Triple categories in lifting problems

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August 17, 2023

Contents

1 Lifting structures on morphisms, background on double categories

Definition 1 (Lift, lifting structure)**.** Let $\mathcal{C}$ be a category. Let $f : a \to a', g : b \to b'$ be morphisms in $\mathcal{C}$.

- Given $h : a \to b, k : a' \to b'$ with $g \circ h = k \circ f$, a *lift* of (h, k) is a morphism $t : b \to a'$ such that $h = t \circ f$ and $k = g \circ t$.
- A *lifting structure* on f, g is a function which assigns to each pair $(h, k) : f \to g$ a lift of h, k.

The set of lifting structures on (f, g) will here be denoted $\mathrm{Lift}(f, g)$.

Definition 2 (Right, left composition of lifting structures)**.** Let a_0, a_1, a_2, b_0, b_1 be objects in $\mathcal{C}$; let $v_0 : a_0 \to a_1, v_1 : a_1 \to a_2$, $w : b_0 \to b_1$. If $\rho \in \mathrm{Lift}(v_0, w)$ and $\pi \in \mathrm{Lift}(v_1, w)$ are lifting structures, their *left composite* is a function $\pi \star^{\ell} \rho \in \mathrm{Lift}(v_1 \circ v_0, w)$ defined by

$$(\pi \star^{\ell} \rho)(h, k) = \pi(\rho(h, k \circ v_1), k) \tag{1}$$

The notion of *right composite* is straightforwardly dual and gives a function

$$\star^r : \mathrm{Lift}(v, w_0) \times \mathrm{Lift}(v, w_1) \to \mathrm{Lift}(v, w_1 \circ w_0) \tag{2}$$

whenever $w_1 \circ w_0$ is defined.

We assume the reader is familiar with double categories. Double categories are all strict; so are double functors.

Recall that a double category is called *flat* if there is at most one 2-cell lying over a given square; such a double category is a kind of two-dimensional version of a poset category.

Definition 3 (Double category of commutative squares)**.** Let $\mathcal{C}$ be a 1-category. The double category $\mathbb{S}\mathrm{q}(\mathcal{C})$ of commutative squares has the same objects as $\mathcal{C}$; its horizontal and vertical morphisms are both the morphisms of $\mathcal{C}$ and inherit the multiplication. The double category $\mathbb{S}\mathrm{q}(\mathcal{C})$ is flat and a 2-cell through a given square exists iff the square commutes.

Definition 4 (Horizontally discrete, vertically discrete double categories)**.** A double category is called horizontally discrete if its only horizontal morphisms are identities. If $\mathcal{C}$ is a 1-category, there is a horizontally discrete double category denoted $\mathbb{V}(\mathcal{C})$ whose objects are those of $\mathcal{C}$ and whose vertical arrows are the arrows of $\mathcal{C}$; all horizontally discrete double categories are essentially of this form for some $\mathcal{C}$.

Similarly, a vertically discrete category has no vertical morphisms other than the identities. $\mathbb{H}(\mathcal{C})$ denotes the vertically discrete double category associated to a 1-category $\mathcal{C}$.

Definition 5 (Horizontal, vertical duality)**.** If $\mathbb{D}$ is a double category, $\mathbb{D}^{\mathrm{op}}$ refers to the horizontal dual of $\mathbb{D}$, whose horizontal morphisms and horizontal 2-cells between vertical arrows are reversed. $\mathbb{D}^{\mathrm{co}}$ refers to the vertical dual of $\mathbb{D}$, whose vertical arrows reverse in direction.

If $\mathbb{A}$ is a double category, the 1-category of objects and horizontal maps of $\mathbb{A}$ is denoted $\mathbb{A}_O$, and the category of vertical arrows and 2-cells between them is denoted $\mathbb{A}_A$.

2 A "tensor product" of two double categories

A **triple category** $\mathcal{E}$ is a data structure which starts with

- objects
- three classes of 1-cells, called "horizontal", "vertical" and "transversal", which can be composed with other 1-cells of the same class
- three classes of 2-cells, called "vertical", "horizontal" and "basic". A vertical cell is one with boundaries which are transversal and vertical. A horizontal cell is one with boundaries which are transversal and horizontal. A basic cell is one with boundaries which are horizontal and vertical arrows.

