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DOTTORATO DI RICERCA IN

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IX CICLO

THE TYPE THEORY OF CATEGORICAL UNIVERSES.

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Introduction

The subject of this thesis is the type theory of categorical universes such as Heyting pretoposes and elementary toposes. These categories are considered universes, since they provide models for classical and intuitionistic set theory. The reason to look for a type theory of such universes arises from the desire to compare them with Martin-Löf 's constructive type theory, since in all these frameworks intuitionistic mathematics can be modelled.

Beginning in the seventies, Martin-Löf proposed his Constructive Type Theory as a set theory where intuitionistic mathematics can be formalized. In this approach, "constructive" means predicative and "computable"; indeed every proof within type theory corresponds to a program [NPS90]. An example of constructive mathematics fully developed in this framework is formal topology [Sam87].

On the categorical side, in the sixties, Lawvere aimed at giving a purely categorical foundation of mathematics. He wanted to axiomatize the category of sets, replacing set membership by composition of functions. Together with Tierney, he produced the notion of an elementary topos resembling the structural properties of a Grothendieck topos, that is a category of set-valued sheaves on a site. The axiomatization they gave made no relevant set-theoretic assumptions. So, while a Grothendieck topos is always an elementary topos, the converse does not hold in general.

Following Lawvere, a topos can be thought as a generalized universe of sets. But the underlying logic of this universe is intuitionistic, not classical in general. Indeed, the truth values of a topos form a Heyting algebra, like the algebra of open sets of a topological space.

In the seventies, Mitchell, Benabou, Joyal and others provided an explicit description of a formal language apt to be interpreted in a topos. This language is typed, because to each term occurring in the formulas a type is assigned. The resulting logic seems to be many-sorted, taking the simple types as sorts. The formulas are the terms of the specific type corresponding to the subobject classifier. A systematic exposition of this theory, as a higher order logic, was given by Lambek and Scott [LS86]. This internal language formalizes the ideas of a topos as a generalized set theory.

In order to model classical set theory, Cole [Col73] and Mitchell [Mit72] found that well-pointed toposes with a natural numbers object and axiom of choice provide models for restricted Zermelo set theory with the axiom of choice, where the comprehension axiom is given only for formulas with bounded quantifiers.

More recently, Joyal and Moerdijk explored how to provide models for the full Zermelo-Fraenkel set theory in a categorical setting [JM95]. They found that, in order to model classic and intuitionistic Zermelo-Fraenkel set theory, it is sufficient to take a Heyting pretopos with a natural numbers object as a categorical universe, and within this to single out a class of "small" maps satisfying suitable axioms.

The notion of pretopos was introduced by Grothendieck: an elementary topos is a pretopos, but the latter is a weaker notion. Makkay and Reyes found that pretoposes can be characterized with respect to the logical categories, which are the necessary structures to interpret the first order, many-sorted, coherent logic, see [MR77]. A Heyting pretopos is obtained by enriching such a category with the necessary structure to interpret first order predicative intuitionistic logic.

In order to compare Martin-Löf's Constructive Type Theory with these categorical frameworks, one possible direction of research is to find typed theories, which corresponds precisely to toposes and to Heyting pretoposes with a natural numbers object. This direction has been explored in the present thesis.

The main issue is to pass from a many-sorted logic to a dependent type theory complete with respect to the class of universes under consideration. Indeed, in a many-sorted logic there is a syntactic distinction between formulas and types. Moreover, the types are not dependent, since they correspond to sorts. On the other hand, in a dependent type theory of Heyting pretoposes or of toposes, such as those proposed in the thesis, the key point is that formulas correspond to particular dependent types. These calculi are formulated in a style which is basically that of Martin-Löf's type theory, so that a more precise comparison with Constructive Type Theory is possible.

Looking at the higher order logic of a topos, the main difference with respect to Constructive Type Theory is that, in a topos, there is a power set construction, which allows impredicative quantification. Instead in Martin-Löf's type theory only predicative constructions are allowed. Moreover, Martin-Löf's type theory has a stronger existential quantifier than the intuitionistic one, so that the axiom of choice is provable. On the contrary, in a topos, the axiom of choice is not always valid; in fact, it implies the principle of excluded middle, and thus the logic of the topos becomes classical, see [Joh77], [MM92]. Therefore, from a constructive point of view, this difference seems to make the two frameworks incompatible.

As a matter of fact, in the first chapter, we will prove that by extending intensional Martin-Löf's type theory with an extensional power set, where subsets are propositional functions, the principle of excluded middle is provable by the axiom of choice, like in topos theory. This extension is made by following the isomorphism "propositions as types".

On the other hand, the dependent type theory for toposes, presented in the fourth chapter of the thesis, reveals that, in a topos, formulas correspond to "mono" dependent types, i.e. types with at most one proof, by following the isomorphism "propositions as mono closed types".

Therefore, it should be possible to extend Martin-Löf's type theory, without falling into classical logic, with the powersets of a topos, by considering as subsets only mono propositional functions.

Besides powersets, also effective quotients based on generic relations can not be added to Martin-Löf's type theory in the presence of uniqueness of propositional equality proofs. Indeed, as explained in the second chapter, by using effective quotients on the first universe of small sets and on the second universe of large sets, we can prove the principle of excluded middle for small sets. To do this, it is sufficient to adapt to this framework the already considered proof that the axiom of choice implies the principle of excluded middle.

A proposal of effective quotients, which are compatible with the powerset of a topos without loosing constructivity, is given by the type theory of Heyting pretoposes, proposed in the third chapter of the thesis. This dependent type theory, complete with respect to Heyting pretoposes with a natural numbers object, corresponds to a first order extensional type theory with product types restricted to mono types and effective quotients restricted to mono equivalence relations.

The categorical semantics, used to interpret the typed calculi of Heyting pretoposes and toposes, is explained in the fifth chapter. This semantics combines together the notion of model given by display maps [HP89], [See84], with the tools provided by contextual categories to interpret substitution correctly [Car86], emphasizing the context formation. In this way, the proofs of completeness, presented in the sixth chapter, are restricted to particular contextual categories.

In the seventh chapter, we show that also the internal language of a Heyting pretopos or a topos is a dependent type theory, obtained by adding the dependent types specific of the universe under consideration. Finally, another application of the type theory is the construction of the free Heyting pretopos and the free topos generated by a category.

By the dependent typed calculi of Heyting pretoposes and toposes, we are ready to get a type-theoretical description of the notion of small map and hence of the categorical models for intuitionistic set theory, as in [JM95]. On the other hand, we could also investigate possible extensions of Martin-Löf's type theory by the type constructors of the dependent typed calculi of Heyting pretoposes and toposes.

The results of the first chapter are contained in [MV96], of the second in [Mai97d], from the third to sixth in [Mai97c] and [Mai97b], and finally, most of the seventh chapter will be published in [Mai97a].

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Chapter 1

Extensional powersets in constructive type theory

Summary An extension of Martin-Löf's intensional set theory is proposed by means of a powerset $\mathcal{P}(S)$, whose elements are the subsets of the set S, defined as propositional functions.

Since the equality among subsets has to be extensional, it turns out that such extension cannot be constructive: any link between the truth of a proposition and the possibility to exhibit one of its proof-element is lost. This fact is not compatible with the usual meaning of intuitionistic set theory. In fact, we will prove that this extension is classic, i.e., for any proposition A, $(A \vee \neg A)$ *true* holds, as a consequence of the intuitionistic axiom of choice.

1.1 Introduction

In [GR94] it is shown that the proof theoretic strength of Martin-Löf's Type Theory [Mar84, NPS90] with restricted well-orders and the universe of the small types is that of a subsystem of second order arithmetic with Δ^1_2 comprehension and bar-induction. Thus, it is natural to wonder whether it is possible to enforce it to a theory with the strength of the full comprehension schema by adding a powerset constructor; in fact this extension is necessary in order to quantify over the subsets of a given set, since in type theory quantification is allowed only over a set.

In the literature, there are already examples of intuitionistic set theories with some kind of powerset constructor. For instance, one can think of a *topos* as a "generalized set theory" by associating with any topos its internal language (cf. [Bel88]). The logic underlying such set theory is the intuitionistic predicate calculus and so any topos can be thought of as an intuitionistic universe of sets.

Then the lack of the rule of excluded middle seems to assure the constructivity of any proof developed within topos theory. The problem to adapt the topos theoretic approach to Martin-Löf's set theory is due to the impredicativity of the former. Indeed, Martin-Löf's set theory is predicative and provides a fully algorithmic way to construct the elements of the sets and the proofs of the propositions on these sets.

Another approach is the Calculus of Construction by Coquand and Huet [Coq90], where the power of a set S can be identified with the type of the functions from S into $Prop$, if we follow the isomorphism propositions as sets and consider the notion of types as in [NPS90]. But, in this case the power of a set is not itself a set and despite of this the quantification over $Prop$ is allowed. Anyway, it can be proved that the strong sum type, which is present in Martin-Löf's type theory, cannot consistently be added to the Calculus of Constructions (see [Coq90]) at the level of propositions, but only at the level of types [Luo90].

