

# Operads with restriction

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## 1 Introduction

Let  $C$  be a small multicategory. A functor  $X : C \rightarrow \mathbf{Set}$  associates to each  $c$  in  $C$  a set  $X(c)$ , and to each multimorphism  $f : c_1, \dots, c_n \rightarrow c$  an associated function  $X(f) : X(c_1) \times \dots \times X(c_n) \rightarrow X(c)$ .  $C$  can be seen as an algebraic theory with many sorts and  $n$ -ary function symbols, and  $X$  a model of the algebraic theory. This perspective has been useful for the study of coherence theorems in algebraic structures. For the sake of this paper, we will understand a “coherence theorem” for a generalized multicategory  $TA \leftarrow X \rightarrow A$  to mean a theorem which explicitly computes some or all of the fibers of the map  $X \rightarrow A \times TA$ , or describes explicitly what the composition law of the multicategory looks like, particularly when the generalized multicategory is presented by generators and relations. For example, the monoidal category operad, which is freely generated by  $\otimes$ ,  $I$ , and the associator and unitor isomorphisms, is equivalent to the terminal operad in  $\mathbf{Cat}$ , whose objects and morphisms in each degree can be explicitly enumerated (there is one object and one morphism in each degree.)

The paper is organized as follows.

- We observe that some common category-theoretic structures, such as categories, functors, natural transformations, and adjunctions, can be characterized as the algebras of certain generalized multicategories (more specifically, of certain virtual double categories)
- We observe that for the special case of adjunctions, this formalism does not quite allow us to prove all coherence theorems that we are interested in. We observe that for virtual double categories, adding restrictions to the virtual double category is enough to state and prove the theorems that we are interested in.
- Motivated by this, we extend the concept of virtual category with chosen restrictions to a generalized multicategory with chosen restrictions, and define what it means for an algebra for the generalized multicategory to respect this additional structure.
- We investigate further examples (???)

## 2 Virtual double categories, case study of adjunctions

Consider the category **Graph** equipped with the free category monad **FreeCat**. In this setting, it is possible to construct generalized multicategories (also known as virtual double categories) whose algebras are, respectively,

- small categories
- triples  $(C, D, F : C \rightarrow D)$ , where  $C, D$  are small categories and  $F$  is a functor between them
- tuples  $(C, D, F : C \rightarrow D, G : C \rightarrow D, \tau : F \Rightarrow G)$
- adjunctions  $(C, D, F, G, \eta, \epsilon)$

Each of these generalized multicategories is useful in part because it provides a natural setting to speak about coherence theorems.

For example, we might define  $\mathcal{O}_{\mathbf{Cat}}$  to be the operad freely generated by the identity symbol in degree 1 and the composition symbol in degree 2, quotiented out by the associativity and left/right unitality axioms for morphism composition. Then it is a coherence theorem to prove that  $\mathcal{O}_{\mathbf{Cat}}$  is isomorphic to the terminal operad in  $(\mathbf{Graph}, \mathbf{FreeCat})$ , and the other two operads have their own associated coherence theorems.

For adjunctions, we will go into some more detail about what is meant, presenting a virtual double category by generators and relations. The virtual double category  $\mathcal{O}_{\mathbf{Adj}}$  has two objects, called  $C, D$  respectively. Its category of vertical arrows is the category freely generated by  $F : C \rightarrow D$  and  $G : D \rightarrow C$ . It has two horizontal arrows,  $\text{Hom}_C : C \rightarrow C$  and  $\text{Hom}_D : D \rightarrow D$ . The 2-cells are generated under composition by the following:

- a “composition” 2-cell,  $\circ : \text{Hom}_C \otimes \text{Hom}_C \rightarrow \text{Hom}_C$ , with left and right boundary  $1_C$ ; and similarly for  $D$
- a “unit” 2-cell  $e : \square_C \rightarrow \text{Hom}_C$  with left and right boundary  $1_C$ , where  $\square_C$  refers to the empty path of horizontal morphisms at  $C$ ; similarly for  $D$
- left/right unitality constraints,  $\circ \cdot [e, 1_{\text{Hom}_C}] = 1_{\text{Hom}_C} = \circ \cdot [1_{\text{Hom}_C}, e] : \text{Hom}_C \rightarrow \text{Hom}_C$ , with left and right boundary  $1_C$ ; and similarly for  $D$
- associativity constraints relating the left-associative and right-associative composition maps  $\text{Hom}_C^{\otimes 3} \rightarrow \text{Hom}_C$ , with left and right boundary  $1_C$ , and similarly for  $D$
- a 2-cell  $F^{arr} : \text{Hom}_C \rightarrow \text{Hom}_D$  with left and right boundary  $F$ ; similarly with  $G^{arr}$
- functoriality constraints  $F^{arr} \cdot [\circ_C] = \circ_F \cdot [F^{arr}, F^{arr}]$  and  $F^{arr} \cdot e = e \cdot \square_F$  (where  $\square_F$  is the empty list of 2-cells with left and right boundary  $F$ ); similarly for  $G$
- A unit map  $\eta : \square_C \rightarrow \text{Hom}_C$  with left boundary  $1_C$  and right boundary  $GF$
- A counit map  $\varepsilon : \square_D \rightarrow \text{Hom}_D$  with left boundary  $FG$  and right boundary  $1_D$
- zigzag equations stating that  $\circ(F^{arr} \cdot \eta, \varepsilon \cdot \square_F) = e \cdot \square_F$  and  $\circ(\eta \cdot \square_G, G^{arr} \cdot \varepsilon) = e \cdot \square_G$

Then it is a coherence theorem to explicitly characterize the 2-cells in this virtual double category. For example, the following results are called coherence theorems.

- there is a unique 2-cell  $\text{Hom}_C^{\otimes n} \rightarrow \text{Hom}_C$  riding the identity vertical cells  $(1_C, 1_C)$  for each natural number  $n$ . One can restate this by saying that there is a homomorphism from the virtual double category  $\mathcal{O}_{\text{Cat}} \rightarrow \mathcal{O}_{\text{Adj}}$  sending the single object of  $\mathcal{O}_C$  to  $C$ , and this map is fully faithful - no new ways of composing morphisms are introduced, and no two existing ways of composing morphisms are identified. The same is true for  $D$ .
- There is a unique 2-cell  $\text{Hom}_C^{\otimes n} \rightarrow \text{Hom}_D$  riding the vertical 1-cells  $(F, F)$  for every  $n$ . Equivalently, there is a homomorphism of virtual double categories  $\mathcal{O}_{\text{Fun}} \rightarrow \mathcal{O}_{\text{Adj}}$  sending the unique non-identity vertical 1-cell of  $\mathcal{O}_{\text{Fun}}$  to  $F$ , and this homomorphism is fully faithful. The same is true for  $G$ .
- The 2-cells  $\square_C \rightarrow \text{Hom}_C$  riding  $(GF)^n$  and  $(GF)^m$  respectively are in bijection with the morphisms in the augmented simplex category from  $n$  to  $m$ , and the composition in the augmented simplex category corresponds to the composition given by  $f, g \mapsto (\circ)_C \cdot [f, g]$ ; a dual theorem holds for maps  $\square_D \rightarrow \text{Hom}_D$  riding  $(FG)^n$  and  $(FG)^m$ .

One motivation for this paper is the observation that not all operations which are definable in an arbitrary adjunction are expressible within this operad, and thus not all coherence relations between definable operations are expressible. For example, we regard it as a “coherence theorem” that there is only one definable operation  $\text{Hom}(x, Gy) \rightarrow \text{Hom}(x, Gy)$  up to provable equality. However, this cannot be stated in our theory. We will solve this by taking the formal completion of our virtual double category “with all restrictions.”

### 3 Formal completion of a VDC with restrictions

Let  $\mathcal{A}$  be a virtual double category. We will describe a construction that freely adjoins restrictions to  $\mathcal{A}$ . One could consider a split or non-split version of the construction; we will define a split version.

