Categorical Probability and the de Finetti theorem

Tobias Fritz

joint work with Tomáš Gonda, Paolo Perrone and Eigil Rischel

March 2021

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now with proof!

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References

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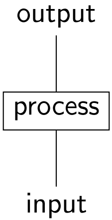
For a broader perspective, see the videos from the online workshop Categorical Probability and Statistics!

Why categorical probability?

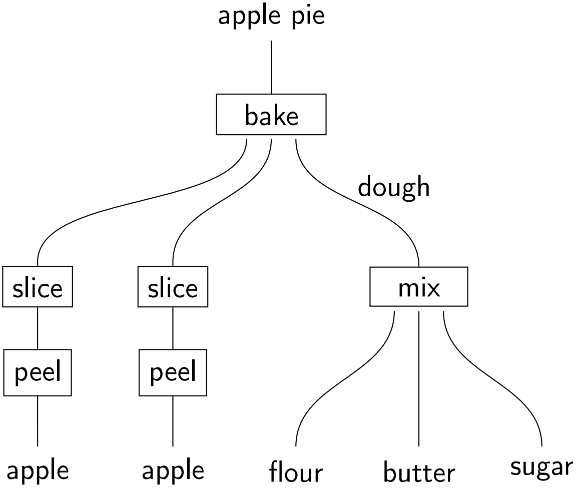
In no particular order:

- Applications to probabilistic programming.
- Prove theorems in greater generality and with more intuitive proofs.
- Reverse mathematics: sort out interdependencies between theorems.
- Ultimately, prove theorems of higher complexity?
- Simpler teaching of probability theory. (String diagrams!)
- Different conceptual perspective on what probability is.

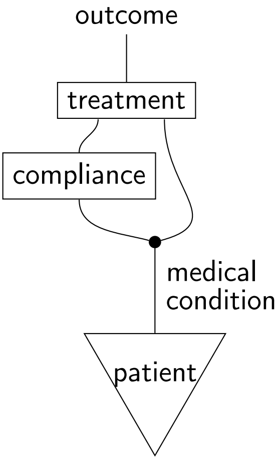
Many of us like to reason about processes in terms of string diagrams:



We can compose processes into networks:



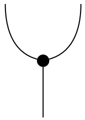
Suppose that we want to reason about **flow of information** in a medical trial. Then we seem to need diagrams like this:



→ Medical condition has an influence on **both** trial compliance and on treatment outcome!

Hence a theory of information flow needs additional pieces of structure:

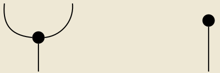
- ▷ **copying information**:



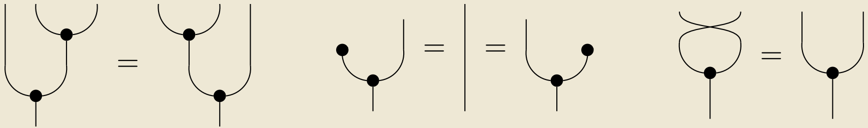
- ▷ **deleting information**:

Definition

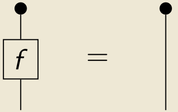
A **Markov category C** is a symmetric monoidal category supplied with **copying** and **deleting** operations on every object,



giving commutative comonoid structures



which interact well with the monoidal structure, and such that



(much can be done without)

A basic example

One of the paradigmatic Markov categories is **FinStoch**, the category of finite sets and **stochastic matrices**: a morphism $f : X \to Y$ is

$$(f(y|x))_{x\in X, y\in Y} \in \mathbb{R}^{X\times Y}$$

with

$$f(y|x) \geq 0, \qquad \sum_y f(y|x) = 1.$$

Composition is the **Chapman-Kolmogorov formula**,

$$(gf)(z|x) := \sum_y g(z|y)\, f(y|x).$$

A morphism $p : 1 \to X$ is a **probability distribution**.

A general morphism $X \to Y$ has many names: **Markov kernel**, probabilistic mapping, communication channel, ...