Of course, there is no reason to expect that a second order construction becomes constructive only because it is added to a theory which is constructive and predicative. And, indeed, we will prove that even the weaker fragment iTT, which contains only the basic type constructors and the *intensional* equality, cannot be extended with a powerset constructor, which is compatible with the usual Martin-Löf's semantical explanation of the connectives and which is the collection of all the subsets of a given set. In fact, by using the so called intuitionistic axiom of choice, it is possible to prove that, for any powerset constructor which satisfies some natural conditions, which we will illustrate in the next section, classical logic arises (see also [Hof95] page. 170, where a similar result is suggested for an extension of the Calculus of Construction with Leibniz equality in the framework of setoids).

1.2 $iTT^P = iTT$ + powersets

In order to express the rules and the conditions that we are going to require on the powerset, it is convenient to recall the main properties of judgements of the form A *true* (see [Mar84]): A *true* holds if and only if there exists a proof-element a such that $a \in A$ holds (for a formal proof see [Val95]). In particular, the following rule is admissible

$$\text{(True Introduction)}\quad \frac{a \in A}{A\ \mathit{true}}$$

as well as all the rules of the intuitionistic predicative calculus with equality, where the judgement A *true* stands for $\vdash A$ (see [Mar84] for the definition of the embedding of the intuitionistic predicative calculus within iTT). Here, we only recall the case of the set of the intensional propositional equality Id (see [NPS90], page. 61) which plays a main role in this chapter (for sake of clearness, supposing A is a set and $a, b \in A$, we will often write $a =_A b$ to mean $\mathsf{Id}(A, a, b)$). The propositional equality is the internalization of the definitional equality between elements of a set. Two objects are definitional equal if they evaluate to the same canonical form. There are two kinds of propositional equality: Id is intensional (see the rules below) and Eq is extensional (see the rules in section 3.2). Intensional propositional equality is entailed by definitional equality, that is two objects are propositionally equal if they are definitionally equal, but the vice versa does not hold. On the contrary, extensional propositional equality is equivalent to definitional equality. The main difference is that in the presence of intensional propositional equality, definitional equality and type checking are decidable, but no more in the presence of extensional propositional equality.

The formation and introduction rules of the set of the intensional propositional equality Id are the following

$$\frac{A\ \mathit{set}\quad a \in A\quad b \in A}{\mathsf{Id}(A, a, b)\ \mathit{set}}\qquad \frac{A = C\ \mathit{set}\quad a = c \in A\quad b = d \in A}{\mathsf{Id}(A, a, b)\ =\ \mathsf{Id}(A, c, d)}$$

$$\frac{A\ \mathit{set}\quad a \in A}{\mathsf{id}(a) \in \mathsf{Id}(A, a, a)}\qquad \frac{A\ \mathit{set}\quad a = b \in A}{\mathsf{id}(a) = \mathsf{id}(b) \in \mathsf{Id}(A, a, a)}$$

whereas the elimination rule is

$$\frac{c \in \mathsf{Id}(A, a, b)\quad \begin{array}{c}[x : A]\\ |\\ d(x) \in C(x, x, \mathsf{id}(x))\end{array}}{\mathsf{idpeel}(c, d) \in C(a, b, c)}$$

and it yields the admissibility of the following two rules on judgements of the form A *true*:

$$\frac{c \in \mathsf{Id}(A, a, b)\quad \begin{array}{c}[x : A]\\ |\\ C(x, x, \mathsf{id}(x))\ \mathit{true}\end{array}}{C(a, b, c)\ \mathit{true}}\qquad \frac{\mathsf{Id}(A, a, b)\ \mathit{true}\quad \begin{array}{c}[x : A]\\ |\\ C(x, x)\ \mathit{true}\end{array}}{C(a, b)\ \mathit{true}}$$

The rules for the set $\mathcal{P}(S)$ depend on the definition of what a subset is within iTT. Following a long tradition, we identify a subset U of S with a propositional function on S, i.e. provided that $U(x)\ \mathit{set}\ [x : S]$, we put $U \equiv (x : S)\ U(x)$, and hence, we say that an element $a \in S$ is an element of U if $U(a)$ is inhabited, i.e. the judgement $U(a)$ *true* holds (cf. [dB80] and [SV95] for a detailed discussion of this topic). We consider a propositional function corresponding to $U(x)$ *set* $[x : S]$, since in Martin-Löf's intensional set theory *propositions* are identified with *sets* and we use *set* for *prop*.

Thus, provided that we want to have an extensional equality between subsets, we are forced to consider equal two subsets U and V of S if and only if $U(x) \leftrightarrow V(x)$ *true* $[x : S]$, i.e. U and V have the same elements.

Extensional equality on subsets, expressed at the level of the collection $(x : S)$ *set* is the crucial point, where classical logic breaks into the system.

Inspired by the previous explanations, here we propose the following formation and introduction rules for $\mathcal{P}(S)$:

Formation

$$\frac{S\ set}{\mathcal{P}(S)\ set} \qquad \frac{S = T}{\mathcal{P}(S) = \mathcal{P}(T)}$$

Introduction

$$\frac{U\ set\ [x \in S]}{\{(x \in S)\ U\} \in \mathcal{P}(S)}$$

We should now formulate the next rules for the type $\mathcal{P}(S)$, i.e. the equality introduction rule, the elimination rule and the equality rule, but the aim of this chapter is to show that it is actually impossible to formulate them, since we would obtain a Heyting semantic for classical logic. Anyhow, it is clear that whatever rules one can give for the type $\mathcal{P}(S)$, some conditions should be satisfied to make $\mathcal{P}(S)$ a suitable representation of the set of all the subsets of the set S. The use of conditions is a device in order to suppose to have type theoretical rules that make the conditions expressed by true-judgements admissible. Since in the presence of these conditions we get a negative result, we conclude that such rules do not exist.

The first condition we require is the equality introduction condition.

Equality introduction condition
Let $U \leftrightarrow V$ *true* $[x \in S]$. Then $\{(x \in S)\ U\} = \{(x \in S)\ V\} \in \mathcal{P}(S)$.

After the previous considerations on the equality between subsets, it is clear that this condition must be satisfied, but, as noted by Peter Aczel after reading a preliminary version of this work, this should not be a formal rule for the type $\mathcal{P}(S)$, since the use of an extensional equality rule for powersets does not fit with the idea of treating the judgemental equalities as definitional, which is basic in iTT.

The elimination and the equality rules are even more problematic, because it is difficult to give a plain application of the standard approach, which allows to obtain the elimination rule out of the introduction rule (see [Mar71]). In fact, the introduction rule does not act over elements of a set but over elements of the collection $((x : S)\ set)_{\leftrightarrow}$. Thus, if one wants to follow for $\mathcal{P}(S)$ the general pattern for a quotient set, he could look for a rule similar to the following:

$$\frac{c \in \mathcal{P}(S) \quad \begin{array}{c}[Y \in (x \in S)\ set] \\ | \\ d(Y) \in C(\{Y\})\end{array} \quad \begin{array}{c}[Y, Z \in (x \in S)\ set, Y(x) \leftrightarrow Z(x)\ true\ [x \in S]] \\ | \\ d(Y) = d(Z) \in C(\{Y\})\end{array}}{\mathsf{P}(c, d) \in C(c)}$$

which, however, requires the use of variables for propositional functions.

Moreover, a standard equality rule should be something similar to the following

$$\frac{U\ set\ [x \in S] \quad \begin{array}{c}[Y \in (x \in S)\ set] \\ | \\ d(Y) \in C(\{Y\})\end{array} \quad \begin{array}{c}[Y, Z \in (x \in S)\ set, Y(x) \leftrightarrow Z(x)\ true\ [x \in S]] \\ | \\ d(Y) = d(Z) \in C(\{Y\})\end{array}}{\mathsf{P}(\{(x \in S)\ U\}, d) = d((x \in S)\ U) \in C(\{(x \in S)\ U\})}$$

These rules are the direct consequence of the introduction rule and the equality introduction condition and they are already not completely within standard type theory. But, the real problem is that they are not sufficient to make $\mathcal{P}(S)$ the set of the subsets of S. For instance, there is no way to obtain a set out of an element of $\mathcal{P}(S)$ and this does not fit with the introduction rule. Thus, to deal with the set $\mathcal{P}(S)$, one should add some rules which link its elements with the elements of the type *set* and with those of the collection $set_\leftrightarrow$, whose elements are propositions but whose equality is induced by logical equivalence, remembering that propositions are identified with sets.

Here, we don't want to propose any particular rule, since we are going to show that there can be no suitable rule, but we simply require that two conditions, which should be a consequence of such rules, are satisfied. The first condition is the following:

Elimination condition
Let $c \in \mathcal{P}(S)$ and $a \in S$. Then there exists a set $a \varepsilon c$.

This condition is suggested by the elimination rule that we have considered. In fact, even if a formal derivation cannot be given until we do not add the suitable rules, a free use of the elimination rule with $C(z) \equiv set_\leftrightarrow$ allows to obtain that $\mathsf{P}(c, (Y)\ Y(a))$ is an element of $set_\leftrightarrow$ and hence, that it is a set that we can identify with the set $a \varepsilon c$. Of course, the above condition is problematic, because it requires the existence of a set but it gives no knowledge about it; in particular, it is not clear if one has to ask for a new set (which are its canonical elements? which are its introduction and elimination rules?) or for an old one (which set should one choose?).

