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Geometrical Gel′fand Models, Tensor Quotients, and Weil Representations

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This is a report on some partial results on different methods of construction of ordinary representations of finite classical groups and related groups. We are especially interested in elementary geometrical methods, which lead to the construction of geometrical Gel′fand models and to the decomposition of multiplicity-free natural (i.e., permutation) representations. We also consider, however, "meta-geometrical" methods as the construction of "contractions" and "tensor quotients" of representations of geometrical nature, which are closely related to the construction of Weil representations and their generalizations.

1. Geometrical Gel′fand Models

1.1. Given a (finite) group G, we call a *Gel′fand model* for G any (complex) representation (M, τ) of G which is isomorphic to the direct sum of all irreducible representations of G.

The underlying philosophy [**G–Z**] is that such a remarkable representation (M, τ) will have "special properties" and some "supplementary structure." This is well illustrated, in the case of compact Lie groups, by $G = \mathrm{SO}(3, \mathbb{R})$ for which a (unitary) Gel′fand model is given by $M = L^2(S^2)$, τ being the corresponding natural representation. The decomposition of this representation involves the classical Legendre polynomials (for a finite analogue see [**SA 1**]).

1.2. For finite (noncommutative) groups G there exists only exceptionally a G-set X such that the associated natural (also called permutation) representation $(L^2(X), \tau)$ is a Gel′fand model for G (here, as usual, we endow X with counting measure).

One such an exceptional example is given by $G = \mathrm{PSL}(2, \mathbb{F}_3)$ and $X = \mathbb{F}_9 - \mathbb{F}_3$ endowed with the homographic (Möbius) action of G. So X is a double covering

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of the finite analogue of Poincaré's upper half plane, which may be realized, in this case, as the set of vertices of the octahedron [**Y**].

More often, for classical groups G of low rank, for instance, we may expect that the following holds.

PROPERTY I. *There are G-sets $X_1, \ldots, X_r$ such that each natural representation $L^2(X_i)$ $(1 \leq i \leq r)$ is multiplicity-free and the amalgamated sum over the constants*

$$M = L^2(X_1) \underset{\mathbf{C}}{\oplus} \cdots \underset{\mathbf{C}}{\oplus} L^2(X_r)$$

is a Gel′fand model for our group G. We say then that M is a geometric Gel′fand model for G, of length r.

NOTATION. In what follows we will denote amalgamated sum over the constants by $\tilde{\oplus}$ instead of $\underset{\mathbf{C}}{\oplus}$.

A weaker version of Property I is

PROPERTY II. *The Gel′fand model M of G is a generalized natural representation.*

We say in this case that M is a *weakly geometric Gel′fand model for G*. This means in fact that M is the difference of two natural representations of G. In terms of characters, Property II states that the sum χ_M of all irreducible characters of G is a generalized permutation character.

REMARK. A given group G may admit geometrical Gel′fand models of different lengths (even in the simplest abelian noncyclic case of the Vierergruppe). We will call length of G and denote by $l(G)$ the minimum length of a geometrical Gel′fand model for G (if G admits no geometrical Gel′fand model we put $l(G) = \infty$).

It is of interest to determine which classes of groups admit geometrical (or weakly geometrical) Gel′fand models. We first give some positive results.

1.3 *The case of the Heisenberg group $H(n, k)$.* Let k be the field $\mathbb{F}_q$. Recall that the Heisenberg group $H(n, k)$ in n degrees of freedom is defined as the set $k^n \times k^n \times k$ endowed with the multiplication

$$(x, y; r)(z, w; s) = (x + z, y + w; r + s + \langle x, w \rangle)$$

for all $x, y, z, w \in k^n$, $r, s \in k$, where $\langle\ ,\ \rangle$ denotes the canonical scalar product in k^n.

PROPOSITION 1. *The Heisenberg group $H(n, k)$ admits a geometric Gel′fand model M of length $q + 1$, of the form*

$$M = \widetilde{\bigoplus_{t \in k \cup \{\infty\}}} L^2(X_t),$$

where the $H(n, k)$-sets X_t are defined as follows:

X_∞ is the set $k^n \times k$ endowed with the (right) action

$$(x, r) \cdot (y, z; s) = (x + y, r + s + \langle x, z \rangle)$$

for all $x, y, z \in k^n$, $r, s \in k$;

X_t $(t \in k)$ *is the set* k^n *endowed with the right action*

$$x \cdot (y, z; s) = x + ty + z,$$

for all $x, y, z \in k^n$, $s \in k$.

