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John C. Baez,[1,a] David Weisbart,[1,b] and Adam M. Yassine[2,c]

AFFILIATIONS

[1] Department of Mathematics, University of California, Riverside, Riverside, California 92521, USA
[2] Department of Mathematics and Actuarial Science, The American University in Cairo, New Cairo, Egypt

[a] baez@math.ucr.edu
[b] **Author to whom correspondence should be addressed:** weisbart@math.ucr.edu
[c] adam.yassine@aucegypt.edu

ABSTRACT

Generalized span categories provide a framework for formalizing mathematical models of open systems in classical mechanics. We introduce categories `LagSy` and `HamSy` that, respectively, provide a categorical framework for the Lagrangian and Hamiltonian descriptions of open classical mechanical systems. The morphisms of `LagSy` and `HamSy` correspond to such open systems, and the composition of morphisms models the construction of systems from subsystems. The Legendre transformation gives rise to a functor from `LagSy` to `HamSy` that translates from the Lagrangian to the Hamiltonian perspective.

I. INTRODUCTION

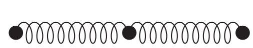
Category theory provides a formalism for unifying ideas across a wide spectrum of disciplines. The last few decades have seen the emergence of applied category theory.[1,2] One prominent program in this subject is to describe "open" systems—that is, systems that can interact with their surroundings—as morphisms in appropriate categories, where composition describes how open systems can be combined to form larger systems.

The idea of describing open systems as morphisms arose from extended topological quantum field theory, where the manifold describing space can be built up by composing cobordisms, manifolds with boundary that describe smaller regions of space.[3–6] It was later applied in a more down-to-earth way to electrical circuits,[7,8] Markov processes,[9] and a wide variety of dynamical systems.[10–12] The morphisms in these categories are often spans or cospans with an extra structure, and there are now several formalisms for constructing such categories.[13]

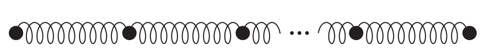
Our goal here is to apply this idea to Lagrangian and Hamiltonian mechanics and describe the Legendre transformation as a functor from a category with open Lagrangian systems as morphisms to a similar category of open Hamiltonian systems. Since the study of classical systems involves solving differential equations that describe paths on general Riemannian and symplectic manifolds, it is in some ways more complicated than the examples treated earlier. The current work investigates some previously unidentified structures that appear critical to the study of open systems in classical mechanics.

The systems under consideration have a state space that is either the tangent bundle to a Riemannian manifold in the Lagrangian description or a symplectic manifold in the Hamiltonian description.[14] A path in the state space models the motion of the system. The state space of any subsystem is a quotient space of that of the entire system. For Lagrangian systems, we require that the quotient maps be surjective Riemannian submersions. For Hamiltonian systems, we require that they be surjective Poisson maps between symplectic manifolds.

A simple example is this system with three point masses attached by springs, where all motion is along the same line:

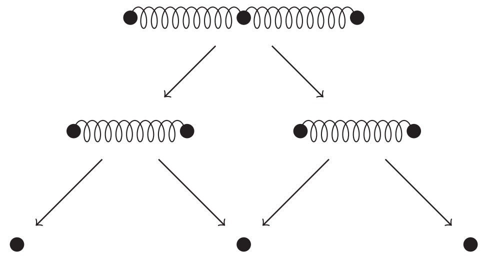


We can build a complicated system by attaching additional point masses and springs in series:

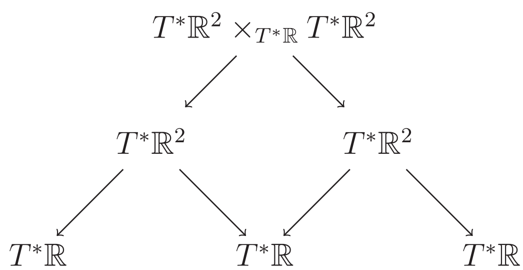


View a pair of point masses attached by a spring as a fundamental component, or a subsystem, of one of these more complicated systems. These subsystems are then "open systems," in the sense that both forces internal to the subsystem and external forces of the larger system govern the dynamics of the subsystems.

We may build such a larger system out of two subsystems by identifying the right mass of the subsystem on the left with the left mass of the subsystem on the right, as follows:



Here we depict the state spaces of these systems from a Hamiltonian perspective:



Each of the maps is a canonical projection and a surjective Poisson map between symplectic manifolds. At the lowest level are the state spaces of the three distinct masses. If each mass moves along a line, then each system has $T^*\mathbb{R}$, the cotangent bundle to $\mathbb{R}$, as its state space. At the middle level are two spring-mass systems, each with a state space given by $T^*\mathbb{R}^2$. On the top level, the total system consists of three masses interacting in series. The state space for this total system is a fibered product of two copies of the symplectic manifold $T^*\mathbb{R}^2$ over the manifold $T^*\mathbb{R}$.

The fibered product is a six dimensional symplectic manifold, whereas the Cartesian product of the state spaces is an eight dimensional symplectic manifold. While the fibered product is an embedded submanifold of the product, it will not be a symplectic submanifold when endowed with the symplectic structure that it requires to be the state space of the given classical system. The Lagrangian setting is similar but uses tangent bundles rather than cotangent bundles as the state spaces. The fibered product together with its canonical projections encapsulates the physical meaning of identifying the right mass of the left spring-mass system with the left mass of the right spring-mass system. Both Dazord in Ref. 15 and Marle in Ref. 16 had similar insights with respect to studying constrained systems, which are similar to the systems given above in the sense that the masses that connect our systems can be thought of as a geometric constraint. In fact, Dazord explicitly uses fibered products to construct the configuration and state spaces for certain constrained systems.

Suppose that X, Y, and Z are sets and f and g are functions that, respectively, map X and Y to the set Z. Henceforth, denote by ρ_X and ρ_Y the respective canonical projections

$$\rho_X : X \times Y \to X \quad \text{and} \quad \rho_Y : X \times Y \to Y,$$

and denote by π_X and π_Y the respective restrictions of ρ_X and ρ_Y to the fibered product $X \times_Z Y$, which is the subset of $X \times Y$ consisting of all elements on which f is equal to g. The fibered product in the category `Set`, whose objects are sets and whose morphisms are functions, has certain universal properties recalled in Sec. II. The connection between these universal properties and the construction of span categories for modeling classical mechanical systems is a central theme of the current investigation.

A span in the category `Set` is a pair of functions with the same source. The fibered product together with the span (π_X, π_Y) gives a prescription for composing spans in `Set`. Bénabou proved in Ref. 17 that if $\mathscr{C}$ is a category with pullbacks, then there is a bicategory, $\text{Span}(\mathscr{C})$, whose objects, morphisms, and 2-morphisms are the respective objects, spans, and maps of spans in $\mathscr{C}$. To avoid unnecessary complications, we view this bicategory as a category, a *span category*, by ignoring the bicategory structure and taking isomorphism classes of spans in $\mathscr{C}$, to be defined in Sec. II, as the morphisms. Fibered products define a composition of isomorphism classes of certain spans in `Set` that seems strikingly similar to the way in which classical mechanical systems compose.

We propose that open classical systems are morphisms in an appropriate span category, where the composition of morphisms using pullbacks describes the composition of classical systems. This formalization of classical mechanics should deepen our understanding of the foundations of classical mechanics and may also offer a way to automate the modeling of classical mechanical systems. Modeling open classical mechanical systems necessitates working with spans in categories other than `Set`, where the fibered product lacks the universal properties that it has in `Set`.

It is natural to view a classical system as an isomorphism class of spans in the category of Riemannian manifolds with surjective Riemannian submersions in the Lagrangian setting or as an isomorphism class of spans in the category of symplectic manifolds with surjective Poisson maps in the Hamiltonian setting. However, Sec. II demonstrates that neither of these categories has pullbacks. Thus, the work of Bénabou does not apply. For this same reason, it does not appear that the work of Fong[18,19] as corrected by Courser[13,20] can be straightforwardly modified from its cospan setting to a span setting that is useful to the present discussion. Derived geometry[21,22] would let us use homotopy pullbacks instead of pullbacks, but in some sense, this is overkill: The fibered products required for the current paper will exist and be smooth manifolds; only the universality condition of a pullback fails.

Section II recalls previous work required for handling this problem. Suppose that $\mathscr{C}$ and $\mathscr{C}'$ are categories and $\mathcal{F}$ is a functor from $\mathscr{C}$ to $\mathscr{C}'$. Weisbart and Yassine defined in Ref. 23 the notion of an "$\mathcal{F}$-pullback" of a cospan in $\mathscr{C}$ and the "span tightness" of the functor $\mathcal{F}$. They proved that if the functor $\mathcal{F}$ is span tight, then $\mathrm{Span}(\mathscr{C}, \mathcal{F})$ is a category, a "generalized span category," whose objects are the objects in $\mathscr{C}$ and whose morphisms are isomorphism classes of spans in $\mathscr{C}$. Composition in this generalized span category is defined using $\mathcal{F}$-pullbacks. Generalized span categories determine the kinematical properties of open classical systems in the Hamiltonian setting and of "free" open systems in the Lagrangian setting—that is, systems where all the energy is kinetic.

In Sec. III, we introduce the notion of an "augmented" span, which allows us to introduce nonzero Hamiltonians and add potentials to Lagrangians. In Sec. IV, we construct the augmented generalized span categories `HamSy` and `LagSy`. In the Hamiltonian setting, the augmentation determines the dynamical evolution of the system. In the Lagrangian setting, the augmentation determines the potential for the classical system, hence its dynamics as well. The categories `LagSy` and `HamSy` provide frameworks for studying open classical systems from the Lagrangian and Hamiltonian perspectives, respectively. Section V introduces a functor $\mathscr{L} : \mathtt{LagSy} \to \mathtt{HamSy}$. This functor, a version of the Legendre transformation, translates from the Lagrangian to the Hamiltonian perspective.

In future work, we hope to compare the present approach to the theory of port-Hamiltonian systems, an approach to open systems in classical mechanics widely used in engineering.[24]

II. SPANS AND GENERALIZED SPAN CATEGORIES

A. Spans and span categories

Refer to Ref. 23 for further discussion of the material presented here. We review for the reader's convenience some of the definitions and basic results from Ref. 23 that the current discussion requires.

A *span* in a category $\mathscr{C}$ is a pair of morphisms in $\mathscr{C}$ with the same source, and a *cospan* in $\mathscr{C}$ is a pair of morphisms in $\mathscr{C}$ with the same target. For any span S in $\mathscr{C}$, write

$$S = (s_L, s_R),$$

where S_L, S_R, and S_A are objects in $\mathscr{C}$,

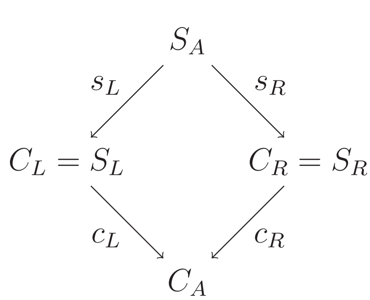
$$s_L : S_A \to S_L \quad \text{and} \quad s_R : S_A \to S_R.$$

Utilize the same notation if S is a cospan, but where s_L and s_R, respectively, map S_L and S_R to S_A. For any span or cospan S in $\mathscr{C}$, refer, respectively, to the objects S_A, S_L, and S_R in $\mathscr{C}$ as the apex, left foot, and right foot of S.

Definition 2.1. A span S in $\mathscr{C}$ is *paired with* a cospan C in $\mathscr{C}$ if

$$C_L = S_L, \quad C_R = S_R, \quad \text{and} \quad c_L \circ s_L = c_R \circ s_R.$$

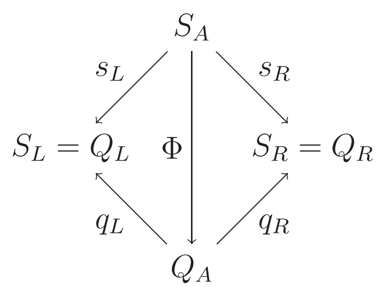
The pairing of a span S with a cospan C has a diagrammatical interpretation, namely, that the following diagram commutes:



Suppose that S and Q are spans in $\mathscr{C}$ with S_L equal to Q_L and S_R equal to Q_R. A *span morphism in* $\mathscr{C}$ *from* S *to* Q is a morphism Φ in $\mathscr{C}$ from S_A to Q_A such that

$$s_L = q_L \circ \Phi \quad \text{and} \quad s_R = q_R \circ \Phi,$$

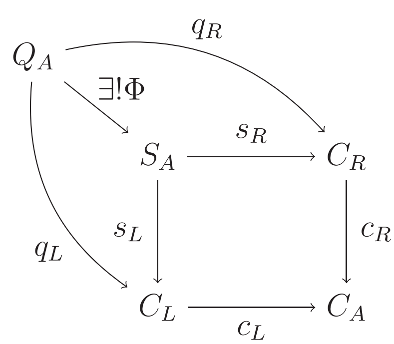
meaning that the following diagram commutes:



A *span isomorphism in* $\mathscr{C}$ *from* S *to* Q is a span morphism that is additionally an isomorphism.

