Logical Relations (and more)

in Higher-order Mathematical Operational Semantics

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Chocola, May 2024

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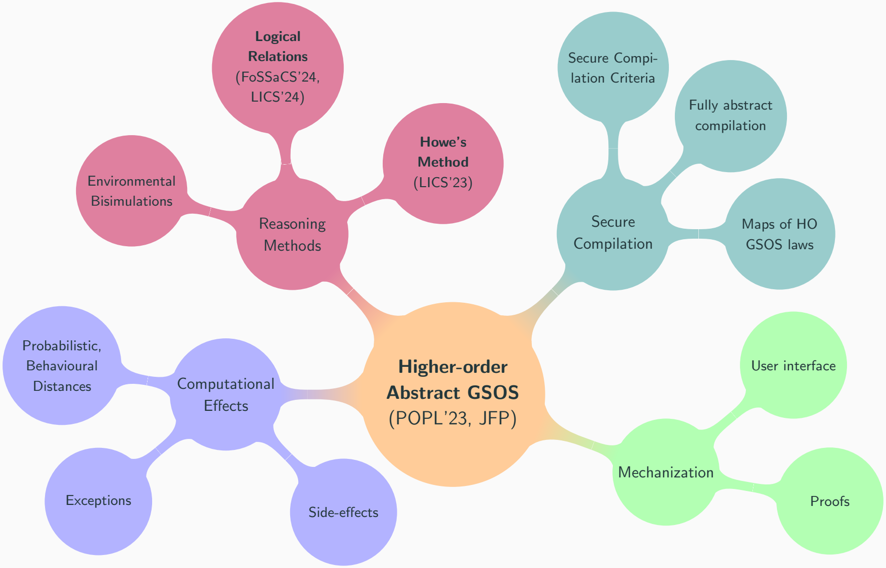
Higher-Order Mathematical Operational Semantics (or HO Abstract GSOS)



Overview

HO-MOS or
Higher-order Abstract GSOS

Relational Reasoning

Step-indexed Logical Relations

HO-MOS or
Higher-order Abstract GSOS

GSOS rules

Definition (GSOS rule)

$$\frac{\{x_i \xrightarrow{a} y_{ij}^a\}_{1\leq j\leq n_i^a}^{1\leq i\leq \mathrm{ar}(\mathsf{f}),\, a\in A_i} \qquad \{x_i \stackrel{b}{\not\to}\}_{b\in B_i}^{1\leq i\leq \mathrm{ar}(\mathsf{f})}}{\mathsf{f}(x_1,\ldots,x_{\mathrm{ar}(\mathsf{f})}) \xrightarrow{c} t}$$

where $\mathsf{f} \in \overline{\Sigma}$, A_i, B_i range over subsets of L and $n_i^a \in \mathbb{N}$ and $c \in$ L. Variables x_i and y_{ij}^a are all distinct and are the only variables appearing in t.

Example rule

$$\frac{p \xrightarrow{a} p'}{p \,||\, q \xrightarrow{a} p' \,||\, q}$$

GSOS rules

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where $\mathrm{f} \in \overline{\Sigma}$, A_i, B_i range over subsets of L and $n_i^a \in \mathbb{N}$ and $c \in \mathrm{L}$. Variables x_i and y_{ij}^a are all distinct and are the only variables appearing in t.

Example rule

generic → p, generic → p'

$$\frac{p \xrightarrow{a} p'}{p \,||\, q \xrightarrow{a} p' \,||\, q}$$

Abstract GSOS (Mathematical Operational Semantics)

Let endofunctors $\Sigma, B\colon \mathcal{C} \to \mathcal{C}$ in some distributive category $\mathcal{C}$ and assume that the free monad over Σ, $\Sigma^\star$, exists.

Definition (Turi and Plotkin [1])

A GSOS law of Σ (modelling the syntax of the system) over B (modelling the behaviour) is a natural transformation [1]

$$\rho_X\colon \Sigma(X \times BX) \to B\Sigma^\star X.$$

[1] Roughly a parametrically polymorphic function.

Abstract GSOS

Theorem (Turi and Plotkin [1])

Let finitary signature $\overline{\Sigma}$, with associated endofunctor $\Sigma \colon \mathsf{Set} \to \mathsf{Set}$, and a finite set of actions L. GSOS specifications of $\overline{\Sigma}$ over L are in a bijective correspondence with GSOS laws of Σ over $(\mathcal{P}_{\mathsf{f}} X)^{\mathsf{L}}$.

$$\frac{p \xrightarrow{a} p'}{p \,||\, q \xrightarrow{a} p' \,||\, q}$$

$$\cong$$

$$\rho_X \colon \coprod_{\mathsf{f} \in \overline{\Sigma}} (X \times (\mathcal{P}_{\mathsf{f}} X)^{\mathsf{L}})^{\mathsf{ar}(\mathsf{f})} \to (\mathcal{P}_{\mathsf{f}} \Sigma^{*} X)^{\mathsf{L}}$$