The proof is a straightforward verification. □

1.4. *The case of* $\mathrm{PGL}(2, \mathbb{F}_q)$: *Generic spaces and geometrical Gel'fand models.* Since, in general, the one-dimensional representations of a group G are easily obtained by decomposing the natural representation $L^2(G_{\mathrm{ab}})$ associated to its abelianized group G_{ab} regarded as a G-set, we are led to the following.

DEFINITION 1. *A G-set X is called a generic space for G iff*

(i) *every irreducible representation* π *of* G *with* $\dim \pi > 1$ *appears exactly once in the natural representation* $L^2(X)$ *of* G;

(ii) $L^2(X)$ *contains no one-dimensional representation of* G *besides the unit representation.*

THEOREM 1. *Let K denote the unique quadratic extension of our finite field k. Then a generic space for* $G = \mathrm{PGL}(2, k)$ *is the projective line* $\mathbb{P}_1(K)$, *endowed with the restriction to G of the usual projective action of* $\mathrm{PGL}(2, K)$. *In other words, a geometric Gel'fand model M for G is given by*

$$M = L^2(K - k)\tilde{\oplus}L^2(\mathbb{P}_1(k))\tilde{\oplus}L^2(k^\times/(k^\times)^2),$$

where G acts homographically on $K - k$, by the usual projective action on $\mathbb{P}_1(k)$, *and by the determinant* mod *squares on* $k^\times/(k^\times)^2$.

PROOF. The G-orbits in $\mathbb{P}_1(K)$ correspond to $K - k$ and $\mathbb{P}_1(K)$. The decomposition of $L^2(\mathbb{P}_1(k))$ gives the unit representation $\mathbf{1}$ and the Steinberg representation $\mathrm{St}_1^{(q)}$ associated to the character 1 of $k^\times$. The representation $L^2(K - k)$ is clearly multiplicity-free, since we have a *symmetric* invariant Δ (a "hyperbolic pseudo-distance") which classifies G-orbits in $(K - k) \times (K - k)$, to wit

$$\Delta(z, w) = N(z - w)/N(z - \bar{w}) \qquad (z, w \in K - k). \tag{1}$$

With the help of the character table of G or $\mathrm{GL}(2, k)$ (see [**PS**] or [**SA 2**]), one easily checks that $L^2(K - k)$ contains every irreducible representation of G with the exception of the Steinberg representation $\mathrm{St}_1^{(q)}$ and the nontrivial one-dimensional representation $\pi_\varepsilon^{(1)} = \varepsilon \circ \det$ in case q is odd (where ε then denotes the nontrivial character of $k^\times$ of square 1). Finally, the abelianized group $k^\times/(k^\times)^2$ of G, nontrivial only if q is odd, affords the missing representation $\varepsilon \circ \det$ in that case. □

REMARKS. (i) The G-space $K - k$ may be understood as a double covering of the finite analogue of Poincaré's upper half plane, which is obtained as the quotient of $K - k$ by the action of the Galois group.

(ii) We have

$$L^2(K - k)\tilde{\oplus}L^2(\mathbb{P}_1(k)) \simeq \mathrm{St}_1^{(q)} \otimes \mathrm{St}_1^{(q)} \simeq \operatorname*{Res}_{\mathrm{PGL}(2,K)\downarrow G} \mathrm{St}_1^{(q^2)},$$

where we denote by $\mathrm{St}_1^{(|k|)}$ the Steinberg representation of $\mathrm{PGL}(2,k)$, associated to the character 1 of $k^\times$, for a finite field k.

1.5. *The case of* $G = \mathrm{PGL}(2,k)$*: Spherical functions.*

1.5.1. We now turn to a brief description of the corresponding spherical functions for G. We concentrate on the generic orbit $X = K - k$ and we assume that q is odd. We then have $K = k(\theta)$ with $\theta^2 = t_0$, where t_0 is a fixed nonsquare in $k^\times$. We write $z = R(z) + \theta I(z)$ with $R(z), I(z) \in k$, for all $z \in K$. We choose θ as origin in X and put

$$C_r = \{z \in X \mid \Delta(z,\theta) = r\} \qquad (r \in k \cup \{\infty\},\ r \neq 1)$$

(recall that we define $\Delta(z,\bar{z}) = \infty$ and that $\Delta(z,w) \neq 1$ for all $z, w \in X$). We then have $|C_r| = q+1$ for $r \neq 0, \infty$ and $|C_0| = |C_\infty| = 1$. As usual, we give the spherical functions as functions on the range $(k - \{1\}) \cup \{\infty\}$ of the classifying invariant Δ.