Proposition 2.2. For any span isomorphism Φ*, the inverse* Φ^{-1} *is also a span isomorphism. Furthermore, any composite of span morphisms is again a span morphism.*

Definition 2.3. A span S in $\mathscr{C}$ is a *pullback* of a cospan C in $\mathscr{C}$ if it is paired with C and if for any other span Q in $\mathscr{C}$ that is also paired with C, there exists a unique span morphism Φ in $\mathscr{C}$ from Q to S:



Note that the diagram formed by pairing a span S with a cospan C, where S is a pullback of C, is a pullback diagram or a pullback square as often discussed in the literature.

Definition 2.4. A category $\mathscr{C}$ *has pullbacks* if for any cospan C in $\mathscr{C}$ there is a span S in $\mathscr{C}$ that is a pullback of C and S is unique up to a span isomorphism in $\mathscr{C}$.

Denote by `Top` the category whose objects are topological spaces and whose morphisms are continuous functions. The categories `Set` and `Top` are examples of categories that have pullbacks, as discussed in Ref. 23. If C is a cospan in `Set`, then let ρ_L and ρ_R be the canonical projections

$$\rho_L : C_L \times C_R \to C_L \quad \text{and} \quad \rho_R : C_L \times C_R \to C_R.$$

Denote by S_A the fibered product

$$C_L \times_{C_A} C_R := \{(x, y) \in C_L \times C_R : (c_L \circ \rho_L)(x, y) = (c_R \circ \rho_R)(x, y)\}.$$

Take S_L and S_R to be, respectively, equal to C_L and C_R, and let s_L and s_R be the respective restrictions of ρ_L and ρ_R to the set S_A. The span (s_L, s_R) is a pullback of C. If C is a cospan in Top, then S is again a pullback of C in Top, where the topology on S_A is the subspace topology induced by the product topology on $S_L \times S_R$.

B. The categories SympSurj and RiemSurj

Refer to Ref. 25 for further background on Poisson geometry. A *Poisson bracket* on a smooth manifold M is an anticommutative, bilinear function from $C^\infty(M) \times C^\infty(M)$ to $C^\infty(M)$ that satisfies Leibniz's rule and the Jacobi identity. A *Poisson manifold* is the pair consisting of a smooth manifold M and a Poisson bracket on M. Suppose that $(M, \{\cdot,\cdot\}_M)$ and $(N, \{\cdot,\cdot\}_N)$ are Poisson manifolds. For each f in $C^\infty(M)$, the *Poisson vector field associated with* f is the derivation v_f given by

$$v_f(\cdot) = \{\cdot, f\}_M.$$

A smooth map Φ from M to N is a *Poisson map* if for any f and g in $C^\infty(N)$,

$$\{f, g\}_N \circ \Phi \ = \ \{f \circ \Phi, g \circ \Phi\}_M.$$

Symplectic manifolds are the primary objects of study in Hamiltonian mechanics. A *symplectic manifold* is a pair (M, ω_M), where M is a smooth (necessarily even dimensional) manifold and ω_M is a smooth, closed, nondegenerate 2-form on M, a *symplectic 2-form*. Suppose that the dimension of M is $2m$. For each x in M, there is an open set U containing x such that the symplectic 2-form gives rise to *Darboux coordinates* $(q_i, p_i)_{i=1}^m$ on U, coordinates such that

$$\omega_M = \sum_{i=1}^m \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

The symplectic 2-form naturally distinguishes position and momentum coordinates on M and induces an isomorphism Ω_M between the tangent and cotangent bundles. Given tangent vectors v and w in the same fiber of TM, define by $\Omega_M(v)$ the covector

$$\Omega_M(v) = \omega_M(\cdot, v) : w \mapsto \omega_M(w, v).$$

Since ω_M is nondegenerate, the map Ω_M is invertible. For each function f in $C^\infty(M)$, denote by D_f the *symplectic gradient of* f, which is defined by

$$D_f = \Omega_M^{-1}(\mathrm{d}f).$$

Every symplectic manifold has a Poisson structure that it inherits from its symplectic structure in the following way. For any symplectic manifold (M, ω_M), define a Poisson bracket $\{\cdot,\cdot\}_M$ on pairs (f, g) in $C^\infty(M) \times C^\infty(M)$ by

$$\{f, g\}_M = \omega_M(D_f, D_g).$$

The symplectic gradient D_f is the Poisson vector field v_f associated with f, implying that

$$\{f, g\}_M = \omega_M(v_f, v_g).$$

The real valued function Π_M defined by

$$\Pi_M(\mathrm{d}f, \mathrm{d}g) = \{f, g\}_M$$

is a global section of $\left(T^*M \wedge T^*M\right)^*$. The *Poisson bivector of* $(M, \{\cdot,\cdot\}_M)$ is the image of the function Π_M under the canonical isomorphism that takes $\left(T^*M \wedge T^*M\right)^*$ to $\Lambda^2 TM$. To simplify notation, denote henceforth by Π_M the Poisson bivector of $(M, \{\cdot,\cdot\}_M)$. Refer to Ref. 26, p. 30 for Proposition 2.5 and see Ref. 26, p. 44 for a Proof of Proposition 2.6.

Proposition 2.5. A smooth map Φ from $(M, \{\cdot,\cdot\}_M)$ to $(N, \{\cdot,\cdot\}_N)$ is a Poisson map if and only if

$$\mathrm{d}\Phi(\Pi_M) = \Pi_N.$$

Proposition 2.6. Suppose that $(M, \{\cdot,\cdot\}_M)$ is a Poisson manifold and (N, ω_N) is a symplectic manifold. Every Poisson map from M to N is a submersion.

Riemannian manifolds are the primary objects of study in Lagrangian mechanics. The metric on the tangent bundle of a Riemannian manifold gives a kinetic energy associated with a particle moving in the base manifold, which is the configuration space for the system (Ref. 14, p.83–84). A *Riemannian submersion* Φ from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N) is a smooth submersion with the property that if v and w are vector fields tangent to the horizontal space $(\ker(\mathrm{d}\Phi))^{\perp}$, then

$$g_M(v, w) = g_N(\mathrm{d}\Phi(v), \mathrm{d}\Phi(w)).$$

Table I specifies the categories to be henceforth denoted by `Diff`, `SurjSub`, `RiemSurj`, and `SympSurj`.

An example in Ref. 23 shows that the category `SurjSub` does not have pullbacks. Since this example involves manifolds that have trivial Riemannian and symplectic structures and mappings in the respective categories, the categories `RiemSurj` and `SympSurj` also do not have pullbacks.

C. $\mathcal{F}$-pullbacks and span tight functors

Assume henceforth that $\mathscr{C}$ and $\mathscr{C}'$ are categories and that $\mathcal{F}$ is a functor from $\mathscr{C}$ to $\mathscr{C}'$. For any span S in $\mathscr{C}$, denote by $\mathcal{F}(S)$ the span $(\mathcal{F}(s_L), \mathcal{F}(s_R))$ in $\mathscr{C}'$. For any cospan C in $\mathscr{C}$, denote by $\mathcal{F}(C)$ the cospan $(\mathcal{F}(c_L), \mathcal{F}(c_R))$ in $\mathscr{C}'$.

Definition 2.7. The category $\mathscr{C}$ *has $\mathcal{F}$-pullbacks in* $\mathscr{C}'$ if for any cospan C in $\mathscr{C}$, there is a span S in $\mathscr{C}$ that is paired with C and the span $\mathcal{F}(S)$ is a pullback of the cospan $\mathcal{F}(C)$ in $\mathscr{C}'$. In this case, the span S is an *$\mathcal{F}$-pullback* of C.

Note that if $\mathscr{C}'$ is equal to $\mathscr{C}$ and $\mathcal{F}$ is the identity functor, then an $\mathcal{F}$-pullback is simply a pullback.

Definition 2.8. Suppose that $\mathscr{C}$ has $\mathcal{F}$-pullbacks in $\mathscr{C}'$. The functor $\mathcal{F}$ is *span tight* if for any $\mathcal{F}$-pullbacks S and Q of the same cospan, the unique span isomorphism Φ from $\mathcal{F}(S)$ to $\mathcal{F}(Q)$ is $\mathcal{F}(\Psi)$ for some span isomorphism Ψ from S to Q.

Definition 2.9. For any two spans S and Q in $\mathscr{C}$ such that S_R is equal to Q_L and there is a span P in $\mathscr{C}$ that is a pullback of the cospan (s_R, q_L), denote by $S \circ_P Q$ the span in $\mathscr{C}$ given by

$$S \circ_P Q = (s_L \circ p_L, q_R \circ p_R).$$

The span $S \circ_P Q$ is the *composite of S and Q along P*. If P is an $\mathcal{F}$-pullback, then the span $S \circ_P Q$ is an *$\mathcal{F}$-pullback composite of S and Q along P*.

Identify the objects of $\mathrm{Span}(\mathscr{C}, \mathcal{F})$ to be the objects in $\mathscr{C}$ and the isomorphism classes of spans in $\mathscr{C}$ to be the morphisms in $\mathrm{Span}(\mathscr{C}, \mathcal{F})$. If $[S]$ is an isomorphism class of spans in $\mathrm{Span}(\mathscr{C}, \mathcal{F})$, then identify S_R and S_L, respectively, to be the source and target of $[S]$. Define the composition of isomorphism classes of spans by

$$[S^1] \circ [S^2] = [S^1 \circ_P S^2],$$

where $S^1 \circ_P S^2$ is an $\mathcal{F}$-pullback composite of S^1 and S^2. For any object X in $\mathscr{C}$, denote by Id_X the identity morphism from X to X and by I_X the span $(\mathrm{Id}_X, \mathrm{Id}_X)$. Define by $[I_X]$ the identity morphism in $\mathrm{Span}(\mathscr{C}, \mathcal{F})$ from X to X. The following theorem is the main result of Ref. 23:

TABLE I. Table of categories.

Category Name	Objects	Morphisms
`Diff`	Smooth manifolds	Smooth maps
`SurjSub`	Smooth manifolds	Surjective submersions
`RiemSurj`	Riemannian manifolds	Surjective Riemannian submersions
`SympSurj`	Symplectic manifolds	Surjective Poisson maps

Theorem 2.10. *If $\mathcal{F}$ is a span tight functor from $\mathscr{C}$ to $\mathscr{C}'$, then* Span$(\mathscr{C}, \mathcal{F})$ *is a category.*

Suppose that X, Y, and Z are smooth manifolds. Suppose further that (f, g) is a cospan in `SurjSub`, where f and g have respective sources X and Y and mutual target Z. Again denote by ρ_X and ρ_Y the respective projections from $X \times Y$ to X and Y, and let π_X and π_Y be their respective restrictions to the embedded submanifold $X \times_Z Y$. Reference 23 proves Proposition 2.11. Proposition 2.11 and Theorem 2.10 together imply that Span(`SurjSub`, $\mathcal{F}$) is a category, where $\mathcal{F}$ is the forgetful functor from `SurjSub` to `Diff`.

Proposition 2.11. The span (π_X, π_Y) is an $\mathcal{F}$-pullback of (f, g), and so `SurjSub` *has $\mathcal{F}$-pullbacks. Moreover, the functor $\mathcal{F}$ is span tight.*

Since we will need to work in the categories `SympSurj` and `RiemSurj`, we will need to prove a similar result for these categories. Section III will provide such a result.

III. LAGRANGIAN AND HAMILTONIAN SYSTEMS

The description of a Lagrangian or Hamiltonian system, respectively, requires not only the identification of a span in `RiemSurj` or `SympSurj` but also the additional information of a potential or a Hamiltonian, both of which are *augmentations*.

A. Systems as isomorphism classes of augmented spans

We now introduce the notion of an augmentation of a span and cospan, but in the restricted settings that are significant to the current discussion. We will discuss augmentations in greater generality in an upcoming paper.

Definition 3.1. An *augmented manifold* is a pair (M, F_M), where M is a smooth manifold and F_M is a smooth real valued function defined on M. The pair (M, F_M) is an *augmented Riemannian (symplectic) manifold* if M is a Riemannian (symplectic) manifold. Refer to F_M as a *potential (or Hamiltonian)*, denoting it by V_M (or H_M) if M is, respectively, a Riemannian (or symplectic) manifold.