Abstract GSOS

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Let finitary signature $\overline{\Sigma}$, with associated endofunctor $\Sigma\colon \mathsf{Set} \to \mathsf{Set}$, and a finite set of actions L. GSOS specifications of $\overline{\Sigma}$ over L are in a bijective correspondence with GSOS laws of Σ over $(\mathcal{P}_{\mathsf{f}} X)^{\mathsf{L}}$.

$$\frac{p \xrightarrow{a} p'}{p \mid\mid q \xrightarrow{a} p' \mid\mid q}$$

$$\cong$$

$$\rho_X\colon \coprod_{\mathsf{f} \in \overline{\Sigma}} (X \times (\mathcal{P}_{\mathsf{f}} X)^{\mathsf{L}})^{\mathsf{ar}(\mathsf{f})} \to (\mathcal{P}_{\mathsf{f}} \Sigma^{*} X)^{\mathsf{L}}$$

Abstract GSOS

The fascinating part is that GSOS laws gave a precise, concise mathematical representation of what GSOS specifications *are*.

They are certain natural transformations.

Abstract GSOS

Operational rules

$$\frac{p \xrightarrow{a} p'}{p \,||\, q \xrightarrow{a} p' \,||\, q}$$

$\cong$

GSOS laws: natural transformations

$$\rho_X \colon \underbrace{\Sigma(X \times BX)}_{\text{premises}} \to \underbrace{B(\Sigma^* X)}_{\text{conclusion}}$$

for functors $\Sigma, B \colon \mathcal{C} \to \mathcal{C}$ representing syntax and behaviour (e.g. $B = \mathcal{P}_{\mathsf{f}}^{\mathsf{L}}$).

(inductively defined) programs — (coinductive) behaviours

- Operational model $\mu\Sigma \to B(\mu\Sigma)$, denotational model $\Sigma(\nu B) \to \nu B$.
- **Key feature:** compositionality, i.e. bisimilarity is a congruence:

$$p_i \sim q_i \quad (i = 1, \ldots, n) \quad \overset{f \in \Sigma}{\Longrightarrow} \quad \mathsf{f}(p_1, \ldots, p_n) \sim \mathsf{f}(q_1, \ldots, q_n).$$

- **Scope**: first-order (CCS, π-calculus, ...), ~~higher-order~~ (λ-calculus, SKI calculus)

Higher-order abstract GSOS?

For all the success of abstract GSOS (variable binding [2], formats [3]–[6], effects [7]–[9], compilers [10]–[12]), higher-order languages have always been the big question mark.

An enduring problem

Turi and Plotkin 1997 [1]

The major challenge ahead is the operational semantics of the languages with variable binders, such as the π-calculus and the λ-calculus.

[...]

Hirschowitz and Lafont 2022 [13]

This approach has been deeply investigated, notably for quantitative languages [3]. However, as of today, attempts to apply it to higher-order (e.g., functional) languages have failed.

$\mathrm{SKI_u}$ **combinator calculus**

$$\frac{}{S \xrightarrow{t} S'(t)} \qquad \frac{}{S'(p) \xrightarrow{t} S''(p,t)} \qquad \frac{}{S''(p,q) \xrightarrow{t} (p\,t)\,(q\,t)}$$

$$\frac{}{K \xrightarrow{t} K'(t)} \qquad \frac{}{K'(p) \xrightarrow{t} p} \qquad \frac{}{I \xrightarrow{t} t}$$

$$\frac{p \to p'}{p\,q \to p'\,q} \qquad \frac{p \xrightarrow{q} p'}{p\,q \to p'}$$

Figure 1: Small-step operational semantics of the $\mathrm{SKI_u}$ calculus, our version of the SKI combinator calculus, invented by Curry [14].

Disclaimer: This is just a convenient example to introduce HO-MOS. The latter can do the λ-calculus, typed or untyped, with simple or recursive types, etc.

$\mathsf{SKI_u}$ combinator calculus

$$\frac{}{S''(p, q) \xrightarrow{t} (p\, t)\,(q\, t)} \qquad \frac{p \to p'}{p\, q \to p'\, q} \qquad \frac{p \xrightarrow{q} p'}{p\, q \to p'}$$

A combinator calculus

$$\frac{}{S''(p, q) \xrightarrow{t} (p\,t)\,(q\,t)} \qquad \frac{p \to p'}{p\,q \to p'\,q} \qquad \frac{p \xrightarrow{q} p'}{p\,q \to p'}$$

combinator

A combinator calculus

$$\frac{}{S''(p,q) \xrightarrow{t} (p\,t)\,(q\,t)} \qquad \frac{p \to p'}{p\,q \to p'\,q} \qquad \frac{p \xrightarrow{q} p'}{p\,q \to p'}$$

combinator

application

A combinator calculus

Labels can be a terms!

$$\frac{}{S''(p, q) \xrightarrow{t} (p\,t)\,(q\,t)} \qquad \frac{p \rightarrow p'}{p\,q \rightarrow p'\,q} \qquad \frac{p \xrightarrow{q} p'}{p\,q \rightarrow p'}$$

combinator

application

GSOS vs combinator calculi

GSOS

$$\frac{p \xrightarrow{a} p'}{p \,||\, q \xrightarrow{a} p' \,||\, q}$$

Is it GSOS?

$$\frac{p \to p'}{p\,q \to p'\,q}$$

$$\frac{p \xrightarrow{q} p'}{p\,q \to p'}$$

$$\frac{}{S''(p, q) \xrightarrow{t} (p\,t)\,(q\,t)}$$

Nope!

