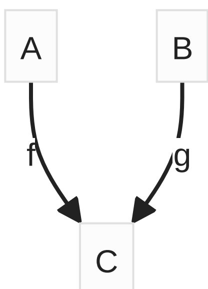
Categorical products

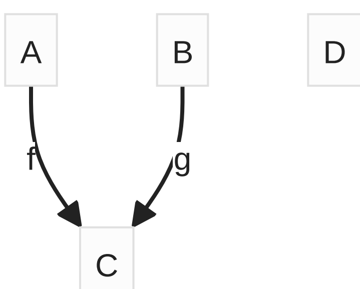
Consider a category $\mathbf{C}$. Now consider two objects in the category, A and B.

Consider a third object, C.

Consider the case where there happens to be an arrow from A to C, and from B to C.



Consider that there is some fourth object D.

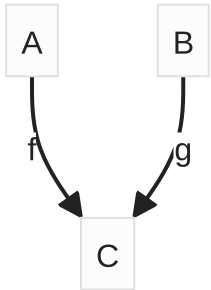


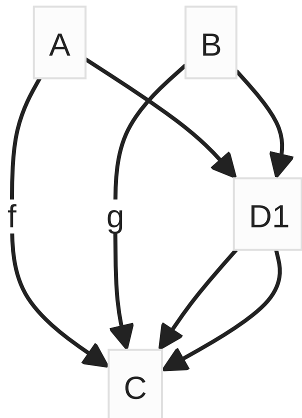
I am going to assume that all this applies even when $A = B = C = D$, for example, which I'll explore later.

The idea of a "product" is that D acts as a "middleman", in the $(A \to C) \wedge (B \to C)$ diagram.

I like inventing new terminology to test my intuition, so I will call this object a "middleman" instead, for now.

A middleman for A and B can always come between A and B and something they both "go to".





An important thing in category theory is that distinct paths between points X and Y do not count as “the same journey” unless explicitly acknowledged so - there is an equivalence relation, but it does not necessarily apply to all paths with the same origin and destination.

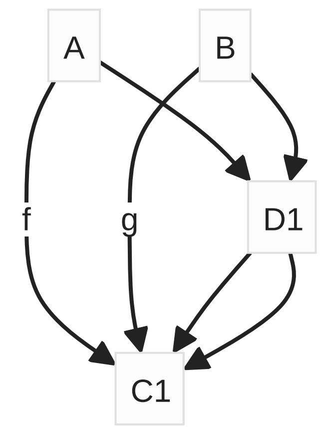
So we have to state that the path from A to $D1$ to C *counts as the same* as the path from A to C.

There are some variations and specific situations here to consider.

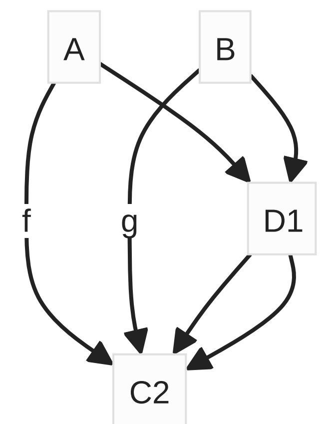
What do we *want*, from this “middleman”?

The first thing is that if A and B have a middleman, that middleman is *always there*, whenever A and B have a shared ‘target’.

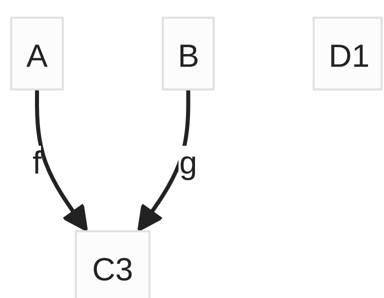
Imagine there is some collection of objects that is “the objects that both A and B have an arrow to”. It includes objects $C1$, $C2$, $C3$, …, from the category.



A true middleman is *always there*, to ‘intercept’ those paths.



If there is any “shared object” (an object both A and B map to) that, for some reason, the “middleman” can’t intercept, then it is not a true middleman.



A middleman is defined as being able to intercept the paths to *all* “shared objects”.

In the above scenario, why can’t $D1$ intercept the path? There could be multiple reasons. If there is no arrow from A to $D1$ at all, then the path from $A \to D1 \to C$ can’t be constructed, at all. Similarly, if there is a path from A to $D1$, but not from $D1$ to C, there is again, no way to get there.

But there is also the case where those paths exist, they just aren’t *equivalent* (as mentioned above).

Why is this middleman useful or interesting?

The general intuition for a middleman is that it “stands for” or “represents” A and B. In a way, it “holds all their information”. I don’t really know, but all I can say for now is, I almost feel like the middleman has a similar role to “naming things”, because to some extent it allows us to refer to multiple things under the denotation reference of a single thing.

One last thing is that if two objects turn out to have a middleman (you checked all their “shared objects” and found there is some object that is a middleman to all of them), there is actually only one possible middleman. According to the way equality of objects is defined in category theory, we would consider all suggested instances of a middleman to count as the “same thing”.

Why? I actually don’t know. I’ll think about it.

My motivation is to show categorically what the collection of all strings over an alphabet “looks like”, in a diagram. My next point of thought is, how are products related to sequences, categorically? A sequence, in set theory, is a bijection from an ordinal set to another set. I need to think about in what ways they are the same or different.