

Generalizing the Rel Construction to Regular Categories

For any [regular category](#) C , one can form a 2-category of relations $\mathbf{Rel}(C)$ in a way analogous to how Rel is defined over Set .

Objects:

The objects of $\mathbf{Rel}(C)$ are objects of C

Morphisms:

The morphisms $r : c \rightarrow d$ in $\mathbf{Rel}(C)$ are defined to be [subobjects](#) of the categorical product $c \times d$,

A subobject is defined as an equivalence class of monomorphisms.

$$R : X \hookrightarrow C \times D$$

$$S : Y \hookrightarrow C \times D$$

$\phi : X \rightarrow Y$, where ϕ is an isomorphism.

Note that $\phi; S : X \rightarrow C \times D$ has the same type as R

So we require that

$$\phi; S = R$$

I assume this condition implies:

$$\phi^{-1}; R = S$$

So we will talk about R and S as particular monics in the category C but we must remember that our actual subobject and hence the morphism in our Rel category is the equivalence class as defined.

Composition:

We form the pullback square:

$$\begin{array}{ccc}
R \times_D S & \xrightarrow{\pi_2} & S \\
\downarrow \pi_1 & \lrcorner & \downarrow i_S \\
& & D \times E \\
& & \downarrow \pi_1 \\
R & \xrightarrow[i_R]{\hookrightarrow} C \times D \xrightarrow{\pi_2} & D
\end{array}$$

The pullback object $R \times_D S$ is almost what we need but it is a subobject of $C \times D \times E$ and we need to turn it into a subobject of $C \times E$. Interestingly, this can be seen as turning a 3-ary relation into a binary relation.

We know that a pullback object is a subobject of the triple product:
i.e, there is a monic from pullback into product:

$$i : R \times_D S \hookrightarrow C \times D \times E$$

We can define a projection of first and third from the triple product:

$$\pi_1 \times \pi_3 : C \times D \times E \rightarrow C \times E$$

Note that this morphism will not generally be monic. For example, in Set , this function would be a many-to-one surjection that projects many triples on to each pair by "forgetting" the middle element.

We compose these to define:

$$i; (\pi_1 \times \pi_3) : R \times_D S \hookrightarrow C \times D \times E \rightarrow C \times E$$

Simplifying the type we get:

$$i; (\pi_1 \times \pi_3) : R \times_D S \rightarrow C \times E$$

Note that because $\pi_1 \times \pi_3$ is not monic, the whole thing will not be monic. Therefore, this does not by itself define a subobject of $C \times E$ which is what we need.

To to this last step we use the feature of regular categories that we can factorize any morphism into an epi-monic composition, with the middle object being definitionally the image of our factored morphism.

$$R \times_D S \rightarrow \text{Img}(i; (\pi_1 \times \pi_3)) \hookrightarrow C \times E$$

At last we can define the composition of $R; S$ in our Rel category as the subobject defined by the inclusion.

i.e.,

$$R; S := \text{Img}(i; (\pi_1 \times \pi_3)) \hookrightarrow C \times E$$

2-Cells

In $\text{Rel}(C)$ all 1-morphisms in $\text{Hom}(A, B)$ are subobjects of common object $A \times B$ in C . Therefore, 1-morphisms are related in a lattice of subobject inclusions. We can take these subobject inclusions and their composition under transitivity of inclusion as 2-cells of the category $\text{Rel}(C)$. This is precisely analogous to how in Rel we have a lattice of relations for each Homset.

Rel(Set) Example

Let's look at how the familiar category of sets and relations known as Rel arises from the general construction as $\text{Rel}(\text{Set})$.

Objects

The objects of $\text{Rel}(\text{Set})$ are simply sets - the objects of the category Set .

Morphisms

In the general construction, morphisms $R : A \rightarrow B$ in $\text{Rel}(C)$ are subobjects of $A \times B$. In Set , subobjects correspond precisely to subsets. Therefore,

A morphism $R : A \rightarrow B$ in $\text{Rel}(\text{Set})$ is a subset $R \subseteq A \times B$.

This matches our standard definition of a relation from A to B as a subset of the Cartesian product.

More abstractly, we're dealing with equivalence classes of monomorphisms into $A \times B$. In Set , monomorphisms are injective functions, and two monomorphisms:

- $i_R : R \hookrightarrow A \times B$
- $i'_R : R' \hookrightarrow A \times B$

represent the same relation if they have the same image in $A \times B$.