For sake of concision, denote by $\mathfrak{M}$ any of the categories listed in Table I.

Definition 3.2. An *augmented (co) span* in $\mathfrak{M}$ is a pair (S, F_S), where S is a (co)span in $\mathfrak{M}$ and F_S is a triple $(F_{S_A}, F_{S_L}, F_{S_R})$ of smooth real valued functions defined, respectively, on S_A, S_L, and S_R. If $\mathfrak{M}$ is `RiemSurj` (or `SympSurj`), then the given augmented span is a *Riemannian (co) span* [or *Poisson (co) span*]. A *classical (co) span* is a (co)span that is either Riemannian or Poisson. If (S, F_S) is a Riemannian (Poisson) span, then refer to F_S as a *potential (or Hamiltonian)* and denote it by V_S (or H_S).

The apex of a Poisson span determines the kinematical properties of the system, and the mapping of the apex to its feet determines the way in which the span composes with other spans and, therefore, how components of systems compose to form more complicated systems. The apex of a Riemannian span determines a free system, and the augmentation will be a potential that determines the interactions in the system. The fundamental object of our study should be an isomorphism class of augmented spans rather than an augmented span because the composition using $\mathcal{F}$-pullbacks is only determined up to isomorphism.

Definition 3.3. Suppose that the classical spans (S, F_S) and (Q, F_Q) are either both Riemannian or both Poisson and that

$$(S_L, F_{S_L}) = (Q_L, F_{Q_L}) \quad \text{and} \quad (S_R, F_{S_R}) = (Q_R, F_{Q_R}).$$

A span morphism Φ from S_A to Q_A is *compatible with* F_S and F_Q if F_{S_A} is equal to $F_{Q_A} \circ \Phi$ and is, in this case, a *morphism of classical spans*. If Φ is additionally an isomorphism, then Φ is an *isomorphism of classical spans* and (S, F_S) and (Q, F_Q) are *isomorphic classical spans*.

The inverse of an isometry is again an isometry. The inverse of a Poisson diffeomorphism is again a Poisson diffeomorphism (Ref. 27, p. 10). Proposition 3.4 follows from these facts.

Proposition 3.4. Suppose that S and Q are spans that are either both in `RiemSurj` *or both in* `SympSurj`*. The inverse of any span isomorphism from S to Q is a span isomorphism from Q to S.*

Denote by $[S, F_S]$ the set of all classical spans that are isomorphic to the classical span (S, F_S). Together with the fact that the composite of classical span isomorphisms is again a classical span isomorphism, Proposition 3.4 implies that isomorphism of classical spans is an equivalence relation; hence, the set $[S, F_S]$ is an equivalence class.

Definition 3.5. A *Lagrangian (or Hamiltonian) system* is an isomorphism class of Riemannian (or Poisson) spans. If $[S, F_S]$ is either a Hamiltonian system or a Lagrangian system, then $[S, F_S]$ is a *classical system*. Classical systems $[S, F_S]$ and $[Q, F_Q]$ are *of the same type* if they are both Hamiltonian systems or both Lagrangian systems.

B. Paths of motion

Suppose that (S, V_S) is a Riemannian span and g_{S_A} is the Riemannian metric on S_A. Denote by ρ_{S_A} the canonical projection from TS_A to S_A. Define the *Lagrangian of* (S, V_S) on TS_A to be the function $\mathcal{L}_S$, where

$$\mathcal{L}_S(v) = \frac{1}{2} g_{S_A}(v, v) - V_{S_A}(\rho_{S_A}(v)) \quad \text{with} \quad v \in TS_A.$$

Definition 3.6. A path in the Riemannian manifold (S_A, g_{S_A}) is a *path of motion of* (S, V_S) if it minimizes the action integral of $\mathcal{L}_S$ under smooth variations with fixed endpoints.

Denote by $\flat_{S_A}$ the function from TS_A to $T^* S_A$ that acts on each v in TS_A by

$$\flat_{S_A}(v) = g_{S_A}(v, \cdot).$$

The nondegeneracy of the metric g_{S_A} implies that the map $\flat_{S_A}$ is invertible. Denote by $\sharp_{S_A}$ the inverse of $\flat_{S_A}$ with

$$\sharp_{S_A} : T^* S_A \to TS_A \quad \text{by} \quad \theta \mapsto v, \quad \text{where} \quad \theta = g_{S_A}(v, \cdot) \quad \text{and} \quad (\theta, v) \in T^* S_A \times TS_A.$$

Denote by $\text{grad}_{S_A}(V_{S_A})$ the vector field

$$\text{grad}_{S_A}(V_{S_A}) = \sharp_{S_A}(\text{d}V_{S_A})$$

and by ∇^{S_A} the Levi-Civita connection on the Riemannian manifold (S_A, g_{S_A}). A standard calculation shows that γ is a path of motion of the Riemannian span (S, V_S) if and only if it satisfies

$$\nabla^{S_A}_{\gamma'} \gamma' + \text{grad}_{S_A}(V_{S_A})|_{\gamma} = 0, \tag{EL}$$

the Euler–Lagrange equations. See Ref. 28 for further explanation of the details in this section.

Definition 3.7. Suppose that (S, H_S) is a Poisson span. Denote by $\{\cdot, \cdot\}_{S_A}$ the Poisson bracket associated with the symplectic form ω_{S_A} on the symplectic manifold S_A. A path γ in S_A is a *path of motion of* (S, H_S) if it is an integral curve of the vector field υ where

$$\upsilon = \{\cdot, H_{S_A}\}_{S_A}.$$

Proposition 3.8. Suppose that (S, F_S) and (Q, F_Q) are classical spans of the same type and Φ is an isomorphism of classical spans taking (S, F_S) to (Q, F_Q). If γ is a path of motion of (S, F_S), then $\Phi \circ \gamma$ is a path of motion of (Q, F_Q). Furthermore, every path of motion of (Q, F_Q) is the image of a path of motion of (S, F_S).

Proof. If (S, V_S) and (Q, V_Q) are Riemannian spans and Φ is an isomorphism from (S, V_S) to (Q, V_Q), then Φ is an isometry from S_A to Q_A and V_{S_A} is equal to $V_{Q_A} \circ \Phi$. Denote by ∇^{S_A} and ∇^{Q_A} the respective Levi-Civita connections on S_A and Q_A. Suppose that p is an element of S_A and that X and Y are tangent vector fields on S_A. The map Φ is an isometry, and so

$$\text{d}\Phi_p\big((\nabla^{S_A}_X Y)(p)\big) = \nabla^{Q_A}_{\text{d}\Phi(X)} \text{d}\Phi(Y)(\Phi(p)) \quad \text{and} \quad \text{d}\Phi\big(\text{grad}_{S_A}(V_{Q_A} \circ \Phi)\big) = \text{grad}_{Q_A}(V_{Q_A}).$$

If γ is a path of motion of (S, V_S), then $\Phi \circ \gamma$ is a curve in Q_A and

$$\begin{aligned}
\nabla^{Q_A}_{(\Phi\circ\gamma)'} (\Phi \circ \gamma)' + \text{grad}_{Q_A}(V_{Q_A})|_{\Phi\circ\gamma} &= \nabla^{Q_A}_{\text{d}\Phi(\gamma')} \big(\text{d}\Phi(\gamma')\big) + \text{grad}_{Q_A}(V_{Q_A})|_{\Phi\circ\gamma} \\
&= \text{d}\Phi_p\Big(\nabla^{S_A}_{\gamma'}(\gamma') + \text{grad}_{S_A}(V_{S_A})|_{\gamma}\Big) \\
&= \text{d}\Phi_p(0) = 0,
\end{aligned}$$

where the fact that γ satisfies (EL) in S_A implies the penultimate equality. The path $\Phi \circ \gamma$ is therefore a path of motion of (Q, V_Q).

If (S, H_S) and (Q, H_Q) are Poisson spans and Φ is an isomorphism from (S, H_S) to (Q, H_Q), then Φ is a Poisson diffeomorphism from S_A to Q_A and H_{S_A} is equal to $H_{Q_A} \circ \Phi$. The curve γ is path of motion of (S, H_S) if and only if it is an integral curve of the vector field $\{\cdot, H_{S_A}\}$. Suppose that α and β are smooth functions on Q_A. Since Φ is Poisson,

$$\text{d}\Phi\big(\{\cdot, \alpha \circ \Phi\}_{S_A}\big)(\beta) = \{\cdot, \alpha \circ \Phi\}_{S_A}(\beta \circ \Phi) = \{\beta \circ \Phi, \alpha \circ \Phi\}_{S_A} = \{\beta, \alpha\}_{Q_A},$$

and so

$$(\Phi \circ \gamma)' = \mathrm{d}\Phi|_{\gamma}\big(\{\cdot, H_{S_A}\}_{S_A}\big)$$
$$= \mathrm{d}\Phi|_{\gamma}\big(\{\cdot, H_{Q_A} \circ \Phi\}_{S_A}\big) = \{\cdot, H_{Q_A}\}_{Q_A}|_{\Phi\circ\gamma}.$$

The curve $\Phi \circ \gamma$ is, therefore, a path of motion of (Q, H_Q).

In both the Riemannian and Poisson settings, the map Φ^{-1} is also an isomorphism of classical spans, and so every path of motion of (Q, F_Q) is the image of a path of motion of (S, F_S). □

C. $\mathcal{F}$-pullbacks of SympSurj and RiemSurj in Diff

Proposition 3.8 implies that an isomorphism class of classical spans determines the dynamics of a classical system. Composing such isomorphism classes requires both the existence of $\mathcal{F}$-pullbacks in these categories, where $\mathcal{F}$ is an appropriate forgetful functor into `Diff`, and the span tightness of the functor $\mathcal{F}$.

Suppose X is a symplectic manifold. The Poisson bivector Π_X of X induces a map $\widetilde{\Pi}_X$ from T^*X to TX that takes any η in T^*X to the vector field $\widetilde{\Pi}_X(\eta)$ with the property that for any ν in T^*X,

$$\nu\big(\widetilde{\Pi}_X(\eta)\big) = \Pi_X(\eta, \nu).$$

Since X is symplectic, the map $\widetilde{\Pi}_X$ is an isomorphism (Ref. 26, p. 17). This isomorphism gives a way to pull back vector fields by surjective Poisson maps, a fact that, along with Proposition 2.6, is critical to the Proof of Theorem 3.9. Theorem 3.9 establishes the existence of a local splitting of the tangent space of a symplectic manifold by a local foliation given by the inverse image of a surjective Poisson map.

Theorem 3.9. *Suppose that X and Z are symplectic manifolds with respective dimensions 2ℓ and $2n$ and that f is a surjective Poisson map from X to Z. Given any z in Z and a choice of Darboux coordinates $(q_i^Z, p_i^Z)_{i=1}^n$ on an open set U containing z and given any x in X with $f(x)$ equal to z, there exist Darboux coordinates $(q_i^X, p_i^X)_{i=1}^{\ell}$ on an open set V containing x such that for any i in $\{1,\ldots,n\}$,*

$$q_i^X = q_i^Z \circ f \quad \text{and} \quad p_i^X = p_i^Z \circ f.$$

Proof. Suppose that x_0 is in X, that U is an open set containing $f(x_0)$, and that $(q_i^Z, p_i^Z)_{i=1}^n$ is a Darboux coordinate system on U. Proposition 2.6 guarantees that f is a surjective submersion; hence, it is an open map, and so there is an open set V' containing x_0 with a Darboux coordinate system $(q_i^X, p_i^X)_{i=1}^{\ell}$ such that $f(V')$ is an open subset of U. Denote by $\mathcal{H}$ the set of all vector fields υ on $f(V')$ for which there is some α in $C^\infty(f(V'))$ such that for any β in $C^\infty(f(V'))$,

$$\upsilon(\beta) = \{\beta, \alpha\}_Z.$$

Denote such a vector field by υ_α. Denote by $f^*(\mathcal{H})$ the set of all vector fields w on V' for which there is an α in $C^\infty(f(V'))$ such that for any h in $C^\infty(V')$,

$$w(h) = \{h, \alpha \circ f\}_X.$$

Denote such a vector field by w_α. For any x in V' and any z in $f(V')$, denote, respectively, by $f^*(\mathcal{H})(x)$ and $\mathcal{H}(z)$ the set of all vector fields in $f^*(\mathcal{H})$ evaluated at x and the set of all vector fields in $\mathcal{H}$ evaluated at z. The bilinearity of the bracket implies that $\mathcal{H}(z)$ and $f^*(\mathcal{H})(x)$ are vectors spaces. Since

$$\upsilon_{-q_i^Z} = \frac{\partial}{\partial p_i^Z} \quad \text{and} \quad \upsilon_{p_i^Z} = \frac{\partial}{\partial q_i^Z},$$

for any z in $f(V')$, the vector space $\mathcal{H}(z)$ spans $T_z(U)$.