Moreover, as a consequence of the suggested equality rule, we require the following equality condition which is just the β-equality for this kind of (second order) application.

Equality condition
Suppose U *set* $[x \in S]$ and $a \in S$ then $a \varepsilon \{(x \in S)\ U\} \leftrightarrow U[x := a]$ *true*

This condition can be proved as above but using the equality rule; in fact, supposing U *set* $[x \in S]$ and $a \in S$, the equality rule allows to obtain that $a \varepsilon \{(x \in S)\ U\}$ and $U[x := a]$ are equal elements of $set_\leftrightarrow$, which yields our condition. This condition cannot be justified from a semantical point of view, since we have no way to recover the proof element for its conclusion; but, this is also the essential feature which allows us to develop our proof in the next section, without furnishing basic type systems with constructors for classical logic.

It is worth noting that no form of η-equality, like for instance

$$\frac{c \in \mathcal{P}(S)}{\{(x \in S)\ x \varepsilon c\} = c \in \mathcal{P}(S)} \quad x \notin VF(c),$$

is required on $\mathcal{P}(S)$. Also for the other sets of type theory, no rule of η-equality is directly required, because its validity can be proved at least within the extensional version of type theory eTT. This theory is obtained from iTT by substituting the intensional equality proposition by the extensional equality proposition $\mathsf{Eq}(A, a, b)$, which allows to deduce $a = b \in A$ from a proof of $\mathsf{Eq}(A, a, b)$. The problem with extensional equality is that it causes to miss the decidability of the equality judgement and for this reason is usually rejected in the present version of the theory. Here, we can also show that η-equality is a consequence in eTT of the suggested elimination rule for $\mathcal{P}(S)$. In fact, let us assume that Y is a subset of S and that $x \in S$, then $Y(x)$ *set* and hence $x \varepsilon \{Y\} \leftrightarrow Y(x)$ *true* holds because of the equality condition. Then it yields $\{(x \in S)\ x \varepsilon \{Y\}\} = \{Y\} \in \mathcal{P}(S)$ and hence $\mathsf{Eq}(\mathcal{P}(S), \{(x \in S)\ x \varepsilon \{Y\}\}, \{Y\})$; thus, if $c \in \mathcal{P}(S)$, by using the elimination rule one obtains $\mathsf{Eq}(\mathcal{P}(S), \{(x \in S)\ x \varepsilon c\}, c)$ and it yields $\{(x \in S)\ x \varepsilon c\} = c \in \mathcal{P}(S)$. Note that the last step is not allowed in iTT^P.

1.3 iTT^P is consistent

It is well known that by adding to iTT just the collection $\mathcal{P}(\mathbb{1})$, whose elements are (the code for) the non-dependent sets, but using an equality between its elements induced by the *intensional* equality between sets, one obtains an inconsistent extension of iTT [Jac89]. On the contrary, we will prove that any extension of iTT with a powerset as proposed in the previous section, i.e. where the equality between two elements of a powerset is induced by the provability equivalence, is consistent or at least it is not inconsistent because of the rules we have proposed on the powerset and the conditions we have required.

The easiest way to prove this result is to prove first that iTT^P can be embedded in the simpler theory iTT^Ω, which contains only the powerset $\Omega \equiv \mathcal{P}(\mathbb{1})$ of all the subsets of the one element set $\mathbb{1}$ and then to show that such a theory is consistent.

Thus, we will have the following formation and introduction rules

$$\Omega\ set \qquad \Omega = \Omega \qquad \frac{U\ set\ [x \in \mathbb{1}]}{\{(x \in \mathbb{1})\ U\} \in \Omega}$$

Moreover, we require that the introduction equality condition, i.e.

$$\text{if } \ U \leftrightarrow V\ true\ [x \in \mathbb{1}] \ \text{ then } \ \{(x \in \mathbb{1})\ U\} = \{(x \in \mathbb{1})\ V\} \in \Omega,$$

holds, while the condition on the set $a \varepsilon c\ set\ [a \in \mathbb{1}, c \in \Omega]$ can be satisfied by putting, for any $c \in \Omega$,

$$a \varepsilon c \equiv c =_\Omega \top_{\mathbb{1}}$$

where $\top_{\mathbb{1}} \equiv \{(x \in \mathbb{1})\ x =_{\mathbb{1}} x\}$; here any reference to the element a disappeared in the definiens, because all the elements in $\mathbb{1}$ are equal. Finally, we require that

$$\text{if } \ U\ set\ [x \in \mathbb{1}] \ \text{ then } \ \{(x \in \mathbb{1})\ U\} =_\Omega \top_{\mathbb{1}} \leftrightarrow U\ true\ [x \in \mathbb{1}]$$

Now, any powerset can be defined by putting

$$\mathcal{P}(S) \equiv S \to \Omega$$

since, for any $U\ set\ [x \in S]$, one obtains an element in $\mathcal{P}(S)$ by putting

$$\{(x \in S)\ U\} \equiv \lambda((x \in S)\ \{(w \in \mathbb{1})\ U\}),$$

where we suppose that w does not appear free in U, which is in fact an element in $S \to \Omega$. Moreover, for any element $c \in \mathcal{P}(S)$, i.e. a function from S into Ω, and any element $a \in S$, one obtains a set by putting

$$a \varepsilon c \equiv (c(a) =_\Omega \top_{\mathbb{1}})$$

which satisfies the required condition.

Thus, any proof of $c \in \bot$ in iTT^P, i.e. any inconsistency in iTT^P, can be reconstructed in this simpler theory.

Therefore, it is sufficient here to show that this new theory is consistent. This will be done by defining an interpretation $\mathcal{I}$ of this theory into Zermelo-Fraenkel set theory with the axiom of choice, ZFC.

The basic idea is to interpret any non-dependent set A of iTT^Ω into a set $\mathcal{I}(A)$ of ZFC and, provided that

$\mathcal{I}(A_1)$ is a set of ZFC,

$\mathcal{I}(A_2)$ is a map from $\mathcal{I}(A_1)$ into the collection of all sets of ZFC,

$\ldots,$

$\mathcal{I}(A_n)$ is a map from the disjoint union

$$\biguplus_{\alpha_1 \in \mathcal{I}(A_1), \ldots, \alpha_{n-2} \in \mathcal{I}(A_{n-2})(\langle \alpha_1, \ldots, \alpha_{n-3} \rangle)} \mathcal{I}(A_{n-1})(\langle \alpha_1, \ldots, \alpha_{n-2} \rangle)$$

into the collection of all sets of ZFC, then the dependent set of iTT^Ω

$$A(x_1, \ldots, x_n)\ set\ [x_1 \in A_1, \ldots, x_n \in A_n(x_1, \ldots, x_{n-1})],$$

i.e. the propositional function $A \in (x_1 \in A_1) \ldots (x_n \in A_n(x_1, \ldots, x_{n-1}))\ set$, is interpreted into a map from the disjoint union

$$\biguplus_{\alpha_1 \in \mathcal{I}(A_1), \ldots, \alpha_{n-1} \in \mathcal{I}(A_{n-1})(\langle \alpha_1, \ldots, \alpha_{n-2} \rangle)} \mathcal{I}(A_n)(\langle \alpha_1, \ldots, \alpha_{n-1} \rangle)$$

into the collection of all sets of ZFC.

Since the axiom of replacement allows to avoid the use of maps into the *collection* of all sets, which can be substituted by indexed families of sets, all the interpretation can be explained within basic ZFC. Anyway, we think that the approach used here is more perspicuous and well suited for the interpretation of a theory like iTT^{Ω}, where propositional functions have to be considered.

The interpretation $\mathcal{I}(a)$ of a closed term $a \in A$, where A is a non-dependent set of iTT^{Ω}, will be an element of the set $\mathcal{I}(A)$, whereas the interpretation of a non-closed term

$$a(x_1, \ldots, x_n) \in A(x_1, \ldots, x_n)\ [x_1 \in A_1, \ldots, x_n \in A_n(x_1, \ldots, x_{n-1})],$$

i.e. the function-element $a \in (x_1 \in A_1) \ldots (x_n \in A_n(x_1, \ldots, x_{n-1}))\ A(x_1, \ldots, x_n)$, is a function $\mathcal{I}(a)$ which, when applied to the element

$$\alpha \in \biguplus_{\alpha_1 \in \mathcal{I}(A_1), \ldots, \alpha_{n-1} \in \mathcal{I}(A_{n-1})(\langle \alpha_1, \ldots, \alpha_{n-2} \rangle)} \mathcal{I}(A_n)(\langle \alpha_1, \ldots, \alpha_{n-1} \rangle)$$

gives the element $\mathcal{I}(a)(\alpha)$ of the set $\mathcal{I}(A)(\alpha)$.

Now, for the basic sets we put: $\mathcal{I}(\bot) \equiv \emptyset$, $\mathcal{I}(\mathbb{1}) \equiv \{\emptyset\}$ and $\mathcal{I}(\mathsf{Bool}) \equiv \{\emptyset, \{\emptyset\}\}$ and there is an obvious interpretation of their elements.

