THE KURAMOTO–SIVASHINKY EQUATION

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The Kuramoto–Sivashinsky equation

$$u_t = -u_{xx} - u_{xxxx} - u_x u$$

applies to a real-valued function of time $t \in \mathbb{R}$ and space $x \in \mathbb{R}$. This equation was introduced as a simple 1-dimensional model of instabilities in flames, but it is mathematically interesting because it describes *Galilean-invariant chaos with an arrow of time.*

It is 'chaotic' because small perturbations can grow exponentially, making the long-term behavior of a solution hard to predict in detail. It has an 'arrow of time' because time reversal

$$(t, x) \mapsto (-t, x)$$

is not a symmetry of this equation. Indeed, in Figure 1 we see that starting from random initial conditions, manifestly time-asymmetric patterns emerge. As we move forwards in time, it looks as if stripes are born and merge, but never die or split. Finally, we say the Kuramoto–Sivashinsky equation is 'Galilean-invariant' because the Galiei group acts as symmetries. This is the group generated by translations in t and x, reflections in x, and **Galilei boosts**, which are transformations to moving coordinate systems:

$$(t, x) \mapsto (t, x - tv).$$

Translations act in the obvious way. Spatial reflections act as follows: if $u(t, x)$ is a solution, so is $-u(t, -x)$. Galilei boosts act in a more subtle way: if $u(t, x)$ is a solution, so is $u(t, x - tv) + v$.

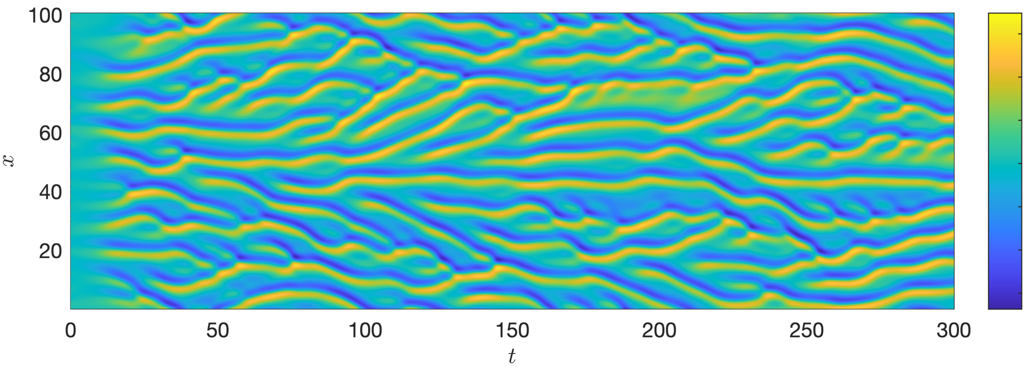

Figure 1 — A solution $u(t, x)$ of the Kuramoto–Sivashinky equation. The variable x ranges over the interval $[0, 100]$ with its endpoints identified. Initial data are independent identically distributed random variables, one at each grid point, uniformly distributed in $[-1, 1]$.

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It is common to study solutions of the Kuramoto–Sivashinky equations that are spatially periodic, so that $u(t,x) = u(t,x+L)$ for some L. We can then treat space as a circle, the interval $[0,L]$ with its endpoints identified. For these spatially periodic solutions, the integral $\int_0^L u(t,x)\,dx$ does not change with time. Applying a Galilean transform adds a constant to this integral. In what follows we restrict attention to solutions where this integral is zero. These are roughly the solutions where the stripes are at rest, on average.

We can learn a surprising amount about these solutions by looking at the linearized equation

$$u_t = -u_{xx} - u_{xxxx}.$$

We can solve this using a Fourier transform

$$u(t,x) = \sum_{0\neq n\in\mathbb{Z}} \hat{u}_n(t)\, e^{ik_n x}$$

where the frequency of the nth mode is $k_n = 2\pi n/L$. We obtain

$$\hat{u}_n(t) = \exp\big((k_n^2 - k_n^4)t\big)\ \hat{u}_n(0).$$

Thus the nth mode grows exponentially with time if and only if $k_n^2 - k_n^4 > 0$, which happens when $0 < |n| < 2\pi L$. This appears to be the cause of chaos even in the nonlinear equation. Indeed, all solutions approach an attractive fixed point if L is small enough, but as we increase L we see the usual transition to chaos via 'period doubling' [3]. Interstingly, the nonlinear term in the Kuramoto–Sivashinsky equation stabilizes the exponentially growing modes: in the language of physics, it tends to transfer power from these modes to high-frequency modes, which decay exponentially.