THEOREM 1 [**GaZ**]. *The spherical functions of $X = K - k$ associated to the principal series of G are the functions $\phi_\alpha (\alpha \in (k^\times)^\wedge)$ given by*

$$\phi_\alpha(0) = 1, \tag{2}$$

$$\phi_\alpha(\infty) = \alpha(-1), \tag{3}$$

$$\phi_\alpha(r) = \frac{1}{q+1} \sum_{z \in C_r} \alpha(I(z)). \tag{4}$$

We have $\phi_\alpha = \phi_{\alpha^{-1}}$ $(\alpha \in (k^\times)^\wedge)$ and
(i) *ϕ_1 generates the unit representation,*
(ii) *ϕ_ε generates the Steinberg representation $\mathrm{St}_\varepsilon^{(q)}$ associated to the "sign character" ε of $k^\times$ which takes the value 1 on squares and -1 on nonsquares.*
(iii) *For $\alpha \neq \alpha^{-1}$ the spherical function ϕ_α generates the generic principal series representation $\pi_\alpha^{(q+1)}$ of G.*

The easiest way to prove Theorem 2 is to notice that the space $X = K - k$ may be endowed with the group structure of the one-dimensional affine group $k^\times \times k^+$ by the bijection $z \mapsto (I(z), R(z))$. Since this group structure is compatible with the action of G, its one-dimensional characters, i.e., the mappings $z \mapsto \alpha(I(z))$ $(z \in K - k)$, for $\alpha \in (k^\times)^\wedge$, will afford spherical functions by averaging on circles, as in (4). This approach is due to [**GaZ**]. Of course one can also check directly that ϕ_α's define characters of the corresponding commuting algebra. □

REMARK. Formula (4) is in fact the finite analogue of Harish-Chandra's description of all spherical functions in the real case. In the finite case, however, the spherical functions corresponding to cuspidal representations of G cannot be obtained in this way. We describe them below.

1.5.2. We first introduce some notation. Let U be the group of elements of norm 1 in the quadratic extension K of the finite field k. We define the epimorphism $\mathcal{U}\colon K^\times \to U$ by

$$\mathcal{U}(z) = zF(z)^{-1} \qquad (z \in K^\times), \tag{5}$$

where F denotes the Frobenius automorphism. We still denote by ε the "sign character" of $k^\times$ extended to k by the convention $\varepsilon(0) = 0$. On the other hand, we denote by ω_0 the "sign character" of U, equal to 1 on squares and to -1 on nonsquares of U.

THEOREM 3. *The spherical functions of X associated to the cuspidal representations of G are the functions ϕ_ω ($\omega \in \hat{U}$, $\omega \neq \omega^{-1}$) given by*

$$\phi_\omega(0) = 1, \tag{6}$$

$$\phi_\omega(\infty) = -\omega(-1), \tag{7}$$

$$\phi_\omega(r) = \frac{1}{q+1} \sum_{u \in U} \varepsilon\left(\mathrm{Tr}\left(u - \frac{1+r}{1-r}\right)\right) \omega_0\omega(u) \tag{8}$$

for $r \in k^\times - \{1\}$.

We have $\phi_\omega = \phi_{\omega^{-1}}$ for all ω and the spherical function ϕ_ω generates the cuspidal irreducible representation $\pi_\omega^{(q-1)}$ of G.

PROOF. Note that $X \simeq G/H$, where H is the subgroup of G consisting of all classes $[m_z]$ ($z \in K^\times$), with $m_z(w) = zw$ ($w \in K$). Therefore the spherical functions of X (w.r.t. the origin θ) may be obtained as H-spherical averages

$$\phi_\chi(x) = |H|^{-1} \sum_{h \in H} \chi(hg_x) \qquad (x \in X), \tag{9}$$

where $g_x \in G$ is such that $g_x \cdot \theta = x$ and χ is an irreducible character of G. Taking for χ a cuspidal character χ_ω ($\omega \in \hat{U}$, $\omega \neq \omega^{-1}$) and noting that

$$\Delta(x, \theta) = 1 - 4I(x)N(I(x) + 1 + R(x)t_0^{-1}\theta)^{-1}$$

for all $x \in X$, $x \neq -\theta$, one obtains from (9)

$$\phi_\omega(r) = \frac{-1}{q^2 - 1} \sum_{z \in K^\times} \sum_{\substack{x \in K^\times \\ \mathrm{Tr}(x) = \mathrm{Tr}(z) \\ N(x) = (1-r)N(z)}} \omega(\mathcal{U}(x)).$$