Composition

In *Rel* composition is defined as:

$$\{(a, c) | \exists b \in B : (a, b) \in R \wedge (b, c) \in S\}$$

Note that here, R and S are understood straightforwardly as sets. So in our more general language we can translate this as:

Given relations $i_R : R \hookrightarrow A \times B$ and $i_S : S \hookrightarrow B \times C$, we want to define composition as:

$$\{(a, c) | \exists b \in B : (a, b) \in \text{Img}(i_R) \wedge (b, c) \in \text{Img}(i_S)\}$$

We need to arrive at this definition via the general construction for $\text{Rel}(\text{Set})$, so here goes.

Step 1. Pullback

Form the pullback $R \times_B S$, which in Set is:

$$R \times_B S := \{(a, b, c) \in A \times B \times C \mid \exists (r : R)(s : S), \ r; i_R; \pi_1 = s; i_S; \pi_2\}$$

We can see that this implies:

$$R \times_B S = \{(a, b, c) \in A \times B \times C \mid (a, b) \in \text{Img}(i_R) \wedge (b, c) \in \text{Img}(i_S)\}$$

Since the b in each pair is projected to itself by π_1 and π_2 respectively. Since the projections always send b to itself, the only thing that determines if the whole morphism $i_R; \pi_1(r)$ and $i_S; \pi_2(s)$ evaluates to b will be whether (a, b) and (b, c) are in their respective image sets.

This gets us close to the traditional definition, but a problem remains. The set $R \times_B S$ is a set of triples. We have to "forget" about B to get a subset of $A \times C$.

Step 2. Image Factorization

We can project out all of A and C with the function

$$\pi_1 \times \pi_3 : A \times B \times C \rightarrow A \times C$$

We also know that $R \times_B S$ is a subset of $A \times B \times C$, so we have the injection that sends every triple in $R \times_B S$ to itself in $A \times B \times C$.

$$i : R \times_B S \hookrightarrow A \times B \times C$$

Now we form a function by composing these:

$$i; (\pi_1 \times \pi_3) : R \times_B S \rightarrow A \times C$$

This function might not be an injection since $\pi_1 \times \pi_3$ is a many-to-one function.

However, we can uniquely factor any function into a surjection (epic arrow) and an injection (monic arrow).

When we do this we define an intermediate set, which is the image the function we factored.

$$R \times_D S \rightarrow \text{Img}(i; (\pi_1 \times \pi_3)) \hookrightarrow C \times E$$

Then the set in the middle is the image of the function $i; (\pi_1 \times \pi_3) : R \times_B S \rightarrow A \times C$

But what does this set actually look like? Since it is an image of function with the codomain $A \times C$ it must be a subset of $A \times C$. We can then take the identity injection which sends every pair to itself to define the composed relation. i.e.,

$$R; S := \text{Img}(i; (\pi_1 \times \pi_3)) \hookrightarrow C \times E$$

To see how this recovers the traditional definition. Consider that above we showed

$$R \times_B S = \{(a, b, c) \in A \times B \times C \mid (a, b) \in \text{Img}(i_R) \wedge (b, c) \in \text{Img}(i_S)\}$$

Now, $i; (\pi_1 \times \pi_3)$ takes this set of triples and projects out the first and third members to form pairs (a, c) . The image of this is therefore the set of pairs (a, c) such that there is a triple (a, b, c) in $R \times_B S$.

$$\text{Img}(i; (\pi_1 \times \pi_3)) = \{(a, c) \in A \times C \mid \exists (a, b, c) \in A \times B \times C, (a, b, c) \in R \times_B S\}$$

Now we can expand the definition of $R \times_B S$ in the above definition.

$$\text{Img}(i; (\pi_1 \times \pi_3)) := \{(a, c) \in A \times C \mid$$

$$\exists (a, b, c) \in A \times B \times C,$$

$$(a, b, c) \in \{(a, b, c) \in A \times B \times C \mid (a, b) \in \text{Img}(i_R) \wedge (b, c) \in \text{Img}(i_S)\}\}$$

However, since we are considering all pairs (a, c) in the outer definition, in this context, we have bound the variables a and c , and we don't need to include them explicitly in the existential quantification. In other words, we can simplify the definition to:

$$\text{Img}(i; (\pi_1 \times \pi_3)) := \{(a, c) \in A \times C \mid \exists b \in B, (a, b) \in \text{Img}(i_R) \wedge (b, c) \in \text{Img}(i_S)\}$$

This is precisely our traditional definition of relation composition.