The Issue With Higher-Order Languages

Higher-order languages require behaviours like

$$BX = X^X.$$

This is not an endofunctor – but

$$B(X, Y) = Y^X$$

is a bifunctor contravariant in X and covariant in Y.

Key idea for higher-order abstract GSOS

endofunctors $B\colon \mathcal{C} \to \mathcal{C}$ $\quad + \quad$ natural transformations

$$\Downarrow$$

bifunctors $B\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ $\quad + \quad$ **dinatural** transformations.

Higher-order GSOS laws

Definition

A *higher-order GSOS law* of $\Sigma\colon \mathcal{C} \to \mathcal{C}$ (modelling the syntax) over $B\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ (modelling higher-order behaviour) is a family of morphisms

$$\rho_{X,Y}\colon \Sigma(X \times B(X,Y)) \to B(X, \Sigma^{\star}(X+Y))$$

dinatural in $X \in \mathcal{C}$ and **natural** in $Y \in \mathcal{C}$.

A higher-order format for combinatory logic

Definition ($\mathcal{HO}$ rules)

$$\frac{(x_j \to y_j)_{j \in W} \quad (x_i \xrightarrow{z} y_i^z)_{i \in \{1,\ldots,n\} \setminus W,\, z \in \{x_1,\ldots,x_n\}}}{\mathsf{f}(x_1,\ldots,x_n) \to t}$$

$$\frac{(x_j \to y_j)_{j \in W} \quad (x_i \xrightarrow{z} y_i^z)_{i \in \{1,\ldots,n\} \setminus W,\, z \in \{x,x_1,\ldots,x_n\}}}{\mathsf{f}(x_1,\ldots,x_n) \xrightarrow{x} t}$$

Example rules (sugared)

$$\frac{}{S''(p,q) \xrightarrow{t} (p\,t)\,(q\,t)} \qquad \frac{p \to p'}{p\,q \to p'\,q} \qquad \frac{p \xrightarrow{q} p'}{p\,q \to p'}$$

Higher-Order Mathematical Operational Semantics

Proposition

For every finitary signature $\overline{\Sigma}$, with associated endofunctor $\Sigma\colon \mathsf{Set} \to \mathsf{Set}$, $\mathcal{HO}$ specifications are in a bijective correspondence with higher-order GSOS laws of Σ over $B(X, Y) = Y + Y^X$.

$$\frac{p \xrightarrow{q} p'}{p\ q \to p'}$$

$$\cong$$

$$\rho_X \colon \coprod_{\mathsf{f} \in \overline{\Sigma}} (X \times (Y + Y^X))^{\mathsf{ar}(\mathsf{f})} \to \Sigma^*(X + Y)$$

Higher-Order Abstract GSOS

Operational rules

$$\dfrac{\dfrac{p \xrightarrow{q} p'}{p\,q \to p'}}{(\lambda x.p)\,q \to p[q/x]}$$

$\cong^\star$

Higher-order GSOS laws: (di-)natural trf.

$$\rho_{X,Y} \colon \underbrace{\Sigma(X \times B(X,Y))}_{\text{premises}} \to \underbrace{B(X, \Sigma^*(X+Y))}_{\text{conclusion}}$$

For combinator calculi, we have

$$\mathcal{C} = \mathsf{Set}$$

$$\Sigma X = 1 + X \times X + \dots$$

$$B(X,Y) = Y + Y^X$$

β-reduction or combinator

Higher-Order Abstract GSOS

Operational rules

$$\frac{p \xrightarrow{q} p'}{p\,q \to p'}$$

$$\frac{}{(\lambda x.p)\,q \to p[q/x]}$$

$\cong^\star$

Higher-order GSOS laws: (di-)natural trf.

$$\rho_{X,Y} \colon \underbrace{\Sigma(X \times B(X,Y))}_{\text{premises}} \to \underbrace{B(X, \Sigma^*(X+Y))}_{\text{conclusion}}$$

For the call-by-name λ-calculus, we have

$$\mathcal{C} = \mathrm{Set}^{\mathbb{F}}$$

$$\Sigma X = V + \delta X + X \times X \quad \text{(Fiore, Plotkin and Turi [15])}$$

$$B(X,Y) = \langle X, Y\rangle \times (Y + Y^X + 1)$$

substitution stucture

β-reduction, λ-expr or stuck

Higher-Order Abstract GSOS

Operational rules

$$\frac{}{(\lambda x.p)\, q \to p[q/x]}$$

$$\frac{p \to p'}{p\, q \to p'\, q}$$

$\cong^\star$

Higher-order GSOS laws: (di-)natural trf.