- a class of 3-cells, called "cubes."

A triple category $\mathcal{E}$ has three distinct category structures on the same common class of objects given by the composition of 1-cells. I will refer to these category structures as $\mathcal{E}_{\mathbf{x}}, \mathcal{E}_{\mathbf{y}}, \mathcal{E}_{\mathbf{z}}$ respectively. It also has three underlying double category structures; $\mathcal{E}_{\mathbf{xy}}$ denotes the double category of horizontal and vertical morphisms, and so on.

2.1 The "tensor product"

Let $\mathbb{J}$, $\mathbb{K}$ be two strict double categories.

This section defines a triple category which is a kind of "tensor product" of $\mathbb{J}$ with $\mathbb{K}$.

- The **objects** of $\mathbb{J} \otimes \mathbb{K}$ are all pairs (j, k) where j is an object in $\mathbb{J}$ and $k \in \mathbb{K}$.
- For its **transversal 1-cells**, a transversal morphism $(j, k) \to (j', k')$ is defined to be a pair of horizontal morphisms $g : j' \to j, f : k \to k'$; thus the transversal category $(\mathbb{J} \otimes \mathbb{K})_{\mathbf{z}}$ is just $\mathbb{J}_O^{\mathrm{op}} \times \mathbb{K}_O$.
- For its **horizontal 1-cells**, a horizontal morphism $(j, k) \to (j', k')$ exists only if $j = j'$, and is of the form $(1_j, w)$ where $w : k \to k'$ is a vertical arrow of $\mathbb{K}$.
- For its **vertical** 1-cells, a vertical morphism $(j, k) \to (j', k')$ exists only if $k = k'$ and is of the form $(v, 1_k)$ where $v : j \to j'$ is a vertical arrow of $\mathbb{J}$.
- For the **horizontal 2-cells**, the 2-cells of the double category $(\mathbb{J} \otimes \mathbb{K})_{\mathbf{xz}}$, if $f : j' \to j$ is a horizontal morphism in $\mathbb{J}$, and $\alpha : w \to w'$ is a 2-cell between vertical arrows w, w' in $\mathbb{K}$, then (f, α) is a 2-cell from the horizontal arrow $(1_j, w)$ to the horizontal arrow $(1_{j'}, w')$, lying over the transversal 1-cells $(f, \operatorname{dom} \alpha)$ and $(f, \operatorname{cod} \alpha)$.
- for the **vertical** 2-cells, the 2-cells of the double category $(\mathbb{J} \otimes \mathbb{K})_{\mathbf{yz}}$, if $g : k \to k'$ is a horizontal arrow in $\mathbb{K}$, and if $\beta : v' \to v$ is a 2-cell between vertical arrows v', v in $\mathbb{J}$, then (β, g) is a 2-cell from the vertical 1-cell $(v, 1_k) \to (v', 1_{k'})$ lying over the transversal 1-cells $(\operatorname{dom}(\beta), g)$ and $(\operatorname{cod} \beta, g)$.
- for the **basic** 2-cells, the 2-cells of the double category $(\mathbb{J} \times \mathbb{K})_{\mathbf{xy}}$, for every vertical arrow $v : j \to j'$ in $\mathbb{J}$ and every vertical arrow $w : k \to k'$ in $\mathbb{K}$, we have a single 2-cell through the square bounded by $(1_j, w)$ and $(1_{j'}, w)$ on top and bottom and $(v, 1_k)$ and $(v, 1_{k'})$ on left and right. We will refer to this 2-cell as (v, w) for short.
- for the **cubes**, a transversal cube from $(v, w) \to (v', w')$ is exactly a pair $\beta : v' \to v$ in $\mathbb{J}_A$, $\alpha : w \to w'$ in $\mathbb{K}_A$. The category of cubes under transversal composition can be identified with $\mathbb{J}_A^{\mathrm{op}} \times \mathbb{K}_A$.

We will omit checking the details here as they are straightforward.

Proposition 1. *The tensor product defines a functor* **Dbl** $\times$ **Dbl** $\to$ **Trpl**.