If  $f : x_1 \rightarrow y_1, g : x_2 \rightarrow y_2$  are vertical 1-cells and  $h : y_1 \rightarrow y_2$  is a horizontal 1-cell, we introduce a new formal 2-cell  $\chi(f, g, h)$  whose left and right boundary are  $f, g$  respectively, whose lower boundary is  $h$  and whose upper boundary is a new horizontal 1-cell  $\kappa(f, g, h)$ . We also introduce a formal splitting map  $\sigma$ ;  $\sigma$  accepts vertical morphisms  $f_1 : x_1 \rightarrow x_2, f_2 : x_2 \rightarrow x_3, g_1 : y_1 \rightarrow y_2, g_2 : y_2 \rightarrow y_3$ , and horizontal 1-cell  $h : x_3 \rightarrow y_3$  and a 2-cell  $s$  whose left and right boundary are  $f_2 \circ f_1, g_2 \circ g_1$  respectively, with lower boundary  $h$ , and returns a new formal 2-cell whose upper boundary agrees with that of  $s$ , whose left and right boundary are  $f_1, f_2$  respectively, and whose lower boundary is  $\kappa(f_2, g_2, h)$ ; we impose the equational constraints that the composite  $\chi(f_1, g_1, \kappa(f_2, g_2, h)) \circ \sigma(f_1, f_2, g_1, g_2, h, s) = s$ , and for any 2-cell  $s'$  with left and right boundary  $f_1, g_1$  respectively, and whose lower boundary is of the form  $\kappa(f_2, g_2, h)$ , the splitting  $\sigma(\chi(f_2, g_2, h) \cdot s') = s'$ . These constraints force  $\chi(f, g, h)$  to be a Cartesian 2-cell.

In order that the cleavage is a splitting, We impose equational constraints that the formally Cartesian cells should respect vertical identity and composition in the evident way, i.e.,

- $\chi(\text{id}_{x_1}, \text{id}_{x_2}, \text{Hom}_C(s_1, s_2))$  is the identity 2-cell on  $\text{Hom}_C(s_1, s_2)$ , and
- $\chi(t_1 \circ r_1, t_2 \circ r_2, \text{Hom}_C(s_1, s_2)) = \chi(t_1, t_2, \text{Hom}_C(s_1, s_2)) \circ \chi(r_1, r_2, \text{Hom}_C(s_1 \circ t_1, s_2 \circ t_2))$ .

This construction is universal among split VDC's and homomorphisms preserving the splitting.

Let **Set** refer to the virtual double category such that

- the objects are sets
- vertical morphisms are functions
- a horizontal 1-cell from  $X$  to  $Y$  is defined to be a family of sets  $\{R_{x,y}\}$  indexed by  $x \in X, y \in Y$
- given a chain of horizontal 1-cells  $\{P_{x,y}^1\}_{x \in A_0, y \in A_1}, \dots, \{P_{x,y}^n\}_{x \in A_{n-1}, y \in A_n}$ , and  $\{Q_{b_0, b_1}\}_{b_0 \in B_0, b_1 \in B_1}$ , and given functions  $f : A_0 \rightarrow B_0$  and  $g : A_n \rightarrow$

$B_n$ , a 2-cell is a function  $h$  that associates to each  $n+1$ -tuple  $(a_0, \dots, a_n)$  and each  $n$ -tuple  $(p_1, \dots, p_n)$  with  $p^i$  in  $P_{a_{i-1}, a_i}^i$ , an element of  $Q_{f(a_0), g(a_n)}$ .

Then **Set** is a split VDC, where for  $f : X_1 \rightarrow X_2, g : Y_1 \rightarrow Y_2, \{Q_{x,y}\}_{x \in X_2, y \in Y_2}$ , we define  $\kappa(f, g, Q)_{x_1, y_1} = Q_{(f(x_1), g(y_1))}$  and  $\chi(f, g, Q)_{x, y}$  as the identity function.

It follows that every functor of virtual double categories  $\mathcal{O}_{\mathbf{Adj}} \rightarrow \mathbf{Set}$  extends uniquely to a functor of virtual double categories preserving the splitting, which indicates that both of these virtual double categories can serve as an algebraic model of the theory of an adjunction.

The completed virtual double category  $\hat{\mathcal{O}}_{\mathbf{Adj}}$  has the same underlying vertical 1-category as  $\mathcal{O}_{\mathbf{Adj}}$ . For its horizontal 1-cells  $x \rightarrow y$ , it has

- $\text{Hom}_C(t_1, t_2)$ , where  $t_1 : x \rightarrow C, t_2 : y \rightarrow C$  are vertical morphisms
- $\text{Hom}_D(t_1, t_2)$ , where  $t_1 : x \rightarrow D, t_2 : y \rightarrow D$  are vertical morphisms

It is now meaningful to talk about the 2-cells  $\text{Hom}_C(1_C, F) \rightarrow \text{Hom}_D(G, 1_D)$  whose left and right boundary are  $(1_C, 1_D)$ . In this setting, it is now possible to state and prove that there are unique “special” cells going in both directions, and that there are no nontrivial endomorphisms of  $\text{Hom}_C(1_C, F)$  riding  $(id_C, id_D)$ .