$$\rho_{X,Y} : \underbrace{\Sigma(X \times B(X,Y))}_{\text{premises}} \to \underbrace{B(X, \Sigma^*(X+Y))}_{\text{conclusion}}$$

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programs

- Operational model $\gamma : \mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$, ~~denotational model~~.

  e.g. $\gamma(t) = t'$ if $t \to t'$ and $\gamma(\lambda x.M) = (e \mapsto M[e/x])$, $(\gamma(I) = \mathrm{id}$ for SKI)

- **Key feature:** compositionality, i.e. bisimilarity is a congruence.

  Proof: more complex than first-order case + needs mild assumptions.

Strong Applicative Bisimilarity

Coalgebraic bisimilarity on operational model $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$

$$=$$

strong applicative bisimilarity.

Example: λ-calculus

closed λ-terms

Greatest relation $\sim \subseteq \Lambda \times \Lambda$ such that for $t_1 \sim t_2$,

$$t_1 \to t_1' \implies t_2 \to t_2' \quad \wedge \; t_1' \sim t_2';$$
$$t_1 = \lambda x.t_1' \implies t_2 = \lambda x.t_2' \;\wedge\; \forall e \in \Lambda.\; t_1'[e/x] \sim t_2'[e/x];$$

$+$ two symmetric conditions

Abstract odelling of Operational Semantics

Concrete/Abstract

Concrete	Abstract
1. Algebraic signature Σ	1. Syntax endofunctor $\Sigma\colon \mathcal{C} \to \mathcal{C}$
2. Program terms $\mu\Sigma$	2. Initial Σ-algebra $\mu\Sigma = \Sigma^*(0)$
3. (Impl.) nature of computation	3. Bifunctor $B\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$
4. Operational rules $\frac{t \to t'}{t \cdot s \to t' \cdot s}$	4. Higher-order GSOS law $\rho_{X,Y}$
5. Oper. model $t \to t'$, $t, t' \in \mu\Sigma$	5. Coalgebra $\gamma\colon \mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$
6. Strong applicative bisimulation	6. $B(\mu\Sigma, -)$-bisimulations

Assuming a suitable category $\mathcal{C}$.

[16]: Congruence of bisimilarity, for free!

Question marks

Concrete/Abstract

8. Howe's closure	8. ???
9. Howe's method	9. ???
10. Logical predicates/relations	10. ???
11. Fundamental Properties	11. ???

We want to mode[l ... ab]ove **generically**, in[...] **[i]ndependent** manner.

Relation Lifting!

Predicate Lifting!

Relational Reasoning

How to do program discourse, categorically

Key concept 1: If $\mathcal{C}$ is our base universe of discourse, we can form the categories $\mathsf{Rel}(\mathcal{C})$ and $\mathsf{Pred}(\mathcal{C})$ of resp. (homogenous) relations and predicates on $\mathcal{C}$. These are the categories of subobjects on rep. $X \times X$ and X.

$$\begin{array}{ccc} R & \dashrightarrow & S \\ {\scriptstyle \langle \mathsf{l}_R, \mathsf{r}_R \rangle} \downarrow & & \downarrow {\scriptstyle \langle \mathsf{l}_S, \mathsf{r}_S \rangle} \\ X \times X & \xrightarrow{f \times f} & Y \times Y \end{array} \qquad \begin{array}{ccc} P & \dashrightarrow & Q \\ {\scriptstyle p} \downarrow & & \downarrow {\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

Key concept 2: We extend the functors to $\mathsf{Rel}(\mathcal{C})$ and $\mathsf{Pred}(\mathcal{C})$, a process that is known as relation (or predicate) lifting [17].

$$\begin{array}{ccc} \mathsf{Rel}(\mathcal{C}) & \xrightarrow{\overline{\Sigma}} & \mathsf{Rel}(\mathcal{C}) \\ {\scriptstyle |-|} \downarrow & & \downarrow {\scriptstyle |-|} \\ \mathcal{C} & \xrightarrow{\Sigma} & \mathcal{C} \end{array} \qquad \begin{array}{ccc} \mathsf{Pred}(\mathcal{C}) & \xrightarrow{\overline{\Sigma}} & \mathsf{Pred}(\mathcal{C}) \\ {\scriptstyle |-|} \downarrow & & \downarrow {\scriptstyle |-|} \\ \mathcal{C} & \xrightarrow{\Sigma} & \mathcal{C} \end{array}$$

Also, write $\mathsf{Pred}_X(\mathcal{C})$, $\mathsf{Rel}_X(\mathcal{C})$ for the lattices of resp. predicates and relations on X.

Act I, Induction. Part 1, Predicates.