Since

$$\varepsilon(\mathrm{Tr}(x)^2 - 4(1-r)^{-1}N(x)) = \varepsilon\left(\mathrm{Tr}(\mathcal{U}(x)) - 2\frac{(1+r)}{(1-r)}\right) \varepsilon(N(x))$$

and $\varepsilon(N(x)) = \omega_0(\mathcal{U}(x))$ for all $x \in K^\times$, formula (8) follows. □

1.6. *Comments.*

(i) The generalization of Theorem 1 doesn't seem to be straightforward. For $G = \mathrm{PGL}(n, k)$ the best candidate to generalize $X = K - k$ would be G/H with $H = \mathrm{PGO}(Q)$, for a nondegenerate quadratic form Q on k^n of minimum Witt index. However, already for $n = 3$ and $n = 4$ there appear small multiplicities (up to 4) for the principal series of G, although cuspidal representations appear with multiplicity 1, for all nondegenerate Q.

(ii) Nongeometrical Gel'fand models have been constructed recently in [K] for $\mathrm{GL}(n, k)$, by extending the construction of Gel'fand-Graev representations.

(iii) $\mathrm{PSL}(2, \mathbf{F}_7)$ is a first example of a classical group which admits a *weakly geometrical* Gel'fand model, but no *geometrical* Gel'fand model (see [**Y**] for details).

(iv) The analogue of Property II is proved in [**P-SA**] for generalized p-adic rigid motion groups. These groups consist of all (affine) isometries of the nth order "pseudo-distance"

$$D(z, w) = N(z - w) \qquad (z, w \in K),$$

where K is now an abelian extension of degree n of the base field k and N is the corresponding norm map. The case of a finite base field k is dealt with in the forthcoming thesis [**Y**].

2. Contraction and tensor quotients

2.1. *Construction of representations by contraction.* Let G be a finite group and $\xi = (E, B, p, \eta^E, \eta^B)$ a complex G-vector bundle, where, as usual, E denotes the total space, B the base, p the projection from E onto B, and η^E (resp. η^B) the action of G on E (resp. B). We will often write just η for η^E and η^B to simplify notation. We then have

$$p(\eta_g(v)) = \eta_g(p(v)) \qquad (g \in G,\ v \in E).$$

We denote by E_b the fiber $p^{-1}(b)$ over $b \in B$.

To obtain a linear representation of G from ξ, by contraction over a base point $b_0 \in B$, one needs a flat $(G-)$ equivariant connection on ξ, in the following sense.

DEFINITION 2. *An equivariant connection on the complex G-vector bundle $\xi = (E, B, p, \eta)$ is a family of linear isomorphisms $\Gamma = \{\gamma_{b,a}\}_{a,b \in B}$ such that*

(i) *each $\gamma_{b,a}$ is a linear isomorphism from the fiber E_a onto the fiber E_b and every $\gamma_{a,a}$ is the identity;*

(ii) *we have*

$$\gamma_{\eta_g(b), \eta_g(a)} \circ \eta_g = \eta_g \circ \gamma_{b,a},$$

for all $a, b \in B$, $g \in G$,

(iii) *we have*

$$\gamma_{c,b} \circ \gamma_{b,a} = \mu_\Gamma(c, b, a) \gamma_{c,a},$$

for all $a, b, c \in B$, for a suitable function $\mu_\Gamma \colon B \times B \times B \to \mathbf{C}^\times$ called the multiplier of Γ.

We say that the connection Γ is flat iff its multiplier μ_Γ is the constant function 1.

In fact, an easy calculation shows that given an equivariant connection Γ on ξ we can contract ξ to a projective representation of G:

PROPOSITION 2. *Let us fix a base point $b \in B$ and suppose an equivariant connection Γ is given on the G-complex vector bundle ξ with base B. A projective representation (V, σ) of G, with multiplier*

$$\mu_\sigma(g, h) = \mu_\Gamma(b, \eta_g(b), \eta_{gh}(b)) \qquad (g, h \in G)$$

may be constructed as follows. Let $V = E_b$ and define σ by

$$\sigma_g(v) = \gamma_{b,\eta_g(b)}(\eta_g(v)) \qquad (g \in G,\ v \in V).$$

We call (V, σ) the representation of G obtained by contraction of ξ over b according to Γ. □

2.2. *Example: Geometric construction of small Weil representations for* $\mathrm{SL}(2,k)$. The classical "atomic" Weil representation of $\mathrm{SL}(2,k)$ (for a finite field k of odd order q) in $L^2(k)$ may be constructed geometrically by the contraction procedure.