Moreover, the sets $\Sigma(A, B)$ and $\Pi(A, B)$ are interpreted respectively in the disjoint union and the indexed product of the interpretation of $B(x)$ indexed on the elements of the interpretation of A.

The disjoint sum set $A + B$ is interpreted in the disjoint union of the interpretation of A and B and the interpretation of the equality proposition $a =_A b$ is the characteristic function of the equality of the interpretation of a and b.

Finally, the interpretation of the set Ω is the set $\{\emptyset, \{\emptyset\}\}$.

Moreover, the judgement $A(x_1, \ldots, x_n)$ *true* $[\Gamma]$ is interpreted in $\mathcal{I}(A)(\gamma) \neq \emptyset$ for every $\gamma \in \mathcal{I}(\Gamma)$, which gives $\mathcal{I}(A) \neq \emptyset$ when A is a non-dependent set of iTT^{Ω}.

The interpretation of all the terms is straightforward; thus, here, we only illustrate the interpretation of the elements of the set Ω:

$$\mathcal{I}(\{(x \in \mathbb{1})\ U(x)\}) \equiv \begin{cases} \emptyset & \text{if } \mathcal{I}(U(*)) = \emptyset \\ \{\emptyset\} & \text{if } \mathcal{I}(U(*)) \neq \emptyset \end{cases}$$

After this definition, for any subset U of $\mathbb{1}$, $\mathcal{I}(\{(x \in \mathbb{1})\ U\} =_{\Omega} \top_{\mathbb{1}} \leftrightarrow U) \neq \emptyset$ by the axiom of choice and hence the equality condition is valid.

It is tedious, but straightforward, to check that all the rules of iTT^{Ω} are valid according to this interpretation and hence that any proof of the judgement $a \in \bot$ within iTT^{Ω}, i.e. any form of inconsistency, would result in a proof that there is some element in $\emptyset$, that is an inconsistency in ZFC.

1.4 iTT^P is classical

We are going to prove that iTT^P gives rise to classical logic, i.e., for any set A, the judgement $A \vee \neg A$ *true* holds. Even if iTT^P is *not* a topos, the proof we show here is obtained by adapting to our framework an analogous result stating that any topos which satisfies the axiom of choice is boolean. Among the various proofs of this result (cf. for instance [LS86],[Bel88]), which goes back to Diaconescu's work, which shows that by adding the axiom of choice to IZF one obtains ZF [Dia75], we choose to translate the proof of Bell [Bel88], because it is very well suited to work in iTT^P, since it is almost completely developed within local set theory instead that in topos theory, except for the use of a choice rule.

In iTT^P, the result is a consequence of the strong elimination rule for disjoint union which allows to prove the so called *intuitionistic axiom of choice*, i.e.

$$((\forall x \in A)(\exists y \in B)\ C(x, y)) \to ((\exists f \in A \to B)(\forall x \in A)\ C(x, f(x)))\ \mathit{true}$$

Let us recall the proof [Mar84]. Assume that $h \in (\forall x \in A)(\exists y \in B)\ C(x, y)$ and that $x \in A$. Then $h(x) \in (\exists y \in B)\ C(x, y)$. Let $\pi_1(-)$ and $\pi_2(-)$ be the first and second projection respectively; then the elimination rule for the set of the disjoint union allows to prove that $\pi_1(h(x)) \in B$ and $\pi_2(h(x)) \in$

$C(x, \pi_1(h(x)))$. Hence, by putting $f \equiv \lambda x.\pi_1(h(x))$, we obtain both $f \in A \to B$ and $\pi_2(h(x)) \in C(x, f(x))$ since, by β-equality, $f(x) \equiv (\lambda x.\pi_1(h(x)))(x) = \pi_1(h(x))$. Finally, we conclude by *true introduction*.

Since in the following we will use mainly the powerset $\mathcal{P}(\mathbb{1})$, we introduce some abbreviations besides $\Omega \equiv \mathcal{P}(\mathbb{1})$ and $\top_{\mathbb{1}} \equiv \{(w \in \mathbb{1})\ w =_{\mathbb{1}} w\}$ that we have already used in section 1.3; let us suppose that U is a set and $w \in \mathbb{1}$ is a variable which does not appear in U, then we put $[U] \equiv \{(w \in \mathbb{1})\ U\}$ and, supposing $p \in \Omega$, we put $\overline{p} \equiv *\varepsilon p$. Moreover, following a standard logical practice, supposing A is a set, we will simply write A to assert the judgement A *true*. It is convenient to state here all the properties of the intensional equality proposition Id that we need in the following. First, we recall some well known results: Id is an equivalence relation; moreover, if A and B are sets and $a =_A c$ and $f =_{A \to B} g$ then $f(a) =_B g(c)$ (for a proof see [NPS90], page 64).

On the other hand, the following properties of Id are specific to the new set Ω. They are similar to the properties that the set Id enjoys when it is used on elements of the set U_0, i.e. the universe of the small sets. In fact, Ω resembles this set, but it differs also both because of the considered equality and because a code for each set is present in Ω, whereas only the codes for the small sets can be found in U_0.

Lemma 1.4.1 *If $p =_\Omega q$ then $\overline{p} \leftrightarrow \overline{q}$.*

Proof. Let $x \in \Omega$; then $\overline{x} \leftrightarrow \overline{x}$ and hence $\overline{p} \leftrightarrow \overline{q}$ is a consequence of $p =_\Omega q$ by Id-elimination.
■

Lemma 1.4.2 $\neg(\mathsf{true} =_{\mathsf{Bool}} \mathsf{false})$.

Proof. Let $x \in \mathsf{Bool}$; then $\mathsf{if}\ x\ \mathsf{then}\ [\mathbb{1}]\ \mathsf{else}\ [\bot] \in \Omega$. Now, suppose that $\mathsf{true} =_{\mathsf{Bool}} \mathsf{false}$, then $\mathsf{if\ true\ then}\ [\mathbb{1}]\ \mathsf{else}\ [\bot] =_\Omega \mathsf{if\ false\ then}\ [\mathbb{1}]\ \mathsf{else}\ [\bot]$, which yields $[\mathbb{1}] =_\Omega [\bot]$ by boole-equality and transitivity.

Thus, by the previous lemma, $\overline{[\mathbb{1}]} \leftrightarrow \overline{[\bot]}$ but $\overline{[\mathbb{1}]} \leftrightarrow \mathbb{1}$ and $\overline{[\bot]} \leftrightarrow \bot$ by the equality condition; hence $\bot$ *true* and thus, by discharging the assumption $\mathsf{true} =_{\mathsf{Bool}} \mathsf{false}$, we obtain the result.
■

We will start, now, the proof of the main result of this section. The trick to internalize the proof in [Bel88] within iTT^P is stated in the following lemma.

Lemma 1.4.3 *For any set A, if A true then $\mathsf{id}(\top_{\mathbb{1}}) \in [A] =_\Omega \top_{\mathbb{1}}$ and hence $[A] =_\Omega \top_{\mathbb{1}}$ and if $[A] =_\Omega \top_{\mathbb{1}}$ then A true.*

Proof. If A *true* then $A \leftrightarrow w =_{\mathbb{1}} w$ *true* $[w \in \mathbb{1}]$ and hence $[A] = \top_{\mathbb{1}} \in \Omega$, which implies $\mathsf{id}(\top_{\mathbb{1}}) \in [A] =_\Omega \top_{\mathbb{1}}$ and hence $[A] =_\Omega \top_{\mathbb{1}}$ *true* by true introduction; on the other hand, if $[A] =_\Omega \top_{\mathbb{1}}$ then, by lemma 1.4.1, $\overline{[A]} \leftrightarrow \overline{\top_{\mathbb{1}}}$, but $\overline{[A]} \leftrightarrow A$ and $* =_{\mathbb{1}} * \leftrightarrow \overline{\top_{\mathbb{1}}}$, by the equality condition, and hence A *true* since $* =_{\mathbb{1}} *$ *true*.
■

Indeed, after this lemma it is possible to obtain, for any proposition A, a logically equivalent proposition, i.e. $[A] =_\Omega \top_{\mathbb{1}}$, such that, if A *true*, the proof element $\mathsf{id}(\top_{\mathbb{1}})$ of $[A] =_\Omega \top_{\mathbb{1}}$ has no memory of the proof element which testifies the truth of A. We will see that this property is essential in the proof of the following theorems. We will use it immediately in the following proposition where, instead of the proposition $\overline{\pi_1(w)} \vee \overline{\pi_2(w)}$ *set* $[w : \Omega \times \Omega]$, we use $[\overline{\pi_1(w)} \vee \overline{\pi_2(w)}] =_\Omega \top_{\mathbb{1}}$ *set* $[w : \Omega \times \Omega]$ in order to avoid that the proof-term in the main statement depends on the truth of the first or of the second disjunct.