Let F be the function

$$F : \mathcal{H} \to f^*(\mathcal{H}) \quad \text{by} \quad F(\upsilon_\alpha) = w_\alpha.$$

The fact that f is Poisson implies that

$$\begin{aligned} \mathrm{d}f(w_\alpha)(\beta) &= w_\alpha(\beta\circ f) \\ &= \{\beta\circ f,\alpha\circ f\}_X \\ &= \{\beta,\alpha\}_Z = v_\alpha(\beta), \end{aligned}$$

and so

$$\mathrm{d}f(F(v_\alpha)) = v_\alpha.$$

Similarly, for any w_α in $f^*(\mathcal{H})$,

$$F(\mathrm{d}f(w_\alpha)) = F(v_\alpha) = w_\alpha.$$

The maps F and $\mathrm{d}f|_{\mathcal{H}}$ are therefore inverses of each other, and so for each x in V', the vector spaces $\mathcal{H}(f(x))$ and $f^*(\mathcal{H})(x)$ are isomorphic. Both of these vector spaces have the same dimension as Z.

For any w_α and $w_{\alpha'}$ in $f^*(\mathcal{H})$, the Jacobi identity implies that

$$\begin{aligned} [w_\alpha, w_{\alpha'}]_{TX} &= w_\alpha(w_{\alpha'}(\beta)) - w_{\alpha'}(w_\alpha(\beta)) \\ &= \{w_{\alpha'}(\beta),\alpha\circ f\}_X - \{w_\alpha(\beta),\alpha'\circ f\}_X \\ &= \{\{\beta\circ f,\alpha'\circ f\}_X,\alpha\circ f\}_X - \{\{\beta\circ f,\alpha\circ f\}_X,\alpha'\circ f\}_X \\ &= \{\beta,\{\alpha'\circ f,\alpha\circ f\}_X\}_X = w_{\{\alpha,\alpha'\}}(\beta), \end{aligned}$$

and so the space of vector fields $f^*(\mathcal{H})$ is closed under the bracket $[\cdot,\cdot]_{TX}$ on TX. The Frobenius Theorem for involutive distributions implies that for any x in V', there is a submanifold W of V' such that $f^*(\mathcal{H})(x)$ is the tangent space T_xW. Since

$$f^*(\mathcal{H})(x)\cap \ker(\mathrm{d}f|_x) = \{0\},$$

the rank-nullity theorem implies that

$$T_xV' = f^*(\mathcal{H})(x)\oplus \ker(\mathrm{d}f|_x).$$

Define the function g from W to Z to be the restriction of f to the submanifold W. The form $g^*(\omega_Z)$ is a closed 2-form on W as it is the pullback of the closed 2-form ω_Z restricted to $f(V')$. Suppose that there is a v in TW such that for all w in TW, $g^*(\omega_Z)(v,w)$ is equal to zero. In this case,

$$0 = g^*(\omega_Z)(v,w) = \omega_Z(\mathrm{d}g(v),\mathrm{d}g(w)),$$

and so

$$\omega_Z(\mathrm{d}g(v),\cdot) = 0$$

since $\mathrm{d}g|_x$ is surjective at each point x of W. Nondegeneracy of ω_Z implies that $\mathrm{d}g(v)$ is equal to zero, and the injectivity of $\mathrm{d}g$ further implies that v is equal to zero. The form $g^*(\omega_Z)$ is, therefore, a symplectic form on W.

For any (η,ζ) in $C^\infty(V')\times C^\infty(V')$,

$$\begin{aligned} f^*(\omega_Z)|_x(w_\eta,w_\zeta) &= \omega_Z(\mathrm{d}f(w_\eta),\mathrm{d}f(w_\zeta))|_{f(x)} \\ &= \omega_Z(v_\eta,v_\zeta)|_{f(x)} \\ &= \{\eta,\zeta\}_Z|_{f(x)} \\ &= \{\eta\circ f,\zeta\circ f\}_X|_x = \omega_X(w_\eta,w_\zeta)|_x, \end{aligned} \tag{1}$$

where the assumption that f is Poisson implies the penultimate equality. The pullback $f^*(\omega_Z)$ is therefore the restriction of ω_X to $TW\times TW$. The manifold W is an embedded symplectic submanifold of V', and so Ref. 29, p.124, Exercise 3.38, implies that there is an open set V of V' that contains x_0 and a Darboux coordinate system $(q_i^X, p_i^X)_{i=1}^{\ell}$ on V such that for any x in V and i strictly larger than n,

$$q_i^X(x) = p_i^X(x) = 0.$$

Define

$$\omega_A = \sum_{i=1}^{n} \mathrm{d}q_i^X \wedge \mathrm{d}p_i^X \quad \text{and} \quad \omega_B = \sum_{i=n+1}^{\ell} \mathrm{d}q_i^X \wedge \mathrm{d}p_i^X$$

so that in the open set V, ω_X is equal to the sum of ω_A and ω_B. The form ω_B is the restriction of ω_X to $(TW \times TW) \cap (TV \times TV)$, and so (1) implies that ω_B is equal to $f^*(\omega_X)$. Furthermore, for any θ in $C^\infty(U)$,

$$\begin{aligned}(f^*(\mathrm{d}q_i^Z))(w_\theta)|_x &= \mathrm{d}q_i^Z(\mathrm{d}f(w_\theta))|_x \\ &= \mathrm{d}q_i^Z(v_\theta)|_{f(x)} \\ &= v_\theta(q_i^Z)|_{f(x)} \\ &= \{q_i^Z, \theta\}_Z|_{f(x)} \\ &= \{q_i^Z \circ f, \theta \circ f\}_X|_x = \mathrm{d}(q_i^Z \circ f) w_\theta|_x.\end{aligned}$$

Every element of TW is of the form w_θ for some θ in $C^\infty(U)$, implying that

$$f^*(\mathrm{d}q_i^Z) = \mathrm{d}(q_i^Z \circ f) \quad \text{and} \quad f^*(\mathrm{d}p_i^Z) = \mathrm{d}(p_i^Z \circ f). \tag{2}$$

Use (2) together with the coordinate representation of ω_Z to obtain the equality

$$f^*(\omega_Z) = \sum_{i=1}^{n} \mathrm{d}(q_i^Z \circ f) \wedge \mathrm{d}(p_i^Z \circ f),$$

which implies that in the open set V,

$$\omega_X = \sum_{i=1}^{n} \mathrm{d}(q_i^Z \circ f) \wedge \mathrm{d}(p_i^Z \circ f) + \sum_{i=n+1}^{\ell} \mathrm{d}q_i^X \wedge \mathrm{d}p_i^X.$$

The coordinate system ϕ on V given by

$$\phi = (q_1^Z \circ f, p_1^Z \circ f, \dots, q_n^Z \circ f, p_n^Z \circ f, q_{n+1}^X, p_{n+1}^X, \dots, q_\ell^X, p_\ell^X)$$

is, therefore, a Darboux coordinate system on V. □

Denote by π_Z the map

$$\pi_Z = f \circ \pi_X = g \circ \pi_Y,$$

where π_X and π_Y are the projections from $X \times_Z Y$ to Z. More generally, for any span Q that is paired with a cospan (f, g), define by q_M the map

$$q_M = f \circ q_L = g \circ q_R.$$

Theorem 3.10. *Suppose that (f, g) is a cospan in* `SympSurj` *with*

$$f : X \to Z \quad \text{and} \quad g : Y \to Z,$$

with 2ℓ, $2m$, $2n$ being the respective dimensions of X, Y, and Z, and suppose that ω_X, ω_Y, and ω_Z are the respective symplectic forms on X, Y, and Z. Suppose that Q is a span in `SympSurj` *that is paired with (f, g), and suppose that Q_A has dimension $2(\ell + m - n)$. The 2-form ω_{Q_A}, given by*

$$\omega_{Q_A} = q_L^*(\omega_X) + q_R^*(\omega_Y) - q_M^*(\omega_Z),$$

is the symplectic form on Q_A*. Moreover, the 2-form* ω*, given by*

$$\omega = \pi_X^*(\omega_X) + \pi_Y^*(\omega_Y) - \pi_Z^*(\omega_Z),$$

is the unique symplectic form on $X \times_Z Y$ *with the property that* (π_X, π_Y) *is paired with* (f, g)*.*

Proof. Suppose that a is in Q_A. Since Z is a symplectic manifold, there is on some open set U_Z containing $q_M(a)$ a Darboux coordinate system Ψ^Z with

$$\Psi^Z = \left(q_k^Z, p_k^Z\right)_{k\in\{1,\ldots,n\}} : U_Z \to \mathbb{R}^{2n}.$$

Since $q_M(a)$ is equal to $f(q_L(a))$, Theorem 3.9 implies that there is an open set U_X containing $q_L(a)$ and a Darboux coordinate system Ψ^X on U_X with

$$\Psi^X = \left(q_i^X, p_i^X, q_k^Z \circ f, p_k^Z \circ f\right)_{\substack{i\in\{1,\ldots,\ell-n\}\\ k\in\{1,\ldots,n\}}} : U_X \to \mathbb{R}^{2\ell}.$$

Similarly, there is an open set U_Y containing $q_R(a)$ and a Darboux coordinate system Ψ^Y on U_Y with

$$\Psi^Y = \left(q_j^Y, p_j^Y, q_k^Z \circ g, p_k^Z \circ g\right)_{\substack{j\in\{1,\ldots,m-n\}\\ k\in\{1,\ldots,n\}}} : U_Y \to \mathbb{R}^{2m}.$$

For each k in $\{1, \ldots, n\}$, the equality of $f \circ q_L$ and $g \circ q_R$ implies that

$$q_k^Z \circ f \circ q_L = q_k^Z \circ g \circ q_R = q_k^Z \circ q_M \quad \text{and} \quad p_k^Z \circ f \circ q_L = p_k^Z \circ g \circ q_R = p_k^Z \circ q_M.$$

Furthermore, there is an open set U containing a with the property that $q_L(U)$ and $q_R(U)$ are, respectively, subsets of U_X and U_Y. Denote, respectively, by $\tilde{q}_i^X, \tilde{p}_i^X, \tilde{q}_j^Y, \tilde{p}_j^Y, \tilde{q}_k^Z$, and $\tilde{p}_k^Z$ the functions $q_i^X \circ q_L, p_i^X \circ q_L, q_j^Y \circ q_R, p_j^Y \circ q_R, q_k^Z \circ q_M$, and $p_k^Z \circ q_M$ acting on Q_A. The map Ψ given by

$$\Psi = \left(\tilde{q}_i^X, \tilde{p}_i^X, \tilde{q}_j^Y, \tilde{p}_j^Y, \tilde{q}_k^Z, \tilde{p}_k^Z\right)_{\substack{i\in\{1,\ldots,\ell-n\}\\ j\in\{1,\ldots,m-n\}\\ k\in\{1,\ldots,n\}}} : U \to \mathbb{R}^{2(\ell+m-n)}$$

is a homeomorphism from U to an open subset of $\mathbb{R}^{2(\ell+m-n)}$ and hence a coordinate system on U that is a Darboux coordinate system. The 2-form ω_{Q_A} is therefore the form

$$\omega_{Q_A} = \sum_{i=1}^{\ell-n} \mathrm{d}\tilde{q}_i^X \wedge \mathrm{d}\tilde{p}_i^X + \sum_{j=1}^{m-n} \mathrm{d}\tilde{q}_j^Y \wedge \mathrm{d}\tilde{p}_j^Y + \sum_{k=1}^{n} \mathrm{d}\tilde{q}_k^Z \wedge \mathrm{d}\tilde{p}_k^Z,$$

proving that if there is a span Q with the given properties, then the symplectic form on Q_A is determined by the cospan (f, g). It does not, however, prove that there is such a span.