Let $P \rightarrowtail \mu\Sigma$ be a predicate on terms (assume a typed syntax, for the heck of it).

Structural induction

1. (Repeat for every operation) For all $t : \tau_1 \rightarrow \tau_2$, $s : \tau_1$ such that $P_{\tau_1 \rightarrow \tau_2}(t)$ and $P_{\tau_1}(s)$, then $P_{\tau_2}(t\,s)$.
2. By induction, for all types τ and terms $t : \tau$, $P_\tau(t)$.

Unary induction proof principle

1. $\overline{\Sigma}(P)$ represents 1-depth terms (operations) whose subterms are in P ($\overline{\Sigma}$ is the canonical lifting). There is a Σ-algebra structure

$$\overline{\Sigma}(P) \leq \iota^\star[P], \text{ where } \iota\colon \Sigma\mu\Sigma \to \mu\Sigma \text{ is the initial } \Sigma\text{-algebra.}$$

2. As initial algebras have no proper subalgebras, $P \cong \mu\Sigma$.

Act I, Induction. Part 2, Relations.

Let $R \rightarrowtail \mu\Sigma \times \mu\Sigma$ be a relation on terms.

Structural induction (Fundamental Property)

1. (Repeat for every operation) For all $t_1, t_2 : \tau_1 \rightharpoonup \tau_2$, $s_1, s_2 : \tau_1$ such that $R_{\tau_1 \rightharpoonup \tau_2}(t_1, t_2)$ and $R_{\tau_1}(s_1, s_2)$, then $R_{\tau_2}(t_2\, s_2, t_2\, s_2)$.
2. Then for all types τ, relation R_τ is reflexive.

Binary induction proof principle

1. $\overline{\Sigma}(R)$ represents pairs of 1-depth terms with subterms in R. If there is

$$\overline{\Sigma}(R) \leq (\iota \times \iota)^\star[R] (\text{that is, } R \text{ is a congruence}),$$

2. then $\Delta \leq R$ because all congruences on an initial algebra are reflexive.

Simple go-to example (untyped syntax this time)

$$B(X,Y) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C} \quad \gamma \colon \mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$$

$$B(X,Y) = Y + Y^X \qquad \gamma(t) = t' \text{ if } t \to t' \text{ and } \gamma(\lambda x.M) = (e \mapsto M[e/x])$$

$$\begin{array}{ccc} \mathrm{Pred}(\mathcal{C})^{\mathrm{op}} \times \mathrm{Pred}(\mathcal{C}) & \xrightarrow{\overline{B}} & \mathrm{Pred}(\mathcal{C}) \\ {\scriptstyle |-|^{\mathrm{op}} \times |-|}\downarrow & & \downarrow{\scriptstyle |-|} \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{B} & \mathcal{C} \end{array} \qquad \begin{array}{ccc} \mathrm{Rel}(\mathcal{C})^{\mathrm{op}} \times \mathrm{Rel}(\mathcal{C}) & \xrightarrow{\overline{B}} & \mathrm{Rel}(\mathcal{C}) \\ {\scriptstyle |-|^{\mathrm{op}} \times |-|}\downarrow & & \downarrow{\scriptstyle |-|} \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{B} & \mathcal{C} \end{array}$$

Let $P, Q \subseteq \mu\Sigma$ be predicates. Then $\overline{B}(P,Q) \subseteq \mu\Sigma + \mu\Sigma^{\mu\Sigma}$ amounts to the following:

$$\overline{B}(P,Q) = \{t \mid Q(t)\} \vee \{f \in \mu\Sigma^{\mu\Sigma} \mid \forall t.\, P(t) \implies Q(f(t))\},$$

aka, inputs in P are mapped to outputs in Q! Let $R, S \subseteq \mu\Sigma \times \mu\Sigma$ be relations. Then $\overline{B}(R,S)$ amounts to the following:

Act II, Bisimulations. Part 1, Predicates.

Let $P, Q \rightarrowtail X$ be predicates on the state space of a coalgebra $h : X \to B(X, X)$. We say that P is a (Q-relative) ($\overline{B}$)-invariant [18] if

$$P \leq h^{\star}[\overline{B}(Q, P)]$$

We say that an invariant P is **logical** if it is relative to itself.

Instantiate on $\gamma : \mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$. A predicate P is logical if the following hold:

1. If $t = \lambda x.s$, then for all e with $P(e)$, then $P(s[e/x])$.
2. If $t \to t'$ then $P(t')$.

The above notion instantiates correctly in other settings (assuming the coalgebra is setup correctly), e.g. typed: the tuple $(t, s) : \tau_1 \times \tau_2$ is in P when $P_{\tau_1}(t)$ and $P_{\tau_2}(s)$.

