mathrm{opcart}"] & B \arrow[r, "p"', "\shortmid" marking] & A \arrow[d, equal] \\
B \arrow[rr, "p"', "\shortmid" marking] & & A
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "p", "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\lambda_p \ \mathrm{opcart}"] & A \arrow[r, "e_A", "\shortmid" marking] & A \arrow[d, equal] \\
B \arrow[rr, "p"', "\shortmid" marking] & & A,
\end{tikzcd}
$$

i.e., $\rho_p \mathbin{⨾} (\beta, \alpha) = \lambda_p$.

In this case, we also call (f, p) a conjoint pair in $\mathbb{X}$ and say that $p: B \nrightarrow A$ is a loose conjoint of $f: A \to B$ in $\mathbb{X}$ and that $f: A \to B$ is a tight conjoint of $p: B \nrightarrow A$ in $\mathbb{X}$. We also say that the tight arrow $f: A \to B$ and the loose arrow $p: B \nrightarrow A$ are conjoints of each other in $\mathbb{X}$.

Example.

Let $\mathbb{X}$ be a virtual equipment.

Let $f: A \to B$ be any tight arrow in $\mathbb{X}$.

Then f admits both a loose companion and a loose conjoint in $\mathbb{X}$ as depicted below.

Specifically, the previous remark shows that the 2-cells
$$
\begin{tikzcd}
A \arrow[r, "e_A", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\nu(1,f)"] & A \arrow[d, "f"] \\
A \arrow[r, "B(1,f)"', "\shortmid" marking] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "B(1,f)", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\overline{B}(1,f) \ \mathrm{opcart}"] & B \arrow[d, equal] \\
B \arrow[r, "e_B"', "\shortmid" marking] & B
\end{tikzcd}
$$

witness $(f, B(1,f))$ as a companion pair in $\mathbb{X}$, and the 2-cells
$$
\begin{tikzcd}
A \arrow[r, "e_A", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\nu(f,1)"] & A \arrow[d, equal] \\
B \arrow[r, "B(f,1)"', "\shortmid" marking] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
B \arrow[r, "B(f,1)", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\overline{B}(f,1) \ \mathrm{cart}"] & A \arrow[d, "f"] \\
B \arrow[r, "e_B"', "\shortmid" marking] & B
\end{tikzcd}
$$

witness $(f, B(f,1))$ as a conjoint pair in $\mathbb{X}$.

Theorem.

Let $\mathbb{X}$ be a virtual equipment.

Suppose we are given a niche configuration in $\mathbb{X}$ of the form
$$
\begin{tikzcd}
A \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "q"', "\shortmid" marking] & D.
\end{tikzcd}
$$

Then the loose restriction $q(g,f): A \nrightarrow C$ of f and g along q is also a loose composite of the triple $(B(1,f), q, D(g,1))$ of loose arrows in $\mathbb{X}$ as depicted below,

$$
A \xrightarrow{B(1,f)} B \xrightarrow{q} D \xrightarrow{D(g,1)} C,
$$

i.e., $\exists$ opcartesian 2-cell ϕ in $\mathbb{X}$ of shape

$$\begin{tikzcd}
A \arrow[r, "B(1,f)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "q", "\shortmid"{marking}] \arrow[d, phantom, "\text{opcart}"] & D \arrow[r, "D(g,1)", "\shortmid"{marking}] & C \arrow[d, equal] \\
A \arrow[rrr, "q(g,f)"', "\shortmid"{marking}] & & & C .
\end{tikzcd}$$

In particular, the triple $(B(1,f), q, D(g,1))$ admits a loose composite.

Hence, given any loose composite $B(1,f) \odot q \odot D(g,1)$ of $(B(1,f), q, D(g,1))$, then the loose arrows $q(g,f) : A \nrightarrow C$ and $B(1,f) \odot q \odot D(g,1) : A \nrightarrow C$ in $\mathbb{X}$ are isomorphic as objects in the category $\mathbb{X}[\![A, C]\!]$,

$$q(g,f) \cong B(1,f) \odot q \odot D(g,1).$$

Proof.

By assumption, $\mathbb{X}$ is a virtual equipment, so by definition, the niche configuration
$\begin{tikzcd} A \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}$
in $\mathbb{X}$ is equipped with a cartesian 2-cell
$\begin{tikzcd} A \arrow[r, "q(g,f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\substack{\overline{q}(g,f) \\ \text{cart}}"] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}$
in $\mathbb{X}$,

and so are the niche configurations
$\begin{tikzcd} A \arrow[d, "f"'] & B \arrow[d, equal] \\ B \arrow[r, "e_B"', "\shortmid"{marking}] & B \end{tikzcd}$
and
$\begin{tikzcd} D \arrow[d, equal] & C \arrow[d, "g"] \\ D \arrow[r, "e_D"', "\shortmid"{marking}] & D \end{tikzcd}$,

yielding the cartesian 2-cells
$\begin{tikzcd} A \arrow[r, "B(1,f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\substack{\overline{B}(1,f) \\ \text{cart}}"] & B \arrow[d, equal] \\ B \arrow[r, "e_B"', "\shortmid"{marking}] & B \end{tikzcd}$
and
$\begin{tikzcd} D \arrow[r, "D(g,1)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\substack{\overline{D}(g,1) \\ \text{cart}}"] & C \arrow[d, "g"] \\ D \arrow[r, "e_D"', "\shortmid"{marking}] & D \end{tikzcd}$,
respectively.

Consider the following composite 2-cell in $\mathbb{X}$:

$$\gamma := \begin{tikzcd}
A \arrow[r, "B(1,f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\substack{\overline{B}(1,f) \\ \text{cart}}"] & B \arrow[r, "q", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[r, "D(g,1)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\substack{\overline{D}(g,1) \\ \text{cart}}"] & C \arrow[d, "g"] \\
B \arrow[r, "e_B"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, "q"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\substack{\lambda_q \\ \text{opcart}}"] & D \arrow[r, "e_D"', "\shortmid"{marking}] & D \arrow[d, equal] \\
B \arrow[r, "e_B"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\substack{\rho_q \\ \text{opcart}}"] & B \arrow[rr, "q"', "\shortmid"{marking}] & & D \arrow[d, equal] \\
B \arrow[rrr, "q"', "\shortmid"{marking}] & & & D .
\end{tikzcd}$$

By the universal property of the cartesian 2-cell
$\begin{tikzcd} A \arrow[r, "q(g,f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\substack{\overline{q}(g,f) \\ \text{cart}}"] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "\shortmid"{marking}] & D \end{tikzcd}$,

∃! 2-cell $\hat{\gamma}$ in $\mathbb{X}$ of the form

$$\begin{tikzcd}
A \arrow[r, "B(1,f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrrd, phantom, "\hat{\gamma}"] & B \arrow[r, "q", "\shortmid"{marking}] & D \arrow[r, "D(g,1)", "\shortmid"{marking}] & C \arrow[d, equal] \\
A \arrow[rrr, "q(g,f)"', "\shortmid"{marking}] & & & C
\end{tikzcd}$$

satisfying $\gamma = \bar{q}(g,f) \circ \hat{\gamma}$, i.e.,

$$\gamma = \begin{tikzcd}
A \arrow[r, "B(1,f)", "\shortmid"{marking}] \arrow[d, equal] \arrow[rrrd, phantom, "\hat{\gamma}"] & B \arrow[r, "q", "\shortmid"{marking}] & D \arrow[r, "D(g,1)", "\shortmid"{marking}] & C \arrow[d, equal] \\
A \arrow[rrr, "q(g,f)"', "\shortmid"{marking}] \arrow[d, "f"'] \arrow[rrrd, phantom, "\substack{\bar{q}(g,f) \\ \mathrm{cart}}"] & & & C \arrow[d, "g"] \\
B \arrow[rrr, "q"', "\shortmid"{marking}] & & & D.
\end{tikzcd}$$

We claim that $\hat{\gamma}$ is opcartesian.

To prove this, let α be any 2-cell in $\mathbb{X}$ of shape

$$\begin{tikzcd}
X_0 \arrow[r, "r_1", "\shortmid"{marking}] \arrow[d, "h"'] & X_1 \cdots X_{k-1} \arrow[r, "r_k", "\shortmid"{marking}] & X_k = A \arrow[r, "B(1,f)", "\shortmid"{marking}] & B \arrow[r, "q", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & D \arrow[r, "D(g,1)", "\shortmid"{marking}] & C = Y_0 \arrow[r, "s_1", "\shortmid"{marking}] & Y_1 \cdots Y_{\ell-1} \arrow[r, "s_\ell", "\shortmid"{marking}] & Y_\ell \arrow[d, "k"] \\
Z_0 \arrow[rrrrrrr, "t"', "\shortmid"{marking}] & & & & & & & Z_1.
\end{tikzcd}$$

We have to show that ∃! 2-cell $\check{\alpha}$ in $\mathbb{X}$ of the form

$$\begin{tikzcd}
X_0 \arrow[r, "r_1", "\shortmid"{marking}] \arrow[d, "h"'] & X_1 \cdots X_{k-1} \arrow[r, "r_k", "\shortmid"{marking}] & X_k = A \arrow[r, "q(g,f)", "\shortmid"{marking}] \arrow[d, phantom, "\check{\alpha}"] & C = Y_0 \arrow[r, "s_1", "\shortmid"{marking}] & Y_1 \cdots Y_{\ell-1} \arrow[r, "s_\ell", "\shortmid"{marking}] & Y_\ell \arrow[d, "k"] \\
Z_0 \arrow[rrrrr, "t"', "\shortmid"{marking}] & & & & & Z_1
\end{tikzcd}$$

s.t. $\alpha = \check{\alpha} \circ (1_{r_1}, \ldots, 1_{r_k}, \hat{\gamma}, 1_{s_1}, \ldots, 1_{s_\ell})$,

i.e.,

$$\alpha = \begin{tikzcd}
X_0 \arrow[r, "r_1", "\shortmid"{marking}] \arrow[d, equal] & X_1 \cdots X_{k-1} \arrow[r, "r_k", "\shortmid"{marking}] \arrow[d, phantom, "="] & X_k = A \arrow[r, "B(1,f)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "q", "\shortmid"{marking}] \arrow[d, phantom, "\hat{\gamma}"] & D \arrow[r, "D(g,1)", "\shortmid"{marking}] & C = Y_0 \arrow[r, "s_1", "\shortmid"{marking}] \arrow[d, equal] & Y_1 \cdots Y_{\ell-1} \arrow[r, "s_\ell", "\shortmid"{marking}] \arrow[d, phantom, "="] & Y_\ell \arrow[d, equal] \\
X_0 \arrow[r, "r_1"', "\shortmid"{marking}] \arrow[d, "h"'] & X_1 \cdots X_{k-1} \arrow[r, "r_k"', "\shortmid"{marking}] & X_k = A \arrow[rrr, "q(g,f)"', "\shortmid"{marking}] & \arrow[d, phantom, "\check{\alpha}"] & & C = Y_0 \arrow[r, "s_1"', "\shortmid"{marking}] & Y_1 \cdots Y_{\ell-1} \arrow[r, "s_\ell"', "\shortmid"{marking}] & Y_\ell \arrow[d, "k"] \\
Z_0 \arrow[rrrrrrr, "t"', "\shortmid"{marking}] & & & & & & & Z_1.
\end{tikzcd}$$

Consider the 2-cells

$$\begin{tikzcd}
A \arrow[r, "e_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[rd, phantom, "\nu(1,f)"] & A \arrow[d, "f"] \\
A \arrow[r, "B(1,f)"', "\shortmid"{marking}] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C \arrow[r, "e_C", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[rd, phantom, "\nu(g,1)"] & C \arrow[d, equal] \\
D \arrow[r, "D(g,1)"', "\shortmid"{marking}] & C.
\end{tikzcd}$$

We form the following composite 2-cell $\check{\alpha}$ in $\mathbb{X}$:

$$\check{\alpha} := \begin{tikzcd}
X_0 \arrow[r, "r_1"] \arrow[dd, equal] \arrow[ddrr, phantom, "="] & X_1 \cdots X_{k-1} \arrow[r, "r_k"] & X_k = A \arrow[r, phantom, "="] \arrow[d, equal] \arrow[dr, phantom, "e_A"] & A \arrow[r, "q(g,f)"] \arrow[d, equal] \arrow[ddr, phantom, "\bar{q}(g,f)"] & C \arrow[r, phantom, "="] \arrow[d, equal] \arrow[dr, phantom, "e_C"] & C = Y_0 \arrow[r, "s_1"] \arrow[d, equal] \arrow[ddrr, phantom, "="] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"] & Y_l \arrow[dd, equal] \\
 & & A \arrow[r, "e_A"'] \arrow[d, equal] \arrow[dr, phantom, "\nu(1,f)"] & A \arrow[d, "f"] & C \arrow[r, "e_C"'] \arrow[d, "g"'] \arrow[dr, phantom, "\nu(g,1)"] & C \arrow[d, equal] & & \\
X_0 \arrow[r, "r_1"'] \arrow[d, "h"'] \arrow[drrrrrrr, phantom, "\alpha"] & X_1 \cdots X_{k-1} \arrow[r, "r_k"'] & X_k = A \arrow[r, "B(1,f)"'] & B \arrow[r, "q"'] & D \arrow[r, "D(g,1)"'] & C = Y_0 \arrow[r, "s_1"'] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"'] & Y_l \arrow[d, "k"] \\
Z_0 \arrow[rrrrrrr, "t"'] & & & & & & & Z_1.
\end{tikzcd}$$

We want to show that α is equal to the composite 2-cell $\check{\alpha} \mathbin{\underset{\vee}{\circ}} (1_{r_1}, \ldots, 1_{r_k}, \hat{\gamma}, 1_{s_1}, \ldots, 1_{s_l})$.

