TOWARDS STRUCTURES OF HIGHER CATEGORICAL STRUCTURES

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Contents

1. Introduction

Small categories forms a 2-category, but when profunctors are also taken into account, they form a double category $\mathbb{P}\mathrm{rof}$. In [CS10], Cruttwell and Shulman defined a unified framework to conceive several notions of *generalised multicategories*, which are kinds of category-like structures, and showed that each of those concepts forms a *virtual double category*, which is a generalisation of double category. On the other hand, there are also some attempts to formalise category theory by means of *augmented virtual double category*, which is also an generalisation of double category [Kou22]. Thus, it is natural to assume that the collection of 1-dimensional structures such as categories forms a 2-dimensional structure. In general, the collection of n-dimensional structure is expected to have an $n+1$-dimensional structure. The main objective of this paper is to give an answer to this expectation.

This paper uses the theory on familial monads introduced by Shapiro in [Sha21, Sha22] as a general framework for defining higher categories. In general, for a cartesian monad T, a generalised notion of category, called *T-category* in this paper, is defined [Bur71, Lei04], whereas in [Sha22], it is asserted that there exists another familial monad $\mathtt{fc}[T]$ whose algebras coincide with T-categories, whenever T is familial. When T is trivial, T-categories are ordinary categories and $\mathtt{fc}[T]$-categories are virtual double categories, hence $\mathtt{fc}[T]$-category is reworded as T-virtual double category. Moreover, for $\mathtt{fc}^{\mathtt{n}}[T]$ obtained by repeating $\mathtt{fc}[-]$, $\mathtt{fc}^{\mathtt{n}}[T]$-categories (=:virtual $n+1$-tuple categories) can be considered an example of concepts of $(n+1)$-dimensional structure.

In Section 4.1, under some assumptions on T, we suggest a definition of the category of *T-simplices*, hence we obtain notions such as *T-simplicial set* and *T-simplicial category*, and show that the category of T-categories is embedded in the category of T-simplicial set.

In Section 4.2, we investigate the virtual double category of T-categories and *T-profunctors*, $\mathbb{P}\mathrm{rof}(T)$, which is an example of the virtual double category of *T-monoids* defined in [CS10], in terms of *pseudo simplicial category*, i.e. pseudo functor from the category of simplices to 2-category of categories. We show that one can define a (2-truncated) pseudo simplicial category of T-categories and T-profunctors, $\overline{\mathbb{P}\mathrm{rof}_2}(T)$, and $\mathbb{P}\mathrm{rof}(T)$ is the free objects with respect to a "nerve" 2-functor from the 2-category of virtual double categories to the 2-category of pseudo simplicial categories, hence $\mathbb{P}\mathrm{rof}(T)$ can be seen as the "*realization*" of $\overline{\mathbb{P}\mathrm{rof}_2}(T)$ as a virtual double category.

Combining those observations, we suggest an definition of the $\mathtt{fc}$-pseudo simplicial category of virtual double categories, and the virtual triple category of virtual double categories as its *realization*.

Moreover, we suggest a way to define the virtual $n+2$-tuple category of virtual $n+1$-tuple categories, in general.

2. Terminology and Preliminaries

A *double category* $\mathbb{X}$ is a category pseudo internal to the 2-category $\mathcal{C}at$ of categories, hence it has the following data:

- a set of *objects* in $\mathbb{X}$

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- for each pair of objects X, Y, a set of *horizontal arrows*, written as slashed arrows like $X \nrightarrow Y$
- for each pair of objects X, Y, a set of *vertical arrows*, written as $X \longrightarrow Y$
- for each square formed by horizontal and vertical arrows, a set of *cells*, written as follows

(2.1)
$$\begin{tikzcd}
X \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "g"] \\
A \arrow[r, "q"', "\shortmid"{marking}] & B
\end{tikzcd}$$

- vertical and horizontal compositions and identities satisfying coherence conditions

We say a cell α above is *horizontal* if f and g are identities. Horizontal arrows and horizontal cells forms a bicategory, which we write $\mathcal{H}(\mathbb{X})$.

By the notations on the left side of the equations below, we mean the cells denoted on the right side;

(2.2)
$$\begin{tikzcd}
X \arrow[rr, "p", "\shortmid"{marking}] \arrow[dr, "f"'] & \gamma & Y \arrow[dl, "g"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\gamma"] & Y \arrow[d, "g"] \\
A \arrow[r, "\mathtt{Id}_A"', "\shortmid"{marking}] & A
\end{tikzcd}
,\quad
\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A \arrow[rr, "q"', "\shortmid"{marking}] & \gamma & B
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "\mathtt{Id}_X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\gamma"] & X \arrow[d, "g"] \\
A \arrow[r, "q"', "\shortmid"{marking}] & B
\end{tikzcd}
,\quad
\begin{tikzcd}
X \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\gamma"] \\
A
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "\mathtt{Id}_X", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\gamma"] & X \arrow[d, "g"] \\
A \arrow[r, "\mathtt{Id}_A"', "\shortmid"{marking}] & A
\end{tikzcd}$$

For a vertical arrow $f : X \longrightarrow A$ in a double category $\mathbb{X}$, a *companion* of f is a horizontal cell $f_* : X \nrightarrow A$ such that there exists two cells, α and β, satisfying conditions below:

(2.3)
$$\begin{tikzcd}
& X \arrow[dr, "f"] \arrow[d, phantom, "\alpha"] & \\
X \arrow[ur, equal] \arrow[rr, "f_*", "\shortmid"{marking}] \arrow[dr, "f"'] & \beta & A \arrow[dl, equal] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[d, bend right, "f"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "="] \\
A
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[r, "f_*", "\shortmid"{marking}] \arrow[d, "f"] \arrow[dr, phantom, "\beta"] & A \arrow[dl, equal] \\
X \arrow[ur, equal] \arrow[r, "f_*"', "\shortmid"{marking}] \arrow[ur, phantom, "\alpha"] & A &
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[r, "f_*", "\shortmid"{marking}] & A \arrow[dl, equal] \\
X \arrow[ur, equal] \arrow[r, "f_*"', "\shortmid"{marking}] \arrow[ur, phantom, "\shortparallel"] & A &
\end{tikzcd}$$

where $=$ and $\shortparallel$ are horizontal and vertical identity cells. A *conjoint* f^* of f is the horizontal dual of companion. For each vertical arrow $f : X \longrightarrow Y$, its companion and conjoint are unique if exist. $\mathbb{X}$ is called an *equipment* if every vertical arrows have companions and conjoints. A cell α (2.1) is *cartesian* if any cell on the right below uniquely factors through α:

(2.4)
$$\begin{tikzcd}
\cdot \arrow[r, "\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\bar{\beta}"] & \cdot \arrow[d, "l"] \\
X \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "g"] \\
A \arrow[r, "q"', "\shortmid"{marking}] & B
\end{tikzcd}
=
\begin{tikzcd}
\cdot \arrow[r, "\shortmid"{marking}] \arrow[d, "fk"'] \arrow[dr, phantom, "\beta"] & \cdot \arrow[d, "gl"] \\
A \arrow[r, "q"', "\shortmid"{marking}] & B
\end{tikzcd}$$

If $\mathbb{X}$ is an equipment, then α is cartesian if and only if the composite below is an isomorphism in $\mathcal{H}(\mathbb{X})$.

(2.5)
$$\begin{tikzcd}
& X \arrow[r, "\shortmid"{marking}] \arrow[dl, equal] \arrow[d, dashed, "f"] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, dashed, "g"] \arrow[dr, equal] & \\
X \arrow[r, "f^*"', "\shortmid"{marking}] & A \arrow[r, "\shortmid"{marking}] & B \arrow[r, "g^*"', "\shortmid"{marking}] & Y
\end{tikzcd}$$

Example 2.1. For each small categories $\mathbf{X}$ and $\mathbf{Y}$, we define a *profunctor* $\mathbf{X} \nrightarrow \mathbf{Y}$ as a functor $\mathbf{X}^{\mathsf{op}} \times \mathbf{Y} \longrightarrow \mathbf{Set}$. There is the double category $\mathbb{P}\mathrm{rof}$ consisting of small categories as its objects, functors as its vertical arrows, and profunctors as its horizontal arrows. A cell in $\mathbb{P}\mathrm{rof}$

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & \mathbf{Y} \arrow[d, "g"] \\
\mathbf{A} \arrow[r, "q"', "\shortmid"{marking}] & \mathbf{B}
\end{tikzcd}$$

is a family of functions $\alpha_{x,y} : p(x, y) \longrightarrow q(f(x), g(y))$ which is natural in $x \in \mathbf{X}$ and $y \in \mathbf{Y}$. $\mathbb{P}\mathrm{rof}$ is in fact an equipment, where for each functor $f : \mathbf{X} \longrightarrow \mathbf{A}$, the companion $f_* : \mathbf{X} \nrightarrow \mathbf{A}$ and conjoint $f^* : \mathbf{A} \nrightarrow \mathbf{X}$ is defined as $\mathbf{A}(f(x), a)$ and $\mathbf{A}(x, f(a))$ respectively. The horizontal composition of $\mathbf{X} \overset{p}{\nrightarrow} \mathbf{Y} \overset{q}{\nrightarrow} \mathbf{Z}$ is given by the following coend formula:

(2.6)
$$q \circ p(x, z) := \int^{y \in \mathbf{Y}} p(x, y) \times q(y, z)$$

By $\mathcal{P}rof$, we mean the horizontal bicategory $\mathcal{H}(\mathbb{P}\mathrm{rof})$. ■

Example 2.2. Let $\mathbf{E}$ be a category with pullbacks. Then there is a double category of *spans* in $\mathbf{E}$, $\mathbb{S}\mathrm{pan}(\mathbf{E})$, defined as follows: objects are those of $\mathbf{E}$, vertical arrows are morphisms of $\mathbf{E}$, and a horizontal arrow $p : X \nrightarrow Y$ is a span $X \longleftarrow |p| \longrightarrow Y$ in $\mathbf{E}$. A cell α like (2.1) is a morphism $\alpha : |p| \longrightarrow |q|$ which makes obvious diagram consisting of f, g, and the legs of p and q commutes. The horizontal composite $q \circ p$ is given by pullback of the right leg of p along the left leg of q. ■

A *virtual double category* $\mathbb{A}$ has the following data:

- data of objects, horizontal arrows, and vertical arrows as in the definition of double category.
- for each square of the form below, a set of *cells*, written as follows

(2.7)
$$\begin{tikzcd}
X_0 \arrow[r, "p_0", "|" marking] \arrow[d, "f"'] & X_1 \arrow[r, "p_1", "|" marking] & \cdots \arrow[r, "|" marking] & X_{n-1} \arrow[r, "p_{n-1}", "|" marking] & X_n \arrow[d, "g"] \\
A_0 \arrow[rrrr, "q"', "|" marking] & & \alpha & & A_1
\end{tikzcd}$$

- vertical arrows has composition, so that objects and vertical arrows form a category.
- only cell composites that preserve the shape of cell above are allowed. In particular we do not have compositions of horizontal arrows. See [CS10] for more detail.

By *arity* of a cell, we mean the length of the sequence of horizontal arrows which is the source of the cell. In the case of (2.7), the arity is n.

Any double category can be seen as a virtual double category in an obvious way; that is, virtual double category can be seen as a generalisation of double category. In the same way as (2.4), we say a cell α of form (2.1) is *cartesian* if any cell of the form on the right below factors as the left below.

(2.8)
$$\begin{tikzcd}
\cdot \arrow[r, "|" marking] \arrow[d, "k"'] & \cdots \arrow[r, "|" marking] & \cdot \arrow[d, "l"] \\
X \arrow[rr, "p", "|" marking] \arrow[d, "f"'] & \bar{\beta} & Y \arrow[d, "g"] \\
A \arrow[rr, "q"', "|" marking] & \alpha & B
\end{tikzcd}
=
\begin{tikzcd}
\cdot \arrow[r, "|" marking] \arrow[d, "fk"'] & \cdots \arrow[r, "|" marking] & \cdot \arrow[d, "gl"] \\
A \arrow[rr, "q"', "|" marking] & \beta & B
\end{tikzcd}$$

On the other hand, we say a horizontal cell on the left below is *weakly opcartesian* if any cell β on the right below factors as follows:

(2.9)
$$\begin{tikzcd}
\cdot \arrow[r, "p^0", "|" marking] \arrow[d, equal] & \cdots \arrow[r, "p^{n-1}", "|" marking] & \cdot \arrow[d, equal] \\
X \arrow[rr, "p", "|" marking] \arrow[d, "f"'] & \alpha & Y \arrow[d, "g"] \\
A \arrow[rr, "q"', "|" marking] & \bar{\beta} & B
\end{tikzcd}
=
\begin{tikzcd}
\cdot \arrow[r, "|" marking] \arrow[d, "f"'] & \cdots \arrow[r, "|" marking] & \cdot \arrow[d, "g"] \\
A \arrow[rr, "q"', "|" marking] & \beta & B
\end{tikzcd}$$

Remark 5.8 and Theorem 5.2 in [CS10] shows that if any composable string of horizontal arrows is the source of a weakly opcartesian cell, and weakly opcartesian cells are closed under vertical composite, then $\mathbb{X}$ is a double category.

We write Δ for the category of simplices, and write $\mathbf{G}_1$ for the category of globes of dimension ≤ 1, which is interpreted as a subcategory of Δ; i.e., $\mathbf{G}_1$ has two objects written as $[0]$ and $[1]$, and has two non-trivial arrows written as $[0] \xrightarrow{\partial_1^1} [1]$ and $[0] \xrightarrow{\partial_0^1} [1]$.

By $\emptyset$, we mean both the empty set and the empty category, and we write $\mathbf{1}$ for the terminal category.

3. Familial Representation

In this section, we briefly review some concepts surrounding *familial representation*, which is introduced in [Sha21]. Note that, since we define a profunctor $\mathbf{C} \nrightarrow \mathbf{D}$ to be a functor $\mathbf{C}^{\mathsf{op}} \times \mathbf{D} \longrightarrow \mathbf{Set}$, we identify a presheaf with a profunctor to the terminal category $\mathbf{1}$.

Definition 3.1. Let $\mathbf{C}$ and $\mathbf{C}'$ be small categories. A ***familial representation*** $F = (S_F, E_F) : \mathbf{C}' \mathrel{-\mkern-3mu f\mkern-3mu\rightarrow} \mathbf{C}$ is a pair of

- a presheaf $S_F : \mathbf{C} \nrightarrow \mathbf{1}$, or equivalently, a discrete fibration $\mathsf{ty}_F : \int S_F \longrightarrow \mathbf{C}$ and
- a profunctor $E_F : \mathbf{C}' \nrightarrow \int S_F$, or equivalently, a functor $E_F[-] : \int S_F \longrightarrow \widehat{\mathbf{C}'}$.

■

Definition 3.2. Let $F : \mathbf{C}' \mathrel{-\mkern-3mu f\mkern-3mu\rightarrow} \mathbf{C}$ and $F' : \mathbf{C}'' \mathrel{-\mkern-3mu f\mkern-3mu\rightarrow} \mathbf{C}'$ be familial representations. We define the ***composite*** $FF' : \mathbf{C}'' \mathrel{-\mkern-3mu f\mkern-3mu\rightarrow} \mathbf{C}$ as follows:

- The total category of $S_{FF'}$ is

(3.1)
$$\int S_{FF'} := \int\Big(\widehat{\mathbf{C}'}(E_F[-], S_{F'})\Big)$$

  where $\widehat{\mathbf{C}'}(E_F[-], S_{F'}) : \left(\int S_F\right)^{\mathsf{op}} \longrightarrow \mathbf{Set}$ is the presheaf induced from $E_F[-] : \int S_F \longrightarrow \widehat{\mathbf{C}'}$ and $S_{F'} \in \widehat{\mathbf{C}'}$. This presheaf is the same as the right extension $\mathrm{rex}_{E_F} S_{F'} : \int S_F \nrightarrow \mathbf{1}$ of $S_{F'} : \mathbf{C}' \nrightarrow \mathbf{1}$ along $E_F : \mathbf{C}' \nrightarrow \int S_F$. We mean by $F\mathsf{ty}_{F'} : \int S_{FF'} \longrightarrow \int S_F$ the discrete fibration corresponding to $\widehat{\mathbf{C}}(E_F[-], S_{F'})$.
- The discrete fibration $\mathsf{ty}_{FF'} : \int S_{FF'} \longrightarrow \mathbf{C}$ is the composite $\mathsf{ty}_F \cdot F\mathsf{ty}_{F'}$ of discrete fibrations.

- $E_{FF'} : \mathbf{C}'' \nrightarrow \int S_{FF'}$ is defined as the composite

$$\begin{tikzcd} \mathbf{C}'' \arrow[r, "E_{F'}", "\shortmid"{marking}] & \int S_{F'} \arrow[r, "E_{F\overline{S_{F'}}}", "\shortmid"{marking}] & \int S_{FF'} \end{tikzcd} \tag{3.2}$$

  of profunctors where $E_{F\overline{S_{F'}}}(\lambda, \kappa) := \operatorname{colim}\left(\int E_F\left[F\mathsf{ty}_{F'}(\kappa)\right] \xrightarrow{\int\kappa} \int S_{F'} \xrightarrow{\mathrm{Hom}(\lambda,-)} \mathbf{Set} \right)$ for each $\lambda \in S_{F'}$ and $\kappa : E_F\left[F\mathsf{ty}_{F'}(\kappa)\right] \longrightarrow S_{F'}$, hence $E_{FF'}(c'', \kappa) \cong \operatorname*{colim}_{x \in \int E_F[F\mathsf{ty}_{F'}(\kappa)]} E_{F'}(c'', \kappa(x))$.

