A NOTE ON PRODUCTS

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ABSTRACT. An equivalent definition of the notion of product in a category is presented. It is stated in terms of an algebraic structure rather than a universal property.

1. Introduction

The usual definition of a product of two objects in a category is stated in terms of a universal property. It is commonly admitted [3] that the connector & of linear logic [1] must be interpreted in the categorical semantics as a product.

Indeed, the three related inference rules make this connector look like a product:

$$\frac{B \vdash C}{A \,\&\, B \vdash C}\ \&_l^1 \qquad \frac{A \vdash C}{A \,\&\, B \vdash C}\ \&_l^2 \qquad \frac{A \vdash B \qquad A \vdash C}{A \vdash B \,\&\, C}\ \&_r$$

One of the most important properties that a logical system expressed in sequent calculus must verify is cut elimination. "A sequent calculus without cut-elimination is like a car without an engine" [2] in the words of Girard.

Cut elimination is a rewriting process that moves the instances of the cut rule

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}\ \text{cut}$$

at the top of the proof, ultimately eleminating them when they arrive at the level of an axiom

$$\frac{}{A \vdash A}\ \text{ax}$$

by the rewriting rules

$$\frac{\dfrac{}{A \vdash A}\ \text{ax} \qquad \begin{array}{c}\pi \\ \vdots \\ A \vdash B\end{array}}{A \vdash B}\ \text{cut} \quad \rightsquigarrow \quad \begin{array}{c}\pi \\ \vdots \\ A \vdash B\end{array}$$

and

$$\frac{\begin{array}{c}\pi \\ \vdots \\ A \vdash B\end{array} \qquad \dfrac{}{B \vdash B}\ \text{ax}}{A \vdash B}\ \text{cut} \quad \rightsquigarrow \quad \begin{array}{c}\pi \\ \vdots \\ A \vdash B\end{array}$$

Categorically, the axiom is interpreted by the identity, the cut rule by composition, and the two preceding cut elimination steps express the unitality of composition.

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The other cut elimination steps which interest us are

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ X \vdash A\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ X \vdash B\end{array}}{X \vdash A \,\&\, B}\,\&_r \qquad \dfrac{\dfrac{}{A \vdash A}\,\mathrm{ax}}{A \,\&\, B \vdash A}\,\&_l^2}{X \vdash A}\,\mathrm{cut} \quad\rightsquigarrow\quad \dfrac{\begin{array}{c}\pi_1\\ \vdots\\ X \vdash A\end{array} \quad \dfrac{}{A \vdash A}\,\mathrm{ax}}{X \vdash A}\,\mathrm{cut} \quad\rightsquigarrow\quad \begin{array}{c}\pi_1\\ \vdots\\ X \vdash A\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ X \vdash A\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ X \vdash B\end{array}}{X \vdash A \,\&\, B}\,\&_r \qquad \dfrac{\dfrac{}{B \vdash B}\,\mathrm{ax}}{A \,\&\, B \vdash B}\,\&_l^1}{X \vdash B}\,\mathrm{cut} \quad\rightsquigarrow\quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ X \vdash B\end{array} \quad \dfrac{}{B \vdash B}\,\mathrm{ax}}{X \vdash B}\,\mathrm{cut} \quad\rightsquigarrow\quad \begin{array}{c}\pi_2\\ \vdots\\ X \vdash B\end{array}$$

$$\dfrac{\begin{array}{c}\pi_0\\ \vdots\\ Y \vdash X\end{array} \quad \dfrac{\begin{array}{c}\pi_1\\ \vdots\\ X \vdash A\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ X \vdash B\end{array}}{X \vdash A \,\&\, B}\,\&_r}{Y \vdash A \,\&\, B}\,\mathrm{cut} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\pi_0\\ \vdots\\ Y \vdash X\end{array} \quad \begin{array}{c}\pi_1\\ \vdots\\ X \vdash A\end{array}}{Y \vdash A}\,\mathrm{cut} \qquad \dfrac{\begin{array}{c}\pi_0\\ \vdots\\ Y \vdash X\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ X \vdash B\end{array}}{Y \vdash B}\,\mathrm{cut}}{Y \vdash A \,\&\, B}\,\&_r$$

Categorically, they must be interpreted respectively as $\langle \pi_1, \pi_2 \rangle; p_1 = \pi_1$, $\langle \pi_1, \pi_2 \rangle; p_2 = \pi_2$ and $\pi_0; \langle \pi_1, \pi_2 \rangle = \langle \pi_0; \pi_1, \pi_1; \pi_2 \rangle$.

Moreover, another equivalence, which is not related to cut elimination, is commonly imposed on proofs. It is named η-equivalence. In the case of the connector &, it is expressed as below:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{ax}}{A \,\&\, B \vdash A}\,\&_l^2 \qquad \dfrac{\dfrac{}{B \vdash B}\,\mathrm{ax}}{A \,\&\, B \vdash B}\,\&_l^1}{A \,\&\, B \vdash A \,\&\, B}\,\&_r \quad \simeq_\eta \quad \dfrac{}{A \,\&\, B \vdash A \,\&\, B}\,\mathrm{ax}$$

Categorically, it must be interpreted as $\langle p_1, p_2 \rangle = 1_{A \times B}$

In the next section, we show that these four equations provide an equivalent definition of binary products in category theory. We also mention terminal objects (although the observation is quite trivial in this case). In the third section, we generalize to arbitrary products.

These observations about products can of course be dualised to apply to coproducts.