The monoidal structure implements **stochastic independence**,

$$(g \otimes f)(xy|ab) := g(x|a)\, f(y|b).$$

The copy maps are

$$\mathrm{copy}_X : X \longrightarrow X \times X, \qquad \mathrm{copy}_X(x_1, x_2|x) = \begin{cases} 1 & \text{if } x_1 = x_2 = x, \\ 0 & \text{otherwise.} \end{cases}$$

The deletion maps are the unique morphisms $X \to 1$.

- Works just the same with "probabilities" taking values in any **semiring** R.

- Taking R to be the **Boolean semiring** $\mathbb{B} = \{0, 1\}$ with

$$1 + 1 = 1$$

  results in the Kleisli category of the nonempty finite powerset monad.

  $\Rightarrow$ We get a Markov category for non-determinism.

- Measure-theoretic probability: Kleisli category of the **Giry monad**.

Outline

In the rest of this talk, I will sketch:

- How to develop (some) theorems of probability theory in terms of Markov categories.
- This includes work in progress on the **de Finetti theorem** characterizing permutation-invariant probability distributions.
- There is a vast landscape of Markov categories, going much beyond probability theory.

We're just at the beginning!

Analogy with topos theory

- Every constructive piece of mathematics holds in every topos, but there still are many toposes!

  Similarly, every theorem of probability proven in terms of Markov categories holds quite generally.

- There is a hierarchy of additional axioms of different strength.

Bold working hypothesis:

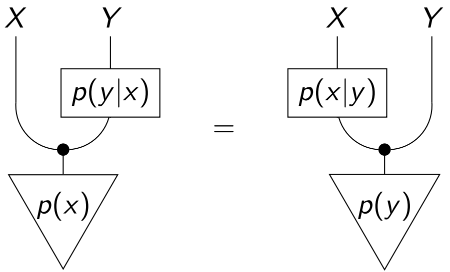
$$
\begin{aligned}
& \text{Theory of Markov categories} \\
\cong\ & \text{General theory of information flow} \\
\cong\ & \text{Generalized probability theory and statistics.}
\end{aligned}
$$

Indeed, for example **Bayesian networks** can be defined in Markov categories (Brendan Fong's 2013 MSc).

In other words, Markov categories are a general setting for talking about cause and effect.

A first theoretical development: Bayesian inversion

Bayes' rule takes the form:



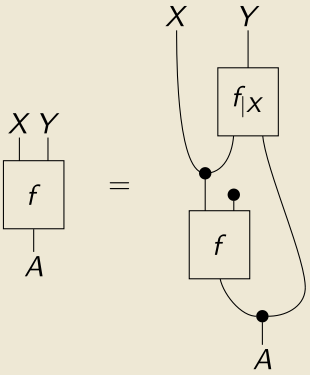
$$p(y|x)\,p(x) \quad = \quad p(x|y)\,p(y)$$

In any Markov category **with conditionals**, there is a sense in which the map $p(y|x) \mapsto p(x|y)$ is a **dagger functor**!

More generally:

Definition

C has conditionals if for $f : A \to X \otimes Y$ there is $f_{|X} : X \otimes A \to Y$ with

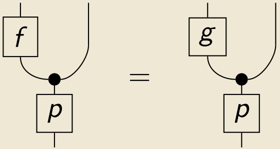


Almost sure equality

Definition

Let $p : A \to X$ and $f, g : X \to Y$.

f and g are **equal** p**-almost surely**, $f =_{p\text{-a.s.}} g$, if

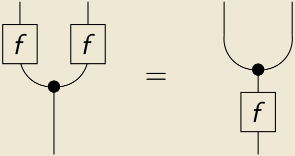


- **Intuition:** f and g behave the same on all inputs produced by p.
- Other concepts (besides equality) also relativize with respect to p-almost surely.