We compute:

$\check{\alpha} \mathbin{\underset{\vee}{\circ}} (1_{r_1}, \ldots, 1_{r_k}, \hat{\gamma}, 1_{s_1}, \ldots, 1_{s_l})$

$$= \begin{tikzcd}
X_0 \arrow[r, "r_1"] \arrow[dddd, equal] & X_1 \cdots X_{k-1} \arrow[r, "r_k"] \arrow[dddd, phantom, "="] & X_k \arrow[r, phantom, "="] & A \arrow[r, "B(1,f)"] \arrow[d, equal] \arrow[drrr, phantom, "\hat{\gamma}"] & B \arrow[r, "q"] & D \arrow[r, "D(g,1)"] & C \arrow[r, phantom, "="] \arrow[d, equal] & Y_0 \arrow[r, "s_1"] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"] \arrow[dddd, phantom, "="] & Y_l \arrow[dddd, equal] \\
 & & & A \arrow[rrr, "q(g,f)"'] \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "e_A"] & & & C \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "e_C"] & & & \\
 & & A \arrow[rr, "e_A"'] \arrow[dd, equal] \arrow[ddrr, phantom, "\nu(1,f)"] & & A \arrow[dd, "f"] \arrow[r, phantom, "\bar{q}(g,f)"] & D \arrow[rr, "e_C"'] \arrow[dd, "g"'] \arrow[ddrr, phantom, "\nu(g,1)"] & & C \arrow[dd, equal] & & \\
 & & & & & & & & & \\
X_0 \arrow[r, "r_1"'] \arrow[d, "h"'] \arrow[drrrrrrrrr, phantom, "\alpha"] & X_1 \cdots X_{k-1} \arrow[r, "r_k"'] & X_k = A \arrow[rr, "B(1,f)"'] & & B \arrow[r, "q"'] & D \arrow[rr, "D(g,1)"'] & & C = Y_0 \arrow[r, "s_1"'] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"'] & Y_l \arrow[d, "k"] \\
Z_0 \arrow[rrrrrrrrr, "t"'] & & & & & & & & & Z_1
\end{tikzcd}$$

$$\underset{\substack{\uparrow \\ \text{construction} \\ \text{of } \hat{\gamma}}}{=} \begin{tikzcd}
X_0 \arrow[r, "r_1"] \arrow[dddd, equal] & X_1 \cdots X_{k-1} \arrow[r, "r_k"] \arrow[dddd, phantom, "="] & X_k \arrow[rr, phantom, "="] & & A \arrow[r, "B(1,f)"] \arrow[dl, equal] \arrow[d, equal] & B \arrow[r, "q"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[r, "D(g,1)"] \arrow[d, equal] & C \arrow[r, phantom, "="] \arrow[d, equal] \arrow[dr, equal] & Y_0 \arrow[r, "s_1"] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"] \arrow[dddd, phantom, "="] & Y_l \arrow[dddd, equal] \\
 & & & A \arrow[r, "e_A"'] \arrow[d, equal] \arrow[dr, phantom, "\nu(1,f)"] \arrow[u, phantom, "e_A"] & A \arrow[r, "B(1,f)"'] \arrow[d, "f"] \arrow[dr, phantom, "\bar{B}(1,f)"] & B \arrow[r, "q"'] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[r, "D(g,1)"'] \arrow[d, equal] \arrow[dr, phantom, "\bar{D}(g,1)"] & C \arrow[r, "e_C"'] \arrow[d, "g"'] \arrow[dr, phantom, "\nu(g,1)"] \arrow[u, phantom, "e_C"] & C \arrow[d, equal] & & \\
 & & & A \arrow[r, "B(1,f)"'] \arrow[dd, equal] \arrow[ddr, phantom, "="] & B \arrow[r, "e_B"'] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, "q"'] \arrow[d, equal] \arrow[drr, phantom, "\lambda_q"] & D \arrow[r, "e_D"'] & D \arrow[r, "D(g,1)"'] \arrow[d, equal] & C \arrow[dd, equal] & & \\
 & & & & B \arrow[r, "e_B"'] \arrow[d, equal] \arrow[drrr, phantom, "\rho_q"] & B \arrow[rr, "q"'] & & D \arrow[d, equal] \arrow[r, phantom, "="] & & & \\
X_0 \arrow[r, "r_1"'] \arrow[d, "h"'] \arrow[drrrrrrrrrr, phantom, "\alpha"] & X_1 \cdots X_{k-1} \arrow[rr, "r_k"'] & & X_k = A \arrow[r, "B(1,f)"'] & B \arrow[rrr, "q"'] & & & D \arrow[r, "D(g,1)"'] & C = Y_0 \arrow[r, "s_1"'] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"'] & Y_l \arrow[d, "k"] \\
Z_0 \arrow[rrrrrrrrrr, "t"'] & & & & & & & & & & Z_1
\end{tikzcd}$$

$$(*)\;=\;\begin{tikzcd}
X_0 \arrow[r, "r_1"] \arrow[ddd, equal] & X_1 \cdots X_{k-1} \arrow[r, "r_k"] & X_k = A \arrow[rr, "B(1,f)"] \arrow[d, equal] \arrow[dr, phantom, "e_A"] & & B \arrow[r, "q"] \arrow[d, equal] & D \arrow[rr, "D(g,1)"] \arrow[d, equal] & & C = Y_0 \arrow[r, "s_1"] \arrow[d, equal] \arrow[dl, phantom, "e_C"] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"] & Y_l \arrow[ddd, equal] \\
& & A \arrow[r, "e_A"'] \arrow[d, equal] \arrow[dr, phantom, "\nu(1,f)"] & A \arrow[r, "B(1,f)"'] \arrow[d, "f"] \arrow[dr, phantom, "\overline{B}(1,f)"] & B \arrow[r, "q"'] \arrow[d, equal] & D \arrow[r, "D(g,1)"'] \arrow[d, equal] \arrow[dr, phantom, "\overline{D}(g,1)"] & C \arrow[r, "e_C"'] \arrow[d, "g"] \arrow[dr, phantom, "\nu(g,1)"] & C \arrow[d, equal] & & \\
& = & A \arrow[r, "B(1,f)"'] \arrow[d, equal] \arrow[drr, phantom, "\lambda_{B(1,f)}"] & B \arrow[r, "e_B"'] & B \arrow[r, phantom, "="] \arrow[d, equal] & D \arrow[r, "e_D"'] \arrow[d, equal] \arrow[drr, phantom, "\rho_{D(g,1)}"] & D \arrow[r, "D(g,1)"'] & C \arrow[d, equal] & = & \\
X_0 \arrow[r, "r_1"'] \arrow[d, "h"'] & X_1 \cdots X_{k-1} \arrow[r, "r_k"'] & X_k = A \arrow[rr, "B(1,f)"'] & & B \arrow[r, "q"'] & D \arrow[rr, "D(g,1)"'] & & C = Y_0 \arrow[r, "s_1"'] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"'] & Y_l \arrow[d, "k"] \\
Z_0 \arrow[rrrrrrrrr, "t"'] \arrow[rrrrrrrrr, phantom, bend left=10, "\alpha"] & & & & & & & & & Z_1
\end{tikzcd}$$

$$=\;\begin{tikzcd}
X_0 \arrow[r, "r_1"] \arrow[dd, equal] & X_1 \cdots X_{k-1} \arrow[r, "r_k"] & X_k = A \arrow[rr, "B(1,f)"] \arrow[d, equal] \arrow[dr, phantom, "e_A"] & & B \arrow[r, "q"] \arrow[d, equal] & D \arrow[rr, "D(g,1)"] \arrow[d, equal] & & C = Y_0 \arrow[r, "s_1"] \arrow[d, equal] \arrow[dl, phantom, "e_C"] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"] & Y_l \arrow[dd, equal] \\
& = & A \arrow[r, "e_A"'] \arrow[d, equal] \arrow[drr, phantom, "\rho_{B(1,f)}"] & A \arrow[r, "B(1,f)"'] & B \arrow[r, "q"'] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[r, "D(g,1)"'] \arrow[d, equal] \arrow[drr, phantom, "\lambda_{D(g,1)}"] & C \arrow[r, "e_C"'] & C \arrow[d, equal] & = & \\
X_0 \arrow[r, "r_1"'] \arrow[d, "h"'] & X_1 \cdots X_{k-1} \arrow[r, "r_k"'] & X_k = A \arrow[rr, "B(1,f)"'] & & B \arrow[r, "q"'] & D \arrow[rr, "D(g,1)"'] & & C = Y_0 \arrow[r, "s_1"'] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"'] & Y_l \arrow[d, "k"] \\
Z_0 \arrow[rrrrrrrrr, "t"'] \arrow[rrrrrrrrr, phantom, bend left=10, "\alpha"] & & & & & & & & & Z_1
\end{tikzcd}$$

by an earlier remark concerning $\nu(1,f)$ and $\nu(g,1)$

$$=\;\begin{tikzcd}
X_0 \arrow[r, "r_1"] \arrow[d, equal] & X_1 \cdots X_{k-1} \arrow[r, "r_k"] & X_k = A \arrow[r, "B(1,f)"] \arrow[d, equal] & B \arrow[r, "q"] \arrow[d, equal] & D \arrow[r, "D(g,1)"] \arrow[d, equal] & C = Y_0 \arrow[r, "s_1"] \arrow[d, equal] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"] & Y_l \arrow[d, equal] \\
X_0 \arrow[r, "r_1"'] \arrow[d, "h"'] & X_1 \cdots X_{k-1} \arrow[r, "r_k"'] & X_k = A \arrow[r, "B(1,f)"'] & B \arrow[r, "q"'] & D \arrow[r, "D(g,1)"'] & C = Y_0 \arrow[r, "s_1"'] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"'] & Y_l \arrow[d, "k"] \\
Z_0 \arrow[rrrrrrr, "t"'] \arrow[rrrrrrr, phantom, bend left=10, "\alpha"] & & & & & & & Z_1
\end{tikzcd}$$

constructions of $\rho_{B(1,f)}$ and $\lambda_{D(g,1)}$

$$=\;\begin{tikzcd}
X_0 \arrow[r, "r_1"] \arrow[d, "h"'] & X_1 \cdots X_{k-1} \arrow[r, "r_k"] & X_k = A \arrow[r, "B(1,f)"] & B \arrow[r, "q"] & D \arrow[r, "D(g,1)"] & C = Y_0 \arrow[r, "s_1"] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"] & Y_l \arrow[d, "k"] \\
Z_0 \arrow[rrrrrrr, "t"'] \arrow[rrrrrrr, phantom, bend left=10, "\alpha"] & & & & & & & Z_1
\end{tikzcd}$$

$$= \alpha_1$$

So $\check{\alpha} \circ (1_{r_1}, \dots, 1_{r_k}, \hat{\gamma}, 1_{s_1}, \dots, 1_{s_l}) = \alpha.$

Proof of (*):