We can now prove:

Proposition 1.4.4 *In iTT^P the following proposition*

$$(\forall z \in \Sigma(\Omega \times \Omega, (w)\ [\overline{\pi_1(w)} \vee \overline{\pi_2(w)}] =_\Omega \top_{\mathbb{1}}))$$
$$(\exists x \in \mathsf{Bool})\ (x =_{\mathsf{Bool}} \mathsf{true} \to \overline{\pi_1(\pi_1(z))}) \wedge (x =_{\mathsf{Bool}} \mathsf{false} \to \overline{\pi_2(\pi_1(z))})$$

is true.

Proof. Suppose $z \in \Sigma(\Omega \times \Omega, (w)\ [\overline{\pi_1(w)} \ \vee \ \overline{\pi_2(w)}] =_\Omega \top_{\mathbb{1}})$ then $\pi_1(z) \in \Omega \times \Omega$ and $\pi_2(z)$ is a proof of $[\overline{\pi_1(\pi_1(z))} \ \vee \ \overline{\pi_2(\pi_1(z))}] =_\Omega \top_{\mathbb{1}}$. Thus, by lemma 1.4.3, $\overline{\pi_1(\pi_1(z))} \ \vee \ \overline{\pi_2(\pi_1(z))}$. The result can now be proved by $\vee$-elimination. In fact, if $\overline{\pi_1(\pi_1(z))}$ *true* then $\mathsf{true} =_{\mathsf{Bool}} \mathsf{true} \to \overline{\pi_2(\pi_1(z))}$; moreover, by lemma 1.4.2, $\neg(\mathsf{true} =_{\mathsf{Bool}} \mathsf{false})$ and hence $\mathsf{true} =_{\mathsf{Bool}} \mathsf{false} \to \overline{\pi_2(\pi_1(z))}$. Thus, we obtain that

$$(\exists x \in \mathsf{Bool})\ (x =_{\mathsf{Bool}} \mathsf{true} \to \overline{\pi_1(\pi_1(z))}) \ \wedge \ (x =_{\mathsf{Bool}} \mathsf{false} \to \overline{\pi_2(\pi_1(z))})$$

On the other hand, by means of a similar proof, we reach the same conclusion starting from the assumption $\overline{\pi_2(\pi_1(z))}$ *true*.
∎

Thus, we can use the intuitionistic axiom of choice to obtain:

Proposition 1.4.5 *In iTT^P the following proposition*

$$\begin{array}{l}(\exists f \in \Sigma(\Omega \times \Omega, (w)\ [\overline{\pi_1(w)} \ \vee \ \overline{\pi_2(w)}] =_\Omega \top_{\mathbb{1}}) \to \mathsf{Bool}) \\ \quad (\forall z \in \Sigma(\Omega \times \Omega, (w)\ [\overline{\pi_1(w)} \ \vee \ \overline{\pi_2(w)}] =_\Omega \top_{\mathbb{1}})) \\ \quad\quad (f(z) =_{\mathsf{Bool}} \mathsf{true} \ \to \ \overline{\pi_1(\pi_1(z))}) \ \wedge \ (f(z) =_{\mathsf{Bool}} \mathsf{false} \ \to \ \overline{\pi_2(\pi_1(z))})\end{array}$$

is true.

Suppose, now, that A is a set; then

$$\langle\langle [A], \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle \in \Sigma(\Omega \times \Omega, (w)\ [\overline{\pi_1(w)} \ \vee \ \overline{\pi_2(w)}] =_\Omega \top_{\mathbb{1}})$$

In fact, $\langle [A], \top_{\mathbb{1}}\rangle \in \Omega \times \Omega$. Moreover $\overline{\top_{\mathbb{1}}}$ *true* and hence $\overline{\pi_1(\langle [A], \top_{\mathbb{1}}\rangle)} \ \vee \ \overline{\pi_2(\langle [A], \top_{\mathbb{1}}\rangle)}$; thus, by lemma 1.4.3, $\mathsf{id}(\top_{\mathbb{1}}) \in [\overline{\pi_1(\langle [A], \top_{\mathbb{1}}\rangle)} \ \vee \ \overline{\pi_2(\langle [A], \top_{\mathbb{1}}\rangle)}] \ =_\Omega \top_{\mathbb{1}}$.

Now, let f be the choice function, i.e. use an $\exists$-elimination rule on the judgement in the proposition 1.4.5; then $f(\langle\langle [A], \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} \mathsf{true} \ \to \ \overline{[A]}$. But

$$(f(\langle\langle [A], \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} \mathsf{true}) \ \vee \ (f(\langle\langle [A], \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} \mathsf{false})$$

since the set Bool is decidable (for a proof see [NPS90], page. 177), and hence, by $\vee$-elimination and a little of intuitionistic logic, one obtains that

$$(1) \quad \overline{[A]} \ \vee \ (f(\langle\langle [A], \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} \mathsf{false})$$

Analogously, one can prove that

$$(2) \quad \overline{[A]} \ \vee \ (f(\langle\langle \top_{\mathbb{1}}, [A]\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} \mathsf{true})$$

Thus, by using distributivity on the conjunction of (1) and (2), one finally obtains

Proposition 1.4.6 *For any set A in iTT^P the following proposition*

$$\begin{array}{l}(\exists f \in \Sigma(\Omega \times \Omega, (w)\ [\overline{\pi_1(w)} \ \vee \ \overline{\pi_2(w)}] =_\Omega \top_{\mathbb{1}}) \to \mathsf{Bool}) \\ \quad \overline{[A]} \ \vee \ ((f(\langle\langle [A], \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} \mathsf{false}) \ \wedge \ (f(\langle\langle \top_{\mathbb{1}}, [A]\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} \mathsf{true}))\end{array}$$

is true.

Let us now assume $\overline{[A]}$ *true*; then, by equality condition A *true*, from which by equality introduction condition, that is by extensionality on subsets, $[A] = \top_{\mathbb{1}} \in \Omega$ and hence

$$\langle\langle [A], \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle =_{\Sigma(\Omega\times\Omega,\ldots)} \langle\langle \top_{\mathbb{1}}, \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle.$$

Thus $f(\langle\langle [A], \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} f(\langle\langle \top_{\mathbb{1}}, \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle)$, where f is obtained by an $\exists$-elimination rule on the judgement in the proposition 1.4.6. With the same assumption, also $f(\langle\langle \top_{\mathbb{1}}, [A]\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} f(\langle\langle \top_{\mathbb{1}}, \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle)$ can be proved in a similar way; hence

$$f(\langle\langle [A], \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} f(\langle\langle \top_{\mathbb{1}}, [A]\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle)$$

Then assuming both $\overline{[A]}$ *true* and

$$(f(\langle\langle[A], \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} \mathsf{false}) \ \wedge \ (f(\langle\langle\top_{\mathbb{1}}, [A]\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} \mathsf{true})$$

one can conclude $\mathsf{true} =_{\mathsf{Bool}} \mathsf{false}$. But, by lemma 1.4.2, $\neg(\mathsf{true} =_{\mathsf{Bool}} \mathsf{false})$. Hence, under the assumption

$$(f(\langle\langle[A], \top_{\mathbb{1}}\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} \mathsf{false}) \ \wedge \ (f(\langle\langle\top_{\mathbb{1}}, [A]\rangle, \mathsf{id}(\top_{\mathbb{1}})\rangle) =_{\mathsf{Bool}} \mathsf{true}),$$

the judgement $\neg\overline{[A]}$ *true* holds. Thus, by using proposition 1.4.6 and a little of intuitionistic logic, we can conclude $(\overline{[A]} \vee \neg\overline{[A]})$ *true* which, by the equality condition, yields $(A \vee \neg A)$ *true*. Thus, provided one can give for the powerset suitable rules which allow our conditions to hold, i.e. which really express the meaning of the powerset, and that meanwhile allow to keep the usual meaning for the judgement C *true*, i.e. C *true* holds if and only if there exists a proof element for the set C, then we would have a proof element for the set $A \vee \neg A$, which is expected to fail.

1.5 Conclusion

To help the reader who knows the proof in [Bel88], it may be useful to explain the differences between the original proof and that presented in the previous section. Our proof is not the plain application of Bell's result to iTT^P, since iTT^P is not a topos. It is possible to obtain a topos out of the extensional theory eTT^P, obtained by adding a powerset constructor to eTT, if one adds to it also the rule of η-equality for powersets, as in the end of section 1.2. But, it is not necessary to be within a topos to reconstruct Diaconescu's result and a weaker theory is sufficient. It is also possible to get the same result, by replacing the equality introduction condition with a weaker rule, stating: if $U \leftrightarrow V$ *true* $[x \in S]$ then there is a proof-term $c(U, V)) \in \{(x \in S)\ U\} =_{\mathcal{P}(S)} \{(x \in S)\ V\}$.

This fact suggests that it is not possible to extend Martin-Löf's type theory, where proof-elements can be provided for any provable set, to an intuitionistic theory of sets fully equipped with powersets, like topos theory, following the isomorphism "propositions as sets" and preserving the constructive meaning of the connectives: one has to choose between predicativity and expressive power.

Chapter 2

Effective quotients in constructive type theory

Summary We extend Martin-Löf's Constructive Set Theory with effective quotient sets and the uniqueness of propositional equality proofs. We prove that in the presence of at least two universes U_0 and U_1, to which the codes of quotient sets are added, the principle of excluded middle holds for small sets. The key point is effectiveness condition, that allows us to recover information on the equivalence relation, from the equality on the quotient set.