The ***identity*** familial representation on $\mathbf{C}$ is defined as the pair $((!_{\mathbf{C}})_*, \mathtt{Id}_{\mathbf{C}})$, where $\mathtt{Id}_{\mathbf{C}}$ is the identity profunctor on $\mathbf{C}$ and $!_{\mathbf{C}} : \mathbf{C} \longrightarrow \mathbf{1}$ is the unique functor to the terminal category $\mathbf{1}$. ■

Remark 3.3. For any presheaf $S : \mathbf{C} \nrightarrow \mathbf{1}$ and any profunctor $E : \mathbf{C}' \nrightarrow \mathbf{C}$, we mean by $\overline{S} : \int S \mathrel{\mathrlap{\;\scriptstyle f}\longrightarrow} \mathbf{C}$ the familial representation $(S, \mathtt{Id}_{\int S})$ and by $\widetilde{E} : \mathbf{C}' \mathrel{\mathrlap{\;\scriptstyle f}\longrightarrow} \mathbf{C}$ we mean $((!_{\mathbf{C}})_*, E)$ where $!_{\mathbf{C}}$ is the unique functor from $\mathbf{C}$ to $\mathbf{1}$. The notation $E_{F\overline{S_{F'}}}$ for the profunctor defined in Definition 3.2 is justified as the E-part of the composite $F\overline{S_{F'}}$. Any familial representation $F : \mathbf{C}' \mathrel{\mathrlap{\;\scriptstyle f}\longrightarrow} \mathbf{C}$ factors as $\overline{S_F}\widetilde{E_F}$. ■

Remark 3.4. On the other hand, a presheaf $X : \mathbf{C} \nrightarrow \mathbf{1}$ can be identified with a familial representation $\emptyset \mathrel{\mathrlap{\;\scriptstyle f}\longrightarrow} \mathbf{C}$ since there is precisely one profunctor whose type is $\emptyset \nrightarrow \int X$. For a familial representation $F : \mathbf{C} \mathrel{\mathrlap{\;\scriptstyle f}\longrightarrow} \mathbf{C}'$, the composite $FX : \emptyset \longrightarrow \mathbf{C}'$ is the presheaf presented as $\coprod_{\lambda \in S_F(-)} \widehat{\mathbf{C}}(E_F[\lambda], X)$. ■

Definition 3.5. A ***morphism of familial representation*** $\phi : F \Rightarrow G$ between parallel familial representations is a pair (ϕ^S, ϕ^E) such that

- $\phi^S : S_F \Rightarrow S_G$ is a morphism of presheaves, and
- ϕ^E is a natural isomorphism $E_F[-] \longrightarrow E_G\left[\int\phi^S(-)\right]$, or equivalently, a cartesian cell below in $\mathbb{P}\mathrm{rof}$.

$$\begin{tikzcd} \mathbf{C}' \arrow[r, "E_F", "\shortmid"{marking}] \arrow[d, equal] & \int S_F \arrow[d, "\int\phi^S"] \arrow[dl, phantom, "\phi^E"] \\ \mathbf{C}' \arrow[r, "E_G"', "\shortmid"{marking}] & \int S_G \end{tikzcd} \tag{3.3}$$

The composite $\psi \circ \phi : F \Rightarrow H$ of two morphisms $\phi : F \Rightarrow G$ and $\psi : G \Rightarrow H$ is defined as the pairwise composite of natural transformations $(\psi^S \circ \phi^S, \psi^E \circ \phi^E)$, where $\psi^E \circ \phi^E$ is the vertical composite of the cells (3.3). There are obvious identity morphisms, hence they form a category $\mathcal{R}ep(\mathbf{C}', \mathbf{C})$ of familial representations from $\mathbf{C}'$ to $\mathbf{C}$. ■

Consider a cell in $\mathbb{P}\mathrm{rof}$

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\shortmid"{marking}] \arrow[d, "f"'] & \mathbf{C} \arrow[d, "g"] \arrow[dl, phantom, "\alpha"] \\ \mathbf{B} \arrow[r, "\shortmid"{marking}] & \mathbf{D} \end{tikzcd} \tag{3.4}$$

and its *conjoint* α^*; the composite of the cells below.

$$\begin{tikzcd} \mathbf{B} \arrow[r, "f^*", "\shortmid"{marking}] \arrow[dr, equal] & \mathbf{A} \arrow[r, "\shortmid"{marking}] \arrow[d, dashed, "f"] & \mathbf{C} \arrow[d, dashed, "g"] \arrow[dl, phantom, "\alpha"] \arrow[dr, equal] & \\ & \mathbf{B} \arrow[r, "\shortmid"{marking}] & \mathbf{D} \arrow[r, "g^*"', "\shortmid"{marking}] & \mathbf{C} \end{tikzcd} \tag{3.5}$$

If f is an identity, then α^* is an isomorphism if and only if α is cartesian. It is straightforward to check cells whose conjoints are isomorphisms are closed under horizontal and vertical compositions.

Definition 3.6 (Whiskerings)**.** Let $\phi : F \Longrightarrow G : \mathbf{C}' \mathrel{\mathrlap{\;\scriptstyle f}\longrightarrow} \mathbf{C}$ and $\phi' : F' \Longrightarrow G' : \mathbf{C}'' \mathrel{\mathrlap{\;\scriptstyle f}\longrightarrow} \mathbf{C}'$ be morphisms of familial representations.

- A morphism $F\phi' : FF' \Longrightarrow FG'$ consists of the following
  - $\int S_{FF'} \xrightarrow{\int(F\phi')^S} \int S_{FG'}$ is derived from the post-composition $\widehat{\mathbf{C}'}(E_F[-], S_{F'}) \xrightarrow{\phi'^S \cdot -} \widehat{\mathbf{C}'}(E_F[-], S_{G'})$.
  - We write

$$\begin{tikzcd} \int S_{F'} \arrow[r, "E_{F\overline{S_{F'}}}", "\shortmid"{marking}] \arrow[d, "\int\phi'^S"'] & \int S_{FF'} \arrow[d, "\int(F\phi')^S"] \arrow[dl, phantom, "\left(F\overline{\phi'^S}\right)^E"] \\ \int S_{G'} \arrow[r, "E_{F\overline{S_{G'}}}"', "\shortmid"{marking}] & \int S_{FG'} \end{tikzcd} \tag{3.6}$$

for the cell whose component

$$\left(F\overline{\phi'^S}\right)^E_{t',\tau} : \underset{x\in\int E_F\,[F\mathsf{ty}_{F'}(\tau)]}{\operatorname{colim}} \int S_{F'}(t',\tau(x)) \longrightarrow \underset{x\in\int E_F\,[F\mathsf{ty}_{F'}(\tau)]}{\operatorname{colim}} \int S_{G'}(\phi'^S(t'),\phi'^S\circ\tau(x))$$

is given by the functor $\int\phi'^S$ on maps. $F\mathsf{ty}_{G'}(\phi'^S\circ\tau) = F\mathsf{ty}_{F'}(\tau)$, hence the right hand side is precisely $E_{F\overline{S_{G'}}}(\phi'^S(t'),\phi'^S\circ\tau)$. This conjoint of this cell is an isomorphism since those colimits commutes with the coend defining the composition $E_{F\overline{S_{F'}}}\circ\int\phi'^{S*}$ of profunctors.
The E-part $(F\phi')^E$ is defined as the composite:

$$\begin{tikzcd}
\mathbf{C}'' \arrow[r, "E_{F'}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\phi'^E"] & \int S_{F'} \arrow[r, "E_{F\overline{S_{F'}}}", "\shortmid"{marking}] \arrow[d, "\int\phi'^S"', "\shortmid"{marking}] \arrow[dr, phantom, "\left(F\overline{\phi'^S}\right)^E"] & \int S_{FF'} \arrow[d, "\int(F\phi')^S"] \\
\mathbf{C}'' \arrow[r, "E_{G'}"', "\shortmid"{marking}] & \int S_{G'} \arrow[r, "E_{F\overline{S_{G'}}}"', "\shortmid"{marking}] & \int S_{FG'}
\end{tikzcd}$$

which is cartesian.

- A morphism $\phi F' : FF' \Rightarrow GF'$ consists of the following
  - $\int S_{FF'} \xrightarrow{\int(\phi F')^S} \int S_{GF'}$ is derived from the pre-composition $\widehat{\mathbf{C}'}(E_F\,[-]\,,S_{F'}) \xrightarrow[\cong]{-\cdot(\phi^E)^{-1}} \widehat{\mathbf{C}'}(E_G\,[\phi^S-]\,,S_{F'})$ which can be interpreted as a cartesian morphism $\widehat{\mathbf{C}'}(E_F\,[-]\,,S_{F'}) \longrightarrow \widehat{\mathbf{C}'}(E_G\,[-]\,,S_{F'})$ in the fibration obtained through the Grothendieck construction of the pseudo-functor $\mathbf{C}\mapsto\widehat{\mathbf{C}} : \mathbf{Cat}^{\mathsf{op}} \longrightarrow \mathcal{CAT}$.
  - We write

$$\begin{tikzcd}
\int S_{F'} \arrow[r, "E_{F\overline{S_{F'}}}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\left(\phi\overline{S_{F'}}\right)^E"] & \int S_{FF'} \arrow[d, "\int(\phi F')^S"] \\
\int S_{F'} \arrow[r, "E_{G\overline{S_{F'}}}"', "\shortmid"{marking}] & \int S_{GF'}
\end{tikzcd} \tag{3.7}$$

for the cartesian cell whose components are the canonical isomorphisms

$$\underset{x\in\int E_F[F\mathsf{ty}_{F'}(\tau)]}{\operatorname{colim}} \int S_{F'}(t',\tau(x)) \xrightarrow[\cong]{} \underset{y\in\int E_G\left[\phi^S(F\mathsf{ty}_{F'}(\tau))\right]}{\operatorname{colim}} \int S_{G'}(t',\tau\circ(\phi^E)^{-1}(y))$$

induced from the isomorphisms $\int E_G\left[\phi^S(F\mathsf{ty}_{F'}(\tau))\right] \xrightarrow{\int(\phi^E)^{-1}_{F\mathsf{ty}_{F'}(\tau)}} \int E_F[F\mathsf{ty}_{F'}(\tau)]$ for each $\tau\in\int S_{FF'}$.
The E-part $(\phi F')^E$ is defined as the composite

$$\begin{tikzcd}
\mathbf{C}'' \arrow[r, "E_{F'}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & \int S_{F'} \arrow[r, "E_{F\overline{S_{F'}}}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\left(\phi\overline{S_{F'}}\right)^E"] & \int S_{FF'} \arrow[d, "\int(\phi F')^S"] \\
\mathbf{C}'' \arrow[r, "E_{F'}"', "\shortmid"{marking}] & \int S_{F'} \arrow[r, "E_{G\overline{S_{F'}}}"', "\shortmid"{marking}] & \int S_{GF'}
\end{tikzcd}$$

■

Proposition 3.7. *Definition 3.2, Definition 3.5, and Definition 3.6 define a bicategory $\mathcal{Rep}$.*

Remark 3.8. For any small category $\mathbf{C}$, $\mathcal{Rep}(\emptyset,\mathbf{C})$ is equivalent to the presheaf category $\widehat{\mathbf{C}}$. This is through the assignment $S\mapsto\overline{S}$ remarked in Remark 3.4. Therefore $\mathcal{Rep}(\emptyset,-)$ induces a pseudo functor $\mathcal{Rep}\longrightarrow\mathcal{CAT}$ which sends small categories to their presheaf categories. Functors between presheaf categories induced by this pseudo functor are called ***familial functors***, and natural transformations between familial functors are in the image of this pseudo functor precisely when they are ***cartesian***, where a natural transformation is said to be cartesian if its naturality squares are pullback squares. Moreover, any familial functor is cartesian, i.e. preserving pullbacks.
We often identify a familial functor $\widehat{\mathbf{C}}\longrightarrow\widehat{\mathbf{D}}$ with its familial representation $\mathbf{C} \overset{f}{\nrightarrow} \mathbf{D}$. ■

Remark 3.9. Recall that any small set of objects in an accessible category is a set of κ-presentable objects for some regular cardinal κ (Corollary 2.3.12 of [MP90]), any presheaf category is accessible, and $\{E_F[\lambda]\,|\,\lambda\in\int S_F\}$ is small. Therefore, any familial functor $F? = \coprod_{\lambda\in S_F(-)}\widehat{\mathbf{C}}(E_F\,[\lambda]\,,?)$ is accessible; that is, it preserves κ-filtered colimits for some κ. ■

Remark 3.10. For any familial representation $F : \mathbf{C}' \overset{f}{\nrightarrow} \mathbf{C}$, ty_F can be seen as the unique morphism $S_F\longrightarrow(!_{\mathbf{C}})_*$ between presheaves, and the notation $F\mathsf{ty}_{F'}$ for the functor defined in Definition 3.2 is justified as the whiskering of $\mathsf{ty}_{F'} : S_{F'} \Rightarrow (!_{\mathbf{C}'})_* : \emptyset \overset{f}{\nrightarrow} \mathbf{C}'$ with $F : \mathbf{C}' \overset{f}{\nrightarrow} \mathbf{C}$. ■

Definition 3.11. A monad in $\mathcal{R}ep$ is identified with the cartesian monad induced by $\mathcal{R}ep(\emptyset, -)$, which is called a ***familial monad***. ■

4. CATEGORICAL STRUCTURES

In this section, we fix a familial monad T on $\mathbf{C}$. Moreover, we suppose the following two conditions for T

- $E_T[\lambda]$ is *connected* [CCT14] in $\widehat{\mathbf{C}}$, which means $\mathbf{Gph}(T)(E_T[\lambda], -) : \widehat{\mathbf{C}} \longrightarrow \mathbf{Set}$ preserves small coproducts.
- $\mathbf{C}$ has no non-trivial endo-morphisms.

For example, the free category monad $\mathtt{fc}$ on $\mathbf{G}_1$ satisfies these condition since $E_{\mathtt{fc}}[n]$ is connected for each $n \in \mathbb{N} = S_{\mathtt{fc}}([1])$, and $\mathbf{G}_1$ has no non-trivial endo-morphisms. See the proof of Proposition 4.12 for more detail.

4.1. T-graphs, T-categories, and T-simplicial sets.

First of all, let us extend the Grothendieck construction to normal lax functors to $\mathcal{P}rof$. This is due to Section 7 of [Ben00].

Definition 4.1. Let $\mathbf{C}$ be a category and $X : \mathbf{C} \longrightarrow \mathcal{P}rof$ be a normal lax functor. The ***Grothendieck construction*** of X is a category $\int\!\!\!\square X$ equipped with a functor $\mathsf{ty}_X : \int\!\!\!\square X \longrightarrow \mathbf{C}$ defined as follows:

- $\mathrm{Obj}(\int\!\!\!\square X) := \coprod_{c \in \mathbf{C}} \mathrm{Obj}(X_c)$.
- For each $c, c' \in \mathbf{C}$, $x \in X_c$, and $x' \in X_{c'}$, $\quad \int\!\!\!\square X(x, x') := \coprod_{f : c \to c' \text{ in } \mathbf{C}} X_f(x, x')$.
  We write $t : x \underset{f}{\longrightarrow} x'$ when $f : c \longrightarrow c'$ in $\mathbf{C}$, $x \in X_c$, $x' \in X_{c'}$, and $t \in X_f(x, x')$.
- For each $t : x \underset{f}{\longrightarrow} x'$ and $t' : x' \underset{f'}{\longrightarrow} x''$, the composite $t't : x \underset{f'f}{\longrightarrow} x''$ in $\int\!\!\!\square X$ is defined by applying the lax functoriality $\mu_{f,f'} : X_{f'} \circ X_f \Rightarrow X_{f'f}$ to $[t, t'] \in \int^{\bar{x}' \in X_{c'}} X_f(x, \bar{x}') \times X_{f'}(\bar{x}', x'') \;\; =: X_{f'} \circ X_f(x, x'')$.

■

Example 4.2. For any presheaf $S : \mathbf{C}^{\mathsf{op}} \longrightarrow \mathbf{Set}$, the corresponding discrete fibration $\int S \longrightarrow \mathbf{C}$ is the same as the Grothendieck construction on $\mathbf{C} \xrightarrow{S^{\mathsf{op}}} \mathbf{Set}^{\mathsf{op}} \hookrightarrow \mathcal{C}at^{\mathsf{op}} \xrightarrow{(-)^*} \mathcal{P}rof$, where, for each $i : c \longrightarrow c'$ in $\mathbf{C}$ and $\lambda, \lambda' \in \int S$, morphism written as $\lambda \underset{i}{\longrightarrow} \lambda'$ in $\int\!\!\!\square S = \int S$ is unique if exists, which is denoted by $\bar{i}_{\lambda'}$. ■

Let $\mathbf{2} := \{0 \longrightarrow 1\}$ be the 2-element chain. Any profunctor $p : \mathbf{C} \nrightarrow \mathbf{D}$ can be seen as a functor $\ulcorner p \urcorner : \mathbf{2} \longrightarrow \mathcal{P}rof$, hence as a category $\int\!\!\!\square \ulcorner p \urcorner$ equipped with a functor $\int\!\!\!\square \ulcorner p \urcorner \longrightarrow \mathbf{2}$ whose pullback along $* \mapsto 0 : \mathbf{1} \longrightarrow \mathbf{2}$ (resp. $* \mapsto 1 : \mathbf{1} \longrightarrow \mathbf{2}$) is $\mathbf{C}$ (resp. $\mathbf{D}$).