To this end, define $\xi = (E, B, p, \eta)$ as follows. The base B is the set of all lines l through the origin in k^2. Fix a nontrivial character ψ of the additive group k^+ of the field k. For each $l \in B$ let H_l be the space of all complex functions f on k^2 such that

$$f(x+y) = \psi(x \wedge y) f(x) \qquad (x \in k^2,\ y \in l),$$

where we identify the exterior product $x \wedge y$ to the determinant of x and y. The total space E is then the disjoint union of all H_l ($l \in B$) and p is the canonical projection sending every $f \in H_l$ onto l. The group $G = \mathrm{SL}(2,k)$ obviously acts on ξ since it preserves determinants, by

$$(\eta_g^E f)(x) = f(g^{-1}(x)) \qquad (g \in G,\ f \in E,\ x \in k^2)$$

and

$$\eta_g^B(l) = g(l) \qquad (g \in G,\ l \in B).$$

Now an equivariant connection $\Gamma = \{\gamma_{l',l}\}_{l,l' \in B}$ is defined on B by $\gamma_{l,l} = \mathrm{Id}$ ($l \in B$) and

$$(\gamma_{l',l} f(x) = q^{-1/2} \sum_{y' \in l'} \psi(-x \wedge y') f(x + y') \tag{10}$$

for all $l, l' \in B$, $x \in k^2$, $f \in H_l$ (where $q = |k|$). Its nontrivial multiplier μ_Γ takes the value 1 on all triples (l, l', l'') such that $|\{l, l', l''\}| \le 2$ and it is given by

$$\mu_\Gamma(l, l', l'') = q^{-1/2} \sum_{x' \in l'} \psi(x'' \wedge x^\circ) \tag{11}$$

when $|\{l, l', l''\}| = 3$, where x° (resp. x'') stands for the component of $x' \in l'$ in l (resp. l'') with respect to the decomposition $k^2 = l \oplus l''$.

Notice that since any triple of distinct lines (l, l', l'') is equivalent modulo G to the triple (l_∞, l_t, l_0), where

$$l_0 = k(1,0), \qquad l_\infty = k(0,1), \qquad l_t = k(1,t),$$

we have

$$\mu_\Gamma(l, l', l'') = \mu_\Gamma(l_0, l_t, l_\infty) = q^{1/2} \sum_{s \in k} \psi(ts^2),$$

and we find the classical quadratic Gauss sum associated to ψ as the multiplier of the connection Γ.

An easy computation gives then the following explicit form of the projective representation σ of G obtained by contracting ξ over l_0 according to the connection Γ.

THEOREM 4 [**C-SA**]. *By sending each* f *in the space* H_{l_0} *of* σ *to the function* $f'\colon s \mapsto f(0,s)$ *on* k, *one defines an isomorphism from* (H_{l_0},σ) *to the representation* $(L^2(k),\sigma')$ *of* G, *where* σ' *is given on the generators*

$$h(a)=\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix},\qquad u(b)=\begin{pmatrix} 1 & b\\ 0 & 1\end{pmatrix},\qquad w=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$$

of G *by the formulas*
(i) $(\sigma'_{h(a)}f')(s)=f'(as)$,
(ii) $\sigma'_{u(b)}f')(s)=\psi(bs^2)f'(s)$,
(iii) $(\sigma'_w f')(s)=q^{-1/2}\sum_{r\in k}\psi(2rs)f'(r)$
for $f'\in L^2(k)$, $a\in k^\times$, $b,s\in k^\times$. $\square$

2.4. *Tensor division of representations.* Tensor division of a representation (W,τ) of G by another representation (U,ρ) of G may be approached as a special case of "contraction." The idea is that if we had $W=U\otimes V$, as representations of G, for some representation (V,σ) of G, replacing $U\otimes V$ by $\mathrm{Hom}_{\mathbf{C}}(U^*,V)$, one gets immediately a trivial G-bundle ξ with base any G-stable set B in U^* and fiber $\overline{W}_{u^*}=W/N_{u^*}$ over $u^*\in B$, where

$$N_{u^*}=\{\Phi\in W\mid \Phi(u^*)=0\},$$

endowed with the obvious flat connection. Conversely contraction may be sometimes looked on as "generalized tensor division."