It suffices to show that the following two composite 2-cells in $\mathbb{X}$ coincide:

$$\begin{tikzcd}
X_0 \arrow[r, "r_1"] \arrow[dd, equal] & X_1 \cdots X_{k-1} \arrow[r, "r_k"] \arrow[dd, phantom, "="] & X_k = A \arrow[r, "B(1,f)"] \arrow[dd, equal] & B \arrow[r, "e_B"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, "q"] \arrow[d, equal] & D \arrow[r, "e_D"] \arrow[d, phantom, "\lambda_q"] & D \arrow[r, "D(g,1)"] \arrow[d, equal] \arrow[dr, phantom, "="] & C = Y_0 \arrow[r, "s_1"] \arrow[dd, equal] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"] \arrow[dd, phantom, "="] & Y_l \arrow[dd, equal] \\
 & & \arrow[r, phantom, "="] & B \arrow[r, "e_B"'] \arrow[d, equal] & B \arrow[rr, "q"'] \arrow[d, phantom, "\rho_q"] & & D \arrow[d, equal] & & & \\
X_0 \arrow[r, "r_1"'] \arrow[d, "h"'] & X_1 \cdots X_{k-1} \arrow[r, "r_k"'] & X_k = A \arrow[r, "B(1,f)"'] & B \arrow[rrr, "q"'] & \arrow[d, phantom, "\alpha"] & & D \arrow[r, "D(g,1)"'] & C = Y_0 \arrow[r, "s_1"'] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"'] & Y_l \arrow[d, "k"] \\
Z_0 \arrow[rrrrrrrrr, "t"'] & & & & & & & & & Z_1
\end{tikzcd}$$

and

$$\begin{tikzcd}
X_0 \arrow[r, "r_1"] \arrow[d, equal] & X_1 \cdots X_{k-1} \arrow[r, "r_k"] \arrow[d, phantom, "="] & X_k = A \arrow[r, "B(1,f)"] \arrow[d, equal] & B \arrow[r, "e_B"] \arrow[d, phantom, "\lambda_{B(1,f)}"] & B \arrow[r, "q"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[r, "e_D"] \arrow[d, equal] & D \arrow[r, "D(g,1)"] \arrow[d, phantom, "\rho_{D(g,1)}"] & C = Y_0 \arrow[r, "s_1"] \arrow[d, equal] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"] \arrow[d, phantom, "="] & Y_l \arrow[d, equal] \\
X_0 \arrow[r, "r_1"'] \arrow[d, "h"'] & X_1 \cdots X_{k-1} \arrow[r, "r_k"'] & X_k = A \arrow[rr, "B(1,f)"'] & & B \arrow[r, "q"'] \arrow[d, phantom, "\alpha"] & D \arrow[rr, "D(g,1)"'] & & C = Y_0 \arrow[r, "s_1"'] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"'] & Y_l \arrow[d, "k"] \\
Z_0 \arrow[rrrrrrrrr, "t"'] & & & & & & & & & Z_1 .
\end{tikzcd}$$

(Remark:

We will not directly make use of the 2-cell α. Yet, we include it to ensure that we have well-defined 2-cells in $\mathbb{X}$ as the loose target of a 2-cell has to be unary.)

By the universal properties of the opcartesian 2-cells
$\begin{tikzcd} & B \arrow[dr, equal] & \\ B \arrow[ur, equal] \arrow[rr, "e_B"'] & \arrow[u, phantom, "e_B \ \mathrm{opcart}"] & B \end{tikzcd}$
and
$\begin{tikzcd} & D \arrow[dr, equal] & \\ D \arrow[ur, equal] \arrow[rr, "e_B"'] & \arrow[u, phantom, "e_D \ \mathrm{opcart}"] & D \end{tikzcd}$,

it is sufficient to prove equality of the two composite 2-cells below:

$$\xi := \begin{tikzcd}
X_0 \arrow[r, "r_1"] \arrow[dddd, equal] & X_1 \cdots X_{k-1} \arrow[r, "r_k"] \arrow[dddd, phantom, "="] & X_k = A \arrow[rr, "B(1,f)"] \arrow[d, equal] & \arrow[d, phantom, "="] & B \arrow[r, "q"] & D \arrow[rr, "D(g,1)"] \arrow[d, equal] \arrow[dr, equal] \arrow[d, phantom, "e_D"] & \arrow[d, phantom, "="] & C = Y_0 \arrow[r, "s_1"] \arrow[d, equal] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"] \arrow[dddd, phantom, "="] & Y_l \arrow[dddd, equal] \\
 & & A \arrow[rr, "B(1,f)"] \arrow[d, equal] & \arrow[d, phantom, "="] & B \arrow[r, "q"] \arrow[dl, equal] \arrow[d, equal] \arrow[d, phantom, "e_B"] & D \arrow[r, "e_D"] \arrow[d, phantom, "="] & D \arrow[r, "D(g,1)"] & C \arrow[d, equal] & & \\
 & & A \arrow[r, "B(1,f)"'] \arrow[dd, equal] & B \arrow[r, "e_B"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, "q"] \arrow[d, equal] & D \arrow[r, "e_D"] \arrow[d, phantom, "\lambda_q"] & D \arrow[r, "D(g,1)"'] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[dd, equal] & & \\
 & & \arrow[r, phantom, "="] & B \arrow[r, "e_B"'] \arrow[d, equal] & B \arrow[rr, "q"'] \arrow[d, phantom, "\rho_q"] & & D \arrow[d, equal] & & & \\
X_0 \arrow[r, "r_1"'] \arrow[d, "h"'] & X_1 \cdots X_{k-1} \arrow[r, "r_k"'] & X_k = A \arrow[r, "B(1,f)"'] & B \arrow[rrr, "q"'] & \arrow[d, phantom, "\alpha"] & & D \arrow[r, "D(g,1)"'] & C = Y_0 \arrow[r, "s_1"'] & Y_1 \cdots Y_{l-1} \arrow[r, "s_l"'] & Y_l \arrow[d, "k"] \\
Z_0 \arrow[rrrrrrrrr, "t"'] & & & & & & & & & Z_1
\end{tikzcd}$$

and

$$\xi := \begin{tikzcd}[column sep=small]
X_0 \arrow[r,"r_1","|"] \arrow[ddd, equal] & X_1 \cdots X_{k-1} \arrow[r,"r_k","|"] & X_k = A \arrow[rr,"{B(1,f)}","|"] \arrow[d, equal] \arrow[dddl, phantom, "="] & & B \arrow[rr,"q","|"] \arrow[d, equal] & & D \arrow[rr,"{D(g,1)}","|"] \arrow[d, phantom, "e_D"] & & C = Y_0 \arrow[r,"s_1","|"] \arrow[d,equal] \arrow[dddr, phantom, "="] & Y_1 \cdots Y_{l-1} \arrow[r,"s_l","|"] & Y_l \arrow[ddd, equal] \\
& & A \arrow[rr,"{B(1,f)}"',"|"] \arrow[d,equal] \arrow[dr, phantom, "="] & & B \arrow[rr,"q"',"|"] \arrow[d, phantom, "e_B"] & & D \arrow[r,"e_D"',"|"] \arrow[d, phantom, "="] & D \arrow[r,"{D(g,1)}"',"|"] & C \arrow[d,equal] & & \\
& & A \arrow[r,"{B(1,f)}"',"|"] \arrow[d,equal] \arrow[drr, phantom, "\lambda_{B(1,f)}"] & B \arrow[r,"e_B"',"|"] & B \arrow[rr,"q"',"|"] \arrow[d,equal] \arrow[drr, phantom, "="] & & D \arrow[r,"e_D"',"|"] \arrow[d,equal] \arrow[drr, phantom, "\rho_{D(g,1)}"] & D \arrow[r,"{D(g,1)}"',"|"] & C \arrow[d,equal] & & \\
X_0 \arrow[r,"r_1"',"|"] \arrow[d,"h"'] & X_1 \cdots X_{k-1} \arrow[r,"r_k"',"|"] & X_k = A \arrow[rr,"{B(1,f)}"',"|"] & & B \arrow[rr,"q"',"|"] \arrow[d, phantom, "\alpha"] & & D \arrow[rr,"{D(g,1)}"',"|"] & & C = Y_0 \arrow[r,"s_1"',"|"] & Y_1 \cdots Y_{l-1} \arrow[r,"s_l"',"|"] & Y_l \arrow[d,"k"] \\
Z_0 \arrow[rrrrrrrrrr,"t"',"|"] & & & & & & & & & & Z_1 .
\end{tikzcd}$$

We compute:

$$\xi = \begin{tikzcd}[column sep=small]
X_0 \arrow[r,"r_1","|"] \arrow[dddd, equal] & X_1 \cdots X_{k-1} \arrow[r,"r_k","|"] & X_k = A \arrow[rr,"{B(1,f)}","|"] \arrow[dddd, phantom, "="] & & B \arrow[rr,"q","|"] \arrow[d, equal] \arrow[dr, phantom, "="] & & D \arrow[rr,"{D(g,1)}","|"] \arrow[d, phantom, "e_D"] & & C = Y_0 \arrow[r,"s_1","|"] \arrow[dddd, phantom, "="] & Y_1 \cdots Y_{l-1} \arrow[r,"s_l","|"] & Y_l \arrow[dddd, equal] \\
& & & & B \arrow[r,"q"',"|"] \arrow[d, equal] \arrow[drr, phantom, "\lambda_q"] & D \arrow[r,"e_D"',"|"] & D \arrow[d,equal] & & & & \\
& & & & B \arrow[rr,"q","|"] \arrow[d, phantom, "e_B"] \arrow[drr, phantom, "="] & & D \arrow[d,equal] & & & & \\
& & & B \arrow[r,"e_B"',"|"] \arrow[d,equal] \arrow[drrr, phantom, "\rho_q"] & B \arrow[rr,"q"',"|"] & & D \arrow[d,equal] & & & & \\
X_0 \arrow[r,"r_1"',"|"] \arrow[d,"h"'] & X_1 \cdots X_{k-1} \arrow[r,"r_k"',"|"] & X_k = A \arrow[r,"{B(1,f)}"',"|"] & B \arrow[rrr,"q"',"|"] \arrow[d, phantom, "\alpha"] & & & D \arrow[rr,"{D(g,1)}"',"|"] & & C = Y_0 \arrow[r,"s_1"',"|"] & Y_1 \cdots Y_{l-1} \arrow[r,"s_l"',"|"] & Y_l \arrow[d,"k"] \\
Z_0 \arrow[rrrrrrrrrr,"t"',"|"] & & & & & & & & & & Z_1
\end{tikzcd}$$

$$= \begin{tikzcd}[column sep=small]
X_0 \arrow[r,"r_1","|"] \arrow[dd, equal] & X_1 \cdots X_{k-1} \arrow[r,"r_k","|"] & X_k = A \arrow[r,"{B(1,f)}","|"] \arrow[dd, phantom, "="] & B \arrow[r,"q","|"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[r,"{D(g,1)}","|"] \arrow[d, equal] & C = Y_0 \arrow[r,"s_1","|"] \arrow[dd, phantom, "="] & Y_1 \cdots Y_{l-1} \arrow[r,"s_l","|"] & Y_l \arrow[dd, equal] \\
& & & B \arrow[r,"q"',"|"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[d, equal] & & & \\
X_0 \arrow[r,"r_1"',"|"] \arrow[d,"h"'] & X_1 \cdots X_{k-1} \arrow[r,"r_k"',"|"] & X_k = A \arrow[r,"{B(1,f)}"',"|"] & B \arrow[r,"q"',"|"] \arrow[d, phantom, "\alpha"] & D \arrow[r,"{D(g,1)}"',"|"] & C = Y_0 \arrow[r,"s_1"',"|"] & Y_1 \cdots Y_{l-1} \arrow[r,"s_l"',"|"] & Y_l \arrow[d,"k"] \\
Z_0 \arrow[rrrrrrr,"t"',"|"] & & & & & & & Z_1
\end{tikzcd}$$

