The triple category of categories

Christian Williams

University of California, Riverside

$\mathbb{C}\mathrm{at}$ has been understood as a double category: categories and functors, profunctors and transformations. Categories and profunctors are monads and bimodules in $\mathrm{Span}(\mathrm{Set})$: a profunctor from $\mathbb{X}$ to $\mathbb{A}$ is a span of sets ob$\mathbb{X} \leftarrow \mathcal{P} \rightarrow$ ob$\mathbb{A}$, giving for each pair of objects a set of "morphisms" $\mathcal{P}(\mathrm{X}, \mathrm{A})$, with a "precompose" action by $\mathbb{X}$ and a "postcompose" action by $\mathbb{A}$.

Yet there is also the richer notion of *two-sided fibration* [5]. The arrow category of $\mathbb{X}$ is a double category, i.e. a monad $\mathbb{X} \leftarrow \vec{\mathbb{X}} \rightarrow \mathbb{X}$ in $\mathrm{Span}(\mathrm{Cat})$, and a two-sided fibration is a bimodule of arrow categories: $\mathbb{X} \leftarrow \mathcal{R} \rightarrow \mathbb{Y}$ gives for each pair a category of morphisms and 2-morphisms $\mathcal{R}(\mathrm{X}, \mathrm{Y})$, with precomposition by squares of $\vec{\mathbb{X}}$ and postcomposition by $\vec{\mathbb{Y}}$. Categories and two-sided fibrations form a tricategory.

We present a *triple category* of categories, in which the three kinds of 1-cell are functor, profunctor, and two-sided fibration. The three kinds of 2-cell are transformation, fibered functor, and *fibered profunctor* — a novel structure, connecting two-sided fibrations along profunctors, which we now define.

Just as arrow categories are monads in $\mathrm{Span}(\mathrm{Cat})$, a profunctor $\mathbb{X} \leftarrow \mathcal{P} \rightarrow \mathbb{A}$ gives an "arrow profunctor" which is a monad in $\mathrm{Span}(\mathrm{Prof})$: a span of profunctors, the apex consisting of commutative squares $\vec{\mathcal{P}}(\mathrm{x}, \mathrm{a}) = \{(p_0 : \mathcal{P}(\mathrm{X}_0, \mathrm{A}_0), p_1 : \mathcal{P}(\mathrm{X}_1, \mathrm{A}_1)) \mid \mathrm{a} \circ p_0 = p_1 \circ \mathrm{x}\}$; composition is that of squares.

$$
\begin{array}{ccccc}
\mathbb{X} & \longleftarrow & \vec{\mathbb{X}} & \longrightarrow & \mathbb{X} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{P} & \longleftarrow & \vec{\mathcal{P}} & \longrightarrow & \mathcal{P} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{A} & \longleftarrow & \vec{\mathbb{A}} & \longrightarrow & \mathbb{A}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{X}_0 & \xrightarrow{\ \mathrm{x}\ } & \mathrm{X}_1 \\
{\scriptstyle p_0}\downarrow & p & \downarrow{\scriptstyle p_1} \\
\mathrm{A}_0 & \xrightarrow{\ \mathrm{a}\ } & \mathrm{A}_1
\end{array}
$$

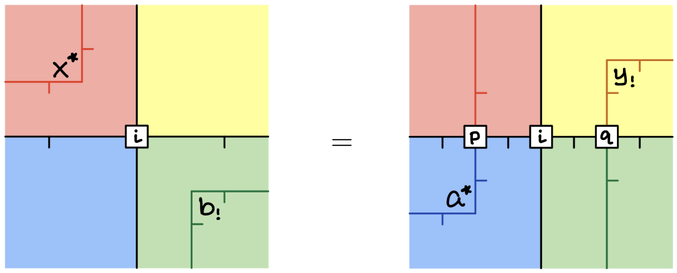
Definition. Let $\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}$ be categories, let $\mathbb{X} \leftarrow \mathcal{P} \rightarrow \mathbb{A}$ and $\mathbb{Y} \leftarrow \mathcal{Q} \rightarrow \mathbb{B}$ be profunctors, and let $\mathbb{X} \leftarrow \mathcal{R} \rightarrow \mathbb{Y}$ and $\mathbb{A} \leftarrow \mathcal{S} \rightarrow \mathbb{B}$ be two-sided fibrations. A **fibered profunctor** over $\mathcal{P}, \mathcal{Q}$ from $\mathcal{R}$ to $\mathcal{S}$ is a profunctor $\mathcal{R} \leftarrow \mathcal{I} \rightarrow \mathcal{S}$ spanning from $\mathcal{P}$ to $\mathcal{Q}$, which is a $\vec{\mathcal{P}}, \vec{\mathcal{Q}}$-bimodule in $\mathrm{Span}(\mathrm{Prof})$.