Determinism

Definition

In a Markov category, a morphism $f : X \to Y$ is **deterministic** if it commutes with copying,

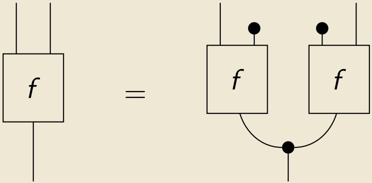


- **Intuition:** Applying f to copies of input = copying the output of f.
- The deterministic morphisms form a cartesian monoidal subcategory.

Conditional independence

Definition

In a Markov category, $f : A \to X \otimes Y$ **displays the conditional independence** $X \perp Y \parallel A$ if



Kleisli categories are Markov categories

Proposition

Let

- ▷ **D** be a category with finite products,
- ▷ P a commutative monad on **D** with $P(1) \cong 1$.

Then the Kleisli category $\mathrm{Kl}(P)$ is a Markov category in the obvious way.

Examples:

- ▷ Kleisli category of the Giry monad, other related monads for measure-theoretic probability.
- ▷ Kleisli category of the non-empty power set monad, which is (almost) **Rel**.

The proposition still holds when **D** is merely a Markov category itself!

Markov categories as Kleisli categories

Definition

A Markov category $\mathbf{C}$ is **representable** if for every $X \in \mathbf{C}$ there is $PX \in \mathbf{C}$ and a natural bijection

$$\mathbf{C}_{\mathrm{det}}(-, PX) \cong \mathbf{C}(-, X).$$

Then P turns out to extend to a commutative monad on $\mathbf{C}_{\mathrm{det}}$, and $\mathbf{C}$ is its Kleisli category!

The counit of the induced comonad is

$$\mathsf{samp}_X : PX \to X,$$

the map that returns a random sample from a distribution.

Detour: random measures

- Suppose that I hand you a coin (which may be biased).
- How much would you bet on the outcome

$$\text{heads}, \quad \text{tails}, \quad \text{tails}$$

when the coin is flipped 3 times?

$\Rightarrow$ Surely the same as you would bet on

$$\text{tails}, \quad \text{tails}, \quad \text{heads}.$$

- If μ is the distribution of your belief on the coin's bias, you would go for odds better than

$$\int p(\text{heads})\, p(\text{tails})^2\, \mu(dp).$$

So p is a **random measure** with distribution μ.

Classical de Finetti theorem

A sequence $(x_n)_{n\in\mathbb{N}}$ of random variables on a space X is **exchangeable** if their distribution is invariant under finite permutations σ,

$$\begin{aligned} &\mathbb{P}\big[x_1 \in S_{\sigma(1)}, \ldots, x_n \in S_{\sigma(n)}\big] \\ = \;&\mathbb{P}[x_1 \in S_1, \ldots, x_n \in S_n]\,. \end{aligned}$$

Theorem

If (x_n) is exchangeable, then there is a measure μ on PX such that

$$\mathbb{P}[x_1 \in S_1, \ldots, x_n \in S_n] = \int p(x_1 \in S_1) \cdots p(x_n \in S_n)\, \mu(dp).$$

Idea: sequence of tosses of a coin with unknown bias!

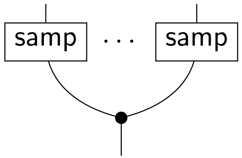
Synthetic de Finetti theorem

Assumption: **C** is an a.s. compatibly representable Markov category with conditionals and countable Kolmogorov products.

Definition

$f : A \to X^{\mathbb{N}}$ is **exchangeable** if it is invariant under composing with finite permutations.