constructions of λ_q and ρ_q

$$= \alpha$$

and

$$
\xi = \begin{array}{c}
X_0 \xrightarrow{r_1} X_1 \cdots X_{k-1} \xrightarrow{r_k} X_k = A \xrightarrow{B(1,f)} B \xrightarrow{q} D \xrightarrow{D(g,1)} C = Y_0 \xrightarrow{s_1} Y_1 \cdots Y_{l-1} \xrightarrow{s_l} Y_l \\
\| \qquad \qquad \| \qquad = \qquad /\!/ \; e_D \; \backslash\!\backslash \qquad = \qquad \| \qquad \qquad \| \\
\| \qquad = \qquad A \xrightarrow{B(1,f)} B \xrightarrow{q} D \xrightarrow{e_D} D \xrightarrow{D(g,1)} C \qquad = \qquad \| \\
\| \qquad \qquad \| \quad = \quad /\!/ \; e_B \; \backslash\!\backslash \qquad \| \qquad \rho_{D(g,1)} \qquad \| \qquad \qquad \| \\
\| \qquad \qquad A \xrightarrow{B(1,f)} B \xrightarrow{e_B} B \;=\; D \xrightarrow{D(g,1)} C \qquad \qquad \| \\
\| \qquad \qquad \| \qquad \lambda_{B(1,f)} \qquad \| \qquad \| \qquad = \qquad \| \qquad \qquad \| \\
X_0 \xrightarrow[r_1]{} X_1 \cdots X_{k-1} \xrightarrow[r_k]{} X_k = A \xrightarrow[B(1,f)]{} B \xrightarrow[q]{} D \xrightarrow[D(g,1)]{} C = Y_0 \xrightarrow[s_1]{} Y_1 \cdots Y_{l-1} \xrightarrow[s_l]{} Y_l \\
h \downarrow \qquad \qquad \qquad \qquad \alpha \qquad \qquad \qquad \qquad \downarrow k \\
Z_0 \xrightarrow[t]{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} Z_1
\end{array}
$$

$$
= \begin{array}{c}
X_0 \xrightarrow{r_1} X_1 \cdots X_{k-1} \xrightarrow{r_k} X_k = A \xrightarrow{B(1,f)} B \xrightarrow{q} D \xrightarrow{D(g,1)} C = Y_0 \xrightarrow{s_1} Y_1 \cdots Y_{l-1} \xrightarrow{s_l} Y_l \\
\| \qquad \qquad \| \qquad = \qquad \| \qquad = \qquad \| \qquad \qquad \| \\
\| \qquad = \qquad A \xrightarrow[B(1,f)]{} B \xrightarrow[q]{} D \xrightarrow[D(g,1)]{} C \qquad = \qquad \| \\
\| \qquad \qquad \| \qquad = \qquad \| \qquad = \qquad \| \qquad \qquad \| \\
X_0 \xrightarrow[r_1]{} X_1 \cdots X_{k-1} \xrightarrow[r_k]{} X_k = A \xrightarrow[B(1,f)]{} B \xrightarrow[q]{} D \xrightarrow[D(g,1)]{} C = Y_0 \xrightarrow[s_1]{} Y_1 \cdots Y_{l-1} \xrightarrow[s_l]{} Y_l \\
h \downarrow \qquad \qquad \qquad \qquad \alpha \qquad \qquad \qquad \qquad \downarrow k \\
Z_0 \xrightarrow[t]{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} Z_1
\end{array}
$$

constructions of $\rho_{D(g,1)}$ and $\lambda_{B(1,f)}$

$$= \alpha .$$

Thus, we obtain $\xi = \alpha = \zeta$, which proves $(*)$.

Uniqueness of $\check{\alpha}$:

Let β be any 2-cell in $\mathbb{X}$ with the same frame as $\check{\alpha}$ satisfying

$\alpha = \beta \circ (1_{r_1}, \dots, 1_{r_k}, \hat{\gamma}, 1_{s_1}, \dots, 1_{s_l})$.

We have:

$$
\check{\alpha} = \begin{array}{c}
X_0 \xrightarrow{r_1} X_1 \cdots X_{k-1} \xrightarrow{r_k} X_k = A \;=\; A \xrightarrow{q(g,f)} C \;=\; C = Y_0 \xrightarrow{s_1} Y_1 \cdots Y_{l-1} \xrightarrow{s_l} Y_l \\
\| \qquad \qquad \| \quad e_A \quad \| \qquad \qquad \| \quad e_C \quad \| \qquad \qquad \| \\
\| \qquad = \qquad A \xrightarrow[e_A]{} A \qquad \bar{q}(g,f) \qquad C \xrightarrow[e_C]{} C \qquad = \qquad \| \\
\| \qquad \qquad \| \quad v(1,f) \quad \downarrow f \qquad \qquad g \downarrow \quad v(g,1) \quad \| \qquad \qquad \| \\
X_0 \xrightarrow[r_1]{} X_1 \cdots X_{k-1} \xrightarrow[r_k]{} X_k = A \xrightarrow[B(1,f)]{} B \xrightarrow[q]{} D \xrightarrow[D(g,1)]{} C = Y_0 \xrightarrow[s_1]{} Y_1 \cdots Y_{l-1} \xrightarrow[s_l]{} Y_l \\
h \downarrow \qquad \qquad \qquad \qquad \alpha \qquad \qquad \qquad \qquad \downarrow k \\
Z_0 \xrightarrow[t]{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} Z_1
\end{array}
$$

$$
\begin{tikzcd}
X_0 \arrow[r,"r_1"] \arrow[ddd,equal] & X_1 \cdots X_{k-1} \arrow[r,"r_k"] \arrow[ddd,phantom,"="] & X_k = A \arrow[r,phantom,"="] \arrow[d,equal] \arrow[dr,phantom,"e_A"] & A \arrow[r,"q(g,f)"] \arrow[d,equal] \arrow[dr,phantom,"\bar q(g,f)"] & C \arrow[r,phantom,"="] \arrow[d,equal] \arrow[dr,phantom,"e_C"] & C = Y_0 \arrow[r,"s_1"] \arrow[d,equal] & Y_1 \cdots Y_{l-1} \arrow[r,"s_l"] \arrow[ddd,phantom,"="] & Y_l \arrow[ddd,equal] \\
 & & A \arrow[r,"e_A"'] \arrow[d,equal] \arrow[dr,phantom,"v(1,f)"] & A \arrow[d,"f"] & C \arrow[r,"e_C"'] \arrow[d,"g"'] \arrow[dr,phantom,"v(g,1)"] & C \arrow[d,equal] & & \\
 & & A \arrow[r,"B(1,f)"'] \arrow[d,equal] & B \arrow[r,"g"'] \arrow[d,phantom,"\hat\gamma"] & D \arrow[r,"D(g,1)"'] & C \arrow[d,equal] & & \\
X_0 \arrow[r,"r_1"'] \arrow[d,"h"'] & X_1 \cdots X_{k-1} \arrow[r,"r_k"'] & X_k = A \arrow[rrr,"q(g,f)"'] & \arrow[d,phantom,"\beta"] & & C = Y_0 \arrow[r,"s_1"'] & Y_1 \cdots Y_{l-1} \arrow[r,"s_l"'] & Y_l \arrow[d,"k"] \\
Z_0 \arrow[rrrrrrr,"t"'] & & & & & & & Z_1
\end{tikzcd}
$$

$=$

$\alpha = \beta \circ (1_{r_1}, \dots, 1_{r_k}, \hat{\gamma}, 1_{s_1}, \dots, 1_{s_l})$,
by assumption on β

$$
\overset{(*)}{=}
\begin{tikzcd}
X_0 \arrow[r,"r_1"] \arrow[d,equal] & X_1 \cdots X_{k-1} \arrow[r,"r_k"] \arrow[d,phantom,"="] & X_k = A \arrow[r,phantom,"="] \arrow[d,equal] & A \arrow[r,"q(g,f)"] \arrow[d,phantom,"="] & C \arrow[r,phantom,"="] & C = Y_0 \arrow[r,"s_1"] \arrow[d,equal] & Y_1 \cdots Y_{l-1} \arrow[r,"s_l"] \arrow[d,phantom,"="] & Y_l \arrow[d,equal] \\
X_0 \arrow[r,"r_1"'] \arrow[d,"h"'] & X_1 \cdots X_{k-1} \arrow[r,"r_k"'] & X_k = A \arrow[rrr,"q(g,f)"'] & \arrow[d,phantom,"\beta"] & & C = Y_0 \arrow[r,"s_1"'] & Y_1 \cdots Y_{l-1} \arrow[r,"s_l"'] & Y_l \arrow[d,"k"] \\
Z_0 \arrow[rrrrrrr,"t"'] & & & & & & & Z_1
\end{tikzcd}
$$

$= \beta$

so $\check{\alpha} = \beta$.

Proof of $(*)$:

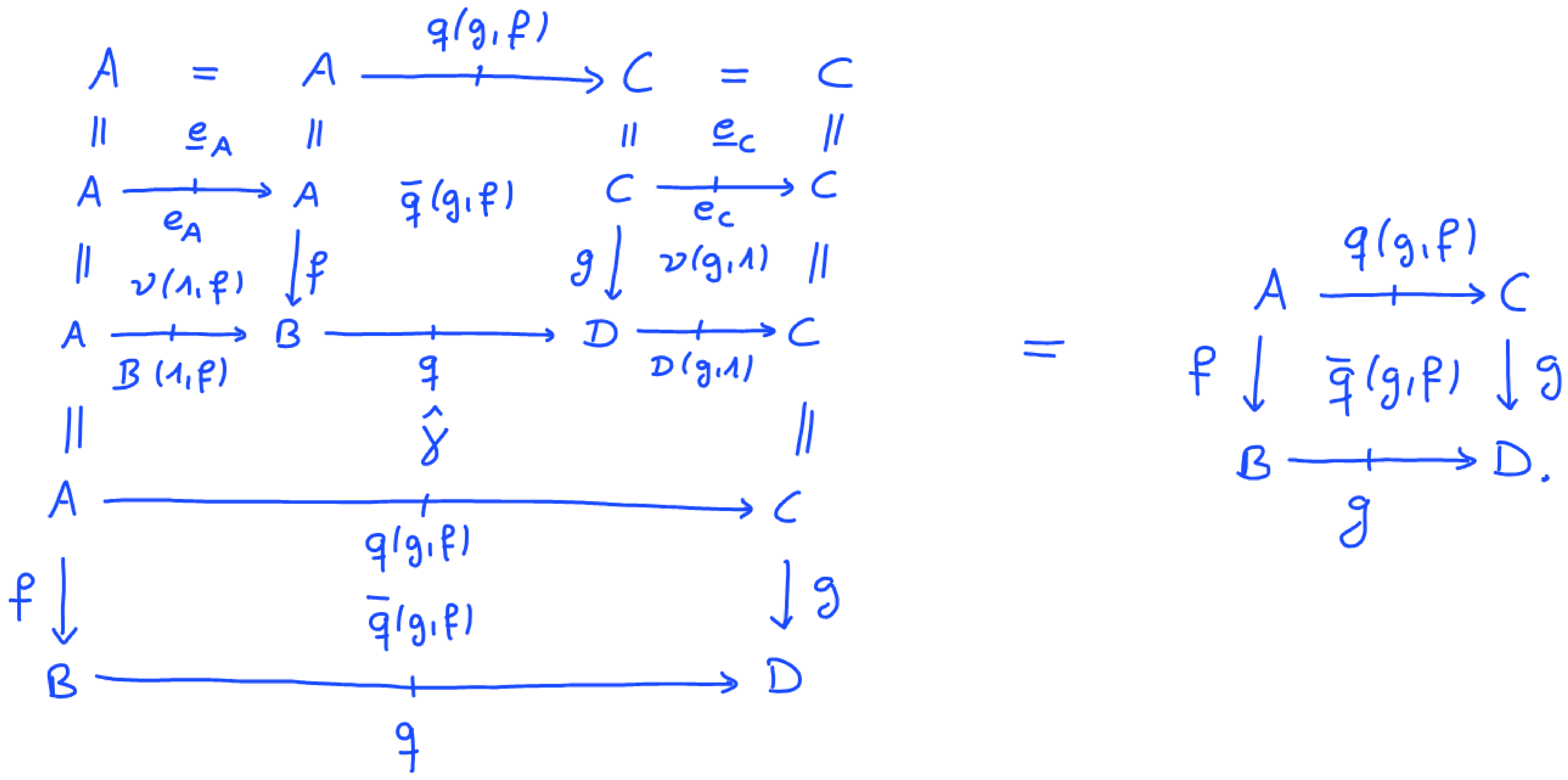
It suffices to show that the composite 2-cell

$$
\begin{tikzcd}
A \arrow[r,phantom,"="] \arrow[d,equal] \arrow[dr,phantom,"e_A"] & A \arrow[r,"q(g,f)"] \arrow[d,equal] \arrow[dr,phantom,"\bar q(g,f)"] & C \arrow[r,phantom,"="] \arrow[d,equal] \arrow[dr,phantom,"e_C"] & C \arrow[d,equal] \\
A \arrow[r,"e_A"'] \arrow[d,equal] \arrow[dr,phantom,"v(1,f)"] & A \arrow[d,"f"] & C \arrow[r,"e_C"'] \arrow[d,"g"'] \arrow[dr,phantom,"v(g,1)"] & C \arrow[d,equal] \\
A \arrow[r,"B(1,f)"'] \arrow[d,equal] & B \arrow[r,"g"'] \arrow[d,phantom,"\hat\gamma"] & D \arrow[r,"D(g,1)"'] & C \arrow[d,equal] \\
A \arrow[rrr,"q(g,f)"'] & & & C
\end{tikzcd}
$$