More precisely, to succeed in tensor dividing (W,τ) by the contragredient $(U^*,\check{\rho})$ of (U,ρ) we will need the following ingredients:
(i) a G-stable set (for instance a G-orbit) B in U;
(ii) for every $u\in B$, a subspace N_u of W so that

$$\tau_g(N_u)=N_{\rho_g(u)}$$

for all $g\in G$, $u\in B$.

Let $\xi=(E,B,p,\eta^E,\eta^B)$ be defined as follows. Its total space E is the disjoint union of all quotients $\overline{W}_u=W/N_u$ ($u\in B$), the projection p sends each $\overline{W}_u$ onto $\{u\}$, the action η^B is the restriction of ρ to B, and the action η^E is given by

$$\eta^E_g(w+N_u)=\tau_g(w)+N_{\rho_g(u)}\in\overline{W}_{\rho_g(u)}$$

for all $g\in G$, $u\in B$, and $w+N_u\in\overline{W}_u$.

If, moreover, B spans U and ξ admits a *flat* equivariant connection Γ, we will say that the representation (V,σ) of G obtained by contraction of ξ over a chosen base point $u_0\in B$ according to Γ is a *generalized tensor quotient of* (W,τ) *by* (U,ρ). Of course this "quotient" will depend strongly on the choice of the connection Γ.

Clearly, under the above hypothesis, we have a natural intertwining operator ϕ from (W,τ) to the representation $(V^B,[\rho,\sigma])$ (where $[\rho,\sigma]_g(f)=\sigma_g\circ f\circ\rho_g^{-1}$, for $g\in G$, $f\colon B\to V$), given by

$$[\phi(w)](u)=\gamma_{u_0,u}(w+N_u).$$

Under a suitable supplementary hypothesis one can prove that ϕ induces an isomorphism from (W, τ) onto the representation $(\mathrm{Hom}_{\mathbf{C}}(U, v), [\sigma, \tau])$ of G. For instance, we have the following:

PROPOSITION 3. *With the above notation and hypothesis, we suppose furthermore that*

(i) *there exists a base B_0 of U, with $u_0 \in B_0 \subset B$, such that*

$$\bigcap_{u \in B_0} N_u = \{0\} \quad \text{and} \quad \left(\bigcap_{u' \in B_0 - \{u\}} Nu' \right) + N_u = W,$$

for all $u \in B_0$;

(ii) *the mapping $u \mapsto \gamma_{u_0,u}(w + N_u)$ from B to $V = W/N_{u_0}$ is linear for every $w \in W$.*

Then the mapping Φ, which sends each $w \in W$ to the unique linear extension f_w to all of U of the section $u \mapsto \gamma_{u_0,u}(w + N_u)$ $(u \in B)$, is an isomorphism from the representation (W, τ) to the representation $(\mathrm{Hom}_{\mathbf{C}}(U, V), [\rho, \sigma])$. □

This proposition is used in **[C-SA]** to construct the principal series representations $\pi_\alpha^{(q+1)}$ $(\alpha \in (k^\times)^\wedge,\ \alpha \neq \alpha^{-1})$ of $G' = \mathrm{SL}(2, k)$ as tensor quotients of the induced representations $\mathrm{Ind}_{T \uparrow G'} \alpha$, where T is the split torus of all diagonal matrices in G'.

2.4. *Weil representations of $G = \mathrm{GL}(2, k)$ as generalized tensor quotients.* The construction in 2.2 can easily be extended to afford the Weil representations of G associated to nondegenerate quadratic planes (which give by decomposition all the irreducible representations of G **[SA 2]** or, more generally, the Weil representations of G associated to nondegenerate quadratic spaces of even dimension.

To this end, let M be an even-dimensional vector space over the finite field $k = \mathbf{F}_q$ (q odd) and $\mathbf{B}$ a nondegenerate symmetric bilinear form on M. The base B of our G-vector bundle ξ will be, as before, the set of all lines l through the origin of k^2. To each $l \in B$, described as the locus of all $x = (x_1, x_2) \in k^2$ such that $rx_1 + sx_2 = 0$ for suitable $r, s \in k$, we associate the subspace M_l of $M^2 = M \oplus M$ consisting of all $x = (x_1, x_2) \in M^2$ such that $rx_1 + sx_2 = 0$. Call Ξ the set of all nontrivial characters ψ of k^+. The fiber E_l over $l \in B$ will be the space of all functions $f\colon M^2 \times \Xi \to \mathbf{C}$ such that

$$f(x + y, \psi) = \psi(\mathbf{A}(x, y)) f(x, \psi)$$

for all $x \in M^2$, $\psi \in \Xi$, and $y \in M_l$, where $\mathbf{A} = \mathbf{A_B}$ denotes the alternating form on M^2 defined by

$$\mathbf{A}(x, y) = \mathbf{B}(x_1, y_2) - \mathbf{B}(x_2, y_1)$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2)$ in M^2.

