1 Quantale of Intervals

Definition 1.1 (Quantale)**.** A **quantale** is a set V equipped with a partial order $\sqsubseteq\, \subseteq V \times V$, a binary operation $\oplus : V \times V \to V$, and a unit $k \in V$ satisfying the following properties:

- $(V, \sqsubseteq)$ is a **complete lattice**, namely, any subset $S \subseteq V$ has a supremum and an infimum, denoted by $\bigsqcup S$ and $\bigsqcap S$ respectively. In particular, V has a least element or bottom (the supremum of the emptyset), denoted by $\bot$, and a greatest element or top (the infimum of the emptyset), denoted by $\top$.
- $(V, \oplus, k)$ is a **monoid**, namely, for any $x, y, z \in V$, $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, and $x \oplus k = x = k \oplus x$.
- $\oplus$ is **join-continuous**, namely, for any $x \in V$ and $S \subseteq V$, $x \oplus \bigsqcup S = \bigsqcup \{x + s \mid s \in S\}$.

Example 1.2 (Extended reals)**.** The real numbers $\mathbb{R}$ equipped with the usual order $\leq$ and addition $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ almost form a quantale. Indeed, $\leq$ is a partial order, and $+$ is associative, has unit 0, and preserves supremum (i.e. it is join-continuous), but $\mathbb{R}$ is not a complete lattice since there is no bottom nor top. One can *fix* this by adding positive and negative infinities to obtain the **extended reals** $[-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$. The order $\leq$ is extended in the obvious way: $-\infty \leq x \leq \infty$ for all $x \in [-\infty, \infty]$. To extend addition, one must take care of ensuring join-continuity. Thus, for any $x \in [-\infty, \infty]$,

$$x + -\infty = x + \bigsqcup \emptyset = \bigsqcup \{x + s \mid s \in \emptyset\} = \bigsqcup \emptyset = -\infty,$$

and moreover, for any finite $x \in \mathbb{R}$,

$$x + \infty = x + \bigsqcup \mathbb{R} = \bigsqcup \{x + s \mid s \in \mathbb{R}\} = \bigsqcup \mathbb{R} = \infty.$$

The remaining undefined addition is $\infty + \infty$, which must be ∞ by join-continuity:

$$\begin{aligned}
\infty + \infty &= \bigsqcup \mathbb{R} + \bigsqcup \mathbb{R} \\
&= \bigsqcup \left\{ \bigsqcup \mathbb{R} + s \mid s \in \mathbb{R} \right\} \\
&= \bigsqcup \left\{ \bigsqcup \{r + s \mid r \in \mathbb{R}\} \mid s \in \mathbb{R} \right\} \\
&= \bigsqcup \left\{ \bigsqcup \mathbb{R} \mid s \in S \right\} \\
&= \bigsqcup \{\infty\} = \infty.
\end{aligned}$$

We can summarize the definition of addition in the quantale of extended reals in the following table (as in [1]) where we write $\oplus$ for addition in the quantale and $+$ for addition of reals (even if we will use $+$ for both in the sequel).

$\oplus$	$-\infty$	r	∞
$-\infty$	$-\infty$	$-\infty$	$-\infty$
s	$-\infty$	$r + s$	∞
∞	$-\infty$	∞	∞

Note that reversing the order $\leq$ does not give a quantale $([-\infty, \infty], \geq, \oplus, 0)$ when $\oplus$ is defined as above because $\oplus$ is not meet-continuous:

$$-\infty \oplus \bigsqcap \varnothing = -\infty \oplus \infty = -\infty \neq \infty = \bigsqcap \varnothing = \bigsqcap \{-\infty + s \mid s \in \varnothing\}.$$

For later, we can define an operation $\boldsymbol{\oplus}$ on $[-\infty, \infty]$ that extends $+$ and is meet-continuous, so that $([-\infty, \infty], \geq, \boldsymbol{\oplus}, 0)$ is a quantale, which we will write $[\infty, -\infty]$. We summarize the definition of $\boldsymbol{\oplus}$ in the table below.