Definition 4.3. For each normal lax functors $X : \mathbf{C} \longrightarrow \mathcal{P}rof$ and $X' : \mathbf{C}' \longrightarrow \mathcal{P}rof$, a ***proarrow*** between normal lax functors $P : X \nrightarrow X'$ consists of a profunctor $\mathsf{dom}(P) : \mathbf{C} \nrightarrow \mathbf{C}'$ and a normal lax functor $|P| : \int\!\!\!\square \ulcorner \mathsf{dom}(P) \urcorner \longrightarrow \mathcal{P}rof$ whose restriction to $\mathbf{C}$ (resp. $\mathbf{C}'$) is equal to X (resp. X'). One can easily check that the Grothendieck construction $\int\!\!\!\square |P| \longrightarrow \int\!\!\!\square \ulcorner \mathsf{dom}(P) \urcorner \longrightarrow \mathbf{2}$ defines another profunctor $\int\!\!\!\square P : \int\!\!\!\square X \nrightarrow \int\!\!\!\square X'$ equipped with a forgetful natural transformation

$$\begin{tikzcd}
\int\!\!\!\square X \arrow[r, "\int\!\!\!\square P", "|" marking] \arrow[d, "\mathsf{ty}_X"'] \arrow[dr, phantom, "\mathsf{ty}_P"] & \int\!\!\!\square X' \arrow[d, "\mathsf{ty}_{X'}"] \\
\mathbf{C} \arrow[r, "\mathsf{dom}(P)"', "|" marking] & \mathbf{C}'
\end{tikzcd} \tag{4.1}$$

which is called the *Grothendieck construction for the proarrow* P. ■

In detail, a proarrow $P : X \nrightarrow X'$ consists of

- a profunctor $\mathsf{dom}(P) : \mathbf{C} \nrightarrow \mathbf{C}'$
- for each $c \in \mathbf{C}$ and $c' \in \mathbf{C}'$, a functor $P_{c,c'} : \mathsf{dom}(P)(c, c') \longrightarrow \mathcal{P}rof(X(c), X'(c'))$
- for each $g \in \mathsf{dom}(P)(c, c')$, $f : \bar{c} \longrightarrow c$ in $\mathbf{C}$, and $f' : c' \longrightarrow \bar{c}'$ in $\mathbf{C}'$, natural transformations

$$\begin{tikzcd}
X(\bar{c}) \arrow[r, "X(f)", "|" marking] \arrow[d, equal] & X(c) \arrow[r, "P_{c,c'}(g)", "|" marking] \arrow[d, phantom, "\lambda^P_{f,g}"] & X'(c') \arrow[d, equal] & X(c) \arrow[r, "P_{c,c'}(g)", "|" marking] \arrow[d, equal] & X'(c') \arrow[r, "X'(f')", "|" marking] \arrow[d, phantom, "\rho^P_{g,f'}"] & X'(\bar{c}') \arrow[d, equal] \\
X(\bar{c}) \arrow[rr, "P_{\bar{c},c'}(g \cdot f)"', "|" marking] & & X'(c') & X(c) \arrow[rr, "P_{c,\bar{c}'}(f' \cdot g)"', "|" marking] & & X'(\bar{c}')
\end{tikzcd} \tag{4.2}$$

satisfying suitable coherence conditions.

Definition 4.4. There exists a pseudo (hence normal lax) functor $T^{\mathbf{G}_1} : \mathbf{G}_1 \longrightarrow \mathcal{P}rof$ defined as the following diagram.

$$\begin{tikzcd} \mathbf{C} \arrow[r, "|", "\mathsf{ty}_T^*", shift left] \arrow[r, "|"', "E_T"', shift right] & \int S_T \end{tikzcd} \tag{4.3}$$

We define the category of T-***globes*** of dimension ≤ 1, $\mathbf{G}_1(T)$, as the Grothendieck construction on $T^{\mathbf{G}_1}$ i.e. $\mathbf{G}_1(T) := \int\!\!\!\square\, T^{\mathbf{G}_1}$. $\mathbf{G}_1(T)$ is presented as follows:

- objects are those in $\mathbf{C}$ and $\int S_T$.
- morphisms generated by the following maps
  - maps in $\mathbf{C}$: each copy of $i : c \longrightarrow c'$ in $\mathbf{C}$
  - maps in $\int S_T$: each copy of $\bar{i}_{\lambda'} : \lambda \underset{i}{\longrightarrow} \lambda'$ in $\int S_T$
  - $a : c \longrightarrow \lambda$ for each $a \in E_T(c, \lambda)$
  - *target* maps: $\tau_\lambda : \mathsf{ty}_\lambda \longrightarrow \lambda$ for each $\lambda \in \int S_T$.

  subject to the following
  - any commutative diagram in $\mathbf{C}$ and $\int S_T$ commutes
  - for each $i : c' \longrightarrow c$ in $\mathbf{C}$, $\bar{j}_{\lambda'} : \lambda \underset{j}{\longrightarrow} \lambda'$ in $\int S_T$, and $a \in E_T(c, \lambda)$, the following diagram commutes

$$\begin{tikzcd} c' \arrow[r, "E_T(i{,}\bar{j}_{\lambda'})(a)"] \arrow[d, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & \lambda' \\ c \arrow[r, "a"'] & \lambda \arrow[u, "j"] \end{tikzcd} \tag{4.4}$$

  - for each $\lambda \underset{j}{\longrightarrow} \lambda'$ in $\int S_T$, the following diagram commutes

$$\begin{tikzcd} \mathsf{ty}_T(\lambda) \arrow[r, "\tau_\lambda"] \arrow[d, "j"'] \arrow[dr, phantom, "\circlearrowleft"] & \lambda \arrow[d, "j"] \\ \mathsf{ty}_T(\lambda') \arrow[r, "\tau_{\lambda'}"'] & \lambda' \end{tikzcd} \tag{4.5}$$

Therefore, $\mathbf{G}_1(T)$ has no non-trivial endo-morphisms if $\mathbf{C}$ is so. ■

Definition 4.5. A T-***graph*** is a presheaf on $\mathbf{G}_1(T)$. We write $\mathbf{Gph}(T)$ for the presheaf category $\widehat{\mathbf{G}_1(T)}$. ■

Remark 4.6. For any ordinary monad T on a (possibly large) category $\mathbf{E}$, a T-***graph*** is an endo-span $TX_0 \longleftarrow X_1 \longrightarrow X_0$. If T is familial then the two definitions of T-graph coincides:
Given a presheaf X on $\mathbf{G}_1(T)$, we obtain a span $TX_0 \longleftarrow X_1 \longrightarrow X_0$ in $\widehat{\mathbf{C}}$ as follows:

- $\mathsf{ty}_{X_0} : \int X_0 \longrightarrow \mathbf{C}$ is the pullback of $\mathsf{ty}_X : \int X \longrightarrow \mathbf{G}_1(T)$ along the canonical inclusion $\mathbf{C} \hookrightarrow \mathbf{G}_1(T)$.
- $\int X_1$ is the domain of the pullback $\mathsf{ty}'_X : \int X_1 \longrightarrow \int S_T$ of $\mathsf{ty}_X : \int X \longrightarrow \mathbf{G}_1(T)$ along the canonical inclusion $\int S_T \hookrightarrow \mathbf{G}_1(T)$, and ty_{X_1} is the composite of ty'_X and $\mathsf{ty}_T : \int S_T \longrightarrow \mathbf{C}$.
- for each $\lambda \in \int S_T$ and $\xi \in \int X_1$ over λ with respect to ty'_{X_1}, $\mathtt{src}(\xi) \in \widehat{\mathbf{C}}(E_T[\lambda], X_0) \subset \int TX_0$ is defined as $\mathtt{src}(\xi)_c : a \mapsto a^*\xi$ for each $c \in \mathbf{C}$, where, for each $a \in E_T(c, \lambda)$, $a^*\xi$ is the outcome of reindexing ξ along $a : c \underset{\partial_1^1}{\longrightarrow} \lambda$ in $\mathbf{G}_1(T)$ with respect to ty_X.
- for each $\lambda \in \int S_T$ and $\xi \in \int X_1$ over λ with respect to ty'_{X_1}, $\mathtt{tgt}(\xi) \in \int X_0$ is defined as ${\tau_\lambda}^*\xi$ which is the outcome of reindexing ξ along $\tau_\lambda : \mathsf{ty}_T(\lambda) \underset{\partial_0^1}{\longrightarrow} \lambda$ in $\mathbf{G}_1(T)$ with respect to ty_X.

■

Let Δ_a be the ***augmented simplex category***, which is presented as follows:

- objects are finite ordinals $[n] := \{0, \ldots, n\}$ $\quad (n \geq -1, \quad [-1] = \emptyset)$
- morphisms generated by the following two kinds of maps
  - *face maps*: $\partial_i^n : [n-1] \longrightarrow [n]$ $\qquad (n \geq 0$ and $i \in [n])$
  - *degeneracy maps*: $\sigma_i^n : [n+1] \longrightarrow [n]$ $\quad (n \geq 0$ and $i \in [n])$

  subject to the following *simplicial identities*

$$\partial_j^{n+1}\partial_i^n = \partial_i^{n+1}\partial_{j-1}^n \qquad i < j \tag{4.6}$$

$$\sigma_j^n\sigma_i^{n+1} = \sigma_i^n\sigma_{j+1}^{n+1} \qquad i \leq j \tag{4.7}$$

$$\sigma_j^{n-1}\partial_i^n = \begin{cases} \partial_i^{n-1}\sigma_{j-1}^{n-2} & i < j \\ \mathtt{id}_{[n-1]} & i \in \{j, j+1\} \\ \partial_{i-1}^{n-1}\sigma_j^{n-2} & i > j+1 \end{cases} \tag{4.8}$$

Δ_a is a monoidal category with the tensor product $[n] \oplus [m] := [n+m+1]$. By a *simplex*, we mean an object of Δ_a. The ***simplex category*** Δ is the full subcategory of Δ_a consisting of simplices of dimension greater than -1. A face map $\partial^n_i : [n-1] \longrightarrow [n]$ is said to be ***inner*** if i is neither 0 nor n. The subcategory of Δ generated from all inner faces and degeneracies is written as $\Delta_{\mathtt{inn}}$.

Proposition 4.7. *There exists a surjective-on-object and full functor* $\Delta_{\mathtt{inn}} \twoheadrightarrow \Delta_a^{\mathsf{op}}$ *which sends*

- $[n]$ *to* $[n-1]$,
- ∂^{n+2}_{i+1} *to* σ^n_i, *and*
- σ^n_i *to* ∂^n_i

for each $n \geq 0$ *and* $i \in [n]$.

The assignment $F \mapsto S_F$ extends to a functor $S : \mathcal{R}ep(\mathbf{C}', \mathbf{C}) \longrightarrow \mathcal{R}ep(\emptyset, \mathbf{C})$ which can be seen as the pre-composition by $(!_{\mathbf{C}'})_* : \emptyset \overset{f}{\nrightarrow} \mathbf{C}'$. On the other hand, T is a monoid in the monoidal category $\mathcal{R}ep(\mathbf{C}, \mathbf{C})$, hence induces a (strong monoidal) functor $\ulcorner T \urcorner : \Delta_a \longrightarrow \mathcal{R}ep(\mathbf{C}, \mathbf{C})$. Thus we obtain a pseudo functor $T^\Delta|_{\mathtt{inn}}$ defined as the composite of the following (pseudo) functors

$$\Delta_{\mathtt{inn}} \longrightarrow\!\!\!\!\!\rightarrow \Delta_a^{\mathsf{op}} \xrightarrow{\ulcorner T \urcorner^{\mathsf{op}}} \mathcal{R}ep(\mathbf{C}, \mathbf{C})^{\mathsf{op}} \xrightarrow{S^{\mathsf{op}}} \mathcal{R}ep(\emptyset, \mathbf{C})^{\mathsf{op}} \xrightarrow{\int^{\mathsf{op}}} \mathcal{C}at^{\mathsf{op}} \xrightarrow{(-)^*} \mathcal{P}rof \tag{4.9}$$

Proposition 4.8. *By setting* $T^\Delta(\partial^{n+1}_0) := (T^n \mathsf{ty}_T)^*$ *and* $T^\Delta(\partial^{n+1}_{n+1}) := E_{T\overline{S_{T^n}}}$ *for each* $n \geq 0$, $T^\Delta|_{\mathtt{inn}}$ *extends to a pseudo functor* $T^\Delta : \Delta \longrightarrow \mathcal{P}rof$.

Proof. For each $n \geq 0$ and $i \in [n]$, $T^\Delta(\partial^{n+2}_{i+1}) = (T^{n-i}\mu^S T^i)^* : \int S_{T^{n+1}} \nrightarrow \int S_{T^{n+2}}$ and $T^\Delta(\sigma^n_i) = (T^{n-i}\eta^S T^i)^* : \int S_{T^{n+1}} \nrightarrow \int S_{T^n}$. Of the remaining simplicial identities, those not involving ∂^{n+1}_{n+1} are trivial, considering identities such as $\mathsf{ty}_T \circ \mu^S = \mathsf{ty}_{T^2}$, $\mathsf{ty}_T \circ \eta^S = \mathtt{id}_{\mathbf{C}}$, and $\mathsf{ty}_T \circ T\mathsf{ty}_T = \mathsf{ty}_{T^2}$.

- The isomorphism corresponding to $\partial^{n+2}_{n+1}\partial^{n+1}_{i+1} = \partial^{n+2}_{i+1}\partial^{n+1}_{n+1}$ $(n > i \geq 0)$ is given by a cell whose conjoint is an isomorphism, written as $\left(T\overline{(T^{n-i}\mu T^i)^S}\right)^E$, which appears when one defines the post-whiskering of $T^{n-i}\mu T^i$ by T, see (3.6).
- In the same way, the isomorphism corresponding to $\sigma^{n+1}_j \partial^{n+3}_{n+3} = \partial^{n+1}_{n+1}\sigma^n_j$ $(n+1 > j)$ is given by $\left(T\overline{(T^{n-j}\eta T^j)^S}\right)^E$.
- The isomorphism corresponding to $\sigma^n_n \partial^{n+1}_{n+1} = \mathtt{id}_{[n]}$ is given by the cell appearing when one defines pre-whiskering, $\left(\eta \overline{S_{T^n}}\right)^E$, see (3.7).
- $T^n \mathsf{ty}_T$ induces a morphism of familial representations $\phi : \overline{S_{T^{n+1}}} \Longrightarrow \overline{S_{T^n}}$ (see Remark 3.3). Therefore the post-whiskering $\left(T\overline{\phi^S}\right)^E$ gives rise to an isomorphism for $\partial^{n+2}_{n+1}\partial^{n+1}_0 = \partial^{n+2}_0\partial^{n+1}_{n+1}$.

☐

Definition 4.9. We define the category of T-***simplices***, $\Delta(T)$, as the total category of the Grothendieck construction on T^Δ. A presheaf on $\Delta(T)$ is called a T-***simplicial set***, and we write $\mathbf{SSet}(T)$ for the presheaf category $\widehat{\Delta(T)}$. ■

Notation 4.10. Let X be a T-graph. For each $n \in \mathbb{N}$, $\lambda_{n+1} \in \int S_{T^{n+1}}$, $x_n \in \int T^n X_0$, $x_{n+1} \in \int T^{n+1} X_0$, and $\xi_n \in \int T^n X_1$, we mean by $x_{n+1} \xrightarrow[\lambda_{n+1}]{\xi_n} x_n$ that there exists $\lambda_n \in \int S_{T^n}$ such that the following diagram commutes in $\widehat{\mathbf{C}}$.

(4.10)

$$\begin{tikzcd}
 & & E_{T^n}[\lambda_n] \arrow[dll, bend right, "\lambda_{n+1}"'] \arrow[dl, bend right, "x_{n+1}"'] \arrow[d, "\xi_n"] \arrow[dr, bend left, "x_n"] & \\
S_T & TX_0 \arrow[l, "T\mathsf{ty}_{X_0}"] & X_1 \arrow[l, "\mathtt{src}"] \arrow[r, "\mathtt{tgt}"'] & X_0
\end{tikzcd}$$

When $n = 0$, we say $\xi_0 \in \int X_1$ is a λ_1-***arrow***. A T-graph is completely determined by λ_1-arrows equipped with their types for all $\lambda_1 \in \int S_T$.

A ***path*** of shape (λ_{n+m}, n), or λ_{n+m}-***path*** of length $n > 0$, written as $p : x_{n+m} \xrightarrow[\lambda_{n+m}]{} \!\!\!\!\rightarrow x_m$, is a sequence

$$p = (x_{n+m}, \xi_{n+m-1}, x_{n+m-1}, \ldots, x_{m+1}, \xi_m, x_m) \tag{4.11}$$

such that $x_{i+m+1} \xrightarrow[\lambda_{i+m+1}]{\xi_{i+m}} x_{i+m}$ holds for each $i \in [n-1]$. If $\lambda_m = T^m \mathsf{ty}_{X_0}(x_m)$ and $m > 0$, we admits unique λ_m-path of length 0 for each $x_m \in \int T^m X_0$, which is written as $()_{x_m} : x_m \xrightarrow[\lambda_m]{} \!\!\!\!\rightarrow x_m$. For each $c \in \mathbf{C}$ and $x_0 \in X_0(c)$, we admits unique c-path $x_0 \xrightarrow[c]{} \!\!\!\!\rightarrow x_0$,.

In short, a λ_{n+m}-path is an element of $\int(T^{n+m-1}X \circ \cdots \circ T^m X)_1$ (over λ_{n+m} in a sense), where $T^{n+m-1}X \circ \cdots \circ T^m X$ is the composite of n spans $T^{i+m}X : T^{i+m+1}X_0 \nrightarrow T^{i+m}X_0$ $(i < n)$, and $(\cdots)_1$ means the root of the span. ■

For any λ_{n+m}-path $p : x_{n+m} \underset{\lambda_{m+n}}{\twoheadrightarrow} x_m$ as above, $\lambda_m := T^m\mathsf{ty}_{T^n X_0}(x_{n+m})$, $c \in \mathbf{C}$, and $a \in E_{T^m}(c, \lambda_m)$, we can define a $\lambda_{m+n}(a)$-path $x_{n+m}(a) \overset{p(a)}{\underset{\lambda_{m+n}(a)}{\twoheadrightarrow}} x_m(a)$, whose component is written as follows:

$$(x_{n+m}(a), \xi_{n+m-1}(a), x_{n+m-1}(a), \ldots, x_{m+1}(a), \xi_m(a), x_m(a))$$

where each of x_{i+m} and ξ_{i+m} is interpreted as a morphism $E_{T^m}[\lambda_m] \longrightarrow T^i X_j$.