In particular, if $U : \mathbb{J} \to \mathbb{S}\mathrm{q}(\mathcal{C})$ and $V : \mathbb{K} \to \mathbb{S}\mathrm{q}(\mathcal{C})$ are given, there is a triple functor $U \otimes V : \mathbb{J} \otimes \mathbb{K} \to \mathbb{S}\mathrm{q}(\mathcal{C}) \otimes \mathbb{S}\mathrm{q}(\mathcal{C})$.

3 A Triple Category of Lifting Structures

Notation: For $\mathcal{C}$ a category, I denote by $\mathrm{dis}(\mathcal{C})$ the discrete category on the objects of $\mathcal{C}$.

I now define the **triple category of lifting data** $\mathrm{Lift}(\mathcal{C})$ associated to a 1-category $\mathcal{C}$, addressing the above structures cumulatively.

The $\mathrm{Lift}(\mathcal{C})$ will be constructed in a very similar way to $\mathbb{S}\mathrm{q}(\mathcal{C}) \otimes \mathbb{S}\mathrm{q}(\mathcal{C})$ so that it is equipped with a projection triple functor $\Pi : \mathrm{Lift}(\mathcal{C}) \to \mathbb{S}\mathrm{q}(\mathcal{C}) \otimes \mathbb{S}\mathrm{q}(\mathcal{C})$. In fact, $\mathrm{Lift}(\mathcal{C})$ will agree with $\mathbb{S}\mathrm{q}(\mathcal{C}) \otimes \mathbb{S}\mathrm{q}(\mathcal{C})$ on all but the the bullet points from section 2.1, except for the last 2. So $\mathrm{Lift}(\mathcal{C})$ and $\mathbb{S}\mathrm{q}(\mathcal{C}) \otimes \mathbb{S}\mathrm{q}(\mathcal{C})$ have the same set of objects, and they have identical horizontal, vertical and transversal 1-categories; they also have $\mathrm{Lift}(\mathcal{C})_{\mathbf{xz}} = \mathbb{S}\mathrm{q}(\mathcal{C}) \otimes \mathbb{S}\mathrm{q}(\mathcal{C})_{\mathbf{xz}}$ and $\mathrm{Lift}(\mathcal{C})_{\mathbf{yz}} = \mathbb{S}\mathrm{q}(\mathcal{C}) \otimes \mathbb{S}\mathrm{q}(\mathcal{C})_{\mathbf{yz}}$. However, the categories $\mathrm{Lift}(\mathcal{C})_{\mathbf{xy}}$ and $\mathbb{S}\mathrm{q}(\mathcal{C}) \otimes \mathbb{S}\mathrm{q}(\mathcal{C})_{\mathbf{xy}}$ (and thus a fortiori the cubes) will disagree in general.

3.1 Objects and 1-cells of $\mathrm{Lift}(\mathcal{C})$

To begin with, $\mathrm{Lift}(\mathcal{C})$ is a class of objects, together with three 1-category structures on that common class of objects.

- The **objects** or 0-cells of $\mathrm{Lift}(\mathcal{C})$ are ordered pairs of objects (a, b) of $\mathcal{C}$.
- For its **transversal 1-cells**, a transversal morphism $(a, b) \to (a', b')$ is defined to be a pair $g : a' \to a, f : b \to b'$; thus the transversal category $\mathrm{Lift}(\mathcal{C})_{\mathbf{z}}$ is just $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$.
- for its **horizontal 1-cells**, $\mathrm{Lift}(\mathcal{C})_{\mathbf{x}}$ is the product $\mathrm{dis}(\mathcal{C}) \times \mathcal{C}$; thus a morphism $(a, b) \to (a', b')$ exists only if $a = a'$, and in this case a morphism consists of a pair $(1_a, f)$ where $f \in \mathrm{Hom}_{\mathcal{C}}(b, b')$.
- for its **vertical 1-cells**, $\mathrm{Lift}(\mathcal{C})_{\mathbf{y}}$ is the product $\mathcal{C} \times \mathrm{dis}(\mathcal{C})$; a morphism $(a, b) \to (a', b')$ exists only if $b = b'$, and in this case, a morphism is exactly a pair $(g, 1_b)$ where $g \in \mathrm{Hom}_{\mathcal{C}}(a, a')$.