Bisimulations, logical relations and step-indexing [19]

Let $h : X \to B(X, X)$ be a coalgebra and $\tilde{h} : X \to B(X, X)$ be a weakening of h (think $\to$ vs its saturation/closure $\Rightarrow$). We say that:

1. A relation $R \rightarrowtail X \times X$ is a **bisimulation** if $R \leq (h \times \tilde{h})^{\star}[\overline{B}(\Delta, R)]$.
2. A relation $R \rightarrowtail X \times X$ is a **logical relation** if $R \leq (h \times \tilde{h})^{\star}[\overline{B}(R, R)]$.
3. An ordinal-indexed family of relations $(R^{\alpha} \rightarrowtail X \times X)_{\alpha}$ is a **step-indexed logical relation** if it forms a decreasing chain (i.e. $R^{\alpha} \leq R^{\beta}$ for all $\beta < \alpha$) and satisfies

$$R^{\alpha+1} \leq (h \times \tilde{h})^{\star}[\overline{B}(R^{\alpha}, R^{\alpha})] \quad \text{for all } \alpha.$$

Bisimulations, logical relations and step-indexing [19]

Let $h : X \to B(X, X)$ be a coalgebra and $\tilde{h} : X \to B(X, X)$ be a weakening of h (think $\to$ vs its saturation/closure $\Rightarrow$). We say that:

1. A relation R on X is a **($\overline{B}$-)bisimulation** (for $h, \tilde{h}$) if $R \leq (h \times \tilde{h})^\star[\overline{B}(\Delta, R)]$.
2. A relation R on X is a **($\overline{B}$-)logical relation** (for $h, \tilde{h}$) if $R \leq (h \times \tilde{h})^\star[\overline{B}(R, R)]$.
3. An ordinal-indexed family of relations $(R^\alpha \rightarrowtail X \times X)_\alpha$ is a **($\overline{B}$-)step-indexed logical relation** (for $h, \tilde{h}$) if it forms a decreasing chain (i.e. $R^\alpha \leq R^\beta$ for all $\beta < \alpha$) and satisfies

$$R^{\alpha+1} \leq (h \times \tilde{h})^\star[\overline{B}(R^\alpha, R^\alpha)] \quad \text{for all } \alpha.$$

Simple example

$$B(X,Y) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C} \quad \gamma \colon \mu\Sigma \to \mathcal{P}(B(\mu\Sigma, \mu\Sigma))$$

$$B(X,Y) = Y + Y^X \qquad \gamma(t) = \{t'\} \text{ if } t \to t' \text{ and } \gamma(\lambda x.M) = \{(e \mapsto M[e/x])\}$$

We will use the asymmetric Egli-Milner relation lifting for $\mathcal{P}B, \widetilde{\mathcal{P}B}$.

Notation (reminder and introduction)

- Write $t \xrightarrow{e} t'$ if $t = \lambda x.M$ and $t' = M[e/x] = \gamma(t)(e)$.
- Write $t \Rightarrow t'$ if $t \to t_1 \to \cdots \to t_n \to t'$.
- Write $t \overset{e}{\Rightarrow} t'$ if $t \to t_1 \to \cdots \to t_n \to t''$ and $t'' \xrightarrow{e} t'$.
- The system $\Rightarrow$ is modelled by $\tilde{\gamma} : \mu\Sigma \to \mathcal{P}(B(\mu\Sigma, \mu\Sigma))$ (technically $\Rightarrow$ is a notation for $\tilde{\gamma}$).

Act II, Bisimulations. Part 2, Relations.

Simple example

$$B(X,Y) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C} \quad \gamma \colon \mu\Sigma \to \mathcal{P}(B(\mu\Sigma, \mu\Sigma))$$

$$B(X,Y) = Y + Y^X \qquad \gamma(t) = \{t'\} \text{ if } t \to t' \text{ and } \gamma(\lambda x.M) = \{(e \mapsto M[e/x])\}$$

We will use the asymmetric Egli-Milner relation lifting for $\mathcal{P}B, \widetilde{\mathcal{P}B}$.

Let $\tilde{\gamma}$ be the closure of γ under β reductions. A relation $R \subseteq \mu\Sigma \times \mu\Sigma$ is a $(\widetilde{\mathcal{P}B})$-logical relation (for $\gamma, \tilde{\gamma}$) if for all t, s with $R(t,s)$, the following are true:

- If $t \to t'$ then $s \Rightarrow s'$ and $R(t', s')$.
- For all e_1, e_2 with $R(e_1, e_2)$, if $t \xrightarrow{e_1} t'$, then $s \xRightarrow{e_2} s'$ and $R(t', s')$.

Logical preorder! Kind of concurrent flavor when $\to, \Rightarrow$ is used instead of $\Downarrow, \Downarrow$.

Simple example

$$B(X,Y) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C} \quad \gamma\colon \mu\Sigma \to \mathcal{P}(B(\mu\Sigma, \mu\Sigma))$$

$$B(X,Y) = Y + Y^X \qquad \gamma(t) = \{t'\} \text{ if } t \to t' \text{ and } \gamma(\lambda x.M) = \{(e \mapsto M[e/x])\}$$

We will use the asymmetric Egli-Milner relation lifting for $\mathcal{P}B, \widetilde{\mathcal{P}B}$.