Proposition 3.6 of Ref. 23 implies that $X \times_Z Y$ is a smooth manifold of dimension $2(\ell + m - n)$. Suppose v is in $T_a(X \times_Z Y)$, and for any w in $T_a(X \times_Z Y)$, $\omega(v, w)$ is zero. There are coefficients $a^i, b^i, c^j, e^j, s^k, t^k$ such that, using Einstein summation convention,

$$v = a^i \partial\tilde{q}_i^X + b^i \partial\tilde{p}_i^X + c^j \partial\tilde{q}_j^Y + e^j \partial\tilde{p}_j^Y + s^k \partial\tilde{q}_k^Z + t^k \partial\tilde{p}_k^Z.$$

For a fixed i,

$$-\omega(v, \partial\tilde{q}_i^X) = b^i = 0.$$

A similar calculation shows that all of the given coefficients are zero, implying that v is equal to zero and so ω is nondegenerate. The form ω is the sum of pullbacks of smooth closed forms, and so smooth and closed itself, hence symplectic. The construction of ω ensures that the smooth surjections π_X and π_Y are Poisson maps on the symplectic manifold $(X \times_Z Y, \omega)$; hence, (π_X, π_Y) is paired with (f, g). □

Theorem 3.11. *Suppose that (f,g) is a cospan in* `RiemSurj` *with*

$$f : X \to Z \quad \text{and} \quad g : Y \to Z$$

and that g_X, g_Y, and g_Z are the metric tensors on X, Y, and Z, respectively. The tensor $g_{X\times_Z Y}$, given by

$$g_{X\times_Z Y} = \pi_X^*(g_X) + \pi_Y^*(g_Y) - \pi_Z^*(g_Z),$$

is the unique metric tensor on $X \times_Z Y$ such that the span (π_X, π_Y) is paired with (f,g).

Proof. Since every surjective Riemannian submersion is a surjective submersion, the fibered product $X \times_Z Y$ is a smooth manifold. If $g_{X\times_Z Y}$ is positive definite, then $(X \times_Z Y, g_{X\times_Z Y})$ is a Riemannian manifold since $g_{X\times_Z Y}$ is a symmetric tensor as a sum of pullbacks of symmetric tensors. It suffices to show that $g_{X\times_Z Y}$ is nondegenerate.

Follow the Proof of Theorem 3.10 using the splitting of the tangent spaces

$$TX = (\ker(\mathrm{d}f))^\perp \oplus (\ker(\mathrm{d}f)) \quad \text{and} \quad TY = (\ker(\mathrm{d}g))^\perp \oplus (\ker(\mathrm{d}g))$$

rather than the previous appeal to Theorem 3.9 to obtain an expression for $g_{X\times_Z Y}$ in local coordinates. Together with this local coordinate representation of $g_{X\times_Z Y}$, the fact that the maps π_X, π_Y, and π_Z are surjective Riemannian submersions implies that $g_{X\times_Z Y}$ is nondegenerate. The proof is similar to the Proof of Theorem 3.10, and so the details are left to the reader to verify. □

Note that the symplectic form on $X \times_Z Y$ in Theorem 3.10 is not the pullback by the inclusion map of the symplectic form on $X \times Y$ to the manifold $X \times_Z Y$. While the pullback form is symplectic, the span (π_X, π_Y) will no longer be a span in `SympSurj` when $X \times_Z Y$ is endowed instead with the pullback form. The analogous statements about the potential choices for the metric tensor are true in the Riemannian setting.

Theorem 3.12. *The forgetful functors from* `SympSurj` *to* `Diff` *and from* `RiemSurj` *to* `Diff` *are span tight.*

Proof. Suppose that $\mathcal{F}$ is the forgetful functor from `SympSurj` to `Diff`. Since every morphism in `SympSurj` is a surjective submersion, the functor $\mathcal{F}$ maps `SympSurj` to the subcategory `SurjSub` of `Diff`. If (f,g) is a cospan in `SympSurj` and π_X and π_Y are, as defined above, the respective projections from $X \times_Z Y$ to X and Y, then Proposition 2.11 implies that $(\mathcal{F}(\pi_X), \mathcal{F}(\pi_Y))$ is a span in `Diff` that is a pullback of the cospan $(\mathcal{F}(f), \mathcal{F}(g))$. Therefore, `SympSurj` has $\mathcal{F}$-pullbacks in `Diff`. Suppose now that Q is a span in `SympSurj` that is also an $\mathcal{F}$-pullback of (f,g). In this case, the span $\mathcal{F}(Q)$ is a span in `Diff` that is a pullback of $(\mathcal{F}(f), \mathcal{F}(g))$, and so there is a span diffeomorphism Φ from $\mathcal{F}(Q)$ to $\mathcal{F}(X \times_Z Y)$. Since Φ is a span morphism,

$$\mathcal{F}(q_L) \circ \Phi^{-1} = \mathcal{F}(\pi_X), \quad \mathcal{F}(q_R) \circ \Phi^{-1} = \mathcal{F}(\pi_Y), \quad \text{and} \quad \mathcal{F}(f) \circ \mathcal{F}(q_L) \circ \Phi^{-1} = \mathcal{F}(\pi_Z). \tag{3}$$

Denote, respectively, by ω, ω_X, ω_Y, and ω_Z the symplectic forms on $X \times_Z Y$, X, Y, and Z. The equalities of (3) imply that

$$\begin{aligned}
\omega &= \mathcal{F}(\pi_X)^*(\omega_X) + \mathcal{F}(\pi_Y)^*(\omega_Y) - \mathcal{F}(\pi_Z)^*(\omega_Z) \\
&= \left(\mathcal{F}(q_L) \circ \Phi^{-1}\right)^*(\omega_X) + \left(\mathcal{F}(q_R) \circ \Phi^{-1}\right)^*(\omega_Y) - \left(\mathcal{F}(f) \circ \mathcal{F}(q_L) \circ \Phi^{-1}\right)^*(\omega_Z) \\
&= \left(\Phi^{-1}\right)^*\left(\mathcal{F}(q_L)^*(\omega_X) + \mathcal{F}(q_R)^*(\omega_Y) - (\mathcal{F}(f) \circ \mathcal{F}(q_L))^*(\omega_Z)\right) \\
&= \left(\Phi^{-1}\right)^*(\omega_{Q_A}),
\end{aligned}$$

where ω_{Q_A} is the unique 2-form on Q_A such that Q is paired with (f,g). Let Ψ be the map from (Q_A, ω_{Q_A}) to $(X \times_Z Y, \omega)$ that acts as Φ on the underlying manifolds. The map Ψ is, therefore, a diffeomorphism and Ψ^{-1} is a symplectic map; hence, Ψ is a symplectomorphism. Since every symplectomorphism is a Poisson diffeomorphism, Ψ is an isomorphism in the category `SympSurj` with $\mathcal{F}(\Psi)$ equal to Φ (Ref. 30, p. 195).

A similar argument proves the theorem in the case of `RiemSurj`. □

Corollary. If $\mathcal{F}$ is the forgetful functor from $\texttt{SympSurj}$ *to* $\texttt{Diff}$ *(respectively,* $\texttt{RiemSurj}$ *to* $\texttt{Diff}$*), then* $\text{Span}(\texttt{SympSurj}, \mathcal{F})$ *[respectively,* $\text{Span}(\texttt{RiemSurj}, \mathcal{F})$*] is a category.*

While Theorems 2.10 and 3.12 imply that $\text{Span}(\text{SympSurj}, \mathcal{F})$ and $\text{Span}(\text{RiemSurj}, \mathcal{F})$ are categories, where $\mathcal{F}$ is the appropriate forgetful functor into $\texttt{Diff}$, to show that classical systems are morphisms of a category requires additional verifications. Section IV provides the necessary verifications.

IV. CLASSICAL SYSTEMS AS MORPHISMS

This section constructs the categories $\texttt{LagSy}$ and $\texttt{HamSy}$, whose objects are, respectively, augmented Riemannian manifolds or augmented symplectic manifolds and whose morphisms are isomorphism classes of the classical spans appropriate to the given category.

A. The categories HamSy and LagSy

Definition 4.1. The classical system $[S, F_S]$ is *composable* with the classical system $[Q, F_Q]$ if:

(i) both are classical systems of the same type; and
(ii) if (S, F_S) and (Q, F_Q) are respective representatives of the equivalence classes $[S, F_S]$ and $[Q, F_Q]$, then (S_R, F_{S_R}) is equal to (Q_L, F_{Q_L}).

Assume below that the classical system $[S, F_S]$ is composable with $[Q, F_Q]$ and (S, F_S) and (Q, F_Q) are, respectively, representatives of $[S, F_S]$ and $[Q, F_Q]$. To simplify notation, let

$$S_A = X,\ S_L = V,\ S_R = Q_L = Z,\ Q_A = Y,\ \text{and}\ Q_R = W.$$

Again denote by $X \times_Z Y$ the fibered product and by π_X, π_Y, and π_Z the respective projections to X, Y, and Z. Define by $[S, F_S] \circ [Q, F_Q]$ the augmented span given by

$$[S, F_S] \circ [Q, F_Q] = [(s_L \circ \pi_X, q_R \circ \pi_Y), F_{S \circ Q}],$$

where

$$F_{S \circ Q} = (F_X \circ \pi_X + F_Y \circ \pi_Y - F_Z \circ \pi_Z, F_V, F_W).$$

Theorem 4.2. *The Hamiltonian systems are the morphisms in a category,* $\texttt{HamSy}$*, whose objects are augmented symplectic manifolds. The Lagrangian systems are the morphisms in a category,* $\texttt{LagSy}$*, whose objects are augmented Riemannian manifolds.*

Proof. To prove the theorem, it suffices to show that (i) the composition of morphisms in $\texttt{HamSy}$ and in $\texttt{LagSy}$ is well defined; (ii) both $\texttt{HamSy}$ and $\texttt{LagSy}$ have left and right unit laws; and (iii) the composition of morphisms in $\texttt{HamSy}$ and in $\texttt{LagSy}$ is associative. Since $\text{Span}(\texttt{RiemSurj}, \mathcal{F})$ and $\text{Span}(\texttt{SympSurj}, \mathcal{F})$ are categories, to show that $\texttt{HamSy}$ and $\texttt{LagSy}$ are categories, it suffices to show that the augmentations are compatible with the various span isomorphisms that arise in defining the categories $\text{Span}(\texttt{RiemSurj}, \mathcal{F})$ and $\text{Span}(\texttt{SympSurj}, \mathcal{F})$. Suppose that $[S, F_S]$ and $[Q, F_Q]$ are both classical systems, and denote by $\mathcal{F}$ the appropriate forgetful functor from either $\texttt{SympSurj}$ or $\texttt{RiemSurj}$ to $\texttt{Diff}$.

(i) Suppose that $[S', F_{S'}]$ is equal to $[S, F_S]$ and that α is an isomorphism of augmented spans with

$$\alpha : X = S_A \to S'_A.$$

Suppose that $[Q', F_{Q'}]$ is equal to $[Q, F_Q]$ and that β is an isomorphism of augmented spans with

$$\beta : Y = Q_A \to Q'_A.$$

Since (Z, F_Z) is the right foot of (S, F_S) and the left foot of (Q, F_Q),

$$(S'_R, F_{S'_R}) = (Q'_L, F_{Q'_L}) = (Z, F_Z).$$

If P is an $\mathcal{F}$-pullback of (s'_R, q'_L), then there is a span isomorphism Φ with

$$\Phi : X \times_Z Y \to P_A.$$

The augmented span $(S', F_{S'}) \circ_P (Q', F_{Q'})$ is given by

$$(S', F_{S'}) \circ_P (Q', F_{Q'}) = ((s'_L \circ p_L, q'_R \circ p_R), F_{S' \circ_P Q'}),$$

where

$$F_{S' \circ_P Q'} = (F_{S'_A} \circ p_L + F_{Q'_A} \circ p_R - F_Z \circ s'_R \circ p_L, F_V, F_W).$$

Since α and β are isomorphisms of augmented spans,

$$F_{S'_A} \circ \alpha = F_X \quad \text{and} \quad F_{Q'_A} \circ \beta = F_Y.$$

The function Φ is a span isomorphism, and so

$$p_L \circ \Phi = \alpha \circ \pi_X \quad \text{and} \quad p_R \circ \Phi = \beta \circ \pi_Y,$$

and hence,

$$F_{S'_A} \circ p_L \circ \Phi = F_{S'_A} \circ \alpha \circ \pi_X = F_X \circ \pi_X.$$

Similar arguments show that

$$F_{Q'_A} \circ p_R \circ \Phi = F_Y \circ \pi_Y \quad \text{and} \quad F_Z \circ s'_R \circ p_L \circ \Phi = F_Z \circ \pi_Z,$$

and so

$$F_{S \circ Q} = (F_{S' \circ_P Q'}) \circ \Phi. \tag{4}$$

Equality (4) implies that Φ is an augmented span isomorphism; hence, the composition of $[S, F_S]$ and $[Q, F_Q]$ is independent of representative. The composite $[S, F_S] \circ [Q, F_Q]$ is, therefore, a well-defined morphism from (Q_R, F_{Q_R}) to (S_L, F_{S_L}).