This concludes the basic motivation for the paper. In the remainder of this paper introduce a concept of “generalized multicategory with restriction”, which is a generalized multicategory with additional structure. We define an algebra for a generalized multicategory with restriction to be an algebra for the generalized multicategory, respecting the additional restriction structure. In this way we hope that we are able to broaden the class of algebraic structures which can be studied using the operadic point of view.

## 4 Main results

Let  $E, B$  be two categories with finite limits. We draw intuition from the case where  $E$  is **Graph** and  $B$  is **Set**. Let  $p : E \rightarrow B$  be a Grothendieck fibration (for intuition, the forgetful functor), which we will assume to be cloven. Assume that  $p$  preserves finite limits. Let  $T_0 : B \rightarrow B, T_1 : E \rightarrow E$  be Cartesian monads, with  $T_1$  lying strictly above  $T_0$ . We are motivated by the case where  $T_1$  is the free category monad and  $T_0$  is the identity monad.

We will rely here on the notion of “generalized multicategory” due to Leinster, see *Higher categories, Higher operads*.

Let  $A_1, X_1, g_1 : X_1 \rightarrow T_1 A_1, u_1 : X_1 \rightarrow A_1$  be a generalized multicategory in  $E$  relative to  $T_1$ . We will write  $(\cdot)_1$  or  $(-\cdot-)_1$  for the multiplication morphism, and  $e_1$  for the unit morphism. Our basic definition establishes what it means for this generalized multicategory to be a generalized category “with restriction” relative to the fibration  $p$ . (We will see later that a generalized multicategory is always “with restriction” with respect to the unique fibration to the terminal category; so generalized multicategories “with restriction” generalize generalized multicategories.)

Because  $p$  preserves finite limits, it sends the generalized multicategory for  $(E, T_1)$  to a generalized multicategory for  $(B, T_0)$ . We use a subscript 0 to denote the components of the projected operad.

Notationally, if  $u : X \rightarrow Y$  is any morphism in the base category  $B$  and  $S$  is an object in  $E$  over  $Y$ , then the chosen Cartesian lift of  $u$  with codomain  $Y$  is denoted  $\bar{u}$ , the domain of  $\bar{u}$  is called  $u^*Y$ , and if  $f : Z \rightarrow Y$  is a morphism with  $p(f) = u$ , then the unique factoring of  $f$  through  $\bar{u}$  is called  $\hat{f}$ .

Let  $X_1 \otimes X_1$  denote the apex of the operad composed with itself once, i.e.,  $T(u_1) \times_{T_1(A_1)} g_1$ . Let  $\pi_r : X_1 \otimes X_1 \rightarrow X_1$  denote the projection onto the second component. Then in the diagram

$$\begin{array}{ccc} X_1 \otimes X_1 & \xrightarrow{\pi_r} & X_1 \\ \hat{\cdot} \downarrow & & \hat{u}_1 \downarrow \\ (\cdot)_0^* X_1 & & u_0^* A_1 \xrightarrow{\bar{u}_0} A_1 \end{array} \quad (1)$$

we can construct a filler for the bottom row  $c_1 : (\cdot)_0^* X_1 \rightarrow u_0^* A_1$  which is uniquely characterized by the requirement that  $p(c_1) = p(\pi_r)$  and that  $\bar{u}_0 \circ c_1 = u_1 \circ (\cdot)_0$ , by appealing to the fact that  $\bar{u}_0$  is Cartesian and  $u_0 \circ p(\pi_r) = u_0 \circ (\cdot)_0 : X_0 \otimes X_0 \rightarrow X_0 \rightarrow A_0$ .

Now we are able to explain what it means to be a “generalized multicategory with restriction.” We require that in the commutative square we have just constructed, there are morphisms  $\chi$  and  $\sigma$ ,

$$\begin{array}{ccc} X_1 \otimes X_1 & \xrightarrow{\pi_r} & X_1 \\ \hat{\cdot} \downarrow \uparrow \sigma & & \hat{u}_1 \downarrow \uparrow \chi \\ (\cdot)_0^* X_1 & \xrightarrow{c_1} & u_0^* A_1 \end{array} \quad (2)$$

such that

- $\hat{\cdot} \circ \sigma = 1$
- $\hat{u}_1 \circ \chi = 1$
- $\chi \circ c_1 = \pi_r \circ \sigma$
- $\sigma$  is the equalizer of  $\pi_r$  and  $\chi \circ c_1 \circ \hat{\cdot}$

Note that the third condition follows from the first and last.

