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Abelian Property of the Category of U-Complexes

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Abstract

The notion of a chain $U-$complex and chain $(U, U')-$map were introduced by Davvaz and Shabbani as a generalization of a chain complex and a chain map respectively. In this paper we continue their research by proposing a category of $U-$complexes as a generalization of the category of complexes. We show that the category of $U-$complexes is an abelian category.

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1 Introduction

A sequence of $R-$modules and $R-$homomorphism

$$\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \longrightarrow \cdots \quad (1)$$

is called exact sequence if $\mathrm{Im}(d_{n+1}) = d_n^{-1}(0)$. It is a natural question what if 0 is replaced by U_{n-1}, a submodul of X_{n-1}. Davvaz and Parnian [1] modified the definition of exact sequence of modules which is called $U-$exact sequences and generalized some results from existing ones to the modified case. Their research was motivated by the exact sequence of hypergroups which generally has no zero element, introduced by Freni and Elderberry in [2].

Davvaz and Shabbani continued working on this topic and proposed the concept of $U-$complex as a generalization of complex [3]. They defined the concepts of chain $U-$complex, $U-$homology, chain $(U, U')-$map, chain $(U, U')-$homotopy and $\mathcal{U}-$functor and used the concepts to find a generalization of several results in homological algebra.

This paper aims to apply the previous results to examine the concepts category of $U-$complexes. We show that the category of $U-$complexes is an abelian category.

2 Chain of U-Complexes

In this section we review some results introduced by Davvaz and Shabbani.

Definition 2.1 *Given a family $X = (X, U^X, d^X) = (X_n, U_n^X, d_n^X)_{n\in\mathbf{Z}}$ where X_n, U_n are $R-$modules and each of X_n consists U_n and $d_n : X_n \to X_{n-1}$. A chain*

$$(X, U^X, d^X) : \cdots \to X_{n+1} \xrightarrow{d_{n+1}^X} X_n \xrightarrow{d_n^X} X_{n-1} \xrightarrow{d_{n-1}^X} X_{n-2} \longrightarrow \cdots$$

is called U^X-complex if for all $n \in \mathbf{Z}$ we have:

1. $d_n^X d_{n+1}^X(X_{n+1}) \subseteq U_{n-1}^X$ *and*
2. $Im\left(d_n^X\right) \supseteq U_{n-1}^X$

The definition implisitly say that a chain complex is a chain $0-$complex.

Definition 2.2 *Let (X, U^X, d^X) be a U^X-complex and (Y, U^Y, d^Y) be a U^Y-complex. The sequence $f = (f_n : X_n \to Y_n)_{n\in\mathbf{Z}}$ is called chain $(U^X, U^Y)-$map if following diagram is commutative and $f_n(U_n^X) \subseteq U_n^Y$ for each $n \in \mathbf{Z}$.*

$$\begin{array}{ccccccc}
\cdots \longrightarrow & X_{n+1} & \stackrel{d^X_{n+1}}{\longrightarrow} & X_n & \stackrel{d^X_n}{\longrightarrow} & X_{n-1} & \longrightarrow \cdots \\
 & \downarrow_{f_{n+1}} & & \downarrow_{f_n} & & \downarrow_{f_{n-1}} & \\
\cdots \longrightarrow & Y_{n+1} & \stackrel{d^Y_{n+1}}{\longrightarrow} & Y_n & \stackrel{d^Y_n}{\longrightarrow} & Y_{n-1} & \longrightarrow \cdots
\end{array}$$

Proposition 2.3 *Let (X, U^X, d^X) be a U^X-complex such that $d_n^X d_{n+1}^X(X_{n+1}) = U_{n-1}^X$ and (Y, U^X, d^Y) is a chain U^Y-complex. If $f = (f_n : X_n \to Y_n)_{n\in\mathbf{Z}}$ is a chain map then it is also a chain $(U^X, U^Y)-$map.*

Definition 2.4 *Let Let (X, U^X, d^X) and (Y, U^X, d^Y) be a chain U^X-complex and U^Y-complex respectively. A chain $(U^X, U^Y)-$map $f = (f_n)_{n\in\mathbf{Z}}$ is an isomorphism if f_n is $R-$modules isomorphism for all $n \in \mathbf{Z}$ and $f^{-1} = (f_n^{-1})_{n\in\mathbf{Z}}$ is a chain $(U^Y, U^X)-$map.*

If there exists an isomorphism from (X, U^X, d^X) to (Y, U^X, d^Y) we say that (X, U^X, d^X) isomorphic to (Y, U^X, d^Y). The isomorphism of chain $U-$complexes is an equivalence relation.

Proposition 2.5 *If chain U^X-complex and U^Y-complex are isomorphic then $U_n^X \simeq U_n^Y$ for all $n \in \mathbf{Z}$.*

3 The Category of U-Complexes

In this section we introduce the concept of a category of $U-$complexes and study its property. Let $\mathcal{A}$ be an abelian category $R-$**Mod**.

Definition 3.1 *The category of $U-$complexes $\mathbf{C}(\mathcal{A}, U)$ is a category whose objects are chain $U-$complexes in $\mathcal{A}$, the morphisms are chain $(U, U')-$map and the composition operation is the usual composition function.*

Theorem 3.2 *The category of U-complexes $\mathbf{C}(\mathcal{A}, U)$ is an abelian category*

Proof

A1 Let (X, U^X, d^X) and (Y, U^X, d^Y) be a chain U^X-complex and U^Y-complex respectively. Assume that $f = (f_n)_{n\in\mathbf{Z}}$ and $g = (g_n)_{n\in\mathbf{Z}}$ are two chain $(U^X, U^Y)-$maps. By defining $f+g = (f_n+g_n)_{n\in\mathbf{Z}}$ it is easy to prove that $Hom_{\mathbf{C}(\mathcal{A},U)}(X, Y)$ is an abelian group and the composition of morphisms

$$Hom_{\mathbf{C}(\mathcal{A},U)}(Y, Z) \times Hom_{\mathbf{C}(\mathcal{A},U)}(X, Y) \to Hom_{\mathbf{C}(\mathcal{A},U)}(X, Z)$$

is bilinier over integer.

