

Representable polygraphs: a combinatorial-topological approach to higher-dimensional algebra and rewriting

Research Proposal

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The aim of this project is to create new bridges between algebraic topology and higher-dimensional algebra and rewriting.

Newly discovered connections between these fields have driven some of the most fruitful research programmes, and deepest advances in mathematics, computer science, logic, and theoretical physics of the last decades. To name a few: the Baez-Dolan cobordism hypothesis [5] with its outline proof by Lurie [36], and the new perspectives it gave to topological quantum field theory [20, 44]; the path-space interpretation of identity types in Martin-Löf type theory [31, 3], leading to the Univalent Foundations programme in the foundations of mathematics [48]; the string-diagrammatic description of quantum algorithms and protocols [1, 49], fuelling the categorical quantum mechanics programme [16]; in algebraic logic and proof theory, the discovery of proof nets [21] and topological correctness criteria for linear logic proofs [17, 40]; and the modern developments in homotopical and homological conditions for computational properties of rewriting systems, within the framework of higher-dimensional rewriting theory [34, 43, 24].

Many of these connections can be traced back to the observation that higher categories [35] — a natural setting for higher-dimensional algebra and rewriting — generalise higher groupoids [11], which, under the homotopy hypothesis [4], correspond to homotopy types. The formal analogy is most striking in the notion of polygraph [12, 41], also known as computad [47], which underlies modern higher-dimensional rewriting theory: a higher category is built by progressively adjoining generating cells, whose border is a pasting of lower-dimensional cells; this is the same as a combinatorial description of a CW complex [39], with “directed cells” replacing the undirected, topological cells. Which compels the question: is it possible to frame higher-dimensional algebra and rewriting as a kind of *directed algebraic topology* [22], working directly with generalisations of topological methods and constructions, and with a version of polygraphs as the chosen notion of *directed space*?

In my paper [27], I argued that the principle of compositionality, essential to basic techniques of algebraic topology (Seifert-van Kampen, Mayer-Vietoris, monoidality of homology functors), has a use in universal algebra and rewriting theory based on polygraphs. I showed that directed versions of topological compositions of spaces capture in great generality such notions as homomorphisms and actions, and the interactions of monoids and comonoids that lead to the theories of Frobenius algebras and of bialgebras. This was informed by my own work on a string-diagrammatic theory of qubits [25], which showed hints of such a decomposition, and by an alternative, non-topological approach to the composition of algebraic theories [33, 10].

However, before polygraphs can be seen as a satisfactory notion of directed space, there are at least two obstacles to overcome:

1. The standard version of polygraphs, based on the algebraic machinery of globular ω -categories [46], suffers from several technical inadequacies, such as the lack of a natural notion of subspace, and the difficulty of describing products. These can be traced back to the fact that general polygraphs do not form a presheaf category [38].
2. The higher categories that are usually of interest in homotopy theory and theoretical physics are weak, that is, satisfy associativity, unitality, and interchange

constraints only “up to coherent homotopies”, while polygraphs present strict ω -categories, known to be a strictly less general notion [45].

In the first part of this project, I intend to pursue a notion of directed space that keeps what works with polygraphs, while overcoming the issues: *representable polygraphs*.

1. My approach to the “regularisation” of polygraphs would have them locally modelled by suitably directed versions of regular CW decompositions of topological disks. The latter notion has a fully combinatorial characterisation in the undirected, topological case [9, 8]; with directed cells, alternative characterisations may become available. This is meant to provide a notion of directed space which is natively globular, yet, like cubical sets [2, 23], allows for easy calculations of products. To my knowledge, this is a previously unexplored application of poset topology [42, 32] to higher category theory.
2. My approach to weakness is based on *coherence via universality* [30], where coherent algebraic structure is subsumed by the existence of universally characterised elements in a wider structure (representability). Representability in this sense provides systematic methods for generating higher coherence equations, and comes with abstract, generalisable proof techniques, as in Hermida’s proof of strictifiability of monoidal categories [29]. Moreover, it has deep connections with proof theory; I am currently writing a paper on the subject with Alex Kavvos [28], expanding on [15, 14].

Related work includes the opetopic definition of higher categories [13] — with its type-theoretic implementation in the *Opetopic* proof assistant [19, 18] — which shares the coherence-via-universality approach, but does not contemplate compositionality, and focusses on a restricted kind of representability. On the other hand, the *Globular* proof assistant, with the associated “quasistrict” categories [6, 7], shares the string-diagrammatic inspiration, favouring many-input, many-output cells, yet follows the orthogonal path of building “minimal” coherent algebraic definitions from the ground up.

The success of this part of the project relies crucially on the interplay of the two aspects — regularity and representability — between each other and with the compositional structure. In my PhD thesis [26], I present a working definition of regular polygraphs, together with indications that, mixed with representability, it could suffice to capture general higher categories.

In the second, open-ended part of this project, I intend to develop the basic tools of directed algebraic topology for representable polygraphs, building on what has been done for different notions of directed space [22], and test them on concrete examples with particular attention to rewriting-theoretic problems, such as homotopical criteria for convergence. Other directions include abstract, compositional proofs of coherence theorems in higher-dimensional algebra, such as the celebrated theorem of Mac Lane about monoidal categories [37], and general proofs of semi-strictifiability of homotopy types, such as a version of Simpson’s conjecture [45].

This project proposal builds on well-established ideas, that have separately been successful in various fields, but have not been considered in this combination before. It is influenced by my experience in the uniquely assorted environment of the Oxford Quantum Group, with Bob Coecke, Samson Abramsky, Jamie Vicary, and others, allowing me to focus both on abstract aspects of higher category theory and computational logic, and concrete problems in higher-dimensional algebra with applications to quantum theory. At RIMS in Kyoto, it would greatly benefit from the advice and expertise of Masahito Hasegawa on categorical aspects of logic and computation, and from the environment of the Computer Science group, a leading hub of logic in computer science.

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