2.1 Introduction

Within the framework of Martin-Löf's set theory, in order to generate some formal topologies the quotient sets are also desirable [NV97]. But, some care is necessary in extending Martin-Löf's set theory by quotient sets, if we want to keep constructivity.

Here, we consider to extend the intensional version of Martin-Löf's type theory with the quotient sets as formulated in [Hof95] and with the addition of effectiveness condition and the uniqueness of propositional equality proofs. We show that with the presence of at least two universes U_0 and U_1, to which the codes of quotient sets are added, then the principle of excluded middle holds for small sets. Precisely, in this extension we can reproduce the proof that the axiom of choice implies the principle of excluded middle, at least for small sets. The key point to do this proof is the application of the equality rule of quotient sets combined together with the effectiveness condition on the quotients of the first two universes under equiprovability.

Of course, an analogous proof can be reproduced in the extensional version of Martin-Löf's type theory with the quotient sets as given in Nuprl [Con86] and always with the addition of effectiveness condition. We know that the effectiveness condition is surely derivable for decidable equivalence relations, but the general effectiveness condition is problematic, because it restores information that has been forgotten in the introduction rule for equality of equivalence classes.

The interest in effectiveness condition arises from mathematical practice of quotient sets. In order to keep effectiveness for quotient sets, an alternative strategy could be to let only quotient sets based on equivalence relations, which are proof-irrelevant, as it is in the type theory of Heyting pretoposes proposed in this thesis.

2.2 Extension of iTT with quotient sets

We extend the intensional Martin-Löf constructive set theory by quotient sets and uniqueness of proofs for the intensional propositional equality as in [Hof95] (page 111), with the following inference rules. We call this extension iTT^Q.

Intensional Quotient set

$$\frac{\begin{array}{l} R(x,y)\ set\ [x \in A, y \in A] \\ c_1 \in R(x,x)[x \in A], \qquad c_2 \in R(y,x)[x \in A, y \in A, z \in R(x,y)] \\ c_3 \in R(x,z)[x \in A, y \in A, z \in A, w \in R(x,y), w' \in R(y,z)] \end{array}}{A/R\ set}$$

I-int.quotient

$$\frac{a \in A}{[a] \in A/R}$$

We specify also the equality between terms of A/R
eq -int.quotient

$$\frac{a \in A \quad b \in A \quad d \in R(a,b)}{\mathsf{Qax}(d) \in \mathsf{Id}(A/R, [a], [b])}$$

E-int.quotient

$$\frac{\begin{array}{l} s \in A/R \quad l(x) \in L([x])[x \in A] \\ h \in \mathsf{Id}(L([y]), \mathsf{sub}(\mathsf{Qax}(d), l(x)), l(y))\ [x \in A, y \in A, d \in R(x,y)] \end{array}}{\mathsf{Q}(l,s) \in L(s)}$$

where *sub* is defined as in [NPS90] page.64 for substitution with equal elements;
C-int.quotient

$$\frac{\begin{array}{l} a \in A \quad l(x) \in L([x])[x \in A] \\ h \in \mathsf{Id}(L([y]), \mathsf{sub}(\mathsf{Qax}(d), l(x)), l(y))\ [x \in A, y \in A, d \in R(x,y)] \end{array}}{\mathsf{Q}(l,[a]) = l(a) \in L([a])}$$

We also want to make equivalence relations effective:
Effectiveness condition

$$\frac{a \in A \quad b \in A \quad \mathsf{Id}(A/R, [a], [b])\ true}{R(a,b)\ true}$$

Moreover, we add the axiom of uniqueness of propositional equality proofs:
Id-Uni I

$$\frac{a \in A \quad p \in \mathsf{Id}(A,a,a)}{\mathsf{iduni}(a,p) \in \mathsf{Id}(\mathsf{Id}(A,a,a), p, \mathsf{id}(a))}$$

The corresponding conversion rule is the following:
Id-Uni conv

$$\frac{a \in A}{\mathsf{iduni}(a, \mathsf{id}(a)) = \mathsf{id}(\mathsf{id}(a)) \in \mathsf{Id}(\mathsf{Id}(A,a,a), \mathsf{id}(a), \mathsf{id}(a))}$$

By Id-Uni and the elimination rule of propositional equality on the proposition

$$\Pi_{w \in \mathsf{Id}(A,x,y)} \mathsf{Id}(\mathsf{Id}(A,x,y), w, z)\ [x \in A, y \in A, z \in \mathsf{Id}(A,x,y)]$$

Streicher proved that under the context $x \in A, y \in A, z \in \mathsf{Id}(A,x,y), w \in \mathsf{Id}(A,x,y)$ this proof-term

$$\mathsf{idpeel}(z, \lambda w \in \mathsf{Id}(A,x,x).\mathsf{iduni}(x,w))(w)$$

is of type

$$\mathsf{Id}(\mathsf{Id}(A,x,y), w, z)\ [x \in A, y \in A, z \in \mathsf{Id}(A,x,y), w \in \mathsf{Id}(A,x,y)]$$

that is the uniqueness of proofs of propositional equality type, called UIP (see [Hof95] page.81).

Remark 2.2.1 The uniqueness of proofs of propositional equality type is definable by pattern-matching, but it is not derivable in the intensional version of Martin-Löf's type theory, as showed by M. Hofmann and T. Streicher (see [Hof95]). In our proof of the principle of excluded middle for small sets, the use of this principle seems crucial.

Remark 2.2.2 Note that in order to do the proof of excluded middle for small sets, we never make use of the elimination rule for the quotient set and of the conversion rule Id-Uni conv.

Moreover, we consider the first universe U_0, whose elements are called small sets, and the second universe U_1, whose elements are called large sets, as in [Mar84] and [NPS90].
We have to add to the rules for these universes the following introduction rule for the codes of quotient sets for $i = 0, 1$:
UQ-I

$$\frac{\begin{array}{ll} a \in U_i & r(x,y) \in U_i\ [x \in T_i(a), y \in T_i(a)] \\ c_1 \in T_i(r(x,x))\ [x \in T_i(a)], & c_2 \in T_i(r(y,x))\ [x \in T_i(a), y \in T_i(a), z \in T_i(r(x,y))] \\ \multicolumn{2}{l}{c_3 \in T_i(r(x,z))\ [x \in T_i(a), y \in T_i(a), z \in T_i(a), w \in T_i(r(x,y)), w' \in T_i(r(y,z))]} \end{array}}{a\hat{/}r \in U_i}$$

with the following conversion rule:

$$T_i(a\hat{/}r) = T_i(a)/T_i(r(x,y))$$

About the properties of judgements of the form A *true* (see [Mar84]) we refer to the section 1.2.

This extension of iTT is consistent, because there exists an interpretation in classical set theory (ZFC) plus two strongly inaccessible cardinals, by interpreting the quotient sets in classical quotient sets and the first two universes respectively in the set of small sets and in the set of large sets.

2.3 Small sets are classical

We are going to prove that for small sets in iTT^Q the principle of excluded middle holds, i.e. for any element of the first universe $a \in U_0$, the judgement $T_0(a) \vee \neg T_0(a)$ *true* holds. By quotienting the two universes under the relation of equiprovability among their elements, we simulate the powersets. The proof we show here is obtained by adapting to our framework the analogous result of section 1.4, that the axiom of choice implies the principle of excluded middle in the presence of extensional powersets.

Therefore, also in iTT^Q the result is a consequence of the strong elimination rule for disjoint union which allows to prove the so called *intuitionistic axiom of choice* as in section 1.4, i.e.

$$((\forall x \in A)(\exists y \in B)\ C(x,y)) \to ((\exists f \in A \to B)(\forall x \in A)\ C(x, f(x)))\ true$$

In order to recover the proof of the principle of excluded middle from the axiom of choice, we quotient the first two universes under the equivalence relation of equiprovability, i.e.

$$T_0(x) \leftrightarrow T_0(y)\ [x \in U_0, y \in U_0] \qquad\qquad T_1(x) \leftrightarrow T_1(y)\ [x \in U_1, y \in U_1]$$

Then we use the following abbreviations for $i = 0, 1$

$$\Omega_i \equiv U_i/\ T_i(x) \leftrightarrow T_i(y)$$

Since there is a code of U_0 in U_1

$$\widehat{U_0} \in U_1$$

then there is inside U_1 the code for Ω_o

$$\widehat{\Omega_0} \equiv \widehat{U_0}\hat{/}\ x \hat{\leftrightarrow} y$$

Indeed, we can derive

$$\widehat{\Omega_0} \in \Omega_1 \qquad \text{and} \qquad T_1(\widehat{\Omega_0}) = \Omega_0$$

Note that we do not distinguish the codes of U_0 and U_1, with t_0 and t_1 as in [Dyb97], in order to make formulas more readable.

The reason to use the first two universes is due to the possibility of deriving

$$\hat{\mathsf{Id}}(\hat{\Omega_o}, z, [\widehat{\top}]) \in U_1 \ [z \in \Omega_0]$$

where $\top$ is the terminal type (see the rule in section 3.2). We use the abbreviation $a =_A b$ for $\mathsf{Id}(A, a, b)$, when it is not coded in a universe.