Note that  $\sigma$  is uniquely determined by  $\chi$  if it exists.

## 5 Example - virtual double categories

Let  $(X_1, A_1, \dots)$  be a virtual double category; let  $V$  denote its underlying category of vertical arrows.  $u_0^* A_1$  is a graph whose nodes are morphisms  $f$  in  $V$ , and where an edge  $f \rightarrow f'$  is a horizontal 1-cell  $\text{cod } f \rightarrow \text{cod } f'$ .  $(\cdot)_0^* X_1$  is a

graph whose nodes are pairs of composable morphisms  $(f, g)$  in  $V$ , and whose edges  $(f_1, g_1) \rightarrow (f_2, g_2)$  are two-cells with left and right boundary  $g_1 \circ f_1$  and  $g_2 \circ f_2$ .  $c_1$  is a graph homomorphism that sends the node  $(f, g)$  to  $g$  and the 2-cell  $\alpha$  to the bottom horizontal 1-cell bounding  $\alpha$ .  $\hat{(\cdot)}$  is an operation which, for each pair of pairs of composable vertical morphisms  $(f_0, g_1), (f_n, g_2)$ , accepts as argument a single 2-cell  $\alpha : g_1 \Rightarrow g_2$  whose source is a list of  $n$  edges and a list of 2-cells  $(\beta_1, \dots, \beta_n)$  where the left boundary of  $\beta_1$  is  $f_0$  and the right boundary of  $\beta_n$  is  $f_n$ ; the composition operator  $\hat{(\cdot)}$  composes the 2-cells, forgetting their factoring, while *remembering* the factoring of the underlying vertical 1-cells. The fact that  $\chi$  is a right inverse to  $\hat{u}_1$  means that  $\chi$  associates to any “cup”  $(f, g, h : \text{cod } f \rightarrow \text{cod } g)$  a 2-dimensional filler, although we don’t specify here the number of morphisms in the arity. The fact that  $\sigma$  is right inverse to  $\hat{(\cdot)}$  means that  $\sigma$  returns a factoring of the 2-cell it is given. The commutativity of the square when going from bottom left to top right indicates that the factoring supplied by  $\sigma$  has its second component determined by  $\chi \circ c_1$ . The equalizer condition implies that whenever  $(\alpha, x)$  is a composable group of 2-cells (with  $\alpha$  a list, and  $x$  a single 2-cell) such that  $x$  is the canonical filler associated to its boundary,  $\alpha$  is uniquely determined by this requirement, as  $(\alpha, x)$  is the canonical factorization of its own composite.

It is not clear that every virtual double category “with restrictions” in our sense, also has restrictions in the usual sense, because in the usual sense the restriction of a horizontal cell along two vertical cells is required to have an upper boundary which is a single morphism rather than a chain of morphisms. We can suggest two ways to rectify this. The first is to alter the definition we have given above so that instead of working with  $X_1 \otimes X_1 = \{(\alpha, x) \mid \alpha \in T_1(X_1), x \in X_1, T_1(u_1)(\alpha) = g_1(x)\}$ , we instead work with  $\{(\alpha, x) \mid \alpha \in X_1, x \in X_1, T_1(u_1)(\eta(\alpha)) = g_1(x)\}$ . The second is to impose the requirement that there is an algebra map  $h : T(A_0) \rightarrow A_0$  such that  $\eta_0 \circ h \circ g_0 = g_0$ , so that it is possible to impose the assumption that  $\chi : u_0^* A_1 \rightarrow X_1$  factors through  $\eta : A_1 \rightarrow T_1 A_1$ .

For now we will accept the inelegance that the choice of function  $\chi$  is a structure rather than a property and may not be unique.