Definition 4.11. A *T-**category*** is a T-graph X equipped with *compositions*, written as $\mathtt{comp}$: for each λ_n-path $p : x_n \underset{\lambda_n}{\twoheadrightarrow} x_0$, $\mathtt{comp}_n$ assigns an $\mu^n(\lambda_n)$-arrow $\mathtt{comp}_n(p) : \mu^n_{X_0}(x_n) \xrightarrow[\mu(\lambda_n)]{} x_0$, where $\mu^n : T^n \longrightarrow T$ is the n-ary multiplication for T. We suppose that $\mathtt{comp}$ satisfies the following coherence condition:

$$\mathtt{comp}_2\left(T\mu^n\left(\mu^m\left(x_{n+m}\right)\right) \xrightarrow[T\mu^n\cdot\mu^m(\lambda_{n+m})]{\mathtt{comp}_n(\mu^m(p))} \mu^m\left(x_m\right) \xrightarrow[\mu^m(\lambda_m)]{\mathtt{comp}_m(q)} x_0 \right) = \mathtt{comp}_{n+m}\left(q \circ p\right) \tag{4.12}$$

for each composable paths $x_{n+m} \overset{p}{\underset{\lambda_{n+m}}{\twoheadrightarrow}} x_m \overset{q}{\underset{\lambda_m}{\twoheadrightarrow}} x_0$, where

- $\mathtt{comp}_n(p) : T^m\mu^n(x_{n+m}) \xrightarrow[T\mu^n(\lambda_{m+n})]{} x_m$ is defined as an arrow obtained by applying $\mathtt{comp}_n$ to all $p(a) : x_{n+m}(a) \xrightarrow[\lambda_{n+m}(a)]{} x_m(a)$ for each $a \in E_{T^m}[\lambda_{n+m-1}]$.
- $\mu^m(p) : \mu^m(x_{n+m}) \underset{\mu^m(\lambda_{m+n})}{\twoheadrightarrow} \mu^m(x_m)$ is defined as a path obtained by μ^m to each component of p as a sequence.

The naturality of μ^m guarantees that

$\mu^m\left(T^m\mu^n\left(x_{n+m}\right)\right) \xrightarrow[\mu^m\cdot T^m\mu^n(\lambda_{n+m})]{\mu^m(\mathtt{comp}_n(p))} \mu^m(x_m)$ is equivalent to $T\mu^n\left(\mu^m\left(x_{n+m}\right)\right) \xrightarrow[T\mu^n\cdot\mu^m(\lambda_{n+m})]{\mathtt{comp}_n(\mu^m(p))} \mu^m(x_m)$. ■

In [Lei99], it is shown that S-categories are cartesian monadic over $\mathbf{Gph}(S)$ provided that S is what is called a *suitable* monad, see [Lei99] or Appendix D of [Lei04]. Although it is not clear that our T is suitable, the proof of this fact in [Lei04] is still valid for monad S on a presheaf category which preserves coproducts but is not necessarily suitable, hence there is a cartesian monad $\mathtt{fc}[T]$ on $\mathbf{Gph}(T)$ whose algebras are T-categories.

In detail, $\mathtt{fc}[T](X)_0$ is defined as X_0 and $\mathtt{fc}[T](X)_1$ is defined as the coproduct $\coprod_{n\in\mathbb{N}}(T^{n-1}X \circ \cdots \circ X)_1$, hence the discrete fibration $\mathsf{ty}'_{\mathtt{fc}[T](X)} : \int\mathtt{fc}[T](X)_1 \longrightarrow \int S_T$ (see Remark 4.6) places λ_m-paths of length m over $\mu^m(\lambda_m) \in \int S_T$ for each $\lambda_m \in \int S_{T^m}$.

Note that since μ^m is cartesian, a λ_{n+m}-path $p : x_{n+m} \underset{\lambda_{m+n}}{\twoheadrightarrow} x_m$ of length n corresponds to a pair

$$\left(\lambda_{n+m}, \quad \mu^m(p) : \mu^m(x_{n+m}) \underset{\mu^m(\lambda_{m+n})}{\twoheadrightarrow} \mu^m(x_m)\right) \tag{4.13}$$

where $\mu^m(p)$ is $\mu^m(\lambda_{m+n})$-path obtained by applying μ^m to each component of p.

Moreover, a λ_{m+1}-arrow $x_{m+1} \xrightarrow[\lambda_{m+1}]{} x_m$ in $\mathtt{fc}[T](X)$ corresponds to a λ'_{m+n}-path $x_{m+n} \underset{\lambda'_{m+n}}{\twoheadrightarrow} x_m$ such that $\mu^n(\lambda'_{m+n}) = \lambda_{m+1}$ holds, since $E_{T^m}[\lambda_m]$ is connected.

Therefore an element of $\int\mathtt{fc}[T]^n(X)_1$ corresponds to an *n-times nested path* whose target is in X_0:

- a 0-times nested path is an arrow $x_{m+1} \xrightarrow[\lambda_{m+1}]{} x_m$ in X and
- an $n+1$-times nested path is a sequence $x_{k_s} \overset{\zeta_{s-1}}{\underset{\lambda_{k_s}}{\longrightarrow}} x_{k_{s-1}} \ldots x_{k_1} \overset{\zeta_0}{\underset{\lambda_{k_1}}{\longrightarrow}} x_m$, where $k_{i+1} \geq k_i$, $k_0 := m$, and ζ_i is an n-times nested path for each $i \in [s-1]$.

The n-ary multiplication $\mathtt{fc}[T]^n(X) \longrightarrow \mathtt{fc}[T](X)$ is given by the concatenation of n times nested paths in X, in particular, the unit $\mathtt{fc}[T]^0(X) \longrightarrow \mathtt{fc}[T](X)$ sends an arrow to a path of length 1.

It is asserted in [Sha22] that $\mathtt{fc}[T]$ is familial if T is so.

In this paper, we directly give the familial representation of $\mathtt{fc}[T]$.

Proposition 4.12. $\mathtt{fc}[T]$ *is familial.*

Proof. There are two pseudo (hence normal lax) functors $T^{\mathbf{G}_1} : \mathbf{G}_1 \longrightarrow \mathcal{P}rof$ and $\ulcorner\mathbf{1}\urcorner : \mathbf{1} \longrightarrow \mathcal{P}rof$ which corresponds to $\mathbf{G}_1(T) \longrightarrow \mathbf{G}_1$ and $\mathbf{1} \longrightarrow \mathbf{1}$ respectively. We define a proarrow (Definition 4.3) $T^{S_{\mathtt{fc}}} : T^{\mathbf{G}_1} \nrightarrow \ulcorner\mathbf{1}\urcorner$ as follows:

- The domain profunctor $\mathsf{dom}(T^{S_{\mathtt{fc}}})$ is the presheaf part $S_{\mathtt{fc}} : \mathbf{G}_1 \nrightarrow \mathbf{1}$ of the familial representation $\mathtt{fc} : \mathbf{G}_1 \overset{\mathtt{f}}{\nrightarrow} \mathbf{G}_1$ of the ordinary free category monad. $S_{\mathtt{fc}}$ is defined as $S_{\mathtt{fc}}([0]) := \{\underline{0}\}$ and $S_{\mathtt{fc}}([1]) := \mathbb{N}$, where $\{\underline{0}\} \cong [0]$ is the terminal set. Note that for each $n \in \mathbb{N}$, the two morphisms $\underline{0} \rightrightarrows n$ in $\int S_{\mathtt{fc}}$ can be interpreted as monotone functions $[\underline{0}] \overset{\ulcorner 0 \urcorner}{\underset{\ulcorner n \urcorner}{\rightrightarrows}} [n]$ which send $\underline{0}$ to 0 and n respectively.

- For each $[\varepsilon] \in \mathbf{G}_1$, define a function $T^{S_{\mathsf{fc}}}_{[\varepsilon]} : S_{\mathsf{fc}}([\varepsilon]) \longrightarrow \mathcal{P}rof(T^{\mathbf{G}_1}([\varepsilon]), \mathbf{1})$ as follows:
  - $T^{S_{\mathsf{fc}}}_{[0]}(\underline{0}) := (!_{\mathbf{C}})_* : \mathbf{C} \nrightarrow \mathbf{1}$
  - $T^{S_{\mathsf{fc}}}_{[1]}(n) : \int S_T \nrightarrow \mathbf{1}$ is the presheaf corresponding to the discrete fibration $\int(\mu^n)^S : \int S_{T^n} \longrightarrow \int S_T$, where $(\mu^n)^S$ is the presheaf part of n-th composition $\mu^n : T^n \longrightarrow T$.
- The left actions (4.2) are uniquely determined through the universality of the terminal presheaf $T^{S_{\mathsf{fc}}}_{[0]}(\underline{0}) = (!_{\mathbf{C}})_* : \mathbf{C} \nrightarrow \mathbf{1}$.

$S_{\mathsf{fc}[T]} := \fint\!\!\!\fint T^{S_{\mathsf{fc}}} : \mathbf{G}_1(T) \nrightarrow \mathbf{1}$ is the Grothendieck construction on this proarrow, see Definition 4.3. Since the E-part of any morphism of familial representations is cartesian, the canonical forgetful natural transformation (4.1) induces a functor $\int S_{\mathsf{fc}[T]} \longrightarrow \int S_{\mathsf{fc}}$ which corresponds to a normal lax functor $\fint T^{S_{\mathsf{fc}}} : \int S_{\mathsf{fc}} \longrightarrow \mathcal{P}rof$ defined as follows:

$$\begin{tikzcd} {[\underline{0}]} \arrow[r, shift left, "\ulcorner 0 \urcorner"] \arrow[r, shift right, "\ulcorner n \urcorner"'] & {[n]} & \mapsto & \mathbf{C} \arrow[r, shift left, "\mathsf{ty}^*_{T^n}", "\shortmid"{marking}] \arrow[r, shift right, "E_{T^n}"', "\shortmid"{marking}] & \int S_{T^n} \end{tikzcd} \tag{4.14}$$

For $E_{\mathsf{fc}[T]} : \mathbf{G}_1(T) \nrightarrow \int S_{\mathsf{fc}[T]}$, we define a proarrow $T^{E_{\mathsf{fc}}} : T^{\mathbf{G}_1} \nrightarrow \fint T^{S_{\mathsf{fc}}}$ as follows

- The domain profunctor $\mathsf{dom}(T^{E_{\mathsf{fc}}})$ is the E-part $E_{\mathsf{fc}} : \mathbf{G}_1 \nrightarrow \int S_{\mathsf{fc}}$ of the familial representation $\mathtt{fc}$. E_{fc} is defined as $E_{\mathsf{fc}}([\varepsilon], a) := \{i \,|\, i + \varepsilon \in [a]\}$ where $[\underline{0}] := [0]$. $i \in E_{\mathsf{fc}}([\varepsilon], a)$ is interpreted as a monotone function $i + - : [\varepsilon] \longrightarrow [a]$ and the left and right actions for E_{fc} are given by pre- and post- compositions.
- For each $[\varepsilon] \in \mathbf{G}_1$ and $a \in \int S_{\mathsf{fc}}$, define a function $T^{E_{\mathsf{fc}}}_{[\varepsilon],a} : E_{\mathsf{fc}}([\varepsilon], a) \longrightarrow \mathcal{P}rof(T^{\mathbf{G}_1}([\varepsilon]), \fint T^{S_{\mathsf{fc}}}(a))$ as $T^{E_{\mathsf{fc}}}_{[\varepsilon],a}(i) := T^{\Delta}(i + -)$, hence in detail,
  - $T^{E_{\mathsf{fc}}}_{[0],0}(0) = T^{E_{\mathsf{fc}}}_{[0],\underline{0}}(0) := \mathtt{Id}_{\mathbf{C}} : \mathbf{C} \nrightarrow \mathbf{C}$ is the identity profunctor on $\mathbf{C}$.
  - $T^{E_{\mathsf{fc}}}_{[1],1}(0) := \mathtt{Id}_{\int S_T} : \int S_T \nrightarrow \int S_T$ is the identity profunctor on $\int S_T$.
  - $T^{E_{\mathsf{fc}}}_{[0],n}(i) : \int S_{T^0} = \mathbf{C} \nrightarrow \int S_{T^n}$ is the composite of profunctors

$$\begin{tikzcd} \mathbf{C} \arrow[r, "E_{T^{n-i}}", "\shortmid"{marking}] & \int S_{T^{n-i}} \arrow[r, "(T^{n-i}\mathsf{ty}_{T^i})^*", "\shortmid"{marking}] & \int S_{T^n} \end{tikzcd} \tag{4.15}$$

  - $T^{E_{\mathsf{fc}}}_{[1],n}(i) : \int S_T \nrightarrow \int S_{T^n}$ is the composite of profunctors

$$\begin{tikzcd} \int S_T \arrow[r, "E_{T^{n-i-1}\overline{S_T}}", "\shortmid"{marking}] & \int S_{T^{n-i}} \arrow[r, "(T^{n-i}\mathsf{ty}_{T^i})^*", "\shortmid"{marking}] & \int S_{T^n} \end{tikzcd} \tag{4.16}$$

  Moreover, the pseudo-functoriality of T^{Δ} defines the (isomorphic) actions since $T^{\mathbf{G}_1}(\partial^1_{1-i}) = T^{\Delta}(i + -) : \mathbf{C} \nrightarrow \int S_T$ and $\fint T^{S_{\mathsf{fc}}}(\ulcorner j \urcorner) = T^{\Delta}(j + -) : \mathbf{C} \nrightarrow \int S_{T^n}$ for each $i \in \{0, 1\}$ and $j \in \{0, n\}$.

$E_{\mathsf{fc}[T]} := \fint\!\!\!\fint T^{E_{\mathsf{fc}}} : \mathbf{G}_1(T) \nrightarrow \int S_{\mathsf{fc}[T]}$ is its Grothendieck construction. In detail, $E_{\mathsf{fc}[T]}$ is generated by the following components:

i) $\ulcorner i \urcorner_{\lambda_{n+1},a} \in E_{\mathsf{fc}[T]}(T\mathsf{ty}_{T^i} \circ \lambda_{n+1}(a), \lambda_{n+1})$ for each $n \geq 1$, $i \in [n-1]$, $\lambda_{n-i} \in \int S_{T^{n-i}}$, $a \in \int E_{T^{n-i}}[\lambda_{n-i}]$, and $\lambda_{n+1} \in \int S_{T^{n+1}} \cong \int S_{T^{n-i}T^{i+1}}$ which can be interpreted as $\lambda_{n+1} : E_{T^{n-i}}[\lambda_{n-i}] \longrightarrow S_{T^{i+1}}$
ii) $a \in E_{\mathsf{fc}[T]}(c, \lambda)$ for each $c \in \mathbf{C}$, $\lambda \in \int S_T$, and $a \in E_T(c, \lambda)$
iii) $\tau_\lambda \in E_{\mathsf{fc}[T]}(\mathsf{ty}_T(\lambda), \lambda)$ for each $\lambda \in \int S_T$
iv) $\ulcorner n-1 \urcorner_{\lambda_n} \in E_{\mathsf{fc}[T]}(T\mathsf{ty}_{T^{n-1}}(\lambda_n), \lambda_n)$ for each $n > 0$ and $\lambda_n \in \int S_{T^n}$
v) $j \in E_{\mathsf{fc}[T]}(c', c)$ for each $c' \xrightarrow{j} c$ in $\mathbf{C} = T^{\mathbf{G}_1}([0]) = \fint T^{S_{\mathsf{fc}}}(\underline{0})$, where c is in $\fint T^{S_{\mathsf{fc}}}(\underline{0})$.
vi) $j \in E_{\mathsf{fc}[T]}(c', \lambda_0)$ for each $c' \xrightarrow{j} \lambda_0$ in $\int S_{T^0} = T^{\mathbf{G}_1}([0]) = \fint T^{S_{\mathsf{fc}}}(0)$ where λ_0 is in $\fint T^{S_{\mathsf{fc}}}(0)$.

which are subject to the following conditions:

- the restriction of $E_{\mathsf{fc}[T]}$ to $\mathbf{G}_1(T) \nrightarrow \mathbf{G}_1(T)$ is identity profunctor, hence morphisms in ii), iii), v), and $\ulcorner 0 \urcorner_{\lambda_1}$ in iv) satisfy obvious commutativity.
- morphisms in vi) defines identity profunctor $T^{\mathbf{G}_1} = \mathbf{C} \nrightarrow \mathbf{C} = \fint T^{S_{\mathsf{fc}}}(0)$
- $E_{\mathsf{fc}[T]}\left(T\mathsf{ty}_{T^i} \circ \lambda_{n+1}(\overline{j}_a), \lambda_{n+1}\right)(\ulcorner i \urcorner_{\lambda_{n+1},a}) = \ulcorner i \urcorner_{\lambda_{n+1},a'}$

  for each $a' \xrightarrow{\overline{j}_a} a$ in $\int E_{T^{n-i}}[\lambda_{n-i}]$ and $\lambda_{n+1} : E_{T^{n-i}}[\lambda_{n-i}] \longrightarrow S_{T^{i+1}}$
- $E_{\mathsf{fc}[T]}\left(T\mathsf{ty}_{T^i} \circ \lambda_{n+1}(a), \overline{j}_{\lambda_{n+1}}\right)(\ulcorner i \urcorner_{\lambda'_{n+1},a'}) = \ulcorner i \urcorner_{\lambda_{n+1},a}$

  for each $\lambda'_{n+1} \xrightarrow{\overline{j}_{\lambda_{n+1}}} \lambda_{n+1}$ in $\int S_{T^{n+1}}$, $a' \in \int E_{T^{n-i}}\left[\lambda'_{n-i}\right]$, $a := E_{T^{n-1}}[\overline{j}_{\lambda_{n+1}}](a') \in \int E_{T^{n-i}}[\lambda_{n-i}]$, $\lambda_{n+1} : E_{T^{n-i}}[\lambda_{n-i}] \longrightarrow S_{T^{i+1}}$, and $\lambda'_{n+1} : E_{T^{n-i}}\left[\lambda'_{n-i}\right] \longrightarrow S_{T^{i+1}}$. Note that $\lambda_{n+1}(a) = \lambda'_{n+1}(a')$ holds by definition of $S_{T^{n-i}T^{i+1}}$.