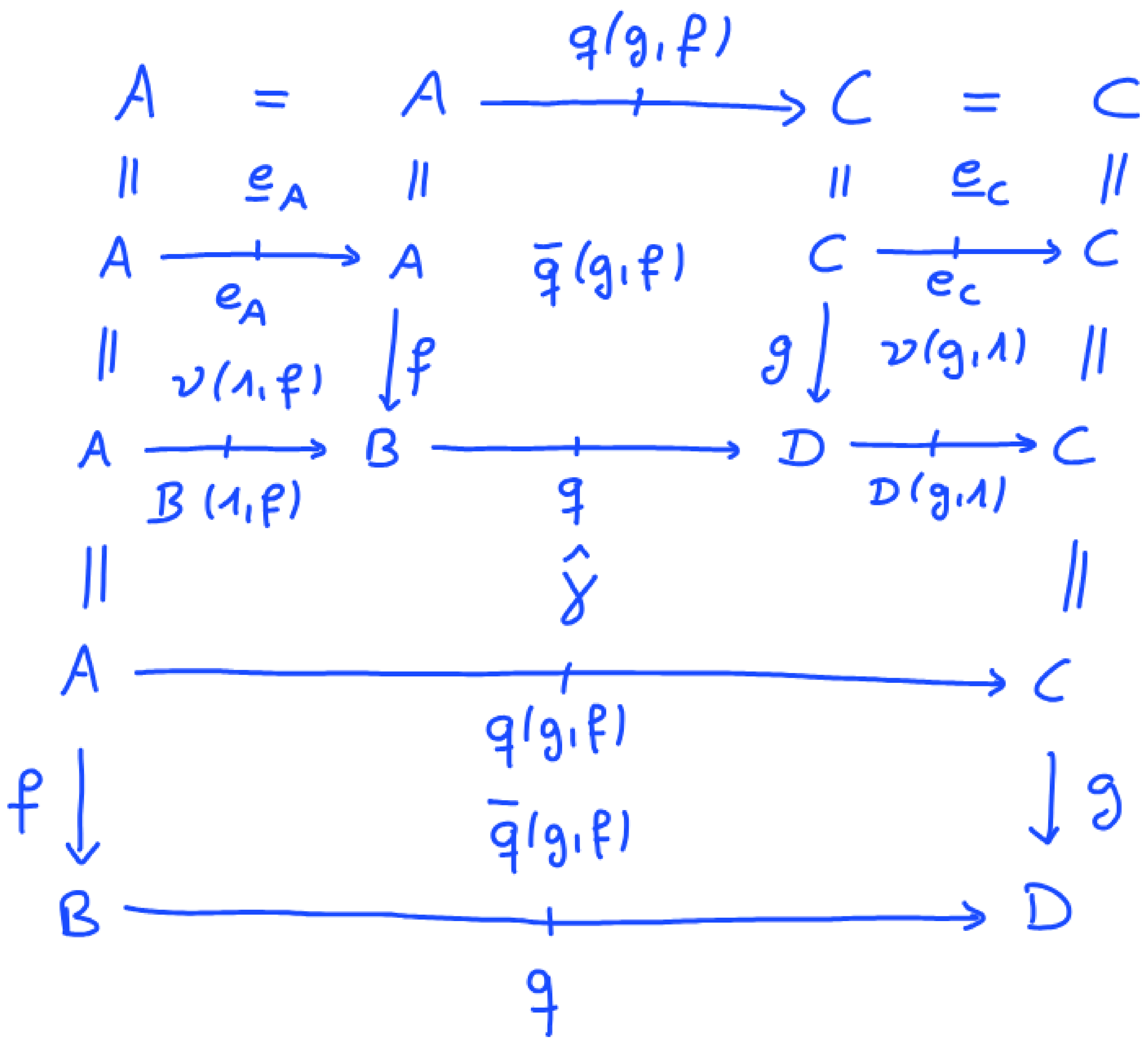
is equal to the identity 2-cell
$$
\begin{tikzcd}
A \arrow[r,"q(g,f)"] \arrow[d,equal] \arrow[dr,phantom,"1_{q(g,f)}"] & C \arrow[d,equal] \\
A \arrow[r,"q(g,f)"'] & C
\end{tikzcd}
$$
of $q(g,f) \colon A \to C$.

By the universal property of the cartesian 2-cell
$$
\begin{tikzcd}
A \arrow[r,"q(g,f)"] \arrow[d,"f"'] \arrow[dr,phantom,"\bar q(g,f)\ \text{cart}"] & C \arrow[d,"g"] \\
B \arrow[r,"g"'] & D
\end{tikzcd},
$$

it is sufficent to show that the post-compositions with $\bar{q}(g,f)$ agree, i.e., that

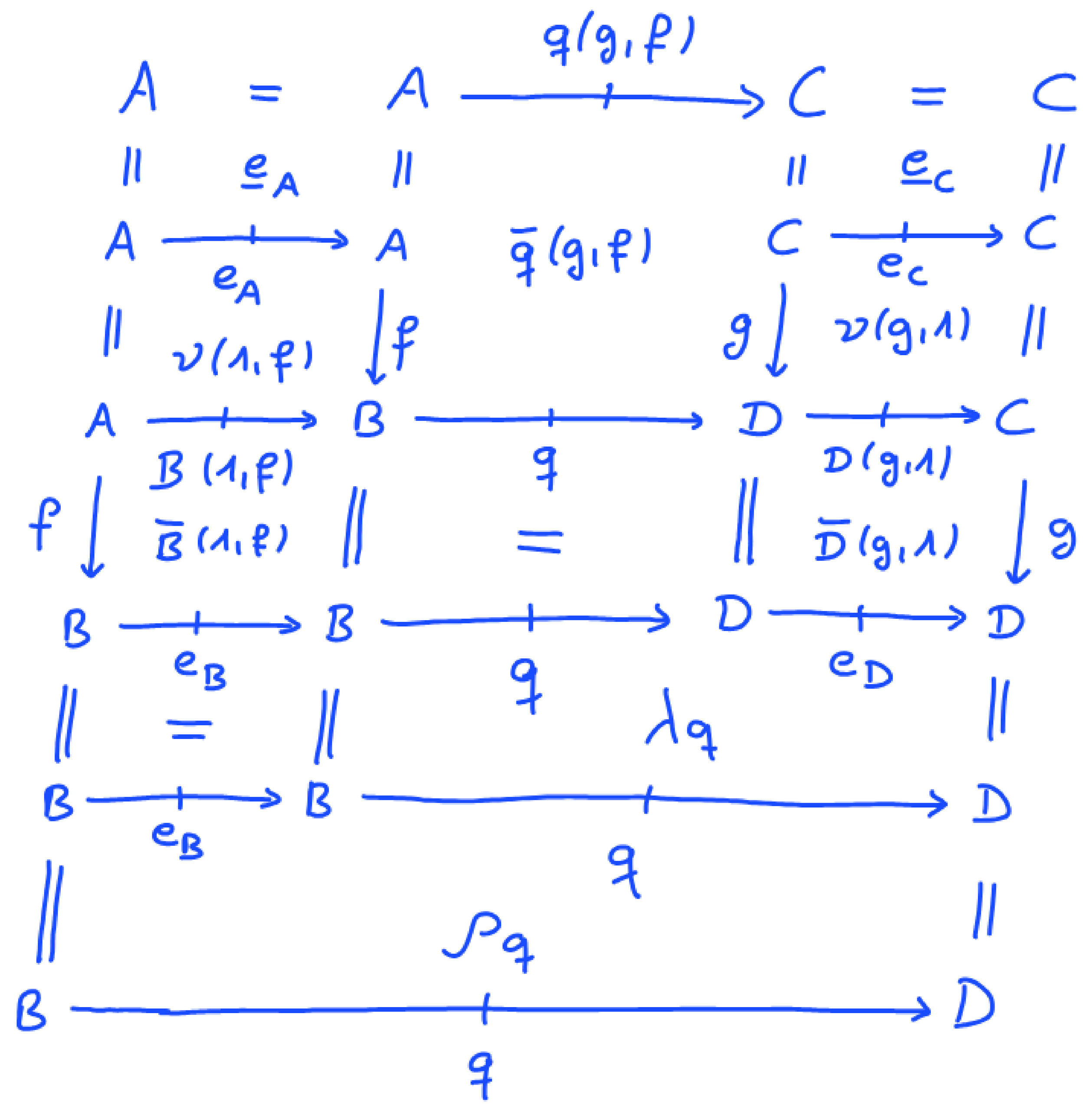


We compute:



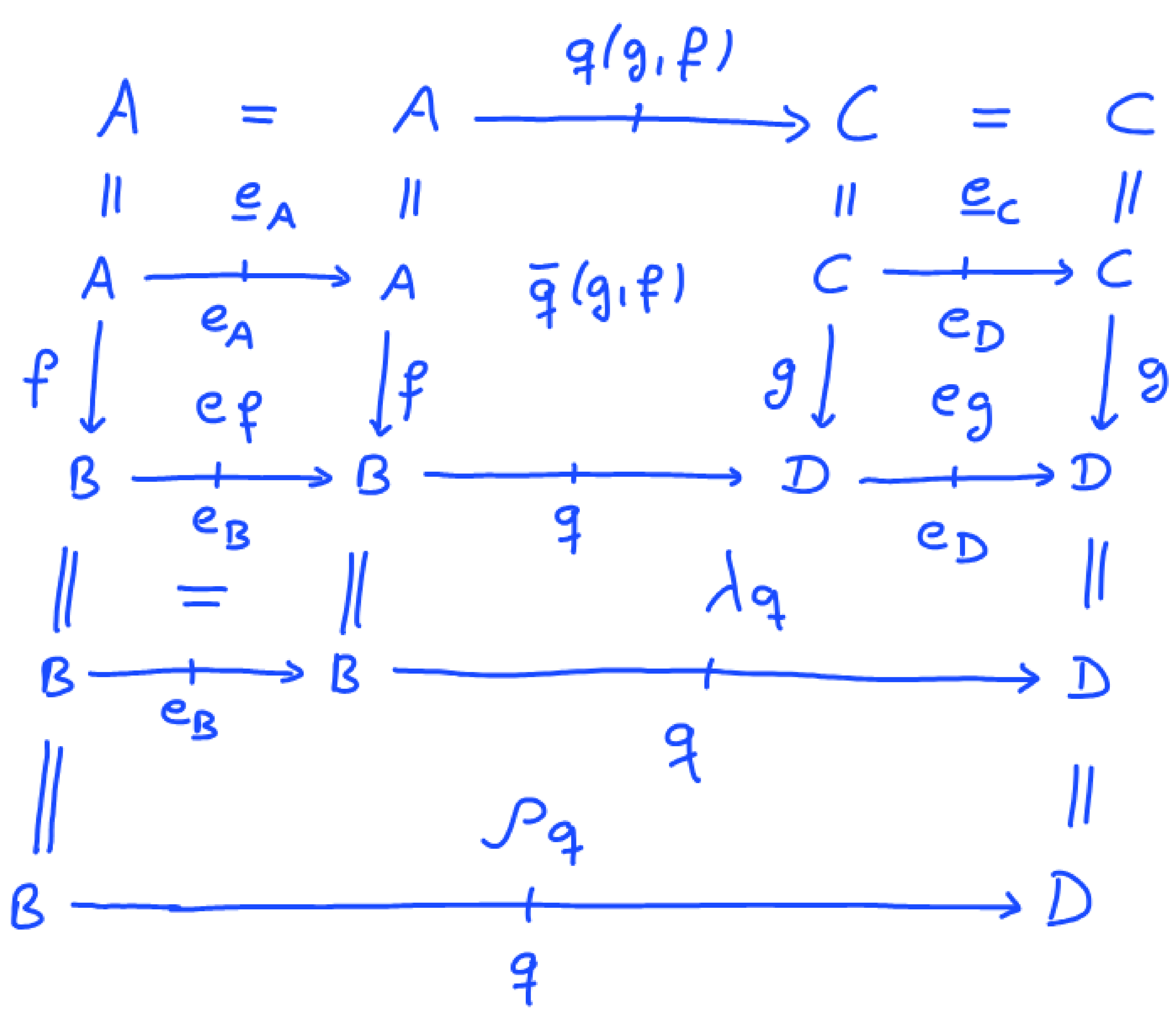
=

construction of $\hat{\gamma}$



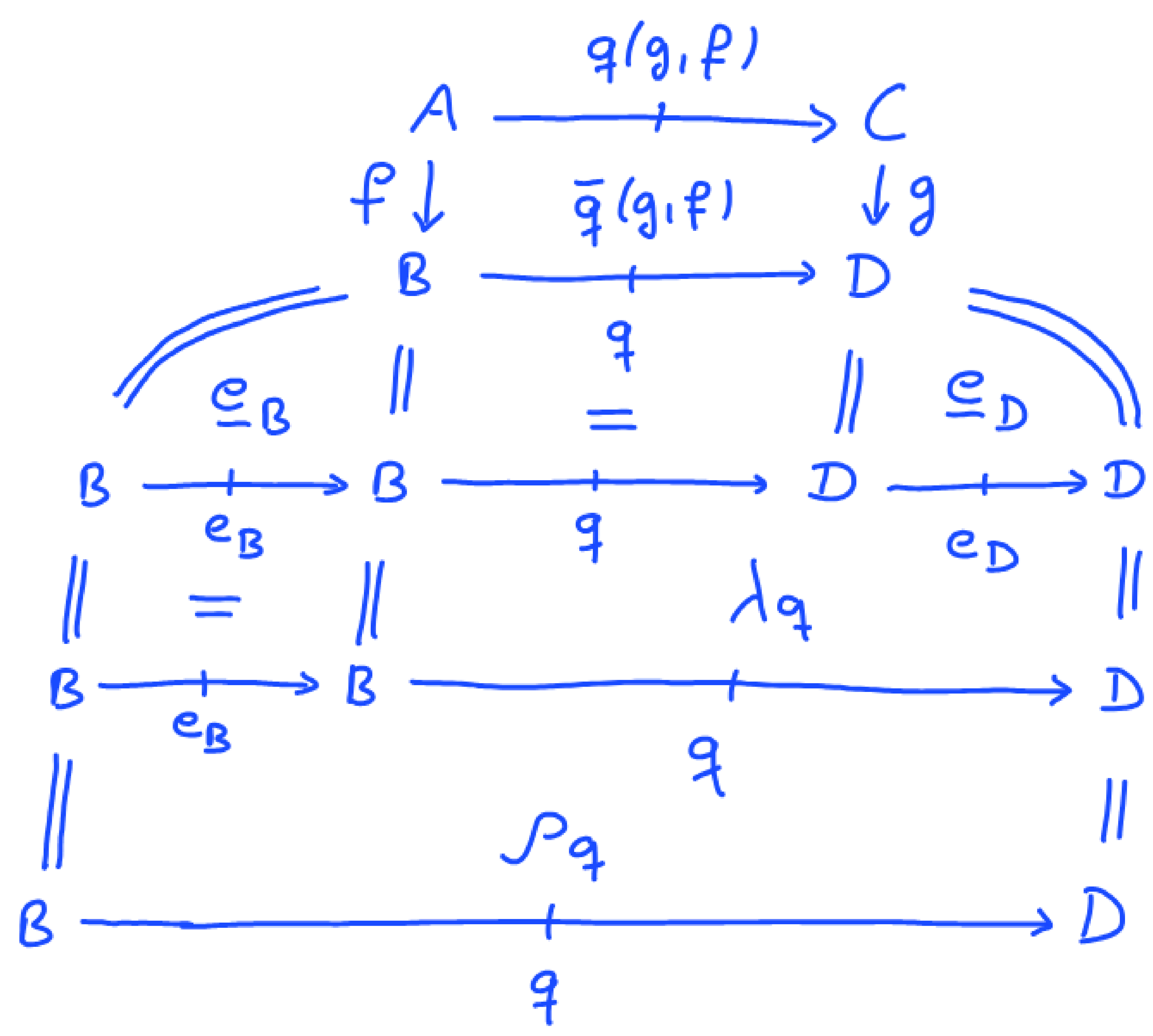
=

constructions of $v(1,f)$ and $v(g,1)$



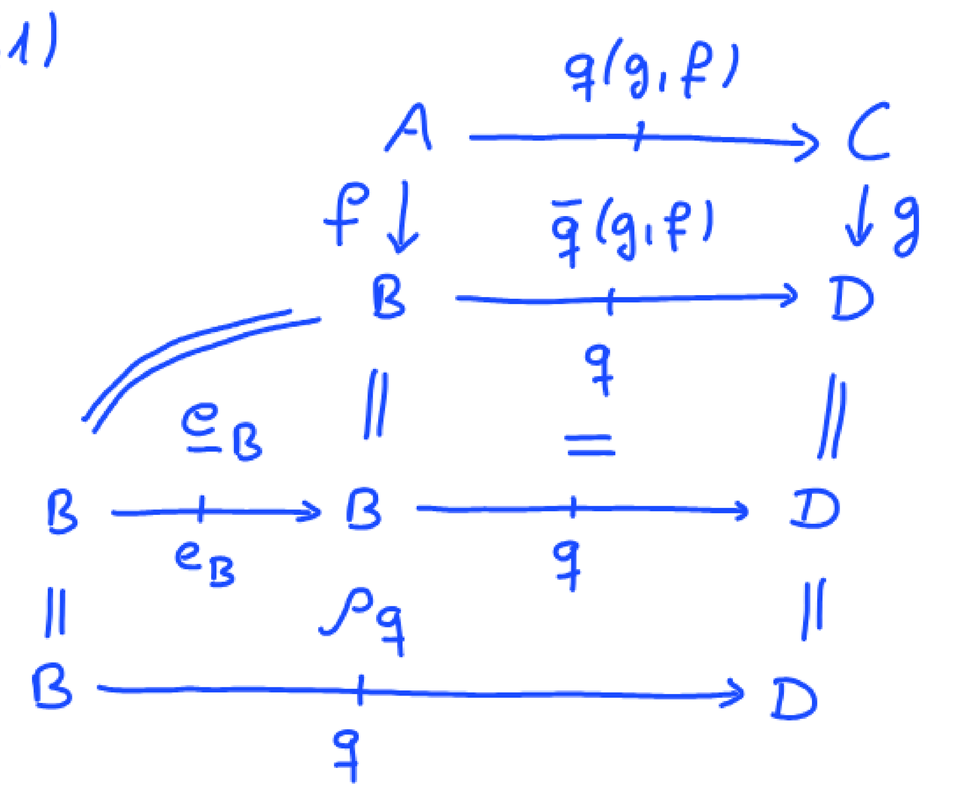
=

constructions of e_f and e_g



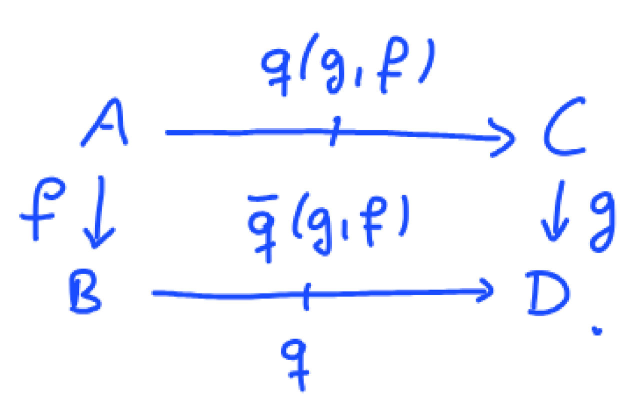
=

construction of λ_q



=

construction of ρ_q



This proves (*).

We may conclude that $\check{\alpha}$ is the unique 2-cell in $\mathbb{X}$ of its form which satisfies $\alpha = \check{\alpha} \circ (1_{r_1}, \dots, 1_{r_k}, \hat{\gamma}, 1_{s_1}, \dots, 1_{s_\ell})$.

This shows that the 2-cell $\hat{\gamma}$ is opcartesian,

$$\begin{tikzcd} A \arrow[r, "B(1,f)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "q", "\shortmid"{marking}] \arrow[d, phantom, "\hat{\gamma}\ \text{opcart}"] & D \arrow[r, "D(g,1)", "\shortmid"{marking}] & C \arrow[d, equal] \\ A \arrow[rrr, "q(g,f)"', "\shortmid"{marking}] & & & C. \end{tikzcd}$$

Hence, it exhibits the loose arrow $q(g,f) : A \nrightarrow C$ as a loose composite of the triple $(B(1,f), q, D(g,1))$.

Corollary.

Let $\mathbb{X}$ be a virtual equipment.

Let $f : A \to B$ and $g : B \to C$ be tight arrows in $\mathbb{X}$.

1) Consider the distinguished loose companion $C(1,gf) : A \nrightarrow C$ of the composite tight arrow $gf : A \to C$ in $\mathbb{X}$.

$$\begin{tikzcd} A \arrow[r, "C(1,gf)", "\shortmid"{marking}] \arrow[d, "gf"'] \arrow[dr, phantom, "\bar{C}(1,gf)\ \text{cart}"] & C \arrow[d, equal] \\ C \arrow[r, "e_C"', "\shortmid"{marking}] & C \end{tikzcd}$$

The loose arrow $C(1,gf) : A \nrightarrow C$ is also a loose composite of the pair $(B(1,f), C(1,g))$ of the distinguished loose companions $B(1,f) : A \nrightarrow B$ of f and $C(1,g) : B \nrightarrow C$ of g, i.e., $\exists$ opcartesian 2-cell in $\mathbb{X}$ of shape

$$\begin{tikzcd} A \arrow[r, "B(1,f)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "C(1,g)", "\shortmid"{marking}] \arrow[d, phantom, "\text{opcart}"] & C \arrow[d, equal] \\ A \arrow[rr, "C(1,gf)"', "\shortmid"{marking}] & & C. \end{tikzcd}$$

In particular, the pair $(B(1,f), C(1,g))$ admits a loose composite in $\mathbb{X}$.

Given any loose composite $B(1,f) \odot C(1,g)$ of $(B(1,f), C(1,g))$, then

$$C(1,gf) \cong B(1,f) \odot C(1,g) \quad \text{in } \mathbb{X}[\![A,C]\!].$$

2) Similarly, consider the distinguished loose conjoint $C(gf,1): C \nrightarrow A$ of the composite tight arrow $gf: A \to C$ in $\mathbb{X}$.

$$\begin{tikzcd} C \arrow[r, "C(gf{,}1)", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\overline{C}(gf{,}1)\ \mathrm{cart}"] & A \arrow[d, "gf"] \\ C \arrow[r, "e_C"', "\shortmid"{marking}] & C \end{tikzcd}$$

The loose arrow $C(gf,1): C \nrightarrow A$ is also a loose composite of the pair $(C(g,1), B(f,1))$ of the distinguished loose conjoints $C(g,1): C \nrightarrow B$ of g and $B(f,1): B \nrightarrow A$ of f, i.e., $\exists$ opcartesian 2-cell in $\mathbb{X}$ of shape

$$\begin{tikzcd} C \arrow[r, "C(g{,}1)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "B(f{,}1)", "\shortmid"{marking}] \arrow[d, phantom, "\mathrm{opcart}"] & A \arrow[d, equal] \\ C \arrow[rr, "C(gf{,}1)"', "\shortmid"{marking}] & & A. \end{tikzcd}$$

In particular, the pair $(C(g,1), B(f,1))$ admits a loose composite in $\mathbb{X}$.