Let $\tilde{\gamma}$ be the closure of γ under β reductions. A family $(R^\alpha \subseteq \mu\Sigma \times \mu\Sigma)_\alpha$ is a step-indexed $(\widetilde{\mathcal{P}B})$-logical relation (for $\gamma, \tilde{\gamma}$) if $\forall \alpha, \beta$ with $\beta < \alpha$, $R^\alpha \leq R^\beta$ and for all α, t, s with $R^{\alpha+1}(t,s)$, the following are true:

- If $t \to t'$ then $s \Rightarrow s'$ and $R^\alpha(t', s')$.
- For all e_1, e_2 with $R^\alpha(e_1, e_2)$, if $t \xrightarrow{e_1} t'$, then $s \overset{e_2}{\Rightarrow} s'$ and $R^\alpha(t', s')$.

This was supposed to be an example on a typed λ-calculus, but I ran out of time while preparing the slides. We can do it on the board, depending on time.

Abstract modelling of Predicates and Relations

Concrete/Abstract

Concrete	Abstract
1. Predicates, relations on terms	1. $P \rightarrowtail \mu\Sigma$, $R \rightarrowtail \mu\Sigma \times \mu\Sigma$
2. Predicate, relational reasoning	2. Complete, well-powered cat. $\mathcal{C}$
3. (P is a) Logical Predicate	3. $P \leq h^{\star}[\overline{B}(P, P)]$
4. (R is a) Logical Relation	4. $R \leq (h \times \tilde{h})^{\star}[\overline{B}(R, R)]$
5. Fundamental Property of Logical Relations	5. Generalized induction $\overline{\Sigma}(R) \leq \iota^{\star}[R] \implies \Delta \leq R$

Recall that relation lifting is algebraic and coalgebraic, and independent of the Higher-order Abstract GSOS framework.

However, the marriage of algebra and coalgebra that HO Abstract GSOS represents extends along their liftings :).

Howe's method in higher-order Abstract GSOS, briefly

Motivation: We need congruence of applicative similarity, not of its strong version.

Plan: Redo Howe's method using our abstract machinery, such that:

- We systematize it into a generic, language-independent method.
- Expose its core ideas, its language-specific part, and then simplify [2].

Key concept: Howe's closure $\hat{R}$: initial algebra (lfp) of an endofunctor on $\mathrm{Rel}_{\mu\Sigma}(\mathcal{C})$!

$$\text{For an applicative simulation } R \rightarrowtail \mu\Sigma \times \mu\Sigma, \quad \hat{R} = \mu S.\, R \vee \iota_\star[\overline{\Sigma}(S)]; R.$$

Results [22]: For $\hat{R}$ to be a bisimulation, just show that "weakened" rules are sound:

$$\frac{t \overset{s}{\Rightarrow} t'}{t\, s \Rightarrow t'} \; \checkmark \qquad \frac{t \Rightarrow t'}{t\, s \Rightarrow t'\, s} \; \checkmark \qquad \text{(Many thanks to dinaturality.)}$$

---

[2] Howe's method is complex, clunky, conceptually mysterious and unclear why it works. Shout-out to people uncovering its mysteries (Dal Lago et al.[20], Borthelle et al. [21], Hirschowiz and Lafont [13]).

Step-indexed Logical Relations

Time for some efficient reasoning in the Higher-order Abstract GSOS framework!

Question marks

Let's go over the typical setting of Logical Relations.

Concrete/Abstract

Concrete	Abstract
8. Operational Semantics	8. HO Abstract GSOS
9. What is a Logical Relation?	9. $R \leq (h \times \tilde{h})^{\star}[\overline{B}(R, R)]$
10. Construct **that** Logical Relation, **the chosen one**	10. `Abstract Construction Missing`
11. Laborious compatibility lemmas	11. ???
12. Reflexivity	12. General induction principle

Constructing step-indexed logical relation, Coalgebraically

In standard settings, (step-indexed) logical relations are defined empirically, on a per-case basis. Our approach systematizes the method. Let's see how:

Step-indexed Henceforth Relation Transformer

Let $B\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ with a relation lifting $\overline{B}$, and let $c, \tilde{c}\colon X \to B(X, X)$ be coalgebras. For every $R \rightarrowtail X \times X$ we define the step-indexed logical relation $(\,\square^{\overline{B},c,\tilde{c},\alpha} R \rightarrowtail X \times X\,)_\alpha$ by transfinite induction (writing $\square^\alpha$ for simplicity):

$$\begin{aligned} \square^0 R &= R, \\ \square^{\alpha+1} R &= \square^\alpha R \wedge (c \times \tilde{c})^\star[\overline{B}(\square^\alpha R, \square^\alpha R)], \\ \square^\alpha R &= \bigwedge_{\beta<\alpha} \square^\beta R \qquad \text{for limit ordinals } \alpha. \end{aligned}$$

Under mild conditions, there exists ν with $\square^{\nu+1} R = \square^\nu R$, which makes $\square^\nu R$ logical. For **the** logical relation, the "chosen one", plug $R = \top = X \times X$.