Following a standard logical practice, supposing A is a proposition, we will simply write A to assert the judgement A *true*.

We recall that, in the presence of U_0, we can derive

$$\neg(\mathsf{true} =_{\mathsf{Bool}} \mathsf{false})$$

We will start now the proof of the main result of this section. One of the key points to internalize the proof in [Bel88] within iTT^Q is stated in the following lemma.

Lemma 2.3.1 *For any set $a \in U_i$, $[a] =_{\Omega_i} [\widehat{\top}]$ iff $T_i(a)$ true for $i = 0, 1$.*

Proof. From $[a] =_{\Omega_i} [\widehat{\top}]$ *true* by effectiveness of quotient sets we get $T_i(a) \leftrightarrow T_i(\widehat{\top})$ *true*, but $T_i(\widehat{\top}) = \top$ so $T_i(a)$ *true*. On the other hand, from $T_i(a)$ *true*, we get $T_i(a) \leftrightarrow T_i(\widehat{\top})$ and by the equality rule on the quotient set we conclude $[a] =_{\Omega_i} [\widehat{\top}]$.
■

Now we consider the following abbreviations: for $z \in \Omega_0$

$$E(z) \equiv \mathsf{Id}(\Omega_0, z, [\widehat{\top}])$$

We can now prove:

Proposition 2.3.2 *In iTT^Q the following proposition*

$$\begin{array}{c}(\forall z \in \Sigma(\Omega_0 \times \Omega_0, (w)\ [\widehat{E(\pi_1(w))} \hat{\vee}\ \widehat{E(\pi_2(w))}] =_{\Omega_1} [\widehat{\top}]) \\ (\exists x \in \mathsf{Bool})\ (x =_{\mathsf{Bool}} \mathsf{true} \ \rightarrow\ E(\pi_1(\pi_1(z)))\ \wedge \\ (x =_{\mathsf{Bool}} \mathsf{false}\ \rightarrow\ E(\pi_2(\pi_1(z)))\end{array}$$

is true.

Proof. Suppose $z \in \Sigma(\Omega_0 \times \Omega_0, (w)\ [\widehat{E(\pi_1(w))} \hat{\vee}\ \widehat{E(\pi_2(w))}] =_{\Omega_1} [\widehat{\top}]$. Then $\pi_1(z) \in \Omega_0 \times \Omega_0$ and $\pi_2(z)$ is a proof of $[\widehat{E(\pi_1(\pi_1(z)))} \hat{\vee}\ \widehat{E(\pi_2(\pi_1(z)))}] =_{\Omega_1} [\widehat{\top}]$. Thus, by lemma 2.3.1 and by the conversion rules for U_1, $E(\pi_1(\pi_1(z)))\ \vee\ E(\pi_2(\pi_1(z)))$. The result can now be proved as in the proposition 1.4.4.
■

Thus, we can use the intuitionistic axiom of choice to obtain:

Proposition 2.3.3 *In iTT^Q the following proposition*

$$\begin{array}{l}(\exists f \in \Sigma(\Omega_0 \times \Omega_0, (w)\ [\widehat{E(\pi_1(w))} \hat{\vee}\ \widehat{E(\pi_2(w))}] =_{\Omega_1} [\widehat{\top}]) \rightarrow \mathsf{Bool}) \\ \quad (\forall z \in \Sigma(\Omega_0 \times \Omega_0, (w)\ [\widehat{E(\pi_1(w))} \hat{\vee}\ \widehat{E(\pi_2(w))}] =_{\Omega_1} [\widehat{\top}]) \\ \quad\quad (f(z) =_{\mathsf{Bool}} \mathsf{true}\ \rightarrow\ E(\pi_1(\pi_1(z)))\ \wedge\ (f(z) =_{\mathsf{Bool}} \mathsf{false}\ \rightarrow\ E(\pi_2(\pi_1(z)))\end{array}$$

is true.

Suppose, now, that $a \in U_0$ is a small set; then

$$\langle\langle [a], [\widehat{\top}]\rangle, \mathsf{Qax}(\langle \lambda y.\star, \lambda y'.\mathsf{inr}(\mathsf{id}([\widehat{\top}]))\rangle)\rangle$$

is of type

$$\Sigma(\Omega_0 \times \Omega_0, (w)\ [\widehat{E(\pi_1(w))} \hat{\vee}\ \widehat{E(\pi_2(w))}] =_{\Omega_1} [\widehat{\top}])$$

where $\star \in \top$. In fact, $\langle [a], [\widehat{\top}] \rangle \in \Omega_0 \times \Omega_0$ and

$$\langle \lambda y.\star, \lambda y'.\mathsf{inr}(\mathsf{id}([\widehat{\top}])) \rangle \in \mathsf{Id}(\Omega_0, [a], [\widehat{\top}]) \vee \mathsf{Id}(\Omega_0, [\widehat{\top}], [\widehat{\top}]) \leftrightarrow \top$$

from which, since

$$\mathsf{Id}(\Omega_0, [a], [\widehat{\top}]) \vee \mathsf{Id}(\Omega_0, [\widehat{\top}], [\widehat{\top}]) \leftrightarrow \top = T_1(\widehat{E([a])} \hat{\vee} \widehat{E([\widehat{\top}])}) \leftrightarrow T_1(\widehat{\top})$$

by the equality rule on the quotient set we get

$$\mathsf{Qax}(\langle \lambda y.\star, \lambda y'.\mathsf{inr}(\mathsf{id}([\widehat{\top}])) \rangle) \in [\widehat{E([a])} \hat{\vee} \widehat{E([\widehat{\top}])}] =_{\Omega_1} [\widehat{\top}]$$

Analogously,

$$\langle \langle [\widehat{\top}], [a] \rangle, \mathsf{Qax}(\langle \lambda y.\star, \lambda y'.\mathsf{inl}(\mathsf{id}([\widehat{\top}])) \rangle) \rangle$$

is of type

$$\Sigma(\Omega_0 \times \Omega_0, (w)\ [\widehat{E(\pi_1(w))} \hat{\vee} \widehat{E(\pi_2(w))}] =_{\Omega_1} [\widehat{\top}])$$

since $\langle [\widehat{\top}], [a] \rangle \in \Omega_0 \times \Omega_0$ and

$$\langle \lambda y.\star, \lambda y'.\mathsf{inl}(\mathsf{id}([\widehat{\top}])) \rangle \in \mathsf{Id}(\Omega_0, [\widehat{\top}], [\widehat{\top}]) \vee \mathsf{Id}(\Omega_0, [a], [\widehat{\top}]) \leftrightarrow \top$$

Let us put for $w \in \Omega_0$

$$\mathsf{q}_1(w) \equiv \mathsf{Qax}(\langle \lambda y.\star, \lambda y'.\mathsf{inr}(\mathsf{id}([\widehat{\top}])) \rangle)$$

and

$$\mathsf{q}_2(w) \equiv \mathsf{Qax}(\langle \lambda y.\star, \lambda y'.\mathsf{inl}(\mathsf{id}([\widehat{\top}])) \rangle)$$

Now, let f be the choice function, i.e. use an $\exists$-elimination rule on the judgement in the proposition 2.3.3; then as in the proof of proposition 1.4.6 $f(\langle \langle [a], [\widehat{\top}] \rangle, \mathsf{q}_1([a]) \rangle) =_{\mathsf{Bool}} \mathsf{true} \rightarrow E([a])$. But

$$(f(\langle \langle [a], [\widehat{\top}] \rangle, \mathsf{q}_1([a]) \rangle) =_{\mathsf{Bool}} \mathsf{true}) \ \vee \ (f(\langle \langle [a], [\widehat{\top}] \rangle, \mathsf{q}_1([a]) \rangle) =_{\mathsf{Bool}} \mathsf{false})$$

since the set Bool is decidable (for a proof see [NPS90], page. 177), and hence, by $\vee$-elimination, lemma 2.3.1 and a little of intuitionistic logic, one obtains that

$$(1) \quad T_0(a) \ \vee \ (f(\langle \langle [a], [\widehat{\top}] \rangle, \mathsf{q}_1([a]) \rangle) =_{\mathsf{Bool}} \mathsf{false})$$

and in an analogous way

$$(2) \quad T_0(a) \ \vee \ (f(\langle \langle [\widehat{\top}], [a] \rangle, \mathsf{q}_2([a]) \rangle) =_{\mathsf{Bool}} \mathsf{true})$$

Thus, by using distributivity on the conjunction of (1) and (2), one finally obtains

Proposition 2.3.4 *For any small set $a \in U_0$ in iTT^Q the following proposition*

$$\begin{array}{l}(\exists f \in \Sigma(\Omega_0 \times \Omega_0, (w)\ [\widehat{E(\pi_1(w))} \hat{\vee} \widehat{E(\pi_2(w))}] =_{\Omega_1} [\widehat{\top}]) \rightarrow \mathsf{Bool}) \\ \quad T_0(a) \vee (f(\langle \langle [a], [\widehat{\top}] \rangle, \mathsf{q}_1([a]) \rangle) =_{\mathsf{Bool}} \mathsf{false}) \ \wedge \ f(\langle \langle [a], [\widehat{\top}] \rangle, \mathsf{q}_2([a]) \rangle) =_{\mathsf{Bool}} \mathsf{true}))\end{array}$$

is true.