$$
\begin{array}{ccccc}
\mathbb{X} & \longleftarrow & \mathcal{R} & \longrightarrow & \mathbb{Y} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{P} & \longleftarrow & \mathcal{I} & \longrightarrow & \mathcal{Q} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{A} & \longleftarrow & \mathcal{S} & \longrightarrow & \mathbb{B}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{X} & \xrightarrow{\ r\ } & \mathrm{Y} \\
{\scriptstyle p}\downarrow & \overset{\|}{\underset{\Downarrow}{i}} & \downarrow{\scriptstyle q} \\
\mathrm{A} & \xrightarrow{\ s\ } & \mathrm{B}
\end{array}
$$

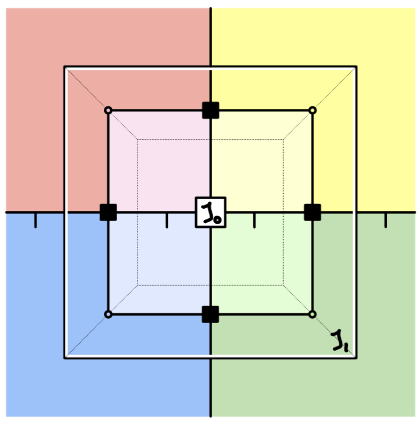
Just as a fibration is a matrix of categories $\mathcal{R}(\mathrm{X}, \mathrm{Y})$, a fibered profunctor is a matrix of profunctors $\mathcal{I}(p, q)$: for each $p : \mathcal{P}(\mathrm{X}, \mathrm{A})$ and $q : \mathcal{Q}(\mathrm{Y}, \mathrm{B})$ there is a profunctor $\mathcal{R}(\mathrm{X}, \mathrm{Y}) \leftarrow \mathcal{I}(p, q) \rightarrow \mathcal{S}(\mathrm{A}, \mathrm{B})$ of "squares"; the actions of $\mathcal{R}$ and $\mathcal{S}$ give "sequential" composition, while the actions of $\mathcal{P}$ and $\mathcal{Q}$ give "parallel" composition.

Note that the structure of $\mathcal{I}$, as a bimodule in $\mathrm{Span}(\mathrm{Prof})$, includes a bimodule structure on $\mathcal{R}$ and $\mathcal{S}$, which may be distinct from those given by assuming $\mathcal{R}$ and $\mathcal{S}$ to be two-sided fibrations. This distinction is immaterial: because arrow categories are "completion" monads, i.e. lax idempotent [5], the bimodule structure of each span is unique up to unique isomorphism.

Intuitively, a fibered profunctor can be understood by its *collage*. A profunctor is a union of categories along a set of morphisms; in the same way, a fibered profunctor is a union of two-sided fibrations along a set of squares, forming a *double category with companions*. The base category is the union of the collages of $\mathcal{P}$ and $\mathcal{Q}$, the hom-category is the union of the collages of $\vec{\mathcal{P}}$ and $\mathcal{J}$ and $\vec{\mathcal{Q}}$, and the actions of $\mathcal{R}$, $\mathcal{J}$, and $\mathcal{S}$ define horizontal composition. The actions of $\mathcal{R}$ and $\mathcal{S}$ are canonically natural with respect to those of $\mathcal{J}$.



So, the naïve view of profunctor elements as morphisms expands to two dimensions: two-sided fibrations give horizontal morphisms, and fibered profunctors give squares. This makes a powerful visual language: fibered profunctors can be drawn as beads, and their transformations can be drawn as *beads within beads*.



A fibered transformation $\mathcal{J}_0 \to \mathcal{J}_1$.

We extend the language of string diagrams [2] to triple categories. Above, the back and front face are fibered profunctors, left and right are transformations, top and bottom are fibered functors, and the cube is a **fibered transformation**: a bimodule transformation $\mathcal{J}_0 \to \mathcal{J}_1$ in $\mathrm{Span}(\mathrm{Prof})$.

Theorem. Categories, functors, profunctors, and two-sided fibrations form a triple category $\mathbb{C}\mathrm{at}_\Omega$.

Composition of two-sided fibrations is associative up to isomorphism, and unital up to equivalence.

Using string diagrams as a template, the language of $\mathbb{C}\mathrm{at}_\Omega$ is three-dimensional, combining the co/end calculus of profunctors and the "co/descent calculus" of two-sided fibrations. It is a rich universe, in which category theory and fibered category theory are unified.

References

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