3.2 Horizontal and Vertical double categories in $\mathrm{Lift}(\mathcal{C})$

We address these separately as they are easier.

- The double category $\mathrm{Lift}_{\mathbf{xz}}(\mathcal{C})$ is flat; if we regard $\mathrm{dis}(\mathcal{C})$ as a subcategory of $\mathcal{C}^{\mathrm{op}}$ then there exists a (unique) 2-cell through a square iff the square commutes in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. This double category can be identified with $\mathbb{H}(\mathcal{C})^{\mathrm{op}} \times$

$\mathbb{S}\text{q}(\mathcal{C})$ (where the *horizontal* 1-cells of $\mathbb{H}(\mathcal{C})^{\text{op}} \times \mathbb{S}\text{q}(\mathcal{C})$ are identified with *transversal* 1-cells in $\text{Lift}_{\mathbf{xz}}$, and the *vertical* 1-cells of $\mathbb{H}(\mathcal{C})^{\text{op}} \times \mathbb{S}\text{q}(\mathcal{C})$ are identified with the *horizontal* 1-cells of $\text{Lift}_{\mathbf{xz}}$.

- The double category $\text{Lift}_{\mathbf{yz}}(\mathcal{C})$ is flat; if we regard $\text{dis}(\mathcal{C})$ as a subcategory of $\mathcal{C}$ then there exists a (unique) 2-cell through a square iff the square commutes in $\mathcal{C} \times \mathcal{C}$. This double category can be identified with $\mathbb{S}\text{q}(\mathcal{C})^{\text{op}} \times \mathbb{H}(\mathcal{C})$, where the **y**-axis is matched to the "vertical" direction and the **z**-axis is matched to the "horizontal" direction. (Note the distinction between $\mathbb{S}\text{q}(\mathcal{C}^{\text{op}})$ and $\mathbb{S}\text{q}(\mathcal{C})^{\text{op}}$; we only want to dualize the horizontal direction.)

We omit verification of the basic properties of a double category here as they are trivial.

3.3 The Basic double category $\text{Lift}_{\mathbf{xy}}(\mathcal{C})$

We already have the objects and horizontal and vertical cells, so it suffices to define the 2-cells and horizontal and vertical composition of 2-cells. Unlike $\text{Lift}_{\mathbf{xz}}(\mathcal{C})$ and $\text{Lift}_{\mathbf{yz}}(\mathcal{C})$, the double category $\text{Lift}_{\mathbf{xy}}(\mathcal{C})$ is not flat in general. Given a square of horizontal and vertical arrows in $\text{Lift}_{\mathbf{xy}}(\mathcal{C})$, if the square does not commute in $\mathcal{C} \times \mathcal{C}$, there are no 2-cells.

If the square does commute, then there are objects a, a', b, b' and morphisms $f : a \to a', g : b \to b'$ in $\mathcal{C}$ such that the commutative square is of the form $(f, 1_{b'}) \circ (1_a, g) = (1_{a'}, g) \circ (f, 1_b)$.

There exists a 2-cell only if the square commutes in $\mathcal{C} \times \mathcal{C}$. In this case, a 2-cell is precisely a lifting structure on f, g.

The *vertical* composition of 2-cells is precisely the left composition of lifting functions defined in 1. A pair of vertically composable basic squares in $\text{Lift}_{\mathbf{xy}}(\mathcal{C})$ is equivalent to a list of objects $a_0, a_1, a_2 b_0, b_1$ in $\mathcal{C}$, and $v_0 : a_0 \to a_1, v_1 : a_1 \to a_2, w =: b_0 \to b_1$ morphisms between them, and lifting functions $\rho \in \text{Lift}(v_0, w), \pi \in \text{Lift}(v, w)$ be 2-cells. Their left-composite as lifting functions is again a 2-cell in the square framed by $v_1 \circ v_0$ and w.

The *horizontal* composition of 2-cells is precisely the right composition of lifting functions 2.

The double category axioms are easy to verify.

