logic in color

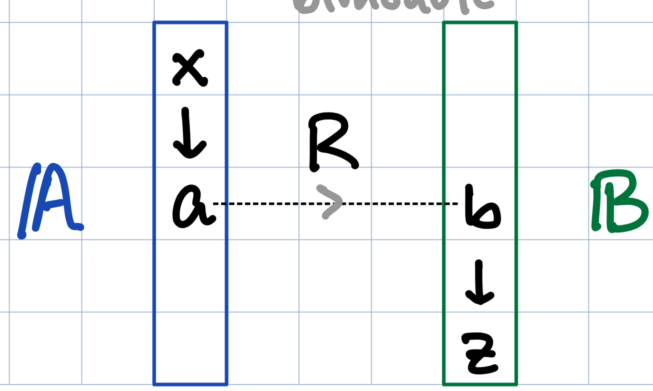
The Yoneda Lemma

(& free discussion)

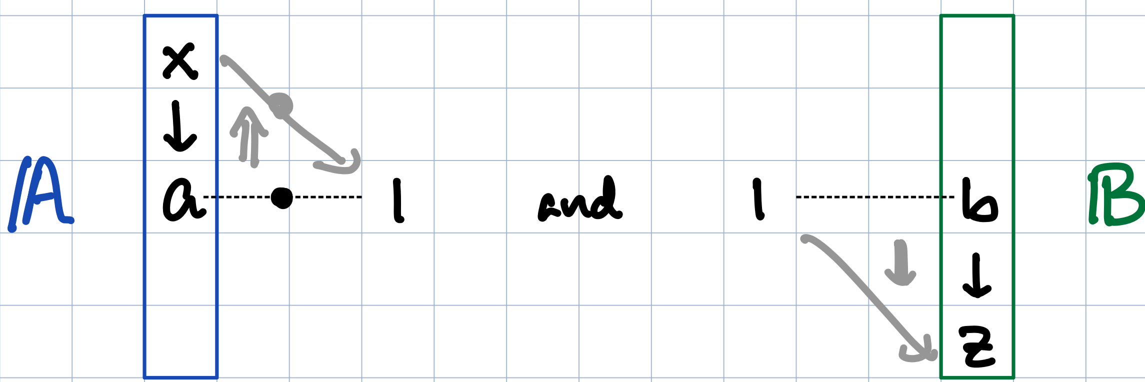
A *predicate* on a set $\mathbb{A}$
is a function $\phi : \mathbb{A} \to \{0,1\}$.
This is the basis of predicate logic.

A relation from $\mathbb{A}$ to $\mathbb{B}$
is a predicate $R : \mathbb{A} \times \mathbb{B} \to \{0,1\}$
and these form $\mathbb{Rel}$.

We saw that relations of sets
generalize to profunctors of categories.
"bimodule" "monad"



So — what's a "predicate" now?
Well, now there are <u>two</u> kinds!



$$\frac{\mathbb{A}^{op} \to \mathrm{Set}}{\text{"negative"}} \; * \qquad\qquad \frac{\mathbb{A} \to \mathrm{Set}}{\text{"positive"}}$$

These are left & right modules,
aka "<u>presheaves</u>" & "<u>copresheaves</u>".

example. Let $\mathbb{G}$ be a category
with one object &
all morphisms invertible. (group)

what is a positive predicate on $\mathbb{G}$?

$$\begin{array}{cccc} \alpha : & \mathbb{G} & \to & \mathrm{Set} \\ & \bullet & \mapsto & X \\ & g \downarrow & & \downarrow \alpha g \\ & \bullet & \mapsto & X \end{array}$$

a G-action
$(G \to [X, X])$

So if a predicate is a $\overset{\text{profunctor}}{\text{judgement}}$ $\mathbb{C}.P.1$,
then the "power set" is the category

$$\Omega\mathbb{C} \equiv [\mathbb{C} \mid 1] \qquad P \vdash Q = \prod c.\, Pc \vdash Qc$$
$$= [\mathbb{C}^{op} \vdash \mathrm{Set}]$$

In $\mathbb{R}\mathrm{el}$, power sets are highly structured
because $0 \vdash 1$ is.
In $\mathbb{C}\mathrm{at}$, the same is true
because Set is nice.
The "category of presheaves" $\Omega\mathbb{C}$ (aka $P(\mathbb{C})$)
is a topos, a setting for predicate logic.

The great news is that every category
embeds into such a rich world.

The Yoneda embedding of $\mathbb{C}$
is defined as follows.

$$Y = \mathbb{C}[-] : \mathbb{C} \vdash \Omega\mathbb{C}$$

$$\begin{array}{ccccc} c & \mapsto & \mathbb{C}(-,c) & : & (c' \mapsto \mathbb{C}(c',c)) \\ f\downarrow & & \mathbb{C}(-,f)\downarrow & & \downarrow f\circ - \\ d & \mapsto & \mathbb{C}(-,d) & & \mathbb{C}(c',d) \end{array}$$

Each $\_\mathbb{C}c$ is a representable* predicate.

example: Cayley's theorem.

$$G \hookrightarrow [G,G]$$
$$g \mapsto g\cdot -$$

$$G\times G \to G$$
$$\sim$$
$$G \to [G,G]$$

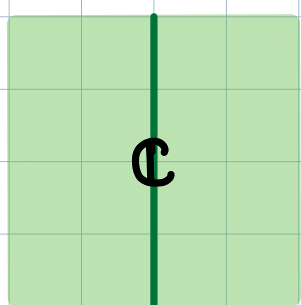
is a nice inclusion.