Given any loose composite $C(g,1) \circ B(f,1)$ of $(C(g,1), B(f,1))$, then the loose arrows $C(gf,1)$, $C(g,1) \circ B(f,1): C \nrightarrow A$ in $\mathbb{X}$ are isomorphic as objects in $\mathbb{X}[\![C,A]\!]$,

$$C(gf,1) \cong C(g,1) \circ B(f,1).$$

Corollary.

Let $\mathbb{X}$ be a virtual equipment.

Let $f: A \to B$ and $g: C \to B$ be tight arrows in $\mathbb{X}$.

We consider the niche configuration $\begin{tikzcd} A \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "e_B"', "\shortmid"{marking}] & B \end{tikzcd}$ and the corresponding distinguished

cartesian 2-cell $\begin{tikzcd} A \arrow[r, "B(g{,}f)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overline{B}(g{,}f)\ \mathrm{cart}"] & C \arrow[d, "g"] \\ B \arrow[r, "e_B"', "\shortmid"{marking}] & B. \end{tikzcd}$

The loose arrow $B(g,f): A \nrightarrow C$ is also a loose composite of the pair $(B(1,f), B(g,1))$ of the companion $B(1,f): A \nrightarrow B$ of f and the conjoint $B(g,1): B \nrightarrow C$ of g, i.e., $\exists$ opcartesian 2-cell in $\mathbb{X}$ of shape

$$\begin{tikzcd} A \arrow[r, "B(1{,}f)", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "B(g{,}1)", "\shortmid"{marking}] \arrow[d, phantom, "\mathrm{opcart}"] & C \arrow[d, equal] \\ A \arrow[rr, "B(g{,}f)"', "\shortmid"{marking}] & & C. \end{tikzcd}$$

In particular, the pair $(B(1,f), B(g,1))$ admits a loose composite in $\mathbb{X}$.

$\forall$ loose composite $B(1,f) \circ B(g,1)$ of $(B(1,f), B(g,1))$, we have

$$B(g,f) \cong B(1,f) \circ B(g,1) \quad \text{in } \mathbb{X}[\![A,C]\!].$$

Theorem.

Let $\mathbb{X}$ be a virtual equipment.

Suppose we are given tight and loose arrows in $\mathbb{X}$ as depicted in the following configuration:

$$\begin{tikzcd} A \arrow[r, "p", "|"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "|"] & D. \end{tikzcd}$$

Then there exists a bijection between 2-cells in $\mathbb{X}$ with frame

$$\begin{tikzcd} A \arrow[r, "p", "|"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "|"] & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r, "B(f{,}1)", "|"] \arrow[d, equal] & A \arrow[r, "p", "|"] & C \arrow[r, "D(1{,}g)", "|"] & D \arrow[d, equal] \\ B \arrow[rrr, "q"', "|"] & & & D. \end{tikzcd}$$

Proof.

We consider the following 2-cells in $\mathbb{X}$:

$$\begin{tikzcd} B \arrow[r, "B(f{,}1)", "|"] \arrow[d, equal] \arrow[dr, phantom, "\overline{B}(f{,}1)\ \mathrm{cart}"] & A \arrow[d, "f"] \\ B \arrow[r, "e_B"', "|"] & B, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "e_A", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\nu(f{,}1)"] & A \arrow[d, equal] \\ B \arrow[r, "B(f{,}1)"', "|"] & A, \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "D(1{,}g)", "|"] \arrow[d, "g"'] \arrow[dr, phantom, "\overline{D}(1{,}g)\ \mathrm{cart}"] & D \arrow[d, equal] \\ D \arrow[r, "e_D"', "|"] & D, \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "e_C", "|"] \arrow[d, equal] \arrow[dr, phantom, "\nu(1{,}g)"] & C \arrow[d, "g"] \\ C \arrow[r, "D(1{,}g)"', "|"] & D. \end{tikzcd}$$

1) Suppose we are given a 2-cell α in $\mathbb{X}$ of the form $\begin{tikzcd} A \arrow[r, "p", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "|"] & D. \end{tikzcd}$

We form the following composite 2-cell in $\mathbb{X}$:

$$S(\alpha) := \begin{tikzcd} B \arrow[r, "B(f{,}1)", "|"] \arrow[d, equal] \arrow[dr, phantom, "\overline{B}(f{,}1)"] & A \arrow[r, "p", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[r, "D(1{,}g)", "|"] \arrow[d, "g"] \arrow[dr, phantom, "\overline{D}(1{,}g)"] & D \arrow[d, equal] \\ B \arrow[r, "e_B"', "|"] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, "q"', "|"] \arrow[d, equal] \arrow[drr, phantom, "\lambda_q"] & D \arrow[r, "e_D"', "|"] & D \arrow[d, equal] \\ B \arrow[r, "e_B"', "|"] \arrow[d, equal] \arrow[drrr, phantom, "\rho_q"] & B \arrow[rr, "q"', "|"] & & D \arrow[d, equal] \\ B \arrow[rrr, "q"', "|"] & & & D. \end{tikzcd}$$

This provides a well-defined map

$$S : \left\{ \text{2-cells in } \mathbb{X} \text{ with frame } \begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "|" marking] & D \end{tikzcd} \right\} \longrightarrow \left\{ \text{2-cells in } \mathbb{X} \text{ with frame } \begin{tikzcd} B \arrow[r, "B(f{,}1)", "|" marking] \arrow[d, equal] & A \arrow[r, "p", "|" marking] & C \arrow[r, "D(1{,}g)", "|" marking] & D \arrow[d, equal] \\ B \arrow[rrr, "q"', "|" marking] & & & D \end{tikzcd} \right\}$$

$$\alpha \mapsto S(\alpha).$$

2) Suppose we are given a 2-cell in $\mathbb{X}$ of the form
$$\begin{tikzcd} B \arrow[r, "B(f{,}1)", "|" marking] \arrow[d, equal] & A \arrow[r, "p", "|" marking] \arrow[d, phantom, "\beta"] & C \arrow[r, "D(1{,}g)", "|" marking] & D \arrow[d, equal] \\ B \arrow[rrr, "q"', "|" marking] & & & D. \end{tikzcd}$$

We form the following composite 2-cell in $\mathbb{X}$:

$$T(\beta) := \begin{tikzcd} & A \arrow[r, "p", "|" marking] \arrow[dl, equal, bend right] \arrow[d, equal] \arrow[dl, phantom, "\varrho_A"] & C \arrow[d, equal] \arrow[dr, equal, bend left] \arrow[dl, phantom, "="] \arrow[dr, phantom, "\varrho_C"] & \\ A \arrow[r, "e_A"', "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\nu(f{,}1)"] & A \arrow[r, "p"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[r, "e_C"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\nu(1{,}g)"] & C \arrow[d, "g"] \\ B \arrow[r, "B(f{,}1)"', "|" marking] \arrow[d, equal] & A \arrow[r, "p"', "|" marking] \arrow[d, phantom, "\beta"] & C \arrow[r, "D(1{,}g)"', "|" marking] & D \arrow[d, equal] \\ B \arrow[rrr, "q"', "|" marking] & & & D. \end{tikzcd}$$

This yields a well-defined map

$$T : \left\{ \text{2-cells in } \mathbb{X} \text{ with frame } \begin{tikzcd} B \arrow[r, "B(f{,}1)", "|" marking] \arrow[d, equal] & A \arrow[r, "p", "|" marking] & C \arrow[r, "D(1{,}g)", "|" marking] & D \arrow[d, equal] \\ B \arrow[rrr, "q"', "|" marking] & & & D \end{tikzcd} \right\} \longrightarrow \left\{ \text{2-cells in } \mathbb{X} \text{ with frame } \begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "q"', "|" marking] & D \end{tikzcd} \right\}$$

$$\beta \mapsto T(\beta).$$

3) We want to show that the maps S and T are inverse to each other.

3.1) $\forall$ 2-cell α as in 1), we have $T(S(\alpha)) = \alpha$:

$$T(S(\alpha)) \underset{\substack{\uparrow \\ \text{definition} \\ \text{of } T}}{=} \begin{tikzcd} & A \arrow[r, "p", "|" marking] \arrow[dl, equal, bend right] \arrow[d, equal] \arrow[dl, phantom, "\varrho_A"] & C \arrow[d, equal] \arrow[dr, equal, bend left] \arrow[dl, phantom, "="] \arrow[dr, phantom, "\varrho_C"] & \\ A \arrow[r, "e_A"', "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\nu(f{,}1)"] & A \arrow[r, "p"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[r, "e_C"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\nu(1{,}g)"] & C \arrow[d, "g"] \\ B \arrow[r, "B(f{,}1)"', "|" marking] \arrow[d, equal] & A \arrow[r, "p"', "|" marking] \arrow[d, phantom, "S(\alpha)"] & C \arrow[r, "D(1{,}g)"', "|" marking] & D \arrow[d, equal] \\ B \arrow[rrr, "q"', "|" marking] & & & D \end{tikzcd} \underset{\substack{\uparrow \\ \text{definition} \\ \text{of } S}}{=} \begin{tikzcd} & A \arrow[r, "p", "|" marking] \arrow[dl, equal, bend right] \arrow[d, equal] \arrow[dl, phantom, "\varrho_A"] & C \arrow[d, equal] \arrow[dr, equal, bend left] \arrow[dl, phantom, "="] \arrow[dr, phantom, "\varrho_C"] & \\ A \arrow[r, "e_A"', "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\nu(f{,}1)"] & A \arrow[r, "p"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[r, "e_C"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\nu(1{,}g)"] & C \arrow[d, "g"] \\ B \arrow[r, "B(f{,}1)"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\bar{B}(f{,}1)"] & A \arrow[r, "p"', "|" marking] \arrow[d, "f"] \arrow[dr, phantom, "\alpha"] & C \arrow[r, "D(1{,}g)"', "|" marking] \arrow[d, "g"] \arrow[dr, phantom, "\bar{D}(1{,}g)"] & D \arrow[d, equal] \\ B \arrow[r, "e_B"', "|" marking] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, "q"', "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\lambda_q"] & D \arrow[r, "e_D"', "|" marking] & D \arrow[d, equal] \\ B \arrow[r, "e_B"', "|" marking] \arrow[d, equal] \arrow[drrr, phantom, "\rho_q"] & B \arrow[rr, "q"', "|" marking] & & D \arrow[d, equal] \\ B \arrow[rrr, "q"', "|" marking] & & & D \end{tikzcd}$$

$\overset{\text{constructions of } \nu(f,1) \text{ and } \nu(1,g)}{=}$

$$
\begin{tikzcd}
 & A \arrow[r, "p"] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[d, equal] \arrow[dr, equal, bend left] \arrow[dr, phantom, "e_C"'] & \\
A \arrow[ur, equal, bend left] \arrow[ur, phantom, "e_A"'] \arrow[r, "e_A"'] \arrow[d, "f"'] \arrow[dr, phantom, "e_f"] & A \arrow[r, "p"] \arrow[d, "f"] \arrow[dr, phantom, "\alpha"] & C \arrow[r, "e_C"'] \arrow[d, "g"] \arrow[dr, phantom, "e_g"] & C \arrow[d, "g"] \\
B \arrow[r, "e_B"'] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, "q"] \arrow[d, equal] \arrow[drr, phantom, "\lambda_q"] & D \arrow[r, "e_D"'] & D \arrow[d, equal] \\
B \arrow[r, "e_B"'] \arrow[d, equal] \arrow[drrr, phantom, "\rho_q"] & B \arrow[rr, "q"'] & & D \arrow[d, equal] \\
B \arrow[rrr, "q"'] & & & D
\end{tikzcd}
$$

$\overset{\text{constructions of } e_f \text{ and } e_g}{=}$

$$
\begin{tikzcd}
 & A \arrow[r, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] & \\
 & B \arrow[r, "q"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[d, equal] \arrow[dr, equal, bend left] \arrow[dr, phantom, "e_D"'] & \\
B \arrow[ur, equal, bend left] \arrow[ur, phantom, "e_B"'] \arrow[r, "e_B"'] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, "q"] \arrow[d, equal] \arrow[drr, phantom, "\lambda_q"] & D \arrow[r, "e_D"'] & D \arrow[d, equal] \\
B \arrow[r, "e_B"'] \arrow[d, equal] \arrow[drrr, phantom, "\rho_q"] & B \arrow[rr, "q"'] & & D \arrow[d, equal] \\
B \arrow[rrr, "q"'] & & & D
\end{tikzcd}
$$

$\overset{\text{construction of } \lambda_q}{=}$

$$
\begin{tikzcd}
 & A \arrow[r, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\
 & B \arrow[r, "q"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[d, equal] \\
B \arrow[ur, equal, bend left] \arrow[ur, phantom, "e_B"'] \arrow[r, "e_B"'] \arrow[d, equal] \arrow[drr, phantom, "\rho_q"] & B \arrow[r, "q"] & D \arrow[d, equal] \\
B \arrow[rr, "q"'] & & D
\end{tikzcd}
$$

$\overset{\text{construction of } \rho_q}{=}$

$$
\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\
B \arrow[r, "q"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[d, equal] \\
B \arrow[r, "q"'] & D
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\
B \arrow[r, "q"'] & D
\end{tikzcd},
$$

so $T(S(\alpha)) = \alpha$.