The group G acts on the total space E (disjoint union of the fibers E_l) by

$$(\eta_g^E f)(x, \psi) = f\left(g^{-1}(x), \psi^{\det(g)}\right) \tag{12}$$

for all $g \in G$, $f \in E$, $x \in M^2$, $\psi \in \Xi$, where we put $\psi^t(s) = (\psi(ts)$ for $s, t \in k^+$. As before $\eta_g^B(l) = g(l)$ for $g \in G$, $l \in B$.

With this set up we have the following.

THEOREM 5. *The G-vector bundle ξ admits a flat equivariant connection $\Gamma = \{\gamma_{l',l}\}_{l,l' \in B}$ given by $\gamma_{l,l} = \mathrm{Id}$ $(l \in B)$ and*

$$(13) \qquad (\gamma_{l',l} f)(x,\psi) = \varepsilon_{\mathbf{B}} |M|^{-1/2} \sum_{y \in M_{l'}} \psi(-A(x,y)) f(x+y)$$

for all $l, l' \in B$, $l \neq l'$, $f \in M_l$, $x \in M^2$, $\psi \in \Xi$, where $\xi_{\mathbf{B}}$ stands for the sign of $\mathbf{B}$ (equal to 1 if $\mathbf{B}$ is an orthogonal sum of hyperbolic planes and to -1 otherwise). Moreover, the representation (V, σ) obtained by contracting ξ over $l_0 = k(1,0)$ according to Γ is the Weil representation of G associated to the quadratic space $(E, \mathbf{B})$ in the sense of [**SA 2**, *Chapter* I].

PROOF. To check that the multiplier μ_Γ of Γ is 1 reduces, after an easy calculation, to checking that

$$(14) \qquad \sum_{y' \in M_{l'}} \psi(\mathbf{A}(y'' \wedge y^\circ)) = \varepsilon_{\mathbf{B}} |M|^{1/2} \qquad (\psi \in \Xi),$$

where, as before, y'' (resp. y°) denotes the component of y' in $M_{l''}$ (resp. M_l) according to the decomposition $M^2 = M_{l''} \oplus M_l$ (for $|\{l, l', l''\}| = 3$). But since (l, l', l'') is equivalent to (l_∞, l_t, l_0) modulo G, we obtain as in 2.2 that the left-hand side of (14) is equal to

$$\sum_{y_1 \in M} \psi(t\mathbf{B}(y_1, y_1)),$$

from which (13) follows immediately. Finally, by realizing V as $L^2(M)$, as in 2.2, comparison with Chapter I of [**SA 2**] completes the proof. □

REMARKS. (i) This construction is in fact the finite analogue of the differential geometric construction of Blattner-Kostant-Souriau-Sternberg in the real case (see [**L-V**]).

(ii) Notice that the orthogonal similarity group $\Gamma = \mathrm{GO}(\mathbf{B})$ of $\mathbf{B}$ acts naturally on $M^2 \times \Xi$ by$((x_1, x_2), \psi) \mapsto ((x_1 h, x_2 h), \psi^{m_h^{-1}})$, where m_h is the multiplier of $h \in \Gamma$. From here we finally get an action of Γ on V commuting with σ, as we should (see [**SA 2**], Chapter I).

(iii) Let us introduce the space $W = L^2(M^2 \times \Xi)$ on which G acts linearly by (12). The spaces E_l $(l \in B)$ may be thought of as quotients of W via the projections P_l defined by

$$(P_l f)(x, \psi) = \sum_{y \in M_l} \psi(-\mathbf{A}(x,y)) f(x+y)$$

for all $f \in W$, $x \in M^2$, $\psi \in \Xi$. Identifying each $l \in B = \mathbb{P}_1(k)$ with the delta function $\delta_l \in L^2(\mathbf{P}_1(k))$, we see that *the Weil representations of G associated to even-dimensional nondegenerate quadratic spaces $(M, \mathbf{B})$ appear as generalized tensor quotients of the natural representation* (12) *of G in $L^2(M^2 \times \Xi)$ by the natural representation of G in $L^2(\mathbb{P}_1(k))$.*

2.5. *Problem: Construction of generalized Weil representations by contraction or (generalized) tensor division.* It would be interesting to recover by the above procedures the "generalized Weil representation" of $G = \mathrm{GL}(3,k)$ associated to the unique cubic extension K of k. This representation, call it (V,ρ), was constructed in [**SA 3**] from Kirillov models. We recall briefly its definition.