Now it is straightforward to check that, for any T-graph X and $\lambda_n \in \int S_{T^n}$, a morphism $\alpha : E_{\mathsf{fc}[T]}[\lambda_n] \longrightarrow X$ corresponds to a λ_n-path $x_n \underset{\lambda_n}{\xrightarrow{\xi_{n-1}}} x_{n-1} \longrightarrow \ldots \underset{\lambda_2}{\longrightarrow} x_1 \underset{\lambda_1}{\xrightarrow{\xi_0}} x_0$ in X as follows:

- If $n \geq 1$,
  - $x_0 := \alpha(\ulcorner n-1 \urcorner_{\lambda_n} \cdot \tau_{\lambda_1})$, where τ_{λ_1} is seen as a morphism in $\mathbf{G}_1(T)$.
  - $x_1 : E_{T^1}[\lambda_1] \longrightarrow X_0$ sends $u \in \int E_{T^1}[\lambda_1]$ to $\alpha(\ulcorner n-1 \urcorner_{\lambda_n} \cdot u)$ for $i < n-1$.
  - $\xi_0 := \alpha(\ulcorner n-1 \urcorner_{\lambda_n})$.
  - $x_{i+1} : E_{T^{i+1}}[\lambda_{i+1}] \longrightarrow X$ sends $\kappa_{\bar{a}}(u) \in E_{T^{i+1}}(c, \lambda_{i+1})$ to $\alpha(\ulcorner n-i-1 \urcorner_{\lambda_n, \bar{a}} \cdot u)$ for $0 < i \leq n-1$, where $\bar{a} \in \int E_{T^i}[\lambda_i]$, $u \in E_T(c, \lambda_{i+1}(\bar{a}))$, and $\kappa_{\bar{a}} : E_T(c, \lambda_{i+1}(\bar{a})) \longrightarrow E_{T^{i+1}}(c, \lambda_{i+1})$ is the coprojection of the colimit defining $E_{TT^i}(c, \lambda_{i+1})$.
  - $\xi_{i+1} : E_{T^{i+1}}[\lambda_{i+1}] \longrightarrow X$ sends $a \in \int E_{T^{i+1}}[\lambda_{i+1}]$ to $\alpha(\ulcorner n-i-2 \urcorner_{\lambda_n, a})$ for $i < n-1$.
- If $n = 0$, $x_0 := \alpha(\mathtt{id}_{\lambda_0})$, where $\mathtt{id}_{\lambda_0}$ is what is introduced in vi above.

Therefore, $(S_{\mathtt{fc}[T]}, E_{\mathtt{fc}[T]})$ actually gives a familial representation of the functor $\mathtt{fc}[T]$. Since $\mathtt{fc}[T]$ is a cartesian monad and cartesian natural transformations between familial functors coincide with morphisms of familial representations (Remark 3.8), this finishes the proof. □

Note that a path in a coproduct $\coprod_{i \in I} X^i$ of T-graphs is contained in X^i for some $i \in I$, since each component x_n or $\xi_n : E_{T^n}[\lambda_n] \longrightarrow \coprod_{i \in I} X^i_j$ $(j = 0, 1)$ is contained in X_i for some $i \in I$ for $E_{T^n}[\lambda_n]$ is connected.

Therefore, since $E_{\mathtt{fc}[T]}[\lambda_m]$ represents λ_m-paths, it is connected for each $\lambda_m \in \int S_{T^m} \subset \int S_{\mathtt{fc}[T]}$, hence $\mathtt{fc}[T]$ satisfies conditions we imposed on T. $\mathtt{fc}^{\mathtt{n}}[T]$ is defined as n-times iteration of $T \mapsto \mathtt{fc}[T]$.

Definition 4.13. The category of T-categories, $\mathbf{Cat}(T)$, is the Eilenberg-Moore category of $\mathtt{fc}[T]$, and morphisms in this category are called ***T-functors***. A ***T-virtual n-tuple category*** is defined as an algebra of $\mathtt{fc}^{\mathtt{n}}[T]$, and we write $\mathbf{V}\text{-}n\text{-}\mathbf{tplCat}(T)$ for the Eilenberg-Moore category of $\mathtt{fc}^{\mathtt{n}}[T]$. In particular, we write $\mathbf{VDblCat}(T)$ if $n = 2$, which is the category of ***T-virtual double categories***. ■

Remark 4.14. When $T = \mathtt{id}_{\mathbf{1}}$, then $\mathbf{Cat}(T) = \mathbf{Cat}$ and T-virtual double categories are virtual double categories defined in [CS10], for example.

On the other hand, for $n > 2$, our notion of virtual n-tuple category (:=$\mathtt{id}_{\mathbf{1}}$-virtual n-tuple category) is *not* consistent with the notion of "virtual widget" proposed in Section 8 of [CS10]. ■

Let $F_{\mathtt{fc}[T]} : \mathbf{Gph}(T) \longrightarrow \mathbf{Cat}(T)$ be the free functor. Recall that for ordinary category, the ordinal $[n]$ as a category is the free category of $E_{\mathtt{fc}}[n] := (0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n) \in \mathbf{Gph}$, and Δ is a full subcategory of $\mathbf{Cat}$. An analogy of this fact for T-category holds:

Theorem 4.15. *$\lambda_n \mapsto F_{\mathtt{fc}[T]}(E_{\mathtt{fc}[T]}[\lambda_n])$ induces a fully faithful, dense functor $\Delta(T) \hookrightarrow \mathbf{Cat}(T)$.*

To show this theorem, we firstly show the following lemma:

Lemma 4.16. *Let $n \leq m$, $\lambda'_m \in S_{T^m}$, and $\lambda_n \in S_{T^n}$. A λ_n-path of length n in the T-graph $E_{\mathtt{fc}[T]}[\lambda'_m]$ precisely corresponds to a pair (l, ζ) of $l \in [m-n]$ and $\zeta \in \mathrm{colim}_{a \in \int E_{T^l}[\lambda'_l]} \int S_{T^n}(\lambda_n, T^{n+l}\mathsf{ty}_{T^{m-n-l}}(\lambda'_m)(a))$, where $\lambda'_l := T^l \mathsf{ty}_{T^{m-l}}(\lambda'_m)$.*

Proof. We write $\lambda_i := T^i \mathsf{ty}_{T^{n-i}}(\lambda_n)$ and $\lambda'_j := T^j \mathsf{ty}_{T^{m-j}}(\lambda'_m)$, so that λ_n and λ'_m are interpreted as maps $\lambda_n : E_{T^i}[\lambda_i] \longrightarrow S_{T^{n-i}}$ and $\lambda'_m : E_{T^j}[\lambda'_j] \longrightarrow S_{T^{m-j}}$ for each $i \in [n]$ and $j \in [m]$. For each $l \in [m-n]$, we show the correspondence between the following data:

i) λ_n-path $x_n \xrightarrow[\lambda_n]{\xi_{n-1}} x_{n-1} \dots x_1 \xrightarrow[\lambda_1]{\xi_0} x_0$ such that x_0 is in $T^{E_{\mathtt{fc}}}_{[0],m}(m-l)(\lambda_0, \lambda'_m)$
ii) $\zeta \in \mathrm{colim}_{a \in \int E_{T^l}[\lambda'_l]} \int S_{T^n}(\lambda_n, \lambda'_{n+l}(a))$

Suppose $\zeta \in \mathrm{colim}_{a \in \int E_{T^l}[\lambda'_l]} \int S_{T^n}(\lambda_n, \lambda'_{n+l}(a))$. Note that $\mathrm{colim}_{a \in \int E_{T^l}[\lambda'_l]} \int S_{T^n}(\lambda_n, \lambda'_{n+l}(a))$ is the same as $T^{\Delta}((m-l-n)+-)(\lambda_n, \lambda'_m)$, where $(m-l-n)+- : [n] \longrightarrow [m]$ is a map in Δ. Now we obtain a morphism of T-graphs $E_{\mathtt{fc}[T]}[\lambda_n] \longrightarrow E_{\mathtt{fc}[T]}[\lambda'_m]$ by the post-composition of $\zeta : \lambda_n \longrightarrow \lambda'_m$ in $\Delta(T)$, but we have already checked that such a morphism corresponds to a λ_n-path in $E_{\mathtt{fc}[T]}[\lambda'_m]$ (see the proof of Proposition 4.12). Moreover, if $n > 0$, since $x_0 := \zeta \cdot \ulcorner n-1 \urcorner_{\lambda_n} \cdot \tau_{\lambda_1}$ and the maps $\lambda_0 \xrightarrow{\tau_{\lambda_1}} \lambda_1 \xrightarrow{\ulcorner n-1 \urcorner_{\lambda_n}} \lambda_n \xrightarrow{\zeta} \lambda'_m$ in $\Delta(T)$ are over $[0] \xrightarrow{1+-} [1] \xrightarrow{(n-1)+-} [n] \xrightarrow{(m-n-l)+-} [m]$, x_0 is in $T^{E_{\mathtt{fc}}}_{[0],m}(m-l)(\lambda_0, \lambda'_m)$. $n = 0$ case is trivial.

It suffices to prove this assignment $\zeta \mapsto (x_n \xrightarrow[\lambda_n]{\xi_{n-1}} x_{n-1} \dots x_1 \xrightarrow[\lambda_1]{\xi_0} x_0)$ defines a bijection between i) and ii), by induction on n. It is trivial ζ itself gives x_0, hence the assignment is a bijection when $n = 0$. In the same way, when $n = 1$, $\zeta = \xi_0$ gives rise to a bijection. Suppose the assignment is a bijection for $n > 0$ and let $x_{n+1} \xrightarrow[\lambda_{n+1}]{\xi_n} x_n \xrightarrow[\lambda_n]{\xi_{n-1}} x_{n-1} \dots x_1 \xrightarrow[\lambda_1]{\xi_0} x_0$ be a λ_{n+1}-path. We suppose there exists $\zeta' \in \mathrm{colim}_{a \in \int E_{T^l}[\lambda'_l]} \int S_{T^n}(\lambda_n, \lambda'_{n+l}(a))$ which corresponds to $x_n \xrightarrow[\lambda_n]{\xi_{n-1}} x_{n-1} \dots x_1 \xrightarrow[\lambda_1]{\xi_0} x_0$. We see ζ' as a morphism $\zeta' : \lambda_n \xrightarrow[0+-]{} \lambda'_{n+l}$ in $\Delta(T)$, which factors as $\lambda_n \xrightarrow[{[n]}]{\bar{\zeta}'} \lambda'_{n+l}(x) \xrightarrow[0+-]{} \lambda'_{n+l}$ for some $x \in \int E_{T^l}[\lambda_l]$. ζ' is also considered the composite $\lambda_n \xrightarrow[0+-]{\zeta'} \lambda'_{n+l} \xrightarrow[(m-n-l)+-]{\tau} \lambda'_m$, where $\lambda'_{n+l} \xrightarrow[(m-n-l)+-]{\tau} \lambda'_m$ is the identity morphism on λ'_{n+l} in $\int S_{T^{n+l}}$, hence is one of the generating elements of

$$T^{\Delta}((m-n-l)+- : [n+l] \longrightarrow [m]) = \textstyle\int S_{T^{n+l}}(\mathtt{id}, T^{n+l}\mathsf{ty}_{T^{m-n-l}})$$

which is sometimes omitted.

Let $b : c_b \xrightarrow[0+-]{} \lambda_n$ in $\Delta(T)$, or equivalently, $b \in E_{T^n}(c_b, \lambda_n)$ for some $c_b \in \mathbf{C}$. There exists $\bar{x}_n(b) : c_b \xrightarrow[0+-]{} \lambda'_{n+l}(x)$ which factors $x_n(b)$ as $c_b \xrightarrow[0+-]{\bar{x}_n(b)} \lambda_{n+l}(x) \xrightarrow[0+-]{} \lambda_{n+l}$, which is defined as follows:

Since we have assumed that x_n is induced from post-composition of ζ', if $n = 1$, $x_n(b) = x_1(b) = \zeta' \cdot b$ holds. Even if $n > 1$, b factors as $c_b \xrightarrow[0+-]{e'} \lambda_n(e) \xrightarrow[0+-]{\ulcorner 0 \urcorner_{\lambda_n, e}} \lambda_n$ for some $e \in \int E_{T^{n-1}}[\lambda_{n-1}]$, hence $x_n(b) = \zeta' \cdot (\ulcorner 0 \urcorner_{\lambda_n, e} \cdot e')$ holds, but moreover $x_n(b) = \zeta' \cdot b$ since the actions for $E_{\mathtt{fc}[T]}$ is defined by restricting composites in $\Delta(T)$. Let $\bar{x}_n(b)$ be the composite $\bar{\zeta}' \cdot b$.

On the other hand, for each $b : c_b \xrightarrow[0+-]{} \lambda_n$, there exist $a_b \in \int E_{T^{n+l}}\left[\lambda'_{n+l}\right]$ and $\tilde{\xi}_n(b) : \lambda_{n+1}(b) \xrightarrow[{[1]}]{} \lambda'_{n+l+1}(a_b)$ such that $\xi_n(b)$ is the composite

$$\lambda_{n+1}(b) \xrightarrow[{[1]}]{\tilde{\xi}_n(b)} \lambda'_{n+l+1}(x) \xrightarrow[0+-]{\ulcorner 0 \urcorner_{\lambda'_{n+l+1}, a_b}} \lambda'_{n+l+1}$$

T^{Δ} sends the commutative square

(4.17)

$$\begin{tikzcd}
{[0]} \arrow[r, "0+-"] \arrow[d, "k+-"'] & {[m]} \arrow[d, "k+-"] \\
{[k]} \arrow[r, "0+-"'] & {[m+k]}
\end{tikzcd}$$

to an isomorphism which makes the following diagram commutes in $\Delta(T)$ (see the proof of Proposition 4.8):

(4.18)

$$\begin{tikzcd}
c \arrow[r, "a", "0+-"'] \arrow[d, "\tau", "k+-"'] & \tilde{\lambda}_m \arrow[d, "\tau", "k+-"'] \\
\tilde{\lambda}_{m+k}(a) \arrow[r, "\ulcorner 0 \urcorner_{\tilde{\lambda}_{k+m}, a}", "0+-"'] & \tilde{\lambda}_{k+m}
\end{tikzcd}$$

for arbitrary $c \in \mathbf{C}$, $\tilde{\lambda}_{m+k} \in \int S_{T^{m+k}}$, and $\tilde{\lambda}_m := T^m \mathsf{ty}_{T^k}(\tilde{\lambda}_{m+k})$.

In summary, we have the following commutative diagrams

(4.19)

$$\begin{tikzcd}
c_b \arrow[r, "\mathsf{ty}_T(\tilde{\xi}_n(b))", "{[0]}"'] \arrow[d, "\tau"'] & c_{a_b} \arrow[r, "a_b", "0+-"'] \arrow[d, "\tau" description, "1+-"'] & \lambda'_{n+l} \arrow[d, "\tau"] \\
\lambda_{n+1}(b) \arrow[r, "\tilde{\xi}_n(b)", "{[1]}"'] \arrow[rr, bend right, "\xi_n(b)"'] & \lambda_{n+l+1}(a) \arrow[r, "\ulcorner 0 \urcorner", "0+-"'] & \lambda'_{n+l+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \lambda'_{n+l}(x) \arrow[dr] & \\
c_b \arrow[ur, "\bar{x}_n(b)"] \arrow[rr, "x_n(b)"] \arrow[d, "\tau"'] & & \lambda'_{n+l} \arrow[d, "\tau"] \\
\lambda_{n+1}(b) \arrow[rr, "\xi_n(b)"] & & \lambda'_{n+l+1}
\end{tikzcd}$$

where the square on the right hand side commutes since x_n is the codomain of the λ_n-arrow ξ_n. Since any map over $1+- : [0] \longrightarrow [n+l+1]$ uniquely factors through $\tau : \lambda'_{n+l} \longrightarrow \lambda'_{n+l+1}$, the following square commutes:

(4.20)

$$\begin{tikzcd}
c_b \arrow[r, "\bar{x}_n(b)"] \arrow[d, "\mathsf{ty}_T(\tilde{\xi}_n(b))"'] & \lambda'_{n+l}(x) \arrow[d] \\
c_{a_b} \arrow[r, "a_b"] & \lambda'_{n+l}
\end{tikzcd}$$

Therefore, $\lambda'_{n+l+1}(x)(\bar{\zeta}' \cdot b) = \lambda'_{n+l+1}(x)(\bar{x}_n(b)) = \lambda'_{n+l+1}(a_b \cdot \mathsf{ty}_T(\tilde{\xi}_n(b))) = S_{T^{n+l}}(\mathsf{ty}_T(\tilde{\xi}_n(b)))(\lambda'_{n+l+1}(a_b)) = \lambda_{n+1}(b)$, where λ_{n+l+1} is interpreted as a morphism $E_{T^l}[\lambda'_l] \longrightarrow S_{T^{n+1}}$ in the first and second terms and as a morphism $E_{T^{n+l}}[\lambda'_{n+l}] \longrightarrow S_T$ in the third term. This means the commutativity of the following:

(4.21)

$$\begin{tikzcd}
E_{T^n}\left[\lambda_n\right] \arrow[r, "\lambda_{n+1}"] \arrow[d, "{E_{T^n}[\bar{\zeta}']}"'] & S_T \\
E_{T^n}\left[\lambda'_{n+1}(x)\right] \arrow[ur, "\lambda'_{n+l+1}(x)"'] &
\end{tikzcd}$$

i.e. $\bar{\zeta}'$ extends to a morphism $\bar{\zeta} : \lambda_{n+1} \longrightarrow \lambda'_{n+l+1}(x)$ in $\int S_{T^{n+1}}$.

Let $\zeta : \lambda_{n+1} \xrightarrow[0+-]{} \lambda'_{n+l+1} \in \operatorname{colim}_{x \in \int E_{T^l}\,[\lambda'_l]} \int S_{T^{n+1}}(\lambda_{n+1}, \lambda'_{n+l}(x))$ be what is represented by $\bar{\zeta}$. It is straightforward to check this precisely induces $\xi_n(b)$ through the post-composition $\zeta \cdot b$. $\square$

proof of Theorem 4.15. For $\bar\lambda_n \in \int S_{T^n}$ and $\lambda'_m \in \int S_{T^m}$, it suffices to show the correspondence between the following data:

i) $\bar\lambda_n$-path $z_n \overset{\zeta_{n-1}}{\underset{\bar\lambda_n}{\longrightarrow}} z_{n-1} \dots z_1 \overset{\zeta_0}{\underset{\bar\lambda_1}{\longrightarrow}} z_0$ in $\mathtt{fc}[T](E_{T^n}[\lambda'_m])$

ii) $p : \bar\lambda_n \longrightarrow \lambda'_m$ in $\Delta(T)$

The discussion preceding Proposition 4.12 shows that a $\bar\lambda_n$-path $z_n \overset{\zeta_{n-1}}{\underset{\bar\lambda_n}{\longrightarrow}} z_{n-1} \dots z_1 \overset{\zeta_0}{\underset{\bar\lambda_1}{\longrightarrow}} z_0$ in $\mathtt{fc}[T]E_{\mathtt{fc}[T]}[\lambda'_m]$ corresponds to a 2-times nested path $x_{k_n} \overset{\zeta_{n-1}}{\underset{\lambda_{k_n}}{\longrightarrow}} x_{k_{n-1}} \dots x_{k_1} \overset{\zeta_0}{\underset{\lambda_{k_1}}{\longrightarrow}} x_0$ in $E_{\mathtt{fc}[T]}[\lambda'_m]$ such that $\mu^{k_n - n+1}(\lambda_{k_n}) = \lambda_n$. We write $\bar\zeta$ for the concatenation $x_{k_n} \xrightarrow[\lambda_{k_n}]{} x_0$, which can be seen as a morphism $\lambda_{k_n} \xrightarrow[0+-]{} \lambda'_{k_n+l} \xrightarrow[f_0+-]{\tau} \lambda'_m$ for some l.