Constructing step-indexed logical relation, Coalgebraically

$$
\begin{aligned}
\mathcal{L}^0_\tau(\Gamma) &= \top_\tau(\Gamma) = \{(t,s) \mid \Gamma \vdash t,s\colon \tau\} \\
\mathcal{L}^{\alpha+1}_\tau &= \mathcal{L}^\alpha_\tau \cap \mathcal{S}_\tau(\mathcal{L}^\alpha,\mathcal{L}^\alpha) \cap \mathcal{E}_\tau(\mathcal{L}^\alpha) \cap \mathcal{V}_\tau(\mathcal{L}^\alpha,\mathcal{L}^\alpha) \\
\mathcal{L}^\alpha_\tau(\Gamma) &= \bigcap_{\beta<\alpha} \mathcal{L}^\alpha_\tau(\Gamma) \quad \text{for limit ordinals } \alpha. \\
\mathcal{S}_\tau(\Gamma)(Q,R) &= \{(t,s) \mid \text{for all } \Delta \text{ and } Q_{\Gamma(x)}(\Delta)(u_x,v_x)\ (x \in |\Gamma|), \\
&\qquad \text{one has } R_\tau(\Delta)(t[\vec{u}], s[\vec{v}])\}, \\
\mathcal{E}_\tau(\Gamma)(R) &= \{(t,s) \mid \text{if } t \to t' \text{ then } \exists s'.\, s \Rightarrow s' \wedge R_\tau(\Gamma)(t',s')\}, \\
\mathcal{V}_{\tau_1 \boxtimes \tau_2}(\Gamma)(Q,R) &= \{(t,s) \mid \text{if } t = \mathsf{pair}_{\tau_1,\tau_2}(t_1,t_2) \text{ then } \exists s_1,s_2.\, s \Rightarrow \mathsf{pair}_{\tau_1,\tau_2}(s_1,s_2) \wedge \\
&\qquad R_{\tau_1}(\Gamma)(t_1,s_1) \wedge R_{\tau_2}(\Gamma)(t_2,s_2)\}, \\
\mathcal{V}_{\mu\alpha.\tau}(\Gamma)(Q,R) &= \{(t,s) \mid \text{if } t = \mathsf{fold}_\tau(t') \text{ then } \exists s'.\, s \Rightarrow \mathsf{fold}_\tau(s') \wedge R_{\tau[\mu\alpha.\tau/\alpha]}(\Gamma)(t',s')\}, \\
\mathcal{V}_{\tau_1 \multimap \tau_2}(\Gamma)(Q,R) &= \{(t,s) \mid \text{for all } Q_{\tau_1}(\Gamma)(e,e'), \\
&\qquad \text{if } t = \lambda x.t' \text{ then } \exists s'.\, s \Rightarrow \lambda x.s' \ \wedge\ R_{\tau_2}(\Gamma)(t'[e/x], s'[e'/x])\}.
\end{aligned}
$$

Some results

Data: Higher-Order GSOS law of Σ over B in a suitable category $\mathcal{C}$, liftings, weakening of the operational model (the coalgebra on terms $\mu\Sigma$) and mild conditions on $\mathcal{C}$.

Main theorem (informal)

Let $R \rightarrowtail \mu\Sigma \times \mu\Sigma$ be a congruence. Assuming a lax-bialgebra condition. If R is a congruence, then for all α, $\Box^{\alpha} R$ is a congruence.

$$\frac{t \overset{s}{\Rightarrow} t'}{t\, s \Rightarrow t'} \quad \checkmark \qquad\qquad \frac{t \Rightarrow t'}{t\, s \Rightarrow t'\, s} \quad \checkmark$$

Corollary

1. *For all α, $\Box^{\alpha}\top$ is a congruence.*
2. *$\Box^{\nu}\top$ is a congruence (and hence reflexive) and, for "reasonable" definitions of contextual equivalence, sound w.r.t. contextual equivalence.*

The point of all this

Accept for a single slide that every higher-order operational semantics is a higher-order GSOS law. You are presented with such semantics and looking for a sensible logical relation, sound for contextual equivalence. What we're saying is that the actual work that needs to be done, the language-dependent part of the problem, is the following:

1. Decide what kinds of relational reasoning you're looking for (define the lifting $\overline{B}$).
2. Check that your notion of "weakening" is sensible w.r.t. the operational semantics.

The intuition is that the standard compatibility lemmas contain lots of boilerplate, contrived proof code that should be "automatic" under reasonable circumstances.

The point of all this

It's not just that higher-order abstract GSOS is cool and efficient. By systematizing (step-indexed) logical relations, we show that, assuming the operational semantics are minimally sane, the evident logical relation should be reflexive and sound w.r.t. contextual equivalence.

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