Denote by N (resp. Tr) the norm (resp. trace) of K over k, call U the kernel of N, and put $E(z) = \mathrm{Tr}(z\bar{z})$ for $z \in K$, where $z \mapsto \bar{z}$ is the Frobenius automorphism. Let $G_2 = \mathrm{GL}(2,k)$, put

$$u_2(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \qquad (s \in k),$$

and, for $x = (x_1, x_2) \in k_*^2 = k^2 - \{0\}$, $t \in k^\times$, put

$$a(x,t) = \begin{pmatrix} 0^- & tx_1^{-1} \\ x_1 & x_2 \end{pmatrix} \quad \text{if } x_1 \neq 0,$$
$$a(x,t) = \begin{pmatrix} tx_2^{-1} & 0 \\ 0 & x_2 \end{pmatrix} \quad \text{if } x_1 = 0.$$

Moreover, introduce the following elements of G:

$$h(a,b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \qquad (a \in G_2,\ b \in k^2),$$
$$w_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Finally, fix a nontrivial character ψ of k^+.

Then the space V consists of all functions f from $K^\times \times G_2$ to $\mathbf{C}$ such that

$$f(z, u_2(s)a) = \psi(sN(z))f(z,a) \qquad (z \in K^\times,\ s \in k^+,\ a \in G_2)$$

and

$$\sum_{u \in U} f(uz, a) = 0.$$

The action ρ is given on a set of generators of G by

$$(\rho_r f)(z,a) = f(r^{-1}z, \mathrm{diag}(r^3, 1)a) \qquad (r \in Z(G) = k^\times),$$
$$(\rho_{h(1,b)} f)(z,a) = \psi((ab)_2) f(z,a) \qquad (b \in k^2),$$
$$(\rho_{h(a,0)} f)(z,a) = f(aa') \qquad (a' \in G_2),$$

$$\begin{aligned}(\rho_{w_{23}} f)&(z, a(x,t)) \\ &= q^{-3} \sum_{\substack{w \in K^\times \\ y \in k_*^2 \\ y_1 \neq 0}} J_0(x_1^{-2} y_1^{-2} t N(x_1 w - y_1 z)) \psi(y_1 x_1^{-1} x_2 + x_1 y_1^{-1} y_2) f(w, a(y,t)) \\ &\quad + q^{-2} \sum_{\substack{w \in K^\times \\ y \in k_*^2, y_1 \neq 0 \\ N(z') y_2^{-2} = -x_1 t^{-1} N(z)}} \psi(-t x_1^{-1} y_2^{-1} E(w \bar{z} \bar{\bar{z}}) f(w, a(y,t))\end{aligned}$$

if $x_1 \neq 0$, and

$$(\rho_{w_{23}} f)(z, a(x,t)) = q^{-2} \sum_{\substack{w \in K^\times \\ y_2 \in k^\times}} \psi(x_2 \mathrm{Tr}(wz^{-1})) f(z, a((tN(z)x_2^{-2}, y_2), t))$$

if $x_1 = 0$, for all $f \in V$, $z \in K^\times$, $x \in k_*^2$, $t \in k^\times$. Here the Bessel function J_0 is defined by

$$J_0(t) = \sum_{\substack{r,s \in k \\ rs = t}} \psi(r - s) \qquad (t \in k^\times),$$

$$J_0(0) = -(q+1).$$

For each $\Omega \in (K^\times)^\wedge$, $\Omega \neq \Omega^q$, we have an irreducible component V_Ω of (V, ρ) consisting of all $f \in V$ such that

$$f(wz, \mathrm{diag}(N(z)^{-1}, 1)a) = \overline{\Omega}(w) f(z, a) \qquad (z, w \in K^\times,\ a \in G_2).$$

In this way we obtain all cuspidal irreducible representations of G, with the only nontrivial isomorphisms $V_\Omega \simeq V_{\Omega^q} \simeq V_{\Omega^{q^2}}$, arising from the natural action of the Galois group Γ of K over k; in fact, we have a perfect multiplicity-free matching with the representations of the orthogonal similarity group $\mathrm{GO}(N) \simeq K^\times \rtimes \Gamma$ of the cubic norm N, as in the quadratic case.

We hope that obtaining this representation in a simpler way, by contraction or (generalized) tensor division, will facilitate its extension to $\mathrm{GL}(n, k)$ and other classical groups.

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