For each sequence $k_n \geq k_{n-1} \geq \cdots k_1 \geq k_0 = 0$, define a monotone function $u_k : [n] \longrightarrow [k_n]$ by $i \mapsto k_n - k_{n-i}$, of which the image by T^Δ, $T^\Delta(u_k) : \int S_{T^n} \nrightarrow \int S_{T^{k_n}}$, is isomorphic to $(\mu^{r_n} \cdot \mu^{r_{n-1}} \cdot \dots \cdot \mu^{r_1})^* : \int S_{T^n} \nrightarrow \int S_{T^{r_n} T^{r_{n-1}} \dots T^{r_1}}$, where $r_{n-i} := k_{i+1} - k_i$ for each $i < n$. Therefore for arbitrary $\lambda_{k_n} \in \int S_{T^{k_n}}$ and $\bar\lambda_s \in \int S_{T^s}$, a map $\bar\lambda_s \xrightarrow[u_k]{} \lambda_{k_n}$ over u_k uniquely factors through a map $\mu^{r_n, \dots, r_1}(\lambda_{k_n}) \xrightarrow[u_k]{\nu_k} \lambda_{k_n}$.

Let $p : \lambda_n \xrightarrow[f]{} \lambda'_m$ be a morphism in $\Delta(T)$ over $f : [n] \longrightarrow [m]$, which sends $i \in [n]$ to $f_i \in [m]$. Let l be $m - f_n$ and k_i be $f_n - f_{n-i}$ for each $i \in [n]$.

We define a 2-times nested path $x_{k_n} \overset{\zeta_{n-1}}{\underset{\lambda_{k_n}}{\longrightarrow}} x_{k_{n-1}} \dots x_{k_1} \overset{\zeta_0}{\underset{\lambda_{k_1}}{\longrightarrow}} x_0$ satisfying $p = \bar\zeta \cdot \nu_k$.

Since $l = m - f_n$, p factors through $\lambda'_{k_n+l} \xrightarrow[f_n+-]{\tau} \lambda'_m$. Moreover, since there is a commutative diagram

$$\begin{tikzcd}
{[k_n]} \arrow[r, "u_k"] \arrow[d, "0+-"'] & {[n]} \arrow[d, "0+-"'] \\
{[k_n + l]} \arrow[r, "\mathtt{id}_l \oplus u_k"] & {[n + l]}
\end{tikzcd} \tag{4.22}$$

p uniquely factors through $T^l \mu^{r_n \dots r_1}(\lambda'_{k_n+l}) \xrightarrow[u_k]{} \lambda'_{k_n+l}$. Therefore, there exists a $x \in \int E_{T^l}[\lambda'_l]$ such that p factors through $\mu^{r_1 \dots r_n} \cdot \lambda'_{k_n+l}(x) \xrightarrow[0+-]{} \lambda'_{k_n+l}$, and since $\mu^{r_n \dots r_1} : \int S_{T^{k_n}} \longrightarrow \int S_{T^n}$ is a discrete fibration, there exists a λ_{k_n} such that $\mu^{r_n \dots r_1}(\lambda_{k_n}) = \bar\lambda_n$ and p factors through $\bar\lambda_n \overset{\nu_k}{\underset{u_k}{\longrightarrow}} \lambda_{k_n}$. This λ_{k_n} does not depends on the choice of x since we have assumed that $\mathbf{C}$ has no non-trivial endo-morphisms and hence so is $\int S_{T^n}$ and all possible candidates of λ_{k_n} is connected through maps over a fixed element $\bar\lambda_n$ with respect to $\mu^{r_n \dots r_1}$, which is a discrete fibration.

Now we have a map $\bar\zeta : \lambda_{k_n} \xrightarrow[0+-]{} \lambda'_{k_n+l}$, which yields a path $x_{k_n} \underset{x}{\twoheadrightarrow} 0$, which is uniquely decomposed to a 2-times nested path since $\mu^\bullet$ are cartesian through the sequence $k_n \geq \cdots k_0 = 0$.

Thus we obtain a fully faithful functor $[-]_T : \Delta(T) \hookrightarrow \mathbf{Cat}(T)$. The induced nerve functor $\mathbf{Cat}(T)([-]_T, ?) : \mathbf{Cat}(T) \longrightarrow \mathbf{SSet}(T)$ is faithful, since maps of T-categories are completely determined by assignments on paths. It is also full since reindexing by inner faces and degeneracies in $\Delta(T)$ means compositions of paths, and a T-graph morphism commutes with those if and only if it is a T-functor. ☐

On the other hand, since

- any familial monad is accessible as an endo-functor on $\widehat{\mathbf{C}}$ (Remark 3.9),
- $\mathtt{fc}[T]$ is familial by Proposition 4.12,
- and the Eilenberg-Moore category for an accessible monad on any locally presentable category is locally presentable (Theorem 5.5.9. of [Bor94]),

$\mathbf{Cat}(T)$ is locally presentable, hence cocomplete as a category. Therefore, $\mathbf{Cat}(T)$ can be seen as a reflective full subcategory of $\mathbf{SSet}(T)$.

For each category X, $(\mathsf{ty}_{T^\vartriangle}(X))_n$ is $\coprod_{a \in X_n} S_{T^n}$, whose elements are pairs of paths in X and $\lambda_n \in \int S_{T^n}$. On the other hand, a λ_{n+1}-arrow $x_{n+1} \xrightarrow[\lambda_{n+1}]{\xi_n} x_n$ in the underlying T-graph of $\mathsf{ty}_{T^\vartriangle}(X)$ is a pair $(|\xi_n|, \lambda_{n+1})$, where $|\xi_n| \in X_1$ is an arrow in X such that for each $a \in \int E_{T^n}[\lambda_n]$ and $a' \in \int E_{T^{n+1}}[\lambda_{n+1}]$, $x_n(a) = \mathtt{src}(|\xi_n|)$ and $x_{n+1}(a') = \mathtt{tgt}(|\xi_n|)$, since $E_{T^n}[\lambda_n]$ is connected. Therefore a path of length n whose target is in X_0 in this T-graph precisely corresponds to an element of $(\mathsf{ty}_{T^\vartriangle}(X))_n$. This shows that $\mathsf{ty}_{T^\vartriangle}(X)$ is a T-category, hence we have a functor $\nabla_T(X) : \mathbf{Cat} \longrightarrow \mathbf{Cat}(T)$.

4.2. **Structures of T-categories.** In the framework of [CS10], for a cartesian monad $\mathbf{S}$ on $\mathbf{E}$, an $\mathbf{S}$-category is an example of what they call a $\mathbb{S}\mathrm{pan}(\mathbf{S})$-monoid, where $\mathbb{S}\mathrm{pan}(\mathbf{S})$ is the outcome of extending $\mathbf{S}$ to a monad on the double category of spans in $\mathbf{E}$, $\mathbb{S}\mathrm{pan}(\mathbf{E})$. In general, for any virtual double category $\mathbb{X}$ and monad S on $\mathbb{X}$, they define another virtual double category of *S-monoids*, $\mathbb{KM}\mathrm{od}(\mathbb{X}, S)$.

Definition 4.17. We write $\mathbb{P}\mathrm{rof}(T)$ for $\mathbb{K}\mathbb{M}\mathrm{od}(\mathbb{S}\mathrm{pan}(\widehat{\mathbf{C}}), \mathbb{S}\mathrm{pan}(T))$. A ***T-profunctor*** is a horizontal cell in $\mathbb{P}\mathrm{rof}(T)$.

$\mathbb{P}\mathrm{rof}(T)$ has units (Proposition 5.5 of [CS10]), hence one obtains its vertical 2-category (Proposition 6.1 of [CS10]), $\mathcal{C}at(T)$, of $\mathbb{P}\mathrm{rof}(T)$. We write $\mathcal{VD}bl\mathcal{C}at(T)$ for $\mathcal{C}at(\mathtt{fc}[T])$ and $\mathcal{V}$-n-$tpl\mathcal{C}at(T)$ for $\mathcal{C}at(\mathtt{fc^n}[T])$. In detail, $\mathbb{P}\mathrm{rof}(T)$ consists of the following data

- objects are T-categories, and vertical arrows are morphisms in $\mathbf{Cat}(T)$, *T-functors.*
- for each T-categories X and Y, a horizontal arrow $p : X \nrightarrow Y$, a T-profunctor, consists of
  - a span $TX_0 \longleftarrow |p| \longrightarrow Y_0$, i.e., a horizontal arrow $TX_0 \overset{p}{\nrightarrow} Y_0$ in $\mathbb{S}\mathrm{pan}(\widehat{\mathbf{C}})$, and
  - left and right actions: cells in the double category $\mathbb{S}\mathrm{pan}(\widehat{\mathbf{C}})$

$$\begin{tikzcd}
T^2X_0 \arrow[r, "TX", "\shortmid"{marking}] \arrow[d, "\mu_{X_0}"] \arrow[drr, phantom, "\lambda^p"] & TX_0 \arrow[r, "p", "\shortmid"{marking}] & Y_0 \arrow[d, equal] & T^2X_0 \arrow[r, "Tp", "\shortmid"{marking}] \arrow[d, "\mu_{X_0}"] \arrow[drr, phantom, "\rho^p"] & TY_0 \arrow[r, "Y", "\shortmid"{marking}] & Y_0 \arrow[d, equal] \\
TX_0 \arrow[rr, "p"', "\shortmid"{marking}] & & Y_0 & TX_0 \arrow[rr, "p"', "\shortmid"{marking}] & & Y_0
\end{tikzcd} \tag{4.23}$$

where $X : TX_0 \nrightarrow X_0$ and $Y : TY_0 \nrightarrow Y_0$ are spans defining underlying T-graphs of X and Y. The compositions of X and Y can be seen as cells in $\mathbb{S}\mathrm{pan}(\widehat{\mathbf{C}})$: e.g.

$$\begin{tikzcd}
T^2X_0 \arrow[r, "TX", "\shortmid"{marking}] \arrow[d, "\mu_{X_0}"] \arrow[drr, phantom, "\mathtt{comp}^2"] & TX_0 \arrow[r, "X", "\shortmid"{marking}] & X_0 \arrow[d, equal] \\
TX_0 \arrow[rr, "X"', "\shortmid"{marking}] & & X_0
\end{tikzcd} \tag{4.24}$$

We suppose that λ^p and ρ^p are compatible with those compositions.
  - a cell

$$\begin{tikzcd}
X^0 \arrow[r, "p^0", "\shortmid"{marking}] \arrow[d, "f"] & X^1 \arrow[r, "p^1", "\shortmid"{marking}] & \cdots \arrow[r, "p^{n-1}", "\shortmid"{marking}] & X^n \arrow[d, "g"] \\
Y^0 \arrow[rrr, "q"', "\shortmid"{marking}] \arrow[urrr, phantom, "\alpha"] & & & Y^1
\end{tikzcd} \tag{4.25}$$

in $\mathbb{P}\mathrm{rof}(T)$ is a cell

$$\begin{tikzcd}
T^nX^0_0 \arrow[r, "T^{n-1}p^0", "\shortmid"{marking}] \arrow[d, "Tf_0\cdot\mu^n"] & T^{n-1}X^1_0 \arrow[r, "T^{n-2}p^1", "\shortmid"{marking}] & \cdots \arrow[r, "\shortmid"{marking}] & X^n_0 \arrow[d, "g_0"] \\
TY^0_0 \arrow[rrr, "q"', "\shortmid"{marking}] \arrow[urrr, phantom, "\alpha"] & & & Y^1_0
\end{tikzcd} \tag{4.26}$$

in $\mathbb{S}\mathrm{pan}(\widehat{\mathbf{C}})$ which is compatible with the "arrow" part $f_1 : X^0_1 \longrightarrow Y^0_1$ and $g_1 : X^n_1 \longrightarrow Y^1_1$ of f and g, and the left and right actions of p^i $(i \in [n-1])$ and q.
We write $p^{n-1} \vee \dots p^1 \vee p^0 : TX^0_0 \nrightarrow X^n_0$ for the composite

$$\begin{tikzcd}
TX^0_0 \arrow[r, "\mu^*", "\shortmid"{marking}] & T^nX^0_0 \arrow[r, "T^{n-1}p^0", "\shortmid"{marking}] & \cdots \arrow[r, "p^{n-1}", "\shortmid"{marking}] & X^n_0
\end{tikzcd} \tag{4.27}$$

in $\mathbb{S}\mathrm{pan}(\widehat{\mathbf{C}})$. It is straightforward to show this span extends to a T-profunctor $X^0 \nrightarrow X^n$ by equipping the left action of X^0 and the right action of X^n. Right and left actions define 2^{n-1} cells of the form

$$\begin{tikzcd}
X^0 \arrow[r, "p^{n-1}\vee\mathtt{Id}_{X^{n-1}}\vee\cdots\vee\mathtt{Id}_{X^2}\vee p^1\vee\mathtt{Id}_{X^1}\vee p^0", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cdot"] & X^n \arrow[d, equal] \\
X^0 \arrow[r, "p^{n-1}\vee\cdots\vee p^1\vee p^0"', "\shortmid"{marking}] & X^n
\end{tikzcd} \tag{4.28}$$

and any cell α can be seen as a cell

$$\begin{tikzcd}
X^0 \arrow[r, "p^{n-1}\vee\cdots\vee p^1\vee p^0", "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\bar{\alpha}"] & X^n \arrow[d] \\
Y^0 \arrow[r, "q"', "\shortmid"{marking}] & Y^1
\end{tikzcd} \tag{4.29}$$

equalizing those cells, where $\mathtt{Id}_Y$, the *identity T-profunctor*, is the span defining underlying graph of Y, $TY_0 \longleftarrow Y_1 \longrightarrow Y_0$, equipped with 2-ary compositions of Y as its left and right actions, for each T-category Y. Since μ^n is cartesian, $\vee$ is associative up to isomorphism: $(p^{n-1} \cdots \vee p^1 \vee p^0 : TX^0_0 \nrightarrow X^n_0) \vee (q^{m-1} \vee \cdots \vee q^1 \vee q^0 : TY^0_0 \nrightarrow Y^m_0)$ is invertible to $(p^{n-1} \cdots p^1 \vee p^0 \vee q^{m-1} \cdots q^1 \vee q^0 : TX^0_0 \nrightarrow Y^m_0)$ both in $\mathbb{S}\mathrm{pan}(T)$ and $\mathbb{P}\mathrm{rof}(T)$, where $Y^0 = X^n$.

$\mathcal{C}at(T)$ consists of the following data:

- The underlying category is $\mathbf{Cat}(T)$.
- A 2-cell $\sigma : f \Longrightarrow g : X \longrightarrow Y$ is a cell

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[rr, "|" marking, "\mathtt{Id}_Y"'] & \sigma & Y \end{tikzcd} \tag{4.30}$$

The identity T-profunctor satisfies some suitable property and called *unit* in [CS10], and this property implies that σ can be seen as a cell

$$\begin{tikzcd} X \arrow[r, "|" marking, "\mathtt{Id}_X"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Y \arrow[r, "|" marking, "\mathtt{Id}_Y"'] \arrow[ur, phantom, "\sigma"] & Y \end{tikzcd} \tag{4.31}$$

■

A T-profunctor $p : X \nrightarrow Y$ can be seen as a T-category $\ulcorner p \urcorner$ defined as follows

- $\ulcorner p \urcorner_0$ is the coproduct $X_0 \sqcup Y_0$
- $\ulcorner p \urcorner_1$ is the coproduct $X_1 \sqcup |p| \sqcup Y_1$, and a λ-arrow $\xi_1 : a_1 \xrightarrow[\lambda]{} a_0$ in $\ulcorner p \urcorner$ is one of the following
  - a λ-arrow $\xi_1 : x_1 \xrightarrow[\lambda]{} x_0$ in X
  - a λ-arrow $\xi_1 : y_1 \xrightarrow[\lambda]{} y_0$ in Y
  - ξ_1 is an element of $\int |p|$, $a_1 = \mathtt{src}_{|p|}(\xi_1)$, and $a_0 = \mathtt{tgt}_{|p|}(\xi_1)$, where $\mathtt{src}_{|p|}$ and $\mathtt{tgt}_{|p|}$ are the left and right legs of the underlying span of p.
- since $E_{T^n}[\lambda_n]$ is connected, a λ_n-path is either contained in X or Y, or of the form

$$x_n \xrightarrow[\lambda_n]{\xi_{n-1}} x_{n-1} \xrightarrow[\lambda_{n-1}]{} \cdots \longrightarrow x_{m+1} \xrightarrow[\lambda_{m+1}]{\bar{\xi}_m} y_m \xrightarrow[\lambda_m]{} \cdots \xrightarrow[\lambda_1]{\xi_0} y_0 \tag{4.32}$$

where all but $\bar{\xi}_m$ are either contained in X or Y. The composite of such a λ_n-path is defined by applying composition of X and Y for its parts contained in X and Y respectively, and applying left and right actions to those composites with $\bar{\xi}_m$.