A2 The zero object in $\mathbf{C}(\mathcal{A}, U)$ is the chain of $0-$complex which all modules are zero.

A3 A coproduct of two objects $X = (X, U^X, d_n^X)$ and $Y = (Y, U^Y, d_n^Y)$ is and object

$$X \oplus Y = \left(X \oplus Y, U^{X \oplus Y}, d^{X \oplus Y}\right) = \left(X_n \oplus Y_n, U_n^{X \oplus Y},\ d_n^{X \oplus Y}\right)_{n \in \mathbf{Z}}$$

where

$$U_n^{X \oplus Y} = \begin{pmatrix} U_n^X \\ U_n^Y \end{pmatrix} \text{ and } d_n^{X \oplus Y} = \begin{pmatrix} d_n^X & \mathbf{0} \\ \mathbf{0} & d_n^Y \end{pmatrix}$$

together with chain $\left(U^X, U^{X \oplus Y}\right)-$map ι_X and chain $\left(U^Y, U^{X \oplus Y}\right)-$map ι_Y satisfying the universal property: for every objects Z in $\mathbf{C}(\mathcal{A}, U)$, chain $\left(U^X, U^Z\right)-$map $\mathtt{f}_X$ and chain $\left(U^Y, U^Z\right)-$map $\mathtt{f}_Y$ there is a unique chain $\left(U^{X \oplus Y}, U^Z\right)-$map $\mathtt{f}$ making following diagram commutative.

$$\begin{array}{ccccc} & & Z_n & & \\ & (\mathtt{f}_X)_n \nearrow & \mathtt{f}_n \uparrow & \nwarrow (\mathtt{f}_Y)_n & \\ X_n & \xrightarrow{(\iota_X)_n} & X_n \oplus Y_n & \xleftarrow{(\iota_Y)_n} & Y_n \end{array} \tag{2}$$

A4 Let $f = (f_n : X_n \to Y_n)_{n \in \mathbf{Z}}$ be a chain $\left(U^X, U^Y\right)-$map, then each f_n is a morphism in $\mathcal{A}$. We show the existence of a cokernel and leave the dual. Since $\mathcal{A}$ is an abelian category, each f_n has a cokernel $C_n = Y_n / \mathrm{Im}\,(f_n)$ in $\mathcal{A}$ together with a morphism $c_n : Y_n \to C_n$ such that $c_n f_n = 0$ satisfying the universal property of cokernel, i.e. there is a unique morphism $d_n^C : C_n \to C_{n-1}$ such that $c_{n-1} d_n^C = d_n^Y c_n$. Let $f = (f_n : X_n \to Y_n)_{n \in \mathbf{Z}}$ be a chain $\left(U^X, U^Y\right)-$map, then each f_n is a morphism in $\mathcal{A}$. Hence the following diagram is commutative.

$$\begin{array}{cccccccc} X & \cdots \to & X_{n+1} & \overset{d_{n+1}^X}{\to} & X_n & \overset{d_n^X}{\to} & X_{n-1} & \cdots \to \\ & & \downarrow_{f_{n+1}} & & \downarrow_{f_n} & & \downarrow_{f_{n-1}} & \\ Y & \cdots \to & Y_{n+1} & \overset{d_{n+1}^Y}{\to} & Y_n & \overset{d_n^Y}{\to} & Y_{n-1} & \cdots \to \\ & & \downarrow_{c_{n+1}} & & \downarrow_{c_n} & & \downarrow_{c_{n-1}} & \\ C & \cdots \to & C_{n+1} & \overset{d_{n+1}^C}{\to} & C_n & \overset{d_n^C}{\to} & C_{n-1} & \cdots \to \end{array} \tag{3}$$

By choosing $U_n^C = U_n^Y /\ \mathrm{Im}\,(f_n)$, it is easy to check that $C = \left(C_n, d_n^C, U_n^C\right)$ is a chain U^C-complex and satisfying the universal property of cokernel for f.

A5 Let $U_n^{\mathrm{Im}(f)} = f_n\left(U_n^X\right)$ and $U_n^{\mathrm{coIm}(f)} = U_n^X / \ker(f_n)$ then $\mathrm{coIm}(f)$ and $\mathrm{Im}(f)$ are objects in $\mathbf{C}(\mathcal{A}, U)$. Consider the natural morphism $\mathrm{coIm}(f) \to \mathrm{coIm}(f)$, since $\mathcal{A}$ is abelian then for every n the natural morphism

g_n : $\mathrm{coIm}(f_n) \to \mathrm{Im}(f_n)$ is an isomophism, hence the invers g_n^{-1} : $\mathrm{coIm}(f_n) \to \mathrm{Im}(f_n)$ is also isomorphism and $g^{-1} = (g_n^{-1})_{n\in\mathbf{Z}}$ is a chain map. As $U_n^{\mathrm{Im}(f)} = f_n\left(U_n^X\right)$ we have $g_n^{-1}\left(U_n^{\mathrm{Im}(f)}\right) = U_n^X$, hence g^{-1} is a chain $\left(U^Y, U^X\right)-$map. By the Definition 2.4, g is an isomorphism of $U-$complexes.

■

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