So, the embedding is simply

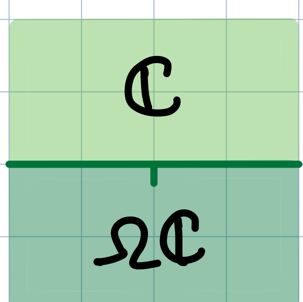
$$c : \mathbb{C} \quad \mapsto \quad \_\mathbb{C}c : [\mathbb{C}|1]$$

this is "currying" the hom of $\mathbb{C}$.

$$\frac{A\times B\to C}{A\to[B\to C]}$$



$$\_\mathbb{C}\_ : \mathbb{C}|\mathbb{C}$$
$$\sim$$
$$\_\mathbb{C}\_ : \mathbb{C}^{op}\times\mathbb{C} \vdash Set$$
$$c\,,\, d \qquad c \rightrightarrows d$$



$$\mathbb{C}[-] : \mathbb{C} \vdash [\mathbb{C}^{op} \vdash Set]$$

that's why ppl like (-)preds

* embedding is covariant

$\Omega\mathbb{C}$ is the "power category".

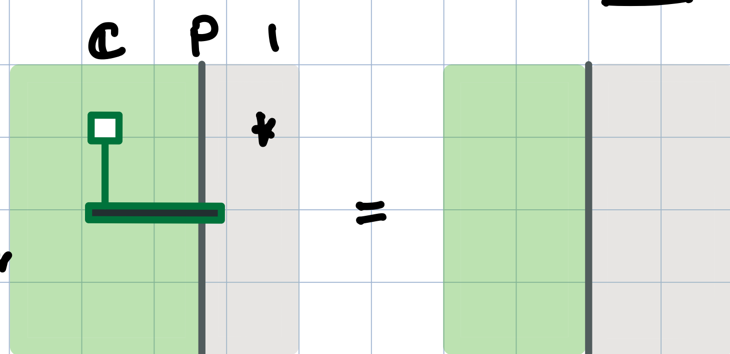
There is a pair of theorems called "coYoneda" + "Yoneda" lemma which are fundamental to CT.

Let $P : \Omega\mathbb{C}$, ie $P : \mathbb{C}^{op} \vdash \mathrm{Set}$

$$\begin{tikzcd} d \arrow[d, "f"'] & \\ c \arrow[r, dashed] & \odot \end{tikzcd}$$

$\boxed{\lambda}$ $\Sigma c.\ \underline{d}\,\mathbb{C}\,c \times c\,P\,1 \approx \underline{Pd}$

predicate P is a $\underline{\text{sum}}$ of representables



simplicial sets $\Delta^{op} \to \mathrm{Set}$

* $\mathbb{C} \circ P \sim P$

$\boxed{Y}$ $\Pi c.\ c\,\mathbb{C}\,\underline{d} \vdash c\,P\,1 \approx P\underline{d}$

* $\Omega\mathbb{C}(\mathbb{C}(-,d), P) \approx Pd$

"$P \sim \mathbb{C} \multimap P$"

A predicate on $\mathbb{C}$ is determined by its representable inferences.

This is useful because it gives a method of constructing & proving.

If we apply Yoneda to $\_ \mathbb{C}\, d$, we have that

$$* \quad \Omega\mathbb{C}(\mathbb{C}(-,c) \vdash \mathbb{C}(-,d)) \overset{\text{bijection}}{\sim} \mathbb{C}(c,d) \quad \text{"full + faithful"}$$

— embedding $\mathbb{C}$ into $\Omega\mathbb{C}$ preserves the "logic" of $\mathbb{C}$.

So proving $\dfrac{x \vdash c}{x \vdash d}$ is equivalent to constructing $t : c \vdash d$.

example. what is the product of predicates?

$P \times Q$

$$\dfrac{\dfrac{X \vdash P \times Q}{X \vdash P \times X \vdash Q}}{\dfrac{\Pi c.\, Xc \vdash Pc \times \Pi c.\, Xc \vdash Qc}{\Pi c.\, Xc \vdash Pc \times Qc}}$$

The other crucial fact:
$Y : \mathbb{C} \vdash \Omega\mathbb{C}$ is the free cocompletion of $\mathbb{C}$.

Like the free monoid on a set,
Y turns objects $c : \mathbb{C}$ into rep's $\_\mathbb{C}c$
and "freely forms sums of them".

So we saw that every predicate is a sum:

$$P \approx \Sigma c.\ \_\mathbb{C}c \times Pc$$

& this completely characterizes Y.

Let $\mathbb{Z}$ be a category with all colimits
& $f : \mathbb{C} \vdash \mathbb{Z}$ a functor.
Then $\exists!$ functor which preserves colimits

$$\Sigma Y.f : \Omega\mathbb{C} \vdash \mathbb{Z}$$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "f"] \arrow[d, "Y"'] & \mathbb{Z} \\
\Omega\mathbb{C} \arrow[ur, dotted, "\Sigma Y.f"'] &
\end{tikzcd}$$

Discussion

What aspects of CT/logic
- do you find interesting/enjoyable?
  confusing/mysterious?
- do you hope to <u>learn</u>?

What do you think of "CT as logic"?
the pictures?
the language?

Linear logic ~ "$*$-autonomous categories"
+ more...

$\times, +$ Rel $A \mid B$

$*$

the language of $\mathbb{V}$Cat
is <u>made of</u> the language of $\mathbb{V}$

(the language of a topos)

hasn't been done for $\underbrace{\overset{\Sigma}{(\mathbb{V}, \otimes, \mathrm{coeq})}}$ $\star$