(ii) Let $[S, F_S]$ be a morphism with the source (S_R, F_{S_R}) and target (S_L, F_{S_L}). Let $(I_{S_R}, F_{I_{S_R}})$ be a representative of the identity augmented span with source (S_R, F_{S_R}) and target (S_R, F_{S_R}). The equality

$$[S] \circ [I_{S_R}] = [S]$$

follows from the fact that both Span(SympSurj, $\mathcal{F}$) and Span(RiemSurj, $\mathcal{F}$) are categories. Let the span P be an $\mathcal{F}$-pullback of (s_R, Id_{S_R}), where

$$P_L = P_A = S_A,\ P_R = S_R,\ p_L = \mathrm{Id}_{S_A},\ \text{and}\ p_R = s_R.$$

The equalities

$$\begin{aligned} F_{P_A} &= F_{S_A} \circ p_L + F_{S_R} \circ s_R - F_{S_R} \circ s_R \circ p_L \\ &= F_{S_A} \circ \mathrm{Id}_{S_A} + F_{S_R} \circ s_R - F_{S_R} \circ s_R \circ \mathrm{Id}_{S_A} = F_{S_A} \end{aligned}$$

imply that there is an augmented span isomorphism from $(S, F_S) \circ (\mathrm{I}_{S_R}, F_{S_R})$ to (S, F_S), and so

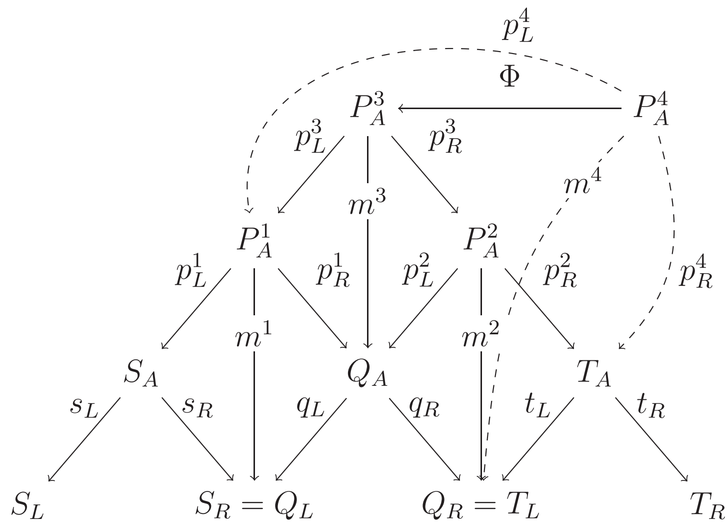
$$[S, F_S] \circ [\mathrm{I}_{S_R}, F_{S_R}] = [S, F_S].$$

A similar argument shows that

$$[\mathrm{I}_{S_L}, F_{S_L}] \circ [S, F_S] = [S, F_S].$$

Therefore, `HamSy` and `LagSy` have left and right unit laws.

(iii) Refer to the following diagram for the naming of the maps below, where all spans paired with a given cospan are augmented $\mathcal{F}$-pullbacks of the given cospan and the diagram is commutative:



Let (P^3, F_{P^3}) be an $\mathcal{F}$-pullback of (p_R^1, p_L^2), and let (P^4, F_{P^4}) be an $\mathcal{F}$-pullback of $(q_R \circ p_R^1, t_L)$.

To prove (iii), show first that there is an augmented span isomorphism from the augmented span $\big((S, F_S) \circ_{(P^1, F_{p^1})} (Q, F_Q)\big) \circ_{(P^4, F_{p^4})} (T, F_T)$ to the augmented span (P, F_P) that is given by the composite $\big((S, F_S) \circ_{(P^1, F_{p^1})} (Q, F_Q)\big) \circ_{(P^3, F_{p^3})} \big((Q, F_Q) \circ_{(P^2, F_{p^2})} (T, F_T)\big)$. A similar argument will show that there is an augmented span isomorphism from the augmented span $(S, F_S) \circ ((Q, F_Q) \circ (T, F_T))$ to (P, F_P), and the result follows by the fact that inverses and compositions of augmented span isomorphisms are augmented span isomorphisms. Since Lemma 5.3 of Ref. 23 proves the existence of a span isomorphism between the non-augmented spans, it suffices to show that this span isomorphism is compatible with the augmentations for the two composite spans.

The commutativity of the diagram above and the definition of the composition of augmented spans together imply that

$$\begin{aligned}
F_{P_A^4} &= F_{P_A^1} \circ p_L^4 + F_{T_A} \circ p_R^4 - F_{Q_R} \circ m^4 \\
&= F_{P_A^1} \circ p_L^3 \circ \Phi + F_{T_A} \circ p_R^2 \circ p_R^3 \circ \Phi - F_{Q_R} \circ m^2 \circ p_R^3 \circ \Phi. \\
&= \left(F_{P_A^1} \circ p_L^3 + F_{T_A} \circ p_R^2 \circ p_R^3 - F_{Q_R} \circ m^2 \circ p_R^3\right) \circ \Phi \\
&= \left(F_{P_A^1} \circ p_L^3 + \left(F_{T_A} \circ p_R^2 - F_{Q_R} \circ m^2\right) \circ p_R^3\right) \circ \Phi \\
&= \left(F_{P_A^1} \circ p_L^3 + \left(F_{Q_A} \circ p_L^2 - F_{Q_A} \circ p_L^2 + F_{T_A} \circ p_R^2 - F_{Q_R} \circ m^2\right) \circ p_R^3\right) \circ \Phi \\
&= \left(F_{P_A^1} \circ p_L^3 + \left(F_{Q_A} \circ p_L^2 + F_{T_A} \circ p_R^2 - F_{Q_R} \circ m^2\right) \circ p_R^3 - F_{Q_A} \circ p_L^2 \circ p_R^3\right) \circ \Phi \\
&= \left(F_{P_A^1} \circ p_L^3 + \left(F_{Q_A} \circ p_L^2 + F_{T_A} \circ p_R^2 - F_{Q_R} \circ m^2\right) \circ p_R^3 - F_{Q_A} \circ m^3\right) \circ \Phi \\
&= \left(F_{P_A^1} \circ p_L^3 + F_{P^2} \circ p_R^3 - F_{Q_A} \circ m^3\right) \circ \Phi = F_{P_A^3} \circ \Phi.
\end{aligned}$$

Therefore, the span isomorphism Φ is compatible with the augmentations F_{P^4} and F_{P^3}. □

B. Motivating example

Suppose that the spring-mass system with three masses given in Sec. I has masses m_1, m_2, and m_3, respectively, as the left, middle, and right masses of the system. Suppose further that the spring constants of the left and right springs are, respectively, k_1 and k_2. The spring-mass system with three masses is a composite of two spring-mass systems with two masses each. We now discuss a category theoretic construction of a model for the composite system with its subsystems.

Let $[S, V_S]$ be a Lagrangian system describing the left spring-mass system and $[Q, V_Q]$ be a Lagrangian systems describing the right spring-mass system. Denote both S_R and Q_L by Z, since S_R is equal to Q_L, and by V_Z the augmentation on Z. Take a representative (S, V_S) of the Langrangian system $[S, V_S]$ to be the Riemannian span with the manifold S_A equal to $\mathbb{R}^2$ and the manifolds S_L and Z equal to $\mathbb{R}$. Let g_1 be the standard Riemannian metric on $\mathbb{R}$. Let ρ_L and ρ_R be the canonical projections on $\mathbb{R}^2$ with

$$\rho_L(q_1, q_2) = q_1 \quad \text{and} \quad \rho_R(q_1, q_2) = q_2.$$

Denote by g_2 the standard Riemannian metric on $\mathbb{R}^2$. Endow S_L with the Riemannian metric g_{S_L} and Z with the Riemannian metric g_Z, where g_{S_L} and g_Z are given by

$$g_{S_L} = m_1 g_1 \quad \text{and} \quad g_Z = m_2 g_1.$$

Define by g_{S_A} the metric on $\mathbb{R}^2$ given for all v and w in $T_{(q_1,q_2)}\mathbb{R}^2$ by

$$g_{S_A}(v, w) = g_{S_L}(\mathrm{d}\rho_L(v), \mathrm{d}\rho_L(w)) + g_Z(\mathrm{d}\rho_R(v), \mathrm{d}\rho_R(w)).$$

Denote, respectively, by s_L and s_R the functions from S_A to S_L and from S_A to Z that act on underlying manifolds as the projections ρ_L and ρ_R. The augmentation V_S is the triple of maps

$$V_S = (V_{S_A}, V_{S_L}, V_Z) \quad \text{with} \quad V_{S_A}(q_1, q_2) = \frac{k_1}{2}(q_1 - q_2)^2, \; V_{S_L} \equiv 0, \; \text{and } V_Z \equiv 0.$$

Define similarly the Riemannian span (Q, V_Q), but with the Riemannian metric g_{Q_R} on Q_R and the augmentations V_{Q_A} and V_{Q_R} given by

$$g_{Q_R} = m_3 g_1, \; V_{Q_A}(q_2, q_3) = \frac{k_2}{2}(q_2 - q_3)^2, \; \text{and } V_{Q_R} \equiv 0.$$

Define by g_{Q_A} the metric on $\mathbb{R}^2$ given for all v and w in $T_{(q_2,q_3)}\mathbb{R}^2$ by

$$g_{Q_A}(v, w) = g_Z(\mathrm{d}\rho_L(v), \mathrm{d}\rho_L(w)) + g_{Q_R}(\mathrm{d}\rho_R(v), \mathrm{d}\rho_R(w)).$$

Denote by π_L and π_R the respective projections from $S_A \times_Z Q_A$ to S_A and to Q_A and by π_M the map $s_R \circ \pi_L$, which is also the map $q_L \circ \pi_R$. Denote by $g_{S_A \times_Z Q_A}$ the Riemannian metric on $S_A \times_Z Q_A$ given by

$$g_{S_A \times_Z Q_A} = \pi_L^*(g_{S_A}) + \pi_R^*(g_{Q_A}) - \pi_M^*(g_Z).$$

The augmentation $V_{S_A \times_Z Q_A}$ is then given by

$$V_{S_A \times_Z Q_A} = \pi_L^*(V_{S_A}) + \pi_R^*(V_{Q_A}) - \pi_M^*(V_Z).$$

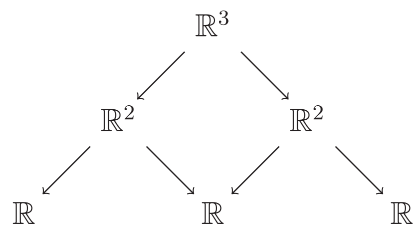
Let Φ be the diffeomorphism from $S_A \times_Z Q_A$ to $\mathbb{R}^3$ given by

$$\Phi(q_1, q_2, q_2, q_3) = (q_1, q_2, q_3)$$

so that $\mathrm{d}\Phi$ is the diffeomorphism from $T(S_A \times_Z Q_A)$ to $T\mathbb{R}^3$ given by

$$\mathrm{d}\Phi(q_1, q_2, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_2, \dot{q}_3) = (q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3).$$

The configuration space of the composite system is, up to an isometry, the fibered product of the configuration spaces of the open subsystems,



Denote by P_A the Riemannian manifold $\mathbb{R}^3$ and by p_L and p_R the maps

$$p_L = s_L \circ \pi_L \circ \Phi^{-1} \quad \text{and} \quad p_R = q_R \circ \pi_R \circ \Phi^{-1}.$$

Denote by V_{P_A} the potential

$$V_{P_A} = V_{S_A \times_Z Q_A} \circ \Phi^{-1}.$$

Define a Riemannian metric g_{P_A} on P_A by

$$g_{P_A} = (\Phi^{-1})^*(g_{S_A \times_Z Q_A}),$$

making Φ an isometry. The Lagrangian for the composite system is $\mathcal{L}_{P_A}$ where for every v in TP_A,

$$\mathcal{L}_{P_A}(v) = \frac{1}{2} g_{P_A}(v, v) - V_{P_A}(\rho_{P_A}(v)).$$

The Lagrangian $\mathcal{L}$ of the system with configuration space given by $\mathbb{R}^3$ is given with respect to coordinate system (q_1, q_2, q_3) by

$$\begin{aligned}\mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) &= \frac{m_1}{2}(\dot{q}_1)^2 + \frac{m_2}{2}(\dot{q}_2)^2 + \frac{m_2}{2}(\dot{q}_2)^2 + \frac{m_3}{2}(\dot{q}_3)^2 - \frac{m_2}{2}(\dot{q}_2)^2 \\ &\quad - \frac{k_1}{2}(q_1 - q_2)^2 - \frac{k_2}{2}(q_2 - q_3)^2 + 0 \quad (\text{since } V_Z \equiv 0) \\ &= \frac{m_1}{2}(\dot{q}_1)^2 + \frac{m_2}{2}(\dot{q}_2)^2 + \frac{m_3}{2}(\dot{q}_3)^2 - \frac{k_1}{2}(q_1 - q_2)^2 - \frac{k_2}{2}(q_2 - q_3)^2.\end{aligned}$$

The Riemannian span (P, V_P) is a representative of the Lagrangian system $[S, V_S] \circ [Q, V_Q]$. The Lagrangian $\mathcal{L}$ on P_A is the Lagrangian for the given system of three masses and two springs with configuration space equal to $\mathbb{R}^3$. We leave the determination of the Hamiltonian system to the reader as it is a straightforward exercise, given the previous discussion and the result of Sec. V.