**Theorem 1.** *Let  $\mathcal{C}$  be a virtual double category. The following are equivalent:*

- $\mathcal{C}$  has (chosen) restrictions in the established sense of virtual double category theory.
- $\mathcal{C}$  has restrictions with respect to the forgetful fibration  $p : \mathbf{Graph} \rightarrow \mathbf{Set}$ , in our sense, i.e., there are morphisms  $\chi, \sigma$  as defined above, subject to the additional constraint that there exists a morphism  $h_1 : u_0^* A_1 \rightarrow A_1$  such that  $g_1 \circ \chi = \eta_1 \circ h_1$ . (Note that this implies that  $g_0$  factors through  $\eta_0$  by  $h_0 = p(\eta_0)$ .)

*Proof.* It is well known that virtual double categories are equivalent to generalized multicategories in  $\mathbf{Graph}$  with respect to the  $\mathbf{FreeCat}$  monad. We regard  $(\mathbf{Graph}, \mathbf{FreeCat})$  as being fibered over  $(\mathbf{Set}, \mathbf{Id})$  by the usual forgetful functor. The forgetful functor  $p$  is a fibration that preserves finite limits; it sends the

virtual double category to an ordinary category  $V = (X_0, g_0 : X_0 \rightarrow T_0 A_0, u_0 : X_0 \rightarrow A_0)$ , where  $A_0$  is the set of objects of the virtual double category,  $X_0$  the set of vertical morphisms,  $g_0$  and  $u_0$  the domain and codomain projections respectfully.

First, note that a splitting  $\chi$  of the map  $\hat{u}_1 : X_1 \rightarrow u_0^* A_1$  is exactly a map that associates to each pair of vertical morphisms  $(f : x_1 \rightarrow y_1, g : x_2 \rightarrow y_2)$  and each horizontal 1-cell  $s : y_1 \rightarrow y_2$  a 2-dimensional “filler”  $\chi(f, g, s)$  with left and right boundary  $f, g$  respectively, and horizontal lower boundary  $s$ . The top boundary of  $\chi(f, g, s)$  may be any chain of composable horizontal cells, but the constraint  $g_1 \circ \chi = \eta_1 \circ h_1$  forces the vertical domain of  $\chi(f, g, s)$  to be a single 1-cell.

We will now prove that the fillers given by  $\chi$  are Cartesian iff there exists  $\sigma$  satisfying the axioms above. Observe that elements of  $(\cdot)_0^* X_1$  are tuples  $((f_1, g_1), (f_2, g_2), k, s)$ , where  $(f_1, g_1)$  and  $(f_2, g_2)$  are pairs of composable vertical 1-cells,  $k : \text{cod } g_1 \rightarrow \text{cod } g_2$  is a horizontal 1-cell, and  $s$  is a 2-cell with left and right boundary  $g_1 \circ f_1$  and  $g_2 \circ f_2$  respectively, and whose lower boundary is  $k$ . A section  $\sigma$  of  $(\cdot)$  is equivalent to a choice of factoring of the given 2-cell  $s$  into composable 2-cells  $([a_1, \dots, a_n], s')$ , where  $s'$  has lower horizontal boundary  $k$ , the upper boundary of  $s'$  is exactly the list of lower boundaries of the  $a_i$ , the left and right boundary of  $s'$  are  $g_1, g_2$  respectively, the left boundary of  $a_1$  is  $f_1$ , and the right boundary of  $a_n$  is  $f_n$ . The constraint  $\chi \circ c_1 = \pi_r \circ \sigma$  states that in the factoring  $\sigma((f_1, g_1), (f_2, g_2), k, s) = ([a_1, \dots, a_n]; s')$ , the morphism  $s'$  is independent of  $s, f_1$  and  $f_2$ ; it is forced to be  $\chi(g_1, g_2, k)$ . Because of our earlier assumption that  $g_1 \circ \chi$  factors through  $\eta_1$ , in the expression  $[a_1, \dots, a_n]$  we always have  $n = 1$ .

Thus, if  $\chi, \sigma$  satisfy the first three bullets of 2, they give a filling 2-cell for every “cup”, and a factorization function that assigns each 2-cell equipped with an appropriate factoring of its boundaries, to a factoring of the associated 2-cell through the designated filler.