Let us now assume $T_0(a)$ *true*; then, by lemma 2.3.1, $[a] =_{\Omega_0} [\widehat{\top}]$ *true* and hence

$$\langle \langle [a], [\widehat{\top}] \rangle, \mathsf{q}_1([a]) \rangle =_{\Sigma(\Omega_0 \times \Omega_0, \ldots)} \langle \langle [\widehat{\top}], [\widehat{\top}] \rangle, \mathsf{q}_1([\widehat{\top}]) \rangle$$

by the elimination rule of the intensional propositional equality with respect to the proposition

$$\langle \langle x, [\widehat{\top}] \rangle, \mathsf{q}_1(x) \rangle =_{\Sigma(\Omega_0 \times \Omega_0, \ldots)} \langle \langle y, [\widehat{\top}] \rangle, \mathsf{q}_1(y) \rangle \ [x \in \Omega_0, y \in \Omega_0]$$

where $\mathsf{q}_1(x) \equiv \mathsf{Qax}(\langle \lambda y.\star, \lambda y'.\mathsf{inr}(\mathsf{id}([\widehat{\top}]))\rangle)$ $[x \in \Omega_0]$ and $\mathsf{q}_2(y) \equiv \mathsf{Qax}(\langle \lambda y.\star, \lambda y'.\mathsf{inl}(\mathsf{id}([\widehat{\top}]))\rangle)$ $[y \in \Omega_0]$. Thus, $f(\langle\langle [a], [\widehat{\top}]\rangle, \mathsf{q}_1([a])\rangle) =_{\mathsf{Bool}} f(\langle\langle [\widehat{\top}], [\widehat{\top}]\rangle, \mathsf{q}_1([\widehat{\top}])\rangle)$, where f is obtained by an $\exists$-elimination rule on the judgement in the proposition 2.3.4. With the same assumption, also

$$f(\langle\langle [\widehat{\top}], [a]\rangle, \mathsf{q}_2([a])\rangle) =_{\mathsf{Bool}} f(\langle\langle [\widehat{\top}], [\widehat{\top}]\rangle, \mathsf{q}_2([\widehat{\top}])\rangle)$$

can be proved in a similar way; hence, since by uniqueness of propositional equality proofs UIP we get a proof-term of

$$\mathsf{q}_1([\widehat{\top}]) =_{[\widehat{E([\widehat{\top}])} \hat{\vee}\ \widehat{E([\widehat{\top}])}] =_{\Omega_1} [\widehat{\top}]} \mathsf{q}_2([\widehat{\top}])$$

we conclude by the elimination rule of propositional equality

$$\langle\langle [\widehat{\top}], [\widehat{\top}]\rangle, \mathsf{q}_1([\widehat{\top}])\rangle =_{\Sigma(\Omega_0 \times \Omega_0, \ldots)} \langle\langle [\widehat{\top}], [\widehat{\top}]\rangle, \mathsf{q}_2([\widehat{\top}])\rangle$$

and therefore

$$f(\langle\langle [a], [\widehat{\top}]\rangle, \mathsf{q}_1([a])\rangle) =_{\mathsf{Bool}} f(\langle\langle [\widehat{\top}], [a]\rangle, \mathsf{q}_2([a])\rangle)$$

Then assuming both $T_0(a)$ *true* and

$$(f(\langle\langle [a], [\widehat{\top}]\rangle, \mathsf{q}_1([a])\rangle) =_{\mathsf{Bool}} \mathsf{false}) \ \wedge \ (f(\langle\langle [\widehat{\top}], [a]\rangle, \mathsf{q}_2([a])\rangle) =_{\mathsf{Bool}} \mathsf{true}) \ true$$

one can conclude $\mathsf{true} =_{\mathsf{Bool}} \mathsf{false}$. But we know that $\neg(\mathsf{true} =_{\mathsf{Bool}} \mathsf{false})$ can be derived. Hence, under the assumption

$$(f(\langle\langle [a], [\widehat{\top}]\rangle, \mathsf{q}_1([a])\rangle) =_{\mathsf{Bool}} \mathsf{false}) \ \wedge \ (f(\langle\langle [\widehat{\top}], [a]\rangle, \mathsf{q}_2([a])\rangle) =_{\mathsf{Bool}} \mathsf{true}),$$

the judgement $\neg T_0(a)$ *true* holds. So, by using proposition 2.3.4 and a little of intuitionistic logic, we can conclude $(T_0(a) \ \vee \ \neg T_0(a))$ *true* that is

$$\Pi_{a \in U_0} \ T_0(a) \ \vee \ \neg T_0(a) \ true$$

In conclusion the key points to reproduce the proof of the principle of excluded middle on small sets are the following:

- we use the axiom of choice, by quantifying on

  $$\Sigma(\Omega_0 \times \Omega_0, (w)\ [\widehat{E(\pi_1(w))} \hat{\vee}\ \widehat{E(\pi_2(w))}] =_{\Omega_1} [\widehat{\top}])$$

  instead of $\Sigma(\Omega_0 \times \Omega_0, (w)\ E(\pi_1(w)) \vee E(\pi_2(w)))$ in order to forget the proof-term of the disjunction;

- we exhibit the proof-term q_1 by means of the equality rule on the quotient set in order to get

  $$\langle\langle [a], [\widehat{\top}]\rangle, \mathsf{q}_1([a])\rangle \ \in \Sigma(\Omega_0 \times \Omega_0, (w)\ [\widehat{E(\pi_1(w))} \hat{\vee}\ \widehat{E(\pi_2(w))}] =_{\Omega_1} [\widehat{\top}])$$

  in order to prove under the assumption $[a] =_{\Omega_0} [\widehat{\top}]$ *true* for $a \in U_0$

  $$\langle\langle [a], [\widehat{\top}]\rangle, \mathsf{q}_1([a])\rangle =_{\Sigma(\Omega_0 \times \Omega_0, \ldots)} \langle\langle [\widehat{\top}], [\widehat{\top}]\rangle, \mathsf{q}_1([\widehat{\top}])\rangle$$

- we need the uniqueness of propositional equality proofs in order to prove

  $$\langle\langle [\widehat{\top}], [\widehat{\top}]\rangle, \mathsf{q}_1([\widehat{\top}])\rangle =_{\Sigma(\Omega_0 \times \Omega_0, \ldots)} \langle\langle [\widehat{\top}], [\widehat{\top}]\rangle, \mathsf{q}_2([\widehat{\top}])\rangle$$

Thus, provided one can give suitable rules, which allow quotient sets and our effectiveness condition to hold, and that meanwhile allow us to keep the usual meaning for the judgement C *true*, i.e. C *true* holds if and only if there exists a proof element for the proposition C, then we would have a proof element for the proposition $\Pi_{a \in U_0} T_0(a) \ \vee \ \neg T_0(a)$, which is expected to fail for small sets, according to an intuitionistic explanation of connectives.

2.4 Extensional quotient sets in extensional type theory

The proof that effectiveness of quotient sets yields classical logic for small sets can also be done for the extensional version of Martin-Löf's Constructive Set Theory, eTT, with the rules for quotient sets, as in Nuprl [Con86], to which we add the effectiveness condition and the introduction and conversion rules of the codes for quotient sets into the first two universes.

About the properties of judgements of the form A *true*, we only recall the case of the set of the extensional propositional equality Eq (see [NPS90]). The formation and introduction rules are the following

$$\frac{A\ set \quad a \in A \quad b \in A}{\mathsf{Eq}(A,a,b)\ set} \qquad \frac{A = C\ set \quad a = c \in A \quad b = d \in A}{\mathsf{Eq}(A,a,b) \ = \ \mathsf{Eq}(C,c,d)}$$

$$\frac{A\ set \quad a \in A}{\mathsf{eq}_A(a) \in \mathsf{Eq}(A,a,a)} \qquad \frac{A\ set \quad a = b \in A}{\mathsf{eq}_A(a) = \mathsf{eq}_A(b) \in \mathsf{Eq}(A,a,a)}$$

whereas the elimination rule is

$$\frac{d \in \mathsf{Eq}(A,a,b)}{a = b \in A}$$

and it yields the admissibility of the following rule on judgements of the form A *true*:

$$\frac{\mathsf{Eq}(A,a,b)\ true}{a = b \in A}$$

In the following, we recall the rules for quotient sets in the extensional type theory:
Quotient set

$$\frac{\begin{array}{ll} R(x,y)\ set\ [x \in A, y \in A] & \\ c_1 \in R(x,x)[x \in A], & c_2 \in R(y,x)[x \in A, y \in A, z \in R(x,y)] \\ c_3 \in R(x,z)[x \in A, y \in A, z \in A, w \in R(x,y), w' \in R(y,z)] & \end{array}}{A/R\ set}$$

I-quotient

$$\frac{a \in A}{[a] \in A/