In general, a description of a composite system requires a prior description of the subsystems. The subsystems need not themselves have descriptions as composite systems, and it remains an open problem to determine the simplest subsystems that are required to construct from them any other system as a composite. If two subsystems that share a common component form a complicated system and if we know how to map the subsystems into two pieces, one of which is the common component, then we can view the complicated system as a composite system in our formalism. We systematically work through a selection of examples in an upcoming paper where we more carefully develop computational tools.

V. THE LEGENDRE FUNCTOR

This section constructs a functor $\mathscr{L}$ from `LagSy` to `HamSy` that preserves the paths of motion.

Suppose that (M, g_M) is a Riemannian manifold of dimension m. The canonical 2-form, ω_{T^*M}, is the exterior derivative of the tautological 1-form and is a symplectic form on T^*M (Ref. 14, p. 202). Denote, respectively, by π_M and ρ_M the canonical projections from T^*M to M and from TM to M. Suppose a is a point of M. There is an open set U of M containing a that is the domain of coordinates $(x_i)_{i\in\{1,\dots,m\}}$. The set of 1-forms $\{\mathrm{d}x_i : i \in \{1, \dots, m\}\}$ trivializes the sub-bundle T^*U. Define for each i the real valued functions p_i^M on T^*U with the property that for all θ in T^*M,

$$\theta = \sum_{i=1}^{m} p_i^M(\theta)\, \mathrm{d}x_i\big|_{\pi_M(\theta)}.$$

The p_i^M are the *momenta* associated with the x_i coordinates. For each i, the function p_i^M is the evaluation map $\mathrm{ev}_{\frac{\partial}{\partial x_i}\big|_{\pi_M(\theta)}}$ that is defined by the equality

$$\mathrm{ev}_{\frac{\partial}{\partial x_i}\big|_{\pi_M(\theta)}}(\theta) = \theta\left(\frac{\partial}{\partial x_i}\bigg|_{\pi_M(\theta)}\right).$$

For each i, define q_i^M by

$$q_i^M = x_i \circ \pi_M.$$

The function given by $\left(q_i^M, p_i^M\right)_{i\in\{1,\ldots,m\}}$ on $\pi_M^{-1}(U)$ is a Darboux coordinate system, that is,

$$\omega_{T^*M} = \sum_{i=1}^{m} \mathrm{d}q_i^M \wedge \mathrm{d}p_i^M.$$

Define for each i the real valued function $\hat{q}_i^M$ on TM with the property that if v is in $\rho_M^{-1}(U)$, then

$$v = \sum_{i=1}^{m} \hat{q}_i^M(v)\,\frac{\partial}{\partial x_i}\bigg|_{\rho_M(v)}.$$

Note that $\hat{q}_i^M$ is the function defined for each v in TU by

$$\hat{q}_i^M(v) = \mathrm{d}x_i|_{\rho_M(v)}(v).$$

Denote ambiguously by q_i^M the function

$$q_i^M = x_i \circ \rho_M$$

on TU. The coordinate system $\left(q_i^M, \hat{q}_i^M\right)$ is a coordinate system on $\rho_M^{-1}(\pi_M(U))$.

The Riemannian metric g_M on TM induces a Riemannian metric on the cotangent bundle T^*M, to be denoted g_M^* and for each a in U defined on the pair (θ_1, θ_2) in $T_a^*M \times T_a^*M$ by

$$g_M^*(\theta_1,\theta_2) = g_M(\sharp_M(\theta_1), \sharp_M(\theta_2)) = \sum_{i,j=1}^{m} g_M^{ij}(a) p_i^M(\theta_1) p_j^M(\theta_2),$$

where g_M^{ij} denotes the (i,j) entry of the inverse of the matrix given by g_M in the $(q_i^M, \hat{q}_i^M)$ coordinates. For all v in TM and θ in T^*M, denote, respectively, by $g_M(\cdot)$ and $g_M^*(\cdot)$ the quadratic forms

$$g_M(v) = g_M(v,v) \quad \text{and} \quad g_M^*(\theta) = g_M^*(\theta,\theta). \tag{5}$$

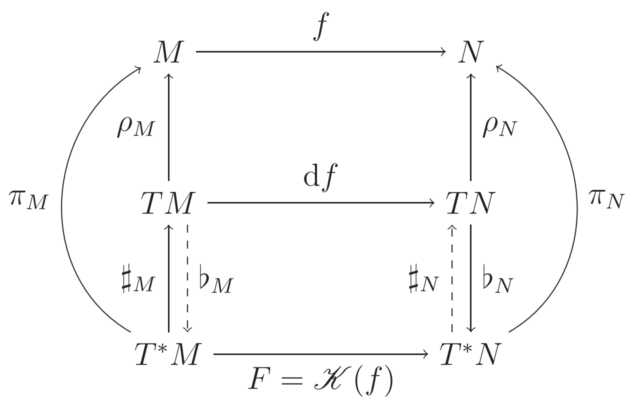
Define $\mathscr{K}$ as a map from Riemannian manifolds to symplectic manifolds by

$$\mathscr{K}(M, g_M) = \left(T^*M, \omega_{T^*M}\right).$$

For any surjective Riemannian submersion f from M to N, define $\mathscr{K}(f)$ by

$$\mathscr{K}(f) = \flat_N \circ \mathrm{d}f \circ \sharp_M.$$

To simplify the notation, denote by F the function $\mathscr{K}(f)$. We depict the various maps here:



Suppose that M and N are Riemannian manifolds of respective dimensions m and n, and suppose further that f is a surjective Riemannian submersion from M to N. For any point p in M, there is a coordinate system $(x_1, \ldots, x_m)$ of M on an open set containing p and a coordinate system $(y_1, \ldots, y_n)$ of N on an open set containing $f(p)$ such that for all i in $\{1, \ldots, n\}$ and k in $\{n+1, \ldots, m\}$,

$$x_i = y_i \circ f \quad \text{and} \quad \frac{\partial}{\partial x_k} \in \ker(\mathrm{d}f).$$

Let j be an index varying in the set $\{1, \ldots, n\}$. For each i and each j, denote, respectively, by q_i^M and q_j^N the functions $x_i \circ \pi_M$ and $y_j \circ \pi_N$ and denote by p_i^M and p_j^N the momenta associated with the coordinate functions x_i and y_j. Use the above notation for the following lemma, as well as for the rest of the section:

Lemma 5.1. For all p_j^M, p_j^N, and F defined as above,

$$p_j^M = p_j^N \circ F.$$

Proof. For all j in $\{1, \ldots, n\}$,

$$\mathrm{d}f\left(\frac{\partial}{\partial x_j}\bigg|_a\right) = \mathrm{d}f\left(\frac{\partial}{\partial(y_j \circ f)}\bigg|_a\right) = \frac{\partial}{\partial y_j}\bigg|_{f(a)}.$$

For all θ in T^*U, there is an element X of TU with θ equal to $g_M(X, \cdot)$. In this case, the form $F(\theta)$ is equal to $g_N(\mathrm{d}f(X), \cdot)$, and so

$$p_j^M(\theta) = \mathrm{ev}_{\frac{\partial}{\partial x_j}\big|_{\pi_M(\theta)}}(\theta) = g_M\left(X, \frac{\partial}{\partial x_j}\bigg|_{\pi_M(\theta)}\right).$$

The function f is a surjective Riemannian submersion, implying that

$$g_M\left(X, \frac{\partial}{\partial(y_j \circ f)}\bigg|_{\pi_M(\theta)}\right) = g_N\left(\mathrm{d}f(X), \mathrm{d}f\left(\frac{\partial}{\partial(y_j \circ f)}\bigg|_{\pi_M(\theta)}\right)\right),$$

and so

$$\begin{aligned} p_j^M(\theta) &= g_N\left(\mathrm{d}f(X), \frac{\partial}{\partial y_j}\bigg|_{f(\pi_M(\theta))}\right) \\ &= g_N\left(\mathrm{d}f(X), \frac{\partial}{\partial y_j}\bigg|_{\pi_N(F(\theta))}\right) \\ &= F(\theta)\left(\frac{\partial}{\partial y_j}\bigg|_{\pi_N(F(\theta))}\right) \\ &= \mathrm{ev}_{\frac{\partial}{\partial y_j}\big|_{\pi_N(F(\theta))}}(F(\theta)) = (p_j^N \circ F)(\theta), \end{aligned}$$

which proves the desired equality. □

Proposition 5.2. For any surjective Riemannian submersion f from a Riemannian manifold M to a Riemannian manifold N, the function $\mathscr{K}(f)$ is a surjective Poisson map.

Proof. Suppose M and N have respective dimensions m and n. The map $\mathscr{K}$ maps Riemannian manifolds to symplectic manifolds, in particular,

$$\mathscr{K}(M) = T^*M \quad \text{and} \quad \mathscr{K}(N) = T^*N.$$

Once again denote by F the map $\mathscr{K}(f)$. Suppose that Π_{T^*M} and Π_{T^*N}, respectively, denote the Poisson bivectors for T^*M and T^*N. For any α and β in $C^\infty(T^*N)$ and any a in T^*M,

$$\begin{aligned}
\mathrm{d}F_a(\Pi_{T^*M})(\mathrm{d}\alpha, \mathrm{d}\beta) &= \Pi_{T^*M}(\mathrm{d}(\alpha \circ F), \mathrm{d}(\beta \circ F))|_a \\
&= \sum_{i=1}^{m} \left(\frac{\partial(\alpha \circ F)}{\partial q_i^M} \frac{\partial(\beta \circ F)}{\partial p_i^M} - \frac{\partial(\beta \circ F)}{\partial q_i^M} \frac{\partial(\alpha \circ F)}{\partial p_i^M} \right)\Bigg|_a \\
&= \sum_{i=1}^{n} \left(\frac{\partial(\alpha \circ F)}{\partial q_i^M} \frac{\partial(\beta \circ F)}{\partial p_i^M} - \frac{\partial(\beta \circ F)}{\partial q_i^M} \frac{\partial(\alpha \circ F)}{\partial p_i^M} \right)\Bigg|_a \\
&= \sum_{i=1}^{n} \left(\frac{\partial(\alpha \circ F)}{\partial(q_i^N \circ F)} \frac{\partial(\beta \circ F)}{\partial(p_i^N \circ F)} - \frac{\partial(\beta \circ F)}{\partial(q_i^N \circ F)} \frac{\partial(\alpha \circ F)}{\partial(p_i^N \circ F)} \right)\Bigg|_a \qquad (6) \\
&= \sum_{i=1}^{n} \left(\frac{\partial\alpha}{\partial q_i^N} \frac{\partial\beta}{\partial p_i^N} - \frac{\partial\beta}{\partial q_i^N} \frac{\partial\alpha}{\partial p_i^N} \right)\Bigg|_{F(a)} = \Pi_{T^*N}(\mathrm{d}\alpha, \mathrm{d}\beta)|_{F(a)},
\end{aligned}$$

where Lemma 5.1 implies the equality in (6). Therefore, $\mathrm{d}F(\Pi_{T^*M})$ is equal to Π_{T^*N}, which implies that F is a Poisson map. The map f is a surjective submersion; therefore, $\mathrm{d}f$ is surjective. The nondegeneracy of g implies that F is also surjective, and so $\mathscr{K}$ maps the morphisms in `RiemSurj` to morphisms in `SympSurj`. □

Lemma 5.3. For any spans S and Q in `RiemSurj` and any span isomorphism Φ from S to Q, the function $\mathscr{K}(\Phi)$ is a span isomorphism from $\mathscr{K}(S)$ to $\mathscr{K}(Q)$.