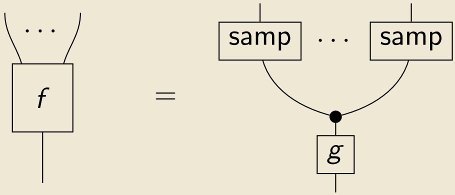
Sampling $\mathbb{N}$ times gives a morphism $PX \to X^{\mathbb{N}}$ given by



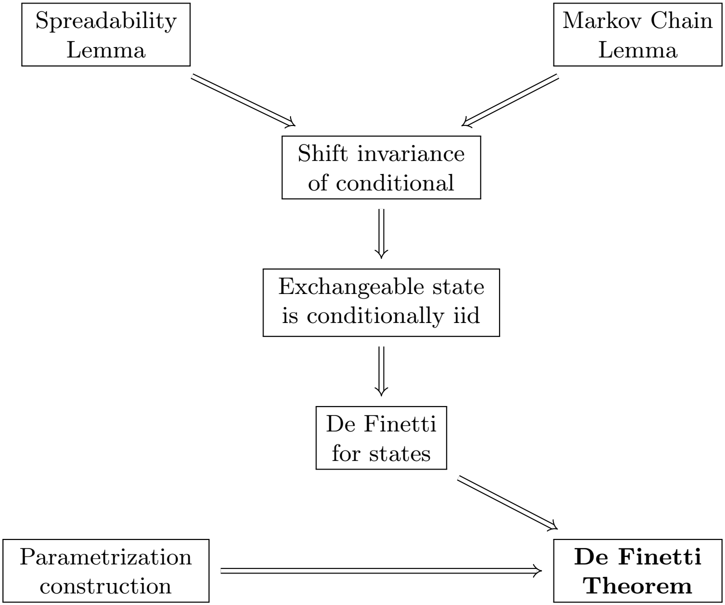
Synthetic de Finetti theorem

Theorem

For every exchangeable $f : A \to X^{\mathbb{N}}$ there is $g : A \to PX$ such that



Structure of proof



Categories of comonoids

Proposition

Let **C** be any symmetric monoidal category. Then the category with:

- Commutative comonoids in **C** as objects,
- Counital maps as morphisms,
- The specified comultiplications as copy maps,

is a Markov category.

A good example is $\mathbf{Vect}_k^{\mathrm{op}}$ for a field k:

- The comonoids correspond to commutative k-algebras of k-valued random variables.
- We obtain **algebraic probability theory** with "random variable transformers" as morphisms (formal opposites of Markov kernels).

Diagram categories and ergodic theory

Proposition

Let $\mathbf{D}$ be any category and $\mathbf{C}$ a Markov category. The category in which

- Objects are functors $\mathbf{D} \to \mathbf{C}_{\mathrm{det}}$,
- Morphisms are natural transformations with components in $\mathbf{C}$.

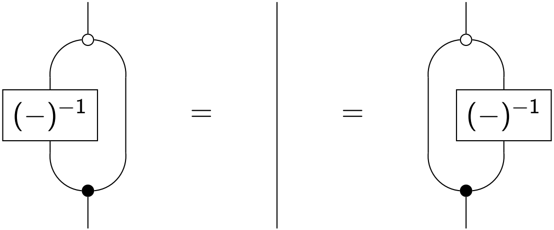
With the poset $\mathbf{D} = \mathbb{Z}$, we get a category of **discrete-time stochastic processes**.

This generalizes an observation going back to (Lawvere, 1962).

We can also take $\mathbf{D} = \mathbf{B}G$ for a group G, resulting in categories of dynamical systems with deterministic dynamics but stochastic morphisms.

Hyperstructures: categorical algebra in Markov categories

A **group** G is a monoid G together with $(-)^{-1} : G \to G$ such that



This equation can be interpreted in any Markov category! (Together with the bialgebra law.)

- ▷ More generally, one can consider models of any algebraic theory in any Markov category.

- ▷ In Kleisli categories of probability-like monads, these are known as **hyperstructures**.

- ▷ Peter Arndt's suggestion:

  Develop categorical algebra for hyperstructures in terms of Markov categories!

Summary

- Markov categories are an emerging formalism providing a general theory of information theory.
- Many qualitative results of probability theory generalize to Markov categories.
- These usually require additional axioms (of various degrees of strength).
- There is a vast unexplored landscape of Markov categories in which these results can be instantiated.
- This is similar to topos theory: a lot of mathematics can be developed constructively and then instantiated in an unexpectedly large number of contexts.