In what follows, given $f : a_0 \to a_1, g : b_0 \to b_1$, we will identify an element of $\text{Lift}(f, g)$ with a 2-cell in $\text{Lift}_{\mathbf{xy}}(\mathcal{C})$ with vertical domain $(1_{a_0}, g)$, vertical codomain $(1_{a_1}, g)$, horizontal domain $(f, 1_{b_0})$, and horizontal codomain $(f, 1_{b_1})$; these 1-cells will not be spelled out explicitly when we refer to a 2-cell $\rho \in \text{Lift}(f, g)$, instead we will simply say "the square framed by f and g."

3.4 Cubes

The cubes of the triple category are again "flat", they exist (uniquely) if their boundary satisfies a coherence condition. Suppose we are given morphisms $a : a_0 \to a_1, b : b_0 \to b_1, s : s_0 \to s_1, t : t_0 \to t_1$; and suppose we are given a 2-cell $\rho \in \text{Lift}(a, b)$ lying over the square in the **xy** plane framed by f, g. Similarly, suppose we are given $\pi \in \mathrm{Lift}(s,t)$ lying over the square in the **xy** plane framed by s,t. Last, suppose we are given $r_0 : s_0 \to a_0, r_1 : s_1 \to a_1, q_0 : b_0 \to t_0, q_1 : b_1 \to t_1$, such that $a \circ r_0 = r_1 \circ s$ and $t \circ q_0 = q_1 \circ b$; in the formalism of double categories, this gives us that

- (r, q_0) is a 2-cell in $\mathrm{Lift}_{\mathbf{yz}}(\mathcal{C}) \cong \mathbb{S}\mathrm{q}(\mathcal{C})^{\mathrm{op}} \times \mathbb{H}(\mathcal{C})$ from (a, b_0) to (s, t_0);
- (r, t_1) is a morphism in $\mathrm{Lift}_{\mathbf{yz}}$ from (a, b_1) to (s, t_1);
- (r_0, q) is a 2-cell in $\mathrm{Lift}_{\mathbf{xz}}(\mathcal{C})$ from (a_0, t) to (s_0, t)
- (r_1, q) is a 2-cell in $\mathrm{Lift}_{\mathbf{xz}}(\mathcal{C})$ from (a_1, t) to (s_1, t)

and in sum, given (ρ, π, r, q), we have have a cube boundary.

Then this cube is inhabited by a (unique) filler iff for all $h : s_0 \to t_0$ and $k : s_1 \to t_1$ with $t \circ h = k \circ s$, we have

$$r_1 \circ \pi(h,k) \circ q_0 = \rho(r_0 \circ h \circ q_0, r_1 \circ k \circ q_1) \tag{3}$$

We will omit the checking of coherence conditions here. The fact that composition in each direction is well-defined is elementary to check; for the various coherence conditions, these are all trivially satisfied because the cube inhabiting the boundary is unique.

This concludes our argument that we have constructed a triple category.

There is an obvious triple functor $\Pi : \mathrm{Lift}(\mathcal{C}) \to \mathbb{S}\mathrm{q}(\mathcal{C}) \otimes \mathbb{S}\mathrm{q}(\mathcal{C})$ which collapses all 2-cells in $\mathrm{Lift}(\mathcal{C})_{\mathbf{xy}}$ down to the unique one, and similarly on cubes.

4 Applications

Proposition 2. *Let $U : \mathbb{J} \to \mathbb{S}\mathrm{q}(\mathcal{C}), V : \mathbb{J} \to \mathbb{S}\mathrm{q}(\mathcal{C})$ be double functors. Then a lifting structure on (U, V) in the sense of Section of 6.1 "Algebraic weak factorization systems I: Accessible awfs" of is exactly equivalent to a lift of $U \otimes V : \mathbb{J} \otimes \mathbb{K} \to \mathbb{S}\mathrm{q}(\mathcal{C}) \otimes \mathbb{S}\mathrm{q}(\mathcal{C})$ along $\Pi : \mathrm{Lift}(\mathcal{C}) \to \mathbb{S}\mathrm{q}(\mathcal{C}) \otimes \mathbb{S}\mathrm{q}(\mathcal{C})$.*