The equalizer condition will then imply that this factorization is unique. Suppose that  $\chi(f, g, k)$  is always Cartesian. A relatively straightforward reading of this as a universal property would be to say that for any object  $Z_1$  and any  $p_1 : Z_1 \rightarrow (\cdot)_0^* X_1$ , there is a unique morphism  $a : Z_1 \rightarrow T(X_1)$  such that  $T(u_1) \circ a = g_1 \circ \chi \circ c_1 \circ p_1$  and  $(\hat{\cdot}) \circ (a, \chi \circ c_1 \circ p_1) = p_1$ . (Applying this to the case where  $p_1$  is the identity morphism, we recover  $\sigma$ .) Then it is evidently equivalent to say that there is a unique  $a' : Z_1 \rightarrow X_1 \otimes X_1$  (with  $a = \pi_\ell \circ a'$ ) such that  $\pi_r \circ a' = \chi \circ c_1 \circ p_1$  and  $(\hat{\cdot}) \circ a' = p_1$ .

So, if  $a' : Z_1 \rightarrow X_1 \otimes X_1$ , letting  $p_1 = (\hat{\cdot}) \circ a'$ , if and  $\pi_r \circ a' = \chi \circ c_1 \circ p_1$ , then  $a'$  is the unique lift of  $p_1$  along  $(\hat{\cdot})$  with this property.

But  $\sigma p_1$  is also a lift of  $p_1$  along  $(\hat{\cdot})$  with this property, so  $a' = \sigma \circ (\hat{\cdot}) \circ a'$ , which gives  $(\hat{\cdot}) \circ a'$  as the desired factoring of  $a'$  through  $\sigma$ , which is unique as  $\sigma$  is a (split) monomorphism. The converse direction is similar.

□

In the case of virtual double categories we were interested in the special

case of restriction where the domain of the filler was of shape  $A_1$  rather than of the more general shape  $T_1(X_1)$ . It is interesting to note that in this case, the factoring of any 2-cell through the Cartesian morphism is also of shape  $X_1$  rather than of the more general shape  $T_1(X_1)$

**Lemma 1.** *Let  $(X_1, A_1, u_1 : X_1 \rightarrow A_1, g_1 : X_1 \rightarrow T_1 A_1)$  be a generalized multicategory “with restriction”, i.e., having morphisms  $(\chi, \sigma)$  as above.*

*Suppose that  $\chi$  factors through  $\eta$  as for virtual double categories with restriction, i.e., there is  $h_1 : u_0^* A_1 \rightarrow A_1$  such that  $\eta_1 \circ h_1 = g_1 \circ \chi$ .*

*Then  $\pi_\ell \circ \sigma : u_0^* A_1 \rightarrow X_1 \otimes X_1 \rightarrow T_1 X_1$  is of the form  $\eta_1 \circ h_2 : (\hat{\cdot})_0^* \rightarrow X_1 \rightarrow T(X_1)$  for a unique morphism  $h_2$ .*

*Proof.* Use that  $\eta$  is a Cartesian natural transformation. □

## 6 Algebras for generalized multicategories with restriction

Let  $s_1 : E_1 \rightarrow A_1$  be an algebra for our generalized multicategory

We will discuss when  $s_1$  respects the “restriction structure”  $\chi$ . In the diagram below, the meaning of the maps  $g_1, \hat{u}_1, \bar{u}_0$ , and  $\chi$  have already been established. Let  $p : g_1 \times_{T_1 A_1} T_1 s_1 \rightarrow E_1$  be the action map of the operad; then  $p'$  is constructed by appeal to the universal property of the pullback, as  $u_1 \circ \pi_{X_1} = s_1 \circ p$  by assumption.

**Definition 1.**  $s_1$  is compatible with the restriction action of  $\chi : u_0^* A_1 \rightarrow X_1$  when there exists a morphism  $\chi'$  as in the diagram below, such that  $\hat{p}_1 \circ \chi' = 1$  and  $\pi_{X_1} \circ \chi' = \chi \circ \pi_{u_0^* A_1}$  is a pullback diagram.

$$\begin{array}{ccccccc}
 T_1 E_1 & \longleftarrow & g_1 \times_{T_1 A_1} T_1 s_1 & \xrightleftharpoons[\chi']{\hat{p}_1} & p_0^* E_1 & \xrightarrow{\bar{p}_0} & E_1 \\
 \downarrow T_1 s_1 & & \downarrow & & \downarrow & & \downarrow s_1 \\
 T_1 A_1 & \xleftarrow{g_1} & X_1 & \xleftarrow{\chi} & u_0^* A_1 & \xrightarrow{\bar{u}_0} & A_1
 \end{array} \tag{3}$$

I will not be explaining further.