Some further directions

- Is there a "most convenient" Markov category **C** for measure-theoretic probability?

  Some desiderata:

  - **C** has conditionals.
  - **C** has Kolmogorov products.
  - **C** has supports.
  - $\mathbf{C}_{\mathrm{det}}$ is cartesian closed.

  I don't know of *any* non-cartesian Markov category with these properties!

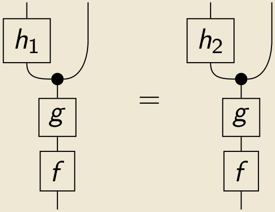
- Many results in probability theory are quantitative.

  $\Rightarrow$ **Do we need enriched Markov categories?**

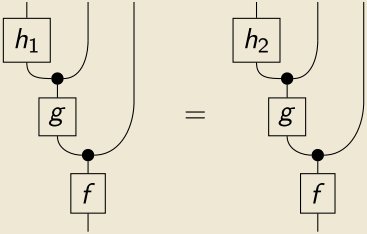
The causality axiom

Definition

C is **causal** if



implies

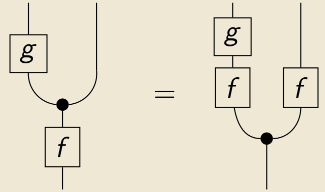


- **Intuition:** The choice between h_1 and h_2 in the "future" of g does not influence the "past" of g.
- Not every Markov category is causal.

The positivity axiom

Definition

C is **positive** if whenever gf is deterministic for composable f and g, then also



- **Intuition:** If a deterministic process has a random intermediate result, then that result can be computed independently from the process.
- Not every Markov category is positive.
- Dario Stein: every causal Markov category is positive!

Definition

Let $(X_i)_{i \in I}$ be a family of objects. The **infinite tensor product**

$$X_I := \bigotimes_{i \in I} X_i$$

is the cofiltered limit of the finite tensor products $X_F := \bigotimes_{i \in F} X_i$, if this limit exists and is preserved by every $- \otimes Y$.

Definition

An infinite tensor product X_I is a **Kolmogorov product** if the limit projections $\pi_F : X_I \to X_F$ are deterministic.

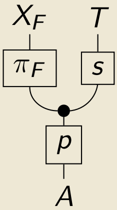
- This additional condition fixes the comonoid structure on X_I.

Theorem (Kolmogorov zero–one law)

Let X_I be a Kolmogorov product of a family $(X_i)_{i \in I}$.

If

- $p : A \to X_I$ makes the X_i independent and identically distributed, and
- $s : X_I \to T$ is such that



  displays $X_F \perp T \,||\, A$ for every finite $F \subseteq I$,

then ps is deterministic.

The classical Hewitt–Savage zero-one law

Theorem

Let $(x_n)_{n\in\mathbb{N}}$ *be independent and identically distributed random variables, and* S *any event depending only on the* x_n *and invariant under finite permutations.*

Then $\boldsymbol{P}(S) \in \{0, 1\}$.

The synthetic Hewitt–Savage zero-one law

Theorem

Let J be an infinite set and $\mathbf{C}$ a causal Markov category. Suppose that:

- The Kolmogorov power $X^{\otimes J} := \lim_{F \subseteq J \text{ finite}} X^{\otimes F}$ exists.
- $p : A \to X^{\otimes J}$ displays the conditional independence $\perp_{i \in J} X_i \parallel A$.
- $s : X^J \to T$ is deterministic.
- For every finite permutation $\sigma : J \to J$, permuting the factors $\tilde{\sigma} : X^{\otimes J} \to X^{\otimes J}$ satisfies

$$\tilde{\sigma} p = p, \qquad s\tilde{\sigma} = s.$$

Then sp is deterministic.

Proof is by string diagrams, but far from trivial!