Proving this last fact is not easy. However, in 1992, Collet, Eckmann, Epstein and Stubbe [1] did this in the process of showing that for any initial data in the Hilbert space

$$\dot{L}^2 = \{u\colon [0,L]\to\mathbb{R} : \int_0^L |u(x)|^2\,dx < \infty,\ \int_0^L u(x)\,dx = 0\}$$

the Kuramoto–Sivashinsky equation has a unique solution for $t \geq 0$, in a suitable sense, and that the norm of this solution eventually becomes less than some constant times $L^{8/5}$. These authors also showed any such solution eventually becomes infinitely differentiable, even analytic [2].

Shortly after this, Temam and Wang went further [4]. They showed that all solutions of the Kuramoto–Sivashinsky equation with initial data in $\dot{L}^2$ approach a *finite-dimensional submanifold* of $\dot{L}^2$ as $t \to +\infty$. They also showed the dimension of this manifold is bounded by a constant times $(\ln L)^{0.2} L^{1.64}$. This manifold, called the **inertial manifold**, describes the 'eventual behaviors' of solutions.

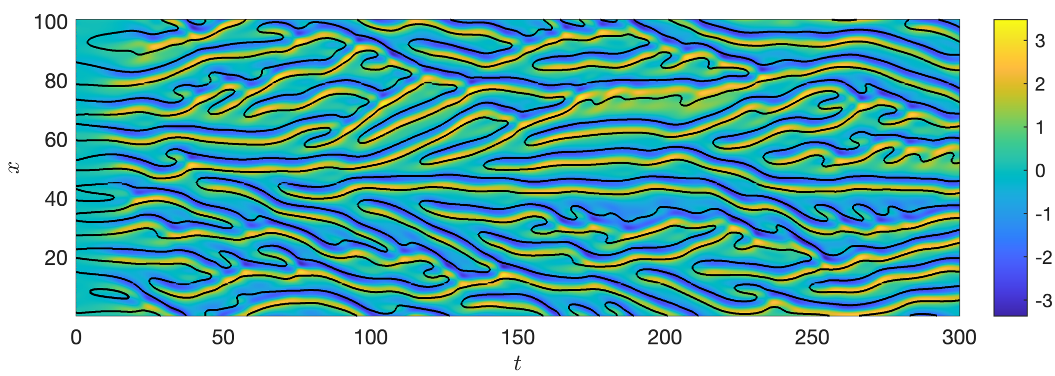

Figure 2 — The same solution of the Kuramoto–Sivashinsky equation, with stripes indicated.

Understanding the eventual behavior of solutions of the Kuramoto–Sivashinsky equation remains a huge challenge. What are the 'stripes' in these solutions? Can we define them precisely? Here is one proposal. Suppose v is the convolution of u_x with a normalized Gaussian with standard deviation of 2. Define a 'stripe' in the solution u to be a region where $v < 0$. For the solution in Figure 1, these stripes are outlined in Figure 2. Note that after the solution nears the inertial manifold, stripes are born and merge as time increases, but they never die or split —at least in this example. We conjecture that this is true 'generically': that is, for all solutions in some open dense subset of the inertial manifold.

Our numerical calculations also indicate that generically, solutions eventually have stripes with an average density that approaches about 0.112 as $L \to +\infty$. This is close to inverse of the wavelength of the fastest-growing mode of the linearized equation, $(2^{3/2}\pi)^{-1} \approx 0.1125$. But we see no reason to think these numbers should be exactly equal, and *proving* the existence of a limiting stripe density is an open problem.

References

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