$\boldsymbol{\oplus}$	$-\infty$	r	∞
$-\infty$	$-\infty$	$-\infty$	∞
s	$-\infty$	$r+s$	∞
∞	∞	∞	∞

Another quantale of extended reals that has often been studied, and sometimes called the Lawvere quantale, is the nonegative extended reals with the revered order: $([0, \infty], \geq, \oplus, 0)$.

Example 1.3 (Quantale of intervals)**.** Given two elements $a \sqsubseteq b$ inside a quantale V, we can define the (closed) **interval** $[a, b]$ as the subset containing all elements above a and below b:

$$[a, b] := \{x \in V \mid a \sqsubseteq x \sqsubseteq b\}.$$

Let $\mathsf{Interval}(V)$ be the set of intervals in V plus the empty subset. We will try to construct a natural quantale structure on this set.

First, the most natural order of subsets is inclusion $[a, b] \subseteq [a', b']$. It turns out this can be defined in terms of the order in V (we abuse $\sqsubseteq$ to denote the orders in V and $\mathsf{Interval}(V)$ and similarly for $\oplus$):

$$[a, b] \sqsubseteq [a', b'] \iff [a, b] \subseteq [a', b'] \iff a \sqsupseteq a' \text{ and } b \sqsubseteq b'.$$

Thus, as a partial order, $\mathsf{Interval}(V)$ is essentially a suborder of $V^{\mathrm{op}} \times V$, where $-^{\mathrm{op}}$ reverses the order. Only the emptyset is not represented as an element of $V^{\mathrm{op}} \times V$, but it can be convenient to identify any pair (a, b) where $a \sqsupseteq b$ and $a \neq b$ as the emptyset. In particular, we find that $\mathsf{Interval}(V)$ is a complete lattice by applying supremums and infimums coordinatewise, making sure to take the order reversal into account. Explicitly,

$$\bigsqcup_i [a_i, b_i] = [\sqcap_i a_i, \sqcup_i b_i] \qquad \bigsqcap_i [a_i, b_i] = [\sqcup_i a_i, \sqcap_i b_i].$$

One can quickly check this by hand. Alternatively, one can see $\mathsf{Interval}(V)$ as a suborder of the powerset of V which is a complete lattice with supremum (resp. infimum) being union (resp. intersection), then recognize the intervals above as the smallest interval containing the union of $[a_i, b_i]$ (resp. the biggest interval contained in the intersecitonof $[a_i, b_i]$).

Next, we would like to define $\oplus$ on intervals and naturally, we want to let

$$[a, b] \oplus [a', b'] := [a \oplus a', b \oplus b'].$$

This yields a monoid with unit $[k, k]$ (recall k is the unit of V). Unfortunately, $\oplus$ is not join-continuous over intervals, unless $\oplus$ is meet-continuous over V:

$$[a,b] \oplus \bigsqcup_i [a_i, b_i] = [a,b] \oplus [\sqcap_i a_i, \sqcup_i b_i] = [a \oplus \sqcap_i a_i, b \oplus \sqcup_i b_i]$$
$$\neq [\sqcap_i a \oplus a_i, \sqcup_i b \oplus b_i] = \bigsqcup_i ([a,b] \oplus [a_i, b_i]) .$$

If there is a binary operation $\oplus$ on V is meet-continuous, and hence makes $(V, \sqsupseteq, \oplus, k)$ a quantale, then we can define the addition of intervals by apply $\oplus$ to the upper bounds and $\oplus$ to the lower bounds:

$$[a,b] \oplus [a', b'] := [a \oplus a', b \oplus b'].$$

Redoing the derivation above, we find that this operation is join-continuous over the quantale because $\oplus$ is meet-continuous. It is also associative with unit $[k, k]$.

In conclusion, we can always define a complete lattice of (closed) intervals inside a quantale, but we need to be careful to make it into a quantale (it is not always possible). We can consider a concrete example of quantale of intervals over the extended reals which is essentially a subquantale of $[\infty, -\infty] \times [-\infty, \infty]$.

References

[1] Giorgio Bacci, Radu Mardare, Prakash Panangaden, and Gordon D. Plotkin. Polynomial lawvere logic. *CoRR*, abs/2402.03543, 2024. URL: `https://doi.org/10.48550/arXiv.2402.03543`, `arXiv:2402.03543`, `doi:10.48550/ARXIV.2402.03543`.