2. An equivalent definition of binary products

2.1. PROPOSITION. *Let $\mathcal{C}$ be category and A, B two objects of $\mathcal{C}$. A product of A and B can be defined as an object $A \times B$ together with:*

- *for every object X, a function $\langle ., . \rangle_X : \mathcal{C}[X, A] \times \mathcal{C}[X, B] \to \mathcal{C}[X, A \times B]$,*
- *a morphism $p_1 : A \times B \to A$,*
- *a morphism $p_2 : A \times B \to B$,*

such that:

- *for every object X, for every morphisms $f : X \to A$ and $g : X \to B$:*

$$\langle f, g \rangle_X ; p_1 = f, \tag{1}$$

- *for every object X, for every morphisms $f : X \to A$ and $g : X \to B$:*

$$\langle f, g\rangle_X ; p_2 = g, \tag{2}$$

- *the following equation is verified:*

$$\langle p_1, p_2\rangle_{A\times B} = 1_{A\times B}, \tag{3}$$

- *for every objects X, Y and morphisms $u : Y \to X$, $f : X \to A$, $g : X \to B$:*

$$\langle u; f, u; g\rangle_Y = u; \langle f, g\rangle_X. \tag{4}$$

Proof. Suppose that $(A \times B, p_1, p_2)$ is a product of A and B. We already know by definition that Equation (1) and Equation (2) are satisfied. Moreover, $1_{A\times B}$ satisfies the equations $1_{A\times B}; p_1 = p_1$ and $1_{A\times B}; p_2 = p_2$. It follows that Equation (3) is satisfied. For every objects X, Y and morphisms $u : Y \to X$, $f : X \to A$, $g : X \to B$, we have $u; \langle f, g\rangle_X ; p_1 = u; f$ and $u; \langle f, g\rangle_X ; p_2 = u; g$. Equation (4) is thus also satisfied. We have thus the data in our proposition. Suppose now given the data in our proposition. Given an object X, two morphisms $f : X \to A$, $g : X \to B$ and a morphism $v : X \to A\times B$ such that $v; p_1 = f$ and $v; p_2 = g$, we obtain by applying $\langle ., .\rangle_X$ that $\langle v; p_1, v; p_2\rangle_X = \langle f, g\rangle_X$. By applying Equation (4), we obtain that $v; \langle p_1, p_2\rangle_{A\times B} = \langle f, g\rangle_X$ and then, by applying Equation (3), that $v = \langle f, g\rangle_X$. It follows from this fact and Equation (1), Equation (2), that $(A \times B, p_1, p_2)$ verifies the required universal property and is a product of A and B. ■

Note that if we want to define a terminal object in our style, we only have to say that it is an object $\top$ together with, for every object X, a morphism $t_X : X \to \top$ such that for every morphism $f : X \to \top$, the equation $f = t_X$ is satisfied.

3. An equivalent definition of arbitrary products

3.1. Proposition. *Let $\mathcal{C}$ be category, I a set and $(A_i)_{i\in I}$ a family of objects of $\mathcal{C}$. A product of the objects A_i can be defined as an object $\prod_{i\in I} A_i$ together with:*

- *for every object X, a function $\langle\ \rangle_X : \prod_{i\in I} \mathcal{C}[X, A_i] \to \mathcal{C}[X, \prod_{i\in I} A_i]$,*
- *for every $i \in I$, a morphism $p_i : \prod_{i\in I} A_i \to A_i$,*

such that:

- *for every object X, for every family of morphisms $(f_i)_{i\in I} \in \prod_{i\in I} \mathcal{C}[X, A_i]$, for every $i \in I$,*

$$\langle f_i, i \in I\rangle_X ; p_i = f_i, \tag{5}$$

- *the following equation is verified:*

$$\langle p_i, i \in I\rangle_{\prod_{i\in I} A_i} = 1_{\prod_{i\in I} A_i}, \tag{6}$$

- *for every objects X, Y, morphisms $u : Y \to X$ and family of morphisms $(f_i)_{i\in I} \in \prod_{i\in I} \mathcal{C}[X, A_i]$,*

$$\langle (u; f_i), i \in I\rangle_Y = u; \langle f_i, i \in I\rangle_X. \tag{7}$$

Proof. Suppose that $(\prod_{i\in I} A_i, (p_i)_{i\in I})$ is a product of the objects A_i. We already know by definition that Equation (5) is satisfied. Moreover, $1_{\prod_{i\in I} A_i}$ satisfies the equation $1_{\prod_{i\in I} A_i}; p_i = p_i$ for every $i \in I$. It follows that Equation (6) is satisfied. For every objects X, Y, morphism $u : Y \to X$ and family of morphisms $(f_i)_{i\in I} \in \prod_{i\in I} \mathcal{C}[X, A_i]$, we have for every $i \in I$, $u; \langle f_i, i \in I\rangle_X; p_i = u; f_i$. Equation (7) is thus also satisfied. We have thus the data in our proposition. Suppose now given the data in our proposition. Given an object X, a family $(f_i)_{i\in I} \in \prod_{i\in I} \mathcal{C}[X, A_i]$ and a morphism $v : X \to \prod_{i\in I} A_i$ such that for every $i \in I$, $v; p_i = f_i$ we obtain by applying $\langle\ \rangle_X$ that $\langle (v; p_i), i \in I\rangle_X = \langle f_i, i \in I\rangle_X$. By applying Equation (7), we obtain that $v; \langle p_i, i \in I\rangle_{\prod_{i\in I} A_i} = \langle f_i, i \in I\rangle_X$ and then, by applying Equation (6), that $v = \langle f_i, i \in I\rangle_X$. It follows from this fact and Equation (5), that $(\prod_{i\in I} A_i, (p_i)_{i\in I})$ verifies the required universal property and is a product of the objects A_i. ∎

References

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