**Theorem 2.** *Let  $\mathcal{C}$  be a virtual double category with restriction. Then algebras respecting the restriction structure of the multicategory are in one to one correspondence with covariant functors  $\mathbf{C} \rightarrow \mathbf{Set}$  sending chosen Cartesian cells to Cartesian cells in  $\mathbf{Set}$ .*

*Proof.* Let  $(X_1, A_1, g_1, u_1, (\cdot)_1, e_1)$  be the generalized multicategory associated to  $\mathcal{C}$ . It is already known that algebras for this multicategory correspond to covariant functors  $\mathcal{C} \rightarrow \mathbf{Set}$  of virtual double categories. What is interesting here is to show that the algebras respecting the restriction structure correspond to functors sending chosen Cartesian 2-cells to Cartesian 2-cells.

Therefore, let  $s_1 : E_1 \rightarrow B_1$  be an algebra which respects the restriction structure  $\chi$ . Let  $(f : x_1 \rightarrow y_1, g : x_2 \rightarrow y_2)$  be two vertical morphisms, and

$h : y_1 \rightarrow y_2$  a horizontal morphism. We have already argued that  $\chi(f, g, h)$  is a Cartesian 2-cell; we will prove that the functor associated to  $s_1$  sends  $\chi(f, g, h)$  to a Cartesian 2-cell. Fix  $w_1 \in E_1(x_1), w_2 \in E_1(x_2)$ ; let  $a \in E_1(h)$  lying over  $E_1(f)(w_1)$  and  $E_1(g)(w_2)$  respectively. Then  $a$  can be viewed as an edge in  $p_0^*E_1$  from  $(w_1, f)$  to  $(w_2, g)$ , and  $\chi'(a)$  is an edge from  $(w_1, f)$  to  $(w_2, g)$  in  $g_1 \times_{T_1 A_1} T_1 s_1$ , equivalently an edge in  $E_1$  from  $w_1$  to  $w_2$  over  $\text{dom } \chi(f, g, h)$ ; the requirement that  $\hat{p}_1(\chi'(a)) = a$  gives that  $\chi'(a)$  is a lift of  $a$  along  $E_1(\chi(f, g, h), w_1, w_2)$ . To see that  $\chi'(a)$  is unique, note that if  $b : w_1 \rightarrow w_2$  is any edge in  $E_1(\text{dom } \chi(f, g, h))$ , then because of the requirement that  $\chi'$  is the pullback of  $\chi$ , there is a (unique) edge  $b'$  from  $(w_1, f)$  to  $(w_2, g)$  in  $p_0^*E_1$  over the edge  $h : f \rightarrow g$  in  $u_0^*A_1$ , such that  $\chi'(b') = b$ ; but then  $\hat{p}_1(b) = \hat{p}_1(\chi'(b')) = b'$ , so if  $\hat{p}_1(b) = a$ ,  $b = \chi'(a)$ .

Let us prove the converse direction. Suppose that we have a functor of virtual double categories  $E_1 : \mathcal{C} \rightarrow \mathbf{Set}$  preserving chosen Cartesian cells. We will construct a map  $\chi'$  satisfying the requirements above. This is equivalent to saying that for each pair of vertical morphisms  $(f : x_1 \rightarrow y_1, g : x_2 \rightarrow y_2)$  in  $X_0$  and  $h : y_1 \rightarrow y_2$  in  $A_1$ , the associated map  $E_1(\chi(f, g, h))$  is fiber-wise a bijection (i.e., a bijection over each  $w_1 \in E_1(x_1), w_2 \in E_1(x_2)$ ). This is what it means for a 2-cell in  $\mathbf{Set}$  to be Cartesian, so we are done.  $\square$

## 7 Remaining work

- Are there interesting examples of algebraic structures which can be expressed as functors from a vdc  $\mathcal{C}$  equipped with a splitting into  $\mathbf{Set}$ , preserving the restriction - but cannot be expressed without the language of restrictions?
- What are some interesting examples of generalized multicategories with restriction?
- Under what conditions can we freely adjoin restrictions to a generalized multicategory, as we can do with virtual double categories?
- What is an example of an algebraic model of “theory” which suffices to define an algebraic theory of small Grothendieck fibrations? (Cloven fibrations can be expressed as pseudo-algebras of a certain 2-monad on  $\mathbf{Cat}$  - how else might they be characterized?)