3.2) $\forall$ 2-cell β as in 2), we have $S(T(\beta)) = \beta$:

$S(T(\beta))$ $\overset{\text{definition of } S}{=}$

$$
\begin{tikzcd}
B \arrow[r, "B(f{,}1)"] \arrow[d, equal] \arrow[dr, phantom, "\bar{B}(f{,}1)"] & A \arrow[r, "p"] \arrow[d, "f"] \arrow[dr, phantom, "T(\beta)"] & C \arrow[r, "D(1{,}g)"] \arrow[d, "g"] \arrow[dr, phantom, "\bar{D}(1{,}g)"] & D \arrow[d, equal] \\
B \arrow[r, "e_B"'] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, "q"] \arrow[d, equal] \arrow[drr, phantom, "\lambda_q"] & D \arrow[r, "e_D"'] & D \arrow[d, equal] \\
B \arrow[r, "e_B"'] \arrow[d, equal] \arrow[drrr, phantom, "\rho_q"] & B \arrow[rr, "q"'] & & D \arrow[d, equal] \\
B \arrow[rrr, "q"'] & & & D
\end{tikzcd}
$$

$\overset{\text{definition of } T}{=}$

$$
\begin{tikzcd}
B \arrow[rr, "B(f{,}1)"] \arrow[ddd, equal] \arrow[ddr, phantom, "\bar{B}(f{,}1)"] & & A \arrow[r, "p"] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[rr, "D(1{,}g)"] \arrow[d, equal] \arrow[dr, equal, bend left] \arrow[dr, phantom, "e_C"'] & & D \arrow[ddd, equal] \arrow[ddl, phantom, "\bar{D}(1{,}g)"] \\
 & A \arrow[ur, equal, bend left] \arrow[ur, phantom, "e_A"'] \arrow[r, "e_A"'] \arrow[d, "f"'] \arrow[dr, phantom, "\nu(f{,}1)"] & A \arrow[r, "p"] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[r, "e_C"'] \arrow[d, equal] \arrow[dr, phantom, "\nu(1{,}g)"] & C \arrow[d, "g"] & \\
 & B \arrow[r, "B(f{,}1)"'] \arrow[d, equal] \arrow[drrr, phantom, "\beta"] & A \arrow[r, "p"'] & C \arrow[r, "D(1{,}g)"'] & D \arrow[d, equal] & \\
B \arrow[r, "e_B"'] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[rrr, "q"'] \arrow[d, equal] \arrow[drrrr, phantom, "\lambda_q"] & & & D \arrow[r, "e_D"'] & D \arrow[d, equal] \\
B \arrow[r, "e_B"'] \arrow[d, equal] \arrow[drrrrr, phantom, "\rho_q"] & B \arrow[rrrr, "q"'] & & & & D \arrow[d, equal] \\
B \arrow[rrrrr, "q"'] & & & & & D
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[rr, "B(f,1)"] \arrow[ddd, equal] & & A \arrow[r, "p"] \arrow[d, equal] \arrow[dl, phantom, "e_A"] \arrow[dr, phantom, "="] & C \arrow[rr, "D(1,g)"] \arrow[d, equal] \arrow[dr, phantom, "e_C"] & & D \arrow[d, equal] \arrow[dl, phantom, "="] \\
& A \arrow[r, "e_A"'] \arrow[d, "f"'] \arrow[l, phantom, "\overline{B}(f,1)"] \arrow[dr, phantom, "\nu(f,1)"] & A \arrow[r, "p"'] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[r, "e_C"'] \arrow[d, equal] \arrow[dr, phantom, "\nu(1,g)"] & C \arrow[r, "D(1,g)"'] \arrow[d, "g"] \arrow[dr, phantom, "\overline{D}(1,g)"] & D \arrow[d, equal] \\
& B \arrow[r, "B(f,1)"'] \arrow[d, equal] \arrow[dr, phantom, "="] & A \arrow[r, "p"'] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[r, "D(1,g)"'] \arrow[d, equal] \arrow[dr, phantom, "\lambda_{D(1,g)}"] & D \arrow[r, "e_D"'] & D \arrow[d, equal] \\
B \arrow[r, "e_B"'] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, "B(f,1)"'] \arrow[d, equal] & A \arrow[r, "p"'] & C \arrow[rr, "D(1,g)"'] \arrow[d, phantom, "\beta"] & & D \arrow[d, equal] \\
B \arrow[r, "e_B"'] \arrow[d, equal] & B \arrow[rrrr, "q"] \arrow[d, phantom, "\rho_q"] & & & & D \arrow[d, equal] \\
B \arrow[rrrrr, "q"'] & & & & & D
\end{tikzcd}
$$

$\overset{=}{\nearrow}$

naturality of λ wrt β (and $1e_D = e_{1_D}$)

$$
\begin{tikzcd}
B \arrow[rr, "B(f,1)"] \arrow[dd, equal] & & A \arrow[r, "p"] \arrow[d, equal] \arrow[dl, phantom, "e_A"] \arrow[dr, phantom, "="] & C \arrow[rr, "D(1,g)"] \arrow[d, equal] \arrow[dr, phantom, "e_C"] & & D \arrow[d, equal] \arrow[dl, phantom, "="] \\
& A \arrow[r, "e_A"'] \arrow[d, "f"'] \arrow[l, phantom, "\overline{B}(f,1)"] \arrow[dr, phantom, "\nu(f,1)"] & A \arrow[r, "p"'] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[r, "e_C"'] \arrow[d, equal] \arrow[drr, phantom, "\rho_{D(1,g)}"] & C \arrow[r, "D(1,g)"'] & D \arrow[d, equal] \\
B \arrow[r, "e_B"'] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, "B(f,1)"'] \arrow[d, equal] & A \arrow[r, "p"'] & C \arrow[rr, "D(1,g)"'] \arrow[d, phantom, "\beta"] & & D \arrow[d, equal] \\
B \arrow[r, "e_B"'] \arrow[d, equal] & B \arrow[rrrr, "q"] \arrow[d, phantom, "\rho_q"] & & & & D \arrow[d, equal] \\
B \arrow[rrrrr, "q"'] & & & & & D
\end{tikzcd}
$$

$\overset{=}{\nearrow}$

by an earlier remark concerning $\nu(1,g)$

$$
\begin{tikzcd}
B \arrow[rr, "B(f,1)"] \arrow[dd, equal] & & A \arrow[r, "p"] \arrow[d, equal] \arrow[dl, phantom, "e_A"] \arrow[dr, phantom, "="] & C \arrow[r, "D(1,g)"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[dd, equal] \\
& A \arrow[r, "e_A"'] \arrow[d, "f"'] \arrow[l, phantom, "\overline{B}(f,1)"] \arrow[dr, phantom, "\nu(f,1)"] & A \arrow[r, "p"'] \arrow[d, equal] \arrow[dr, phantom, "="] & C & \\
B \arrow[r, "e_B"'] \arrow[d, equal] \arrow[dr, phantom, "="] & B \arrow[r, "B(f,1)"'] \arrow[d, equal] & A \arrow[r, "p"'] & C \arrow[r, "D(1,g)"'] \arrow[d, phantom, "\beta"] & D \arrow[d, equal] \\
B \arrow[r, "e_B"'] \arrow[d, equal] & B \arrow[rrr, "q"] \arrow[d, phantom, "\rho_q"] & & & D \arrow[d, equal] \\
B \arrow[rrrr, "q"'] & & & & D
\end{tikzcd}
$$

$\overset{=}{\nearrow}$

construction of $\rho_{D(1,g)}$

$$
\begin{tikzcd}
B \arrow[rr, "B(f,1)"] \arrow[d, equal] \arrow[dr, phantom, "1_{B(f,1)}"] & & A \arrow[r, "p"] \arrow[d, equal] \arrow[dl, phantom, "e_A \text{ opcart}"] \arrow[dr, phantom, "1_p"] & C \arrow[r, "D(1,g)"] \arrow[d, equal] \arrow[dr, phantom, "1_{D(1,g)}"] & D \arrow[dd, equal] \\
B \arrow[r, "B(f,1)"'] \arrow[d, equal] \arrow[dr, phantom, "\overline{B}(f,1) \text{ cart}"] & A \arrow[r, "e_A"'] \arrow[d, "f"'] \arrow[dr, phantom, "\nu(f,1)"] & A \arrow[r, "p"'] \arrow[d, equal] \arrow[dr, phantom, "1_p"] & C & \\
B \arrow[r, "e_B"'] \arrow[d, equal] \arrow[dr, phantom, "\rho_{B(f,1)} \text{ opcart}"] & B \arrow[r, "B(f,1)"'] & A \arrow[r, "p"'] \arrow[d, equal] \arrow[dr, phantom, "1_p"] & C \arrow[r, "D(1,g)"'] \arrow[d, equal] \arrow[dr, phantom, "1_{D(1,g)}"] & D \arrow[d, equal] \\
B \arrow[rr, "B(f,1)"'] \arrow[d, equal] & & A \arrow[r, "p"'] \arrow[d, phantom, "\beta"] & C \arrow[r, "D(1,g)"'] & D \arrow[d, equal] \\
B \arrow[rrrr, "q"'] & & & & D
\end{tikzcd}
$$

$\overset{=}{\nearrow}$

naturality of ρ wrt β (and $1e_B = e_{1_B}$)

$$
\begin{tikzcd}
B \arrow[rr, "B(f,1)"] \arrow[d, equal] \arrow[dr, phantom, "="] & & A \arrow[r, "p"] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[r, "D(1,g)"] \arrow[d, equal] \arrow[ddr, phantom, "="] & D \arrow[dd, equal] \\
B \arrow[r, "B(f,1)"'] \arrow[d, equal] & A \arrow[r, "e_A"'] \arrow[ur, equal, bend left] \arrow[u, phantom, "e_A"] \arrow[d, phantom, "\lambda_{B(f,1)}"] & A \arrow[r, "p"'] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[d, equal] & \\
B \arrow[rr, "B(f,1)"'] \arrow[d, equal] & & A \arrow[r, "p"'] \arrow[d, phantom, "\beta"] & C \arrow[r, "D(1,g)"'] & D \arrow[d, equal] \\
B \arrow[rrrr, "q"'] & & & & D
\end{tikzcd}
$$

$=$ by an earlier remark concerning $\nu(f,1)$

$$
\begin{tikzcd}
B \arrow[r, "B(f,1)"] \arrow[d, equal] \arrow[dr, phantom, "="] & A \arrow[r, "p"] \arrow[d, equal] \arrow[dr, phantom, "="] & C \arrow[r, "D(1,g)"'] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[d, equal] \\
B \arrow[r, "B(f,1)"'] \arrow[d, equal] & A \arrow[r, "p"'] \arrow[d, phantom, "\beta"] & C \arrow[r, "D(1,g)"'] & D \arrow[d, equal] \\
B \arrow[rrr, "q"'] & & & D
\end{tikzcd}
$$

$=$ construction of $\lambda_{B(f,1)}$

$$
=\quad
\begin{tikzcd}
B \arrow[r, "B(f,1)"] \arrow[d, equal] & A \arrow[r, "p"] \arrow[d, phantom, "\beta"] & C \arrow[r, "D(1,g)"] & D \arrow[d, equal] \\
B \arrow[rrr, "q"'] & & & D,
\end{tikzcd}
$$

so $S(T(\beta)) = \beta$.

By 3.1) and 3.2), the maps S and T are inverse to each other and thus mutually inverse bijections.