On the other hand, $\nabla_T[1]$ consists of the following data:

- $(\nabla_T[1])_0$ is the coproduct of two terminal presheaves $1_{\mathbf{C}} \sqcup 1_{\mathbf{C}}$, whose elements are written as pairs $(c, [i])$ where $i = 0, 1$ and $c \in \mathbf{C}$. Since T preserves coproducts, elements in $T(\nabla_T[1])_0$ is also written as (λ, i).
- $(\nabla_T[1])_1$ is the coproduct $S_T \sqcup S_T \sqcup S_T$, whose elements are written as pairs $(\lambda, 00)$, $(\lambda, 01)$, and $(\lambda, 11)$. For each $(\lambda, ij) \in (\nabla_T[1])_1$, $\mathtt{src}(\lambda) = (\lambda, i)$ and $\mathtt{tgt}(\lambda) = (\mathtt{ty}_T(\lambda), j)$.

which is the same data as the identity T-profunctor on the terminal T-category, $\nabla_T[0]$.

Thus, $\nabla_T[1]$ classifies T-profunctors, i.e., a T-profunctor can be seen as a T-functor $\ulcorner p \urcorner \longrightarrow \nabla_T([1])$ whose pullbacks along $\nabla_T(\partial^1_1)$ and $\nabla_T(\partial^1_0)$ are the unique maps $X \longrightarrow 1$ and $Y \longrightarrow 1$ respectively. Moreover, identity T-profunctors are images of the pullback along $\nabla_T(\sigma^0) : \nabla_T[1] \longrightarrow \nabla_T[0]$.

A 2-ary horizontal cell

$$\begin{tikzcd} X^0 \arrow[r, "|" marking, "p_{01}"] \arrow[d, equal] & X^1 \arrow[r, "|" marking, "p_{12}"] & X^2 \arrow[d, equal] \\ X^0 \arrow[rr, "|" marking, "p_{02}"'] & \alpha & X^2 \end{tikzcd} \tag{4.33}$$

can be seen as a T-functor $\ulcorner p \urcorner \longrightarrow \nabla_T[2]$ as follows

- $\ulcorner p \urcorner_0$ is the coproduct $X^1_0 \sqcup X^1_0 \sqcup X^2_0$
- $\ulcorner p \urcorner_1$ is the coproduct $X^0_1 \sqcup X^1_1 \sqcup X^2_1 \sqcup |p_{01}| \sqcup |p_{12}| \sqcup |p_{02}|$, and a λ-arrow $\xi_1 : x_1 \xrightarrow[\lambda]{} x_0$ in $\ulcorner p \urcorner$ is either of the following
  - a λ-arrow $\xi_1 : x_1 \xrightarrow[\lambda]{} x_0$ in X^i for some $i \in [2]$
  - ξ_1 is an element of $\int |p_{ij}|$, $x_1 = \mathtt{src}_{|p_{ij}|}(\xi_1)$, and $x_0 = \mathtt{tgt}_{|p_{ij}|}(\xi_1)$.
- since $E_{T^n}[\lambda_n]$ is connected, a λ_n-path is either of the following:
  - a path contained in X^i for some $i \in [2]$, which is composed in each T-categories.
  - a path of the form

$$x_n \xrightarrow[\lambda_n]{\xi_{n-1}} \cdots \longrightarrow x_{m+1} \xrightarrow[\lambda_{m+1}]{\bar{\xi}_m} y_m \xrightarrow[\lambda_m]{} \cdots \xrightarrow[\lambda_1]{\xi_0} y_0 \tag{4.34}$$

where, $\bar{\xi}_m$ is in $|p_{ik}|$, x_j and ξ_j are in X^i for each $j > m$, and y_{l+1} and ξ_l are in X^k for each $l < m$, for some $i < k \in [2]$. Such a path is composed through the composition of X^i and X^k and left and right actions of p_{ik}.

– a path of the form

$$x_n \xrightarrow[\lambda_n]{\xi_{n-1}} \cdots \longrightarrow x_{m+1} \xrightarrow[\lambda_{m+1}]{\bar{\xi}_m} y_m \xrightarrow[\lambda_m]{} \cdots \xrightarrow[\lambda_{s+1}]{\xi_{s+1}} y_{s+1} \xrightarrow[\lambda_{s+1}]{\bar{\xi}_s} z_s \xrightarrow[\lambda_s]{} \cdots \xrightarrow[\lambda_1]{\xi_0} z_0 \tag{4.35}$$

where, x_j and ξ_j are in X^0 for each $j > m$, $\bar{\xi}_m$ is in $|p_{01}|$, y_{l+1} and ξ_l are in X^1 for each $s < l < m$, $\bar{\xi}_s$ is in $|p_{12}|$, and y_{t+1} and ξ_t are in X^1 for each $t < s$. Such a path composed to a path of the form

$$\mu^{n-m}\mu^m(x_n) \xrightarrow[\mu^{n-m}\mu^m\lambda_n]{\bar{\xi}'} \mu^m(y_m) \xrightarrow[\mu^m(\lambda_m)]{\bar{\xi}''} z_0 \tag{4.36}$$

by applying compositions of X^i $(i \in [2])$ and actions of p_{01} and p_{12}, and then composed to an arrow $\mu^n(x_n) \xrightarrow[\mu^n(\lambda_n)]{} z_0$ contained in $|p_{02}|$ by applying α.

In the same way, one can check that a functor $\ulcorner p \urcorner \longrightarrow \nabla_T\,[n]$ corresponds to $n(n+1)/2$ T-profunctors p_{ij} $(i < j \in [n])$ equipped with horizontal cells

$$\begin{tikzcd} {} \arrow[r, "p_{ij}", "\shortmid" marking] \arrow[d, equal] & {} \arrow[r, "p_{jk}", "\shortmid" marking] & {} \arrow[d, equal] \\ {} \arrow[rr, "p_{ik}"', "\shortmid" marking] \arrow[urr, phantom, "\alpha_{ijk}"] & & {} \end{tikzcd} \tag{4.37}$$

for each $i < j < k \in [n]$ which are coherent in obvious way. The pullback along the unique map of the form $\nabla_T(!_{[n]}) : \nabla_T\,[n] \longrightarrow \nabla_T\,[0]$ sends a T-category X to $n(n+1)/2$ copies of the identity $\mathtt{Id}_X : X \nrightarrow X$ equipped with the canonical cells induced from the 2-ary composition of X.

Example 7.7 of [CS10] shows that $\mathbb{P}\mathrm{rof}(T)$ is in fact a *virtual equipment*; i.e., for each horizontal and vertical arrows p, f and g in the diagram below, there exists a horizontal arrow $q(f,g) : X \nrightarrow Y$, which is called the *restriction* of q along f and g, equipped with an *cartesian* cell (see (2.8))

$$\begin{tikzcd} X \arrow[r, "q(f{,}g)", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\mathtt{cart}"] & Y \arrow[d, "g"] \\ A \arrow[r, "q"', "\shortmid" marking] & B \end{tikzcd} \tag{4.38}$$

Proposition 4.18. *Any path of horizontal arrows in* $\mathbb{P}\mathrm{rof}(T)$ *is a source of an weakly opcartesian cell. Moreover, the composite of cells below is weakly opcartesian if* α *and* β *are so:*

$$\begin{tikzcd} Y \arrow[r, "p", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & X^0 \arrow[r, "p^0", "\shortmid" marking] \arrow[d, equal] \arrow[drr, phantom, "\alpha"] & \cdots \arrow[r, "p^{n-1}", "\shortmid" marking] & X^n \arrow[d, equal] \\ Y \arrow[r, "\shortmid" marking] \arrow[d, equal] \arrow[drrr, phantom, "\beta"] & {} \arrow[rr, "\shortmid" marking] & & {} \arrow[d, equal] \\ {} \arrow[rrr, "(p^{n-1}\circ\cdots\circ p^0)\circ p"', "\shortmid" marking] & & & {} \end{tikzcd} \tag{4.39}$$

Proof. (See (2.9) and proceeding discussions.)
For each path of T-profunctors p^i $(i = 0, \ldots, n-1)$, define a span $p^{n-1} \circ \cdots p^1 \circ p^0$ as the coequalizer of the 2^{n-1} arrows explained in (4.28), interpreted as parallel morphisms in $\mathcal{H}(\mathbb{S}\mathrm{pan}(\widehat{\mathbf{C}}))(TX^0_0, X^n_0)$ of the form

$$p^{n-1} \vee \cdots \vee \mathtt{Id}_{X^2} \vee p^1 \vee \mathtt{Id}_{X^1} \vee p^0 \;\underset{\longrightarrow}{\overset{\longrightarrow}{\vdots\, 2^{n-1}}}\; p^{n-1} \vee \cdots \vee p^1 \vee p^0 \tag{4.40}$$

Since pullbacks in $\widehat{\mathbf{C}}$ preserves coequalizers, for each spans $p : TA_0 \nrightarrow X^0_0$ and $q : TX^n_0 \nrightarrow B_0$, the above coequalizer is preserved by $q \vee - \vee p$. Therefore, by taking $p := \mathtt{Id}_{X^0}$ and $q := \mathtt{Id}_{X^n}$, the left and right actions of p^{n-1} and p^0 induces those for $p^{n-1} \circ \cdots p^1 \circ p^0$ and it becomes a T-profunctor. It is straightforward to check the canonical cell defining $p^{n-1} \circ \cdots p^1 \circ p^0$ is weakly opcartesian.

Since T preserves pullbacks and pullbacks in $\widehat{\mathbf{C}}$ preseves coequalizers, the universality of the coequalizer defining $p^{n-1} \circ \cdots p^1 \circ p^0 \circ p$ is the same as that defining $(p^{n-1} \circ \cdots p^1 \circ p^0) \circ p$, which means that the cell (4.39) is weakly opcartesian. □

Remark 4.19. $\mathbb{P}\mathrm{rof}(T)$ is not a double category in general, which means some composite of weakly opcartesian cells may not be weakly opcartesian. This is because T does not have to preserve coequalizers defining those weakly opcartesian cells. ■

For any two 2-categories $\mathcal{A}$ and $\mathcal{B}$, we write ${}^{2}_{2}[\mathcal{A},\mathcal{B}]$ for the 2-category of 2-functors, 2-natural transformations, and modifications, and ${}^{P}_{P}[\mathcal{A},\mathcal{B}]$ for the 2-category of pseudo functors, pseudo natural transformations, and modifications. If we take $\mathcal{B} := \mathcal{C}at$ and $\mathcal{A}$ to be small, the canonical inclusion ${}^{2}_{2}[\mathcal{A},\mathcal{C}at] \hookrightarrow {}^{P}_{P}[\mathcal{A},\mathcal{C}at]$ is the right adjoint part of a 2-adjoint whose unit is an equivalence; see 4.2 of [Pow89] and [Lac02].

We write $\Delta_2(T)$ for the subcategory of $\Delta(T)$ obtained by taking the pullback of $\Delta(T) \longrightarrow \Delta$ along $\Delta_2 \hookrightarrow \Delta$, where Δ_2 is the full subcategory of Δ which consists of simplices of dimension lower than or equal to 2.

Definition 4.20. We write $\mathcal{SC}at(T)$, $\mathcal{SC}at_2(T)$, $\mathcal{PSC}at(T)$, and $\mathcal{PSC}at_2(T)$ for ${}^{2}_{2}\left[\Delta(T)^{\mathsf{op}},\mathcal{C}at\right]$, ${}^{2}_{2}[\Delta_2(T),\mathcal{C}at]$, ${}^{P}_{P}\left[\Delta(T)^{\mathsf{op}},\mathcal{C}at\right]$, and ${}^{P}_{P}\left[\Delta_2(T)^{\mathsf{op}},\mathcal{C}at\right]$ respectively. Objects in $\mathcal{SC}at(T)$ ($\mathcal{SC}at_2(T)$) are called ***(2-truncated) T-simplicial categories***, while those in $\mathcal{PSC}at(T)$ ($\mathcal{PSC}at_2(T)$) are ***(2-truncated) pseudo T-simplicial categories***. We omit the preposition "T-" when $T = \mathtt{id}_1$. ■

The classical Grothendieck construction shows that a pseudo T-simplicial category can be seen as a fibration over $\Delta(T)$.

The discussion about T-profunctors above suggests that profunctors may be treated in an ordinary pseudo simplicial category, i.e. a fibration over Δ.

Note that since $\mathbf{Cat}(T)$ is finitely complete, its codomain functor $\mathbf{Cat}(T)^{[1]} \longrightarrow \mathbf{Cat}(T)$ is a fibration.

Definition 4.21. The large ***pseudo simplicial category of T-categories***, $\overline{\mathbb{P}\mathrm{rof}}(T)$, is the pullback of the codomain fibration $\mathbf{Cat}(T)^{[1]} \longrightarrow \mathbf{Cat}(T)$ along the functor $\Delta \hookrightarrow \mathbf{Cat} \xrightarrow{\nabla_T} \mathbf{Cat}(T)$. The large ***2-truncated pseudo simplicial category of T-categories***, $\overline{\mathbb{P}\mathrm{rof}}_2(T)$, is defined as the restriction of $\overline{\mathbb{P}\mathrm{rof}}(T)$ to $\Delta_2(T)$. ■

The reflection $\mathbf{SSET} \longrightarrow \mathbf{CAT}$ preserves finite products, see for example Lemma 3.3.13 of [Cis19], hence the 2-category of locally large large 2-categories, $\mathbf{CAT}$-$\mathcal{CAT}$, is a full sub 2-category of the 2-category of large simplicially enriched categories, $\mathbf{SSET}$-$\mathcal{CAT}$. A simplicially enriched category is precisely a simplicial category whose structure maps are identity-on-object functors, and one can check that 2-functors and 2-natural transformations are precisely 1-cells and 2-cells in $\mathcal{SCAT} := {}^{2}_{2}[\Delta^{\mathsf{op}},\mathcal{CAT}]$, i.e. $\mathbf{SSET}$-$\mathcal{CAT}$ is a full sub 2-category of $\mathcal{SCAT}$. We write $\mathcal{N} : \mathbf{CAT}\text{-}\mathcal{CAT} \hookrightarrow \mathcal{SCAT}$ for the composite of those embeddings.

Since a vertically composable n-tuple of natural transformations $(\beta_1, \dots \beta_n)$ in $\mathcal{C}at(T)$ is precisely a map in $\mathbf{Cat}(T)/\nabla_T\,[n]$ between identities, $\mathcal{N}(\mathcal{C}at(T))_n$ is given by the (bijective-on-objects, fully faithful)-factorization of the functor $\mathbf{Cat}(T)/\nabla_T\,[0] \longrightarrow \mathbf{Cat}(T)/\nabla_T\,[n]$ induced by pullback along the unique map $\nabla_T\,[n] \longrightarrow \nabla_T\,[0]$. This means that the 2-category structure of T-categories is induced from $\overline{\mathbb{P}\mathrm{rof}}(T)$.

Let us denote by T-$\mathbf{Alg}$ the Eilenberg-Moore category of T. A T-*algebra* is an element of T-$\mathbf{Alg}$. Given a T-algebra X, one obtains a T-category X^* as follows:

- $(X^*)_0$ is the underlying object $|X| \in \widehat{\mathbf{C}}$.
- $(X^*)_1$ is $T|X|$. $\mathtt{src} := \mathtt{id}_{T|X|} : T|X| \longrightarrow T|X|$, and $\mathtt{tgt} := h_X : T|X| \longrightarrow |X|$ is the structure map of X.
- Now that we obtain a T-graph, a path $x_n \underset{\lambda_n}{\twoheadrightarrow} x_0$ makes sense and corresponds to a sequence $(\lambda_n; x_n, \dots, x_0)$ such that $T^i h_X(x_{i+1}) = x_i$ and $T^n \mathtt{ty}_X(x_n) = \lambda_n$ for each $i < n$. One can easily check that $\mathtt{comp}_n(\lambda_n; x_n, \dots, x_0) := (\mu^n(\lambda_n); \mu^n(x_n), x_0)$ is well defined; i.e. the right hand side is $\mu^n(\lambda_n)$-arrow, which is exactly the same as the condition for h_X to be an algebra.

This construction induces a functor $(-)^* : T\text{-}\mathbf{Alg} \longrightarrow \mathbf{Cat}(T)$.

Therefore, we obtain a 2-functor $\overline{\mathfrak{M}}^{s}_{T} : \mathcal{VD}bl\mathcal{C}at(T) \longrightarrow \mathcal{SC}at_2(T)$ as the $\mathbf{Cat}$-enriched left Kan extension of the 2-Yoneda embedding $\text{よ} : \Delta_2(T) \longrightarrow \mathcal{SC}at_2(T)$ along $[-]^*_T$, which is a 2-functor from locally discrete 2-category $\Delta_2(T)$, i.e., $\overline{\mathfrak{M}}^{s}_{T}(X)$ is $\mathcal{VD}bl\mathcal{C}at(T)([-]^*_T, X)$.

We write $\overline{\mathfrak{M}}_T$ for the composite $\mathcal{VD}bl\mathcal{C}at(T) \xrightarrow{\overline{\mathfrak{M}}^{s}_{T}} \mathcal{SC}at_2(T) \longrightarrow \mathcal{PSC}at_2(T)$.

Remark 4.22. If the 2-category $\mathcal{VD}bl\mathcal{C}at(T)$ is cocomplete as a 2-category, those 2-functors are the right parts of 2-adjoints. This follows from Theorem 4.51 of [Kel82]. ■

When $T = \mathtt{id}_1$, for each $[n] \in \Delta$, $[n]^*$ is a virtual double category defined as follows

- the set of objects is the underlying set of $[n]$.
- there is no non-trivial vertical arrows i.e. the vertical category is discrete.
- for each pair $i \le j$ in $[n]$, there is a unique horizontal arrow $i \overset{ij}{\nrightarrow} j$.
- any possible squares are filled in with a unique cell.

A map $[0]^* \longrightarrow \mathbb{X}$ in $\mathbf{VDblCat}$ is precisely a ***monoid*** in $\mathbb{X}$ (in the sense of [CS10]) and a map $[1]^* \longrightarrow \mathbb{X}$ is what is called a ***module*** between monoids.

Theorem 4.23. $\mathbb{P}\mathrm{rof}(T)$ *is equivalent to a free object of* $\overline{\mathbb{P}\mathrm{rof}}_2(T)$ *with respect to the 2-functor* $\overline{\mathfrak{M}}_{\mathtt{id}_1} : \mathcal{VD}bl\mathcal{C}at \longrightarrow \mathcal{PSCAT}_2$.