Proof. Suppose that Φ is a span isomorphism from S and Q. In this case, $\mathscr{K}(\Phi)$ is Poisson. Since $\mathscr{K}(\Phi)$ is a diffeomorphism and Poisson, it is an isomorphism in the category `SympSurj`. Recall that the isomorphisms in `SympSurj` are Poisson diffeomorphisms, which are symplectomorphisms since the objects in `SympSurj` are symplectic manifolds (Ref. 30, p. 195). Since Φ is a span morphism,

$$s_L = q_L \circ \Phi \quad \text{and} \quad s_R = q_R \circ \Phi,$$

implying that

$$\begin{aligned}
\mathscr{K}(s_L) &= \mathscr{K}(q_L \circ \Phi) \\
&= \flat_{Q_L} \circ \mathrm{d}(q_L \circ \Phi) \circ \sharp_{S_A} \\
&= \flat_{Q_L} \circ \mathrm{d}q_L \circ \mathrm{d}\Phi \circ \sharp_{S_A} \\
&= \flat_{Q_L} \circ \mathrm{d}q_L \circ (\sharp_{Q_A} \circ \flat_{Q_A}) \circ \mathrm{d}\Phi \circ \sharp_{S_A} \\
&= (\flat_{Q_L} \circ \mathrm{d}q_L \circ \sharp_{Q_A}) \circ (\flat_{Q_A} \circ \mathrm{d}\Phi \circ \sharp_{S_A}) = \mathscr{K}(q_L) \circ \mathscr{K}(\Phi).
\end{aligned}$$

A similar argument shows that

$$\mathscr{K}(s_R) = \mathscr{K}(q_R) \circ \mathscr{K}(\Phi),$$

proving that $\mathscr{K}(\Phi)$ is a span morphism. Therefore, for any spans S and Q in `RiemSurj` that are span isomorphic, the spans $\mathscr{K}(S)$ and $\mathscr{K}(Q)$ are also span isomorphic. □

Lemma 5.4. Suppose that (S, V_S) and (Q, V_Q) are Riemannian spans and that Φ is an isomorphism of spans from S to Q. If Φ is additionally an isomorphism of classical spans, then so is $\mathscr{K}(\Phi)$.

Proof. In light of Lemma 5.4, it suffices to show that $\mathscr{K}(\Phi)$ is compatible with the augmentations. For any span isomorphism Φ from S to Q that is compatible with V_S and V_Q,

$$V_{S_A} = V_{Q_A} \circ \Phi.$$

The isomorphism Φ is Riemannian, hence an isometry. Therefore,

$$g^*_{S_A} = g^*_{Q_A} \circ \mathcal{K}(\Phi),$$

and so

$$\begin{aligned} H_{S_A} &= \frac{1}{2} g^*_{S_A} + V_{S_A} \circ \pi_{S_A} \\ &= \frac{1}{2} g^*_{Q_A} \circ \mathcal{K}(\Phi) + V_{Q_A} \circ \pi_{Q_A} \circ \mathcal{K}(\Phi) = H_{Q_A} \circ \mathcal{K}(\Phi). \end{aligned}$$

□

Suppose that (S, V_S) is a Riemannian span, and let $\star$ denote either of the letters A, L, or R. Define $\mathcal{K}(S_\star, V_{S_\star})$ by

$$\mathcal{K}(S_\star, V_{S_\star}) = (\mathcal{K}(S_\star), H_{S_\star}),$$

where for all η in $S_\star$,

$$H_{S_\star}(\eta) = \frac{1}{2} g^*_{S_\star}(\eta) + (V_{S_\star} \circ \pi_{S_\star})(\eta).$$

Each object of `LagSy` is an augmented Riemannian manifold, and so $\mathcal{K}$ maps the objects of `LagSy` to the objects of `HamSy` and the morphisms of `RiemSurj` to the morphisms of `SympSurj`. In this way, $\mathcal{K}$ maps Riemannian spans to Poisson spans. Define $\mathcal{L}$ to be $\mathcal{K}$ on the objects of `LagSy`, and for each morphism $[S, V_S]$ in `LagSy`, define $\mathcal{L}([S, V_S])$ by

$$\mathcal{L}([S, V_S]) = [\mathcal{K}(S, V_S)].$$

Theorem 5.5. *The map $\mathcal{L}$ is a functor from* `LagSy` *to* `HamSy`*. Suppose that π_{S_A} is the canonical projection from T^*S_A to S_A. Suppose that the Lagrangian system $[S, V_S]$ has a path of motion γ on the manifold S_A that is specified by the representative (S, V_S) of $[S, V_S]$, and suppose that γ intersects a point x of S_A at time zero. In this case, the path $\mathcal{K} \circ \gamma$ is a path determined by $\mathcal{L}([S, V_S])$, valued in the symplectic manifold $\mathcal{K}(S_A)$, and $\pi_{S_A} \circ \mathcal{K} \circ \gamma$ also intersects x at time zero.*

Proof. The map $\mathcal{L}$ maps Riemannian manifolds to symplectic manifolds and potentials to Hamiltonians and, therefore, maps the objects of `LagSy` to the objects of `HamSy`. Proposition 5.2 implies that $\mathcal{L}$ maps surjective Riemannian submersions to surjective Poisson maps, and so if S is a span in `RiemSurj`, then $\mathcal{K}(S)$ is a span in `SympSurj`. Lemma 5.4 implies that if (S, F_S) and (Q, F_Q) are isomorphic as Riemannian spans, then $\mathcal{K}(S, F_S)$ and $\mathcal{K}(Q, F_Q)$ are isomorphic as Poisson spans, and so $\mathcal{L}$ is well defined on Lagrangian systems, mapping them to Hamiltonian systems.

Suppose that M is a Riemannian manifold. Denote by $\mathcal{L}_M$ the Lagrangian on TM, where for each v in TM,

$$\mathcal{L}_M(v) = \frac{1}{2} g_M(v, v) - V_M(\rho_M(v)).$$

Denote by H_M the Hamiltonian associated with V_M and by $\{\cdot,\cdot\}_{T^*M}$ the Poisson bracket as given above in the construction of $\mathcal{L}$. It is a standard result in classical mechanics that a path γ on M is a solution to (EL) if and only if it is an integral curve of $\{\cdot, H_M\}_M$ (Ref. 28, p.25, Theorem 3.13). This proves the last two statements of the theorem. To prove that $\mathcal{L}$ is a functor, it suffices to show further that (i) $\mathcal{L}$ preserves the composition and (ii) $\mathcal{L}$ maps identity morphisms to identity morphisms.

To show (i), suppose that $[S, F_S]$ and $[Q, F_Q]$ are Riemannian spans and that $[S, F_S]$ is composable with $[Q, F_Q]$. Suppose that P is an $\mathcal{F}$-pullback of (s_R, q_L), where P_A is the fibered product $S_A \times_{S_R} Q_A$ and p_R and p_L are the respective restrictions of the projections on $S_A \times Q_A$ to S_A and Q_A. The map $\mathcal{K}$ maps $S_A \times_{S_R} Q_A$ to its cotangent bundle $T^*(S_A \times_{S_R} Q_A)$, which is isomorphic in `SympSurj` to the manifold $(T^*S_A) \times_{(T^*S_R)} (T^*Q_A)$. The symplectic form on $T^*(S_A \times_{S_R} Q_A)$ is given by the canonical 2-form, and the symplectic form ω on $(T^*S_A) \times_{(T^*S_R)} (T^*Q_A)$ is given by

$$\omega = \mathcal{K}(p_L)^*(\omega_{T^*S_A}) + \mathcal{K}(p_R)^*(\omega_{T^*Q_A}) - \mathcal{K}(p_L)^*(\mathcal{K}(s_R)^*(\omega_{T^*S_R})).$$

The symplectomorphism Φ from $T^*(S_A \times_{S_R} Q_A)$ to $(T^*S_A) \times_{(T^*S_R)} (T^*Q_A)$ is consistent with the augmentations. Lemma 5.4 implies that

$$\begin{aligned}\mathscr{L}([S,F_S]\circ[Q,F_Q]) &= \mathscr{L}([(S,F_S)\circ_P (Q,F_Q)])\\ &= [\mathscr{K}((S,F_S)\circ_P (Q,F_Q))]\\ &= [\mathscr{K}(S,F_S)\circ_{\mathscr{K}(P)}\mathscr{K}(Q,F_Q)]\\ &= [\mathscr{K}(S,F_S)]\circ[\mathscr{K}(Q,F_Q)] = \mathscr{L}([S,F_S])\circ\mathscr{L}([Q,F_Q]),\end{aligned}$$

where the penultimate equality holds because $\mathscr{K}(P)$ is an $\mathcal{F}$-pullback.

To show (ii), suppose that (X, V_X) is an augmented Riemannian manifold and that Id_X is the identity map from X to X. Denote by I_X the span $(\mathrm{Id}_X, \mathrm{Id}_X)$. The span $\mathscr{K}(\mathrm{I}_X)$ is the pair $(\mathscr{K}(\mathrm{Id}_X), \mathscr{K}(\mathrm{Id}_X))$, where $\mathscr{K}(\mathrm{Id}_X)$ is the identity map Id_{T^*X} from T^*X to T^*X. Furthermore, $\mathscr{K}$ maps the augmentation V_X to the augmentation H_{T^*X}, where

$$H_{T^*X} = \frac{1}{2}g_X^* + V_X\circ\pi_X.$$

Suppose that S is a Poisson span with (S_L, H_{S_L}) equal to (T^*X, H_{T^*X}). Let Q be the $\mathcal{F}$-pullback of the cospan $(\mathscr{K}(\mathrm{Id}_X), s_L)$ with the property that Q_A is the symplectic manifold $T^*X\times_{T^*X}S_A$. The maps q_L and q_R are the respective restrictions to the manifold $T^*X\times_{T^*X}S_A$ of the canonical projections of the manifold $T^*X\times S_A$ to T^*X and S_A and are symplectomorphisms. Since Q is an $\mathcal{F}$-pullback, the augmentation H_{Q_A} on Q_A is given by

$$\begin{aligned}H_{Q_A} &= \left(\frac{1}{2}g_X^* + V_X\circ\pi_X\right)\circ q_L + \left(\frac{1}{2}g_{S_A}^* + V_{S_A}\circ\pi_{S_A}\right)\circ q_R\\ &\quad - \left(\frac{1}{2}g_X^* + V_X\circ\pi_X\right)\circ q_L\circ\mathrm{Id}_{T^*X}\\ &= \left(\frac{1}{2}g_X^* + V_X\circ\pi_X\right)\circ q_L + \left(\frac{1}{2}g_{S_A}^* + V_{S_A}\circ\pi_{S_A}\right)\circ q_R - \left(\frac{1}{2}g_X^* + V_X\circ\pi_X\right)\circ q_L\\ &= \left(\frac{1}{2}g_{S_A}^* + V_{S_A}\circ\pi_{S_A}\right)\circ q_R = H_{S_A}\circ q_R,\end{aligned}$$

and hence,

$$H_{Q_A} = H_{S_A}\circ q_R.$$

The map q_R is, therefore, compatible with the augmentations. Since Q is paired with $(\mathscr{K}(\mathrm{Id}_X), s_L)$,

$$s_L\circ q_R = \mathrm{Id}_X\circ q_L = q_L,$$

and so q_R is a span isomorphism mapping the composite $(\mathscr{K}(\mathrm{Id}_X)\circ q_L, s_R\circ q_R)$ to the span S that is compatible with the augmentations. This compatibility implies that

$$\mathscr{L}([\mathrm{I}_X, V_{\mathrm{I}_X}])\circ[S,H_S] = [\mathscr{K}(\mathrm{I}_X, V_{\mathrm{I}_X})\circ(S,H_S)] = [S,H_S].$$

Similar arguments show that for any Poisson span $(S', H_{S'})$ such that $(S'_R, H_{S'_R})$ is equal to (T^*X, H_{T^*X}),

$$[S', H_{S'}]\circ\mathscr{L}([\mathrm{I}_X, V_{\mathrm{I}_X}]) = [S', H_{S'}],$$

and so $\mathscr{L}([\mathrm{I}_X, V_{\mathrm{I}_X}])$ is the identity map with source and target (T^*X, H_{T^*X}). □

We call the functor $\mathcal{L}$ from `LagSy` to `HamSy` the *Legendre functor*. It is a generalization of the Legendre transformation, which translates from the Lagrangian to the Hamiltonian description of an open system in classical mechanics.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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