Proof. The discussion proceeding Definition 4.17 suggests $\overline{\mathbb{P}\mathrm{rof}}(T)$ can be seen as a strict simplicial category $\overline{\mathbb{P}\mathrm{rof}}(T)'$ up-to-equivalence defined as follows: $\overline{\mathbb{P}\mathrm{rof}}(T)'_n$ is the full subcategory of $\overline{\mathfrak{M}}_{\mathtt{id}_1}(\mathbb{P}\mathrm{rof}(T))_n =$

$\mathcal{VD}bl\mathcal{C}at([n]^*, \mathbb{P}\mathrm{rof}(T))$ consisting of *strictly normal functors*; i.e., maps $[n]^* \longrightarrow \mathbb{P}\mathrm{rof}(T)$ in **VDblCAT** which send $i \xrightarrow{ii} i$ to identity profunctors, and cells

$$\begin{tikzcd}
{} \arrow[r, "ii", "\shortmid"{marking}] \arrow[d, equal] & {} \arrow[r, "ij", "\shortmid"{marking}] \arrow[d, phantom, "\cdot"] & {} \arrow[d, equal] \\
{} \arrow[rr, "ij"', "\shortmid"{marking}] & & {}
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
{} \arrow[r, "ij", "\shortmid"{marking}] \arrow[d, equal] & {} \arrow[r, "jj", "\shortmid"{marking}] \arrow[d, phantom, "\cdot"] & {} \arrow[d, equal] \\
{} \arrow[rr, "ij"', "\shortmid"{marking}] & & {}
\end{tikzcd} \tag{4.41}$$

to cells in $\mathbb{P}\mathrm{rof}(T)$ induced from left and right actions:

$$\begin{tikzcd}
X_i \arrow[r, "\mathtt{Id}_{X_i}", "\shortmid"{marking}] \arrow[d, equal] & X_i \arrow[r, "p_{ij}", "\shortmid"{marking}] \arrow[d, phantom, "\lambda^{p_{ij}}"] & X_j \arrow[d, equal] \\
X_i \arrow[rr, "p_{ij}"', "\shortmid"{marking}] & & X_j
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
X_i \arrow[r, "p_{ij}", "\shortmid"{marking}] \arrow[d, equal] & X_j \arrow[r, "\mathtt{Id}_{X_j}", "\shortmid"{marking}] \arrow[d, phantom, "\rho^{p_{ij}}"] & X_j \arrow[d, equal] \\
X_i \arrow[rr, "p_{ij}"', "\shortmid"{marking}] & & X_j
\end{tikzcd} \tag{4.42}$$

This induces a strict simplicial category since images of maps in Δ sends each $i \xrightarrow{ii} i$ to $j \xrightarrow{jj} j$ for some j.

Let $\overline{\mathbb{P}\mathrm{rof}_2(T)}'$ be the restriction of $\overline{\mathbb{P}\mathrm{rof}(T)}'$ to Δ_2. Now we show that $\mathbb{P}\mathrm{rof}(T)$ is a free object of $\overline{\mathbb{P}\mathrm{rof}_2(T)}'$ with respect to $\overline{\mathfrak{M}}^s_{\mathtt{id}_1}$. Let $\mathbb{X}$ be a virtual double category. For each map $\mathbb{P}\mathrm{rof}(T) \longrightarrow \mathbb{X}$ in $\mathcal{VD}bl\mathcal{C}at$, the post composition gives rise to a map $\overline{\mathbb{P}\mathrm{rof}_2(T)}' \longrightarrow \overline{\mathfrak{M}}^s_{\mathtt{id}_1}(\mathbb{X})$, and this induces a functor

$$\mathcal{VD}bl\mathcal{C}at(\mathbb{P}\mathrm{rof}(T), \mathbb{X}) \longrightarrow \mathcal{SCAT}(\overline{\mathbb{P}\mathrm{rof}(T)}', \overline{\mathfrak{M}}^s_{\mathtt{id}_1}(\mathbb{X}))$$

which is 2-natural in $\mathbb{X}$. In fact, this functor is faithful since any 2-cell in $\mathcal{VD}bl\mathcal{C}at$ is completely determined by its whiskerings with strictly normal functors $[1]^* \longrightarrow \mathbb{P}\mathrm{rof}(T)$. On the other hand, for each map $F : \overline{\mathbb{P}\mathrm{rof}_2(T)}' \longrightarrow \overline{\mathfrak{M}}^s_{\mathtt{id}_1}(\mathbb{X})$, we can construct $\bar{F} : \mathbb{P}\mathrm{rof}(T) \longrightarrow \mathbb{X}$ as follows:

- $\bar{F}$ sends a T-category X to $F_0(\ulcorner X \urcorner)(0)$, where $\ulcorner X \urcorner$ is the strictly normal functor $[0]^* \longrightarrow \mathbb{P}\mathrm{rof}(T)$ representing X, and $F_0(\ulcorner X \urcorner)$ is a map $[0]^* \longrightarrow \mathbb{X}$.
- $\bar{F}$ sends a T-profunctor $X \xrightarrow{p} Y$ to $F_1(\ulcorner p \urcorner)(01)$, where $\ulcorner p \urcorner$ is the strictly normal functor $[1]^* \longrightarrow \mathbb{P}\mathrm{rof}(T)$ representing p, and $F_1(\ulcorner p \urcorner)$ is a map $[1]^* \longrightarrow \mathbb{X}$.
- $\bar{F}$ sends a T-functor $X \xrightarrow{f} Y$ to $F_0(\ulcorner f \urcorner)_0$, where $\ulcorner f \urcorner$ is the 2-cell $\ulcorner f \urcorner : \ulcorner X \urcorner \Longrightarrow \ulcorner Y \urcorner : [0]^* \longrightarrow \mathbb{P}\mathrm{rof}(T)$ representing f.
- In the same way, $\bar{F}$ sends a unary cell

$$\begin{tikzcd}
X \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "g"] \\
A \arrow[r, "q", "\shortmid"{marking}] & B
\end{tikzcd} \tag{4.43}$$

  to $F_1(\ulcorner \alpha \urcorner)_{01}$, where $\ulcorner \alpha \urcorner$ is the 2-cell $\ulcorner \alpha \urcorner : \ulcorner p \urcorner \Longrightarrow \ulcorner q \urcorner : [1]^* \longrightarrow \mathbb{P}\mathrm{rof}(T)$ representing α.
- Let α be an arbitrary cell in $\mathbb{P}\mathrm{rof}(T)$ of the form

$$\begin{tikzcd}
X^0 \arrow[r, "p^0", "\shortmid"{marking}] \arrow[d, "f"] & X^1 \arrow[r, "p^1", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & \cdots \arrow[r, "p^{n-1}", "\shortmid"{marking}] & X^n \arrow[d, "g"] \\
Y^0 \arrow[rrr, "q"', "\shortmid"{marking}] & & & Y^1
\end{tikzcd} \tag{4.44}$$

  Proposition 4.18 shows α uniquely factors through the weakly opcartesian cell written as $p^{n-1} \circ \cdots \circ p^0$. We write $\bar{\alpha}$ for the result of this factorization; i.e., α factors as follows:

$$\begin{tikzcd}
X^0 \arrow[r, "p^0", "\shortmid"{marking}] \arrow[d, "f"] & X^1 \arrow[r, "p^1", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & \cdots \arrow[r, "p^{n-1}", "\shortmid"{marking}] & X^n \arrow[d, "g"] \\
Y^0 \arrow[rrr, "q"', "\shortmid"{marking}] & & & Y^1
\end{tikzcd}
\;=\;
\begin{tikzcd}
X^0 \arrow[r, "p^0", "\shortmid"{marking}] \arrow[d, equal] & X^1 \arrow[r, "p^1", "\shortmid"{marking}] \arrow[d, phantom, "\tilde{p}"] & \cdots \arrow[r, "p^{n-1}", "\shortmid"{marking}] & X^n \arrow[d, equal] \\
X^0 \arrow[rrr, "p^{n-1} \circ \cdots \circ p^0", "\shortmid"{marking}] \arrow[d, "f"] & & \bar{\alpha} & X^n \arrow[d, "g"] \\
Y^0 \arrow[rrr, "q"', "\shortmid"{marking}] & & & Y^1
\end{tikzcd} \tag{4.45}$$

  Since we have already defined where α is sent, we define the cell $\bar{F}(\tilde{p})$ in $\mathbb{X}$ when $n \neq 1$, and $\bar{F}(\alpha)$ is defined as the composite in $\mathbb{X}$.

– If $n = 0$, then the identity T-profunctor is the 0-ary composition i.e., $p^{n-1} \circ \cdots \circ p^0 = \mathtt{Id}_{X^0}$. $\tilde{p}$ is sent to what the following cell in $[0]^*$ is sent to by $F_0(\ulcorner X^0 \urcorner)$:

$$\begin{tikzcd}
 & 0 \arrow[ld, equal] \arrow[rd, equal] & \\
0 \arrow[rr, "00"', "|" marking] & \cdot & 0
\end{tikzcd} \tag{4.46}$$

– If $n > 1$, again by Proposition 4.18, $p^{n-1} \circ \cdots \circ p^0$ is isomorphic to $(\ldots(p^{n-1} \circ p^{n-2}) \circ p^{n-3}) \ldots \circ p^0)$, and $\tilde{p}$ is the composite of $n-1$ opcartesian cells whose sources are of length 2. Therefore what to define is $n = 2$ case. In this case, $\bar{F}(\tilde{p})$ is defined as the image of the unique cell filling the square below in $[2]^*$ by $F_2(\ulcorner \tilde{p} \urcorner) : [2]^* \longrightarrow \mathbb{X}$, where $\ulcorner \tilde{p} \urcorner$ is the strictly normal functor $[2]^* \longrightarrow \mathbb{P}\mathrm{rof}(T)$ representing $\tilde{p}$.

$$\begin{tikzcd}
0 \arrow[r, "01", "|" marking] \arrow[d, equal] & 1 \arrow[r, "12", "|" marking] & 2 \arrow[d, equal] \\
0 \arrow[rr, "02"', "|" marking] & & 2
\end{tikzcd} \tag{4.47}$$

It is straightforward to check this $\bar{F}$ is a morphism in $\mathcal{VD}bl\mathcal{C}at(T)$ and prove surjectiveness of the functor $\mathcal{VD}bl\mathcal{C}at(\mathbb{P}\mathrm{rof}(T), \mathbb{X}) \longrightarrow \mathcal{SC}at(\overline{\mathbb{P}\mathrm{rof}}(T)', \overline{\mathfrak{M}}^s_{\mathtt{id_1}}(\mathbb{X}))$.

Moreover, the construction of $\bar{F}$ suggests that this functor is full since naturality with α above of a transformation between morphisms $\bar{F}$ and $\bar{G}$ in $\mathcal{VD}bl\mathcal{C}at$ is determined by

- naturality with $\bar{\alpha}$, which follows from naturality of $F \Longrightarrow G$ in $\mathcal{SC}at_2$ on [1].
- naturality with $\tilde{p}$, which follows from naturality of $F \Longrightarrow G$ in $\mathcal{SC}at_2$ on [2].

Thus this functor is an isomorphism, which means freeness of $\mathbb{P}\mathrm{rof}(T)$. □

Finally, we suggest a way to define a $n+2$-dimensional structure of virtual $n+1$-tuple categories.

In the Section 3 of [CS10], for each virtual double category $\mathbb{X}$, the virtual double category of monoids in $\mathbb{X}$, $\mathbb{M}\mathrm{od}(\mathbb{X})$, is defined and it is proved that $\mathbb{M}\mathrm{od}$ extends to an endo-functor on **VDblCat**. Moreover, in the Section 5, it is proved that this endo-functor is induced from the pseudo-adjunction between the 2-category of virtual double categories and the category of *unital* virtual double categories.

On the other hand, there are elements written as $\underline{0} \in \int S_{\mathtt{fc}} \subset \mathbf{G}_1(\mathtt{fc})$ and $[1] \in \mathbf{G}_1 \subset \mathbf{G}_1(\mathtt{fc})$, and the virtual double category $[[1]]_{\mathtt{fc}}$ is the smallest virtual double category which contains a horizontal arrow $0 \relbar\joinrel\mapstochar\joinrel\rightarrow 1$, while $[\underline{0}]_{\mathtt{fc}}$ is the smallest virtual double category which contains a vertical arrow $0 \longrightarrow 1$. In particular $\mathbb{M}\mathrm{od}([\underline{0}]_{\mathtt{fc}})$ is the same as $\nabla_{\mathtt{fc}}([1])$, hence a $\mathtt{fc}$-functor $\mathbb{p} : \ulcorner \mathbb{p} \urcorner \longrightarrow \mathbb{M}\mathrm{od}([\underline{0}]_{\mathtt{fc}})$ is the same as a $\mathtt{fc}$-profunctor $\mathbb{X} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{Y}$, which can be seen as a ***vertical profunctor***, since if the restrictions of $\mathbb{p}$ on 0 and 1 are $\mathbb{X}$ and $\mathbb{Y}$ respectively, then it can be seen as a virtual double category which consists of each copy of $\mathbb{X}$ and $\mathbb{Y}$, additional vertical arrows from elements in $\mathbb{X}$ to those in $\mathbb{Y}$, and additional cells containing those vertical arrows. In the same way, we can define a ***horizontal profunctor*** $\mathbb{u} : \mathbb{X} \relbar\joinrel\mapstochar\mapstochar\joinrel\rightarrow \mathbb{Y}$: a virtual double category which consists of each copy of $\mathbb{X}$ and $\mathbb{Y}$, additional horizontal arrows from elements in $\mathbb{X}$ to those in $\mathbb{Y}$, and additional cells containing those horizontal arrows. One can easily check that a horizontal profunctor can be seen as a $\mathtt{fc}$-functor $\ulcorner \mathbb{u} \urcorner \longrightarrow \mathbb{M}\mathrm{od}([[1]]_{\mathtt{fc}})$ in the same way as vertical profunctors.

For each $n \in \int S_{\mathtt{fc}} \subset \mathbf{G}_1(\mathtt{fc})$, the virtual double category $[n]_{\mathtt{fc}}$ classifies n-cells in arbitrary virtual double category, i.e., a map $[n]_{\mathtt{fc}} \longrightarrow \mathbb{X}$ is the same as a n-ary cell in $\mathbb{X}$. Therefore, $\mathbb{M}\mathrm{od}([n]_{\mathtt{fc}})$ has the same objects as $[n]_{\mathtt{fc}}$, but each object has its unit. Hence a $\mathtt{fc}$-functor $\ulcorner \alpha \urcorner \longrightarrow \mathbb{M}\mathrm{od}([n]_{\mathtt{fc}})$ is a virtual double category consisting of $n+1$ horizontal profunctors and 2 vertical profunctors which makes the frame of the square below and additional cells connecting those profunctors.

$$\begin{tikzcd}
\cdot \arrow[r, "||" marking] \arrow[d, "|" marking] & \cdot \arrow[r, "||" marking] & \cdot \cdots \cdot \arrow[r, "||" marking] & \cdot \arrow[r, "||" marking] & \cdot \arrow[d, "|" marking] \\
\cdot \arrow[rrrr, "||" marking] & & \alpha & & \cdot
\end{tikzcd} \tag{4.48}$$

Those define a $\mathtt{fc}$-graph of virtual double categories and vertical and horizontal profunctors.

By considering the analogy of Theorem 4.23, we obtain a definition of the virtual triple category of virtual double categories, as follows:

Definition 4.24. The large $\mathtt{fc}$-***pseudo simplicial category of virtual double categories***, $\overline{2\mathfrak{Prof}}$, is defined as the pullback of the codomain fibration $\mathbf{VDblCat}^{[1]} \longrightarrow \mathbf{VDblCat}$ along the composite

$$\Delta(\mathtt{fc}) \xrightarrow{[-]_{\mathtt{fc}}} \mathbf{VDblCat} \xrightarrow{\mathbb{M}\mathrm{od}} \mathbf{VDblCat} \tag{4.49}$$

The large ***2-truncated*** $\mathtt{fc}$-***pseudo simplicial category of virtual double categories***, $\overline{2\mathfrak{Prof}}_2$, is defined as the restriction of $\overline{2\mathfrak{Prof}}$ to $\Delta_2(\mathtt{fc})$. The ***virtual triple category of virtual double categories***, $2\mathfrak{Prof}$, is the free object of $\overline{2\mathfrak{Prof}}_2$ with respect to the 2-functor $\overline{\mathfrak{M}}_{\mathtt{fc}} : \mathcal{V}\text{-}3\text{-}tpl\mathcal{CAT} \longrightarrow \mathcal{PSCAT}_2(\mathtt{fc})$. ■

This clearly indicates a way to define the virtual $n+2$-tuple category of virtual $n+1$-tuple category: Firstly we define endo-functors $M_n : \mathbf{V}\text{-}n+1\text{-}\mathbf{tplCat} \longrightarrow \mathbf{V}\text{-}n+1\text{-}\mathbf{tplCat}$ which makes virtual $n+1$-tuple categories "*unital*" in some way. Then we can define structures of virtual $n+1$-tuple categories as follows:

- The large $\mathtt{fc}^{\mathtt{n}}$-pseudo simplicial category, $\overline{n+1\mathfrak{Prof}}$, is the pullback of the codomain fibration on $\mathbf{V}\text{-}n+1\text{-}\mathbf{tplCat}$ along

$$\Delta(\mathtt{fc}^{\mathtt{n}}) \xrightarrow{[-]_{\mathtt{fc}^{\mathtt{n}}}} \mathbf{V}\text{-}n+1\text{-}\mathbf{tplCat} \xrightarrow{M_n} \mathbf{V}\text{-}n+1\text{-}\mathbf{tplCat} \tag{4.50}$$

  and let $\overline{n+1\mathfrak{Prof}}_2$ be its 2-truncated version.
- The large virtual $n+2$-tuple category, $n+1\mathfrak{Prof}$, is the free object of $\overline{n+1\mathfrak{Prof}}_2$ with respect to the 2-functor $\overline{\mathfrak{M}}_{\mathtt{fc}^{\mathtt{n}}} : \mathcal{V}\text{-}n+2\text{-}tpl\mathcal{CAT} \longrightarrow \mathcal{PSCAT}_2(\mathtt{fc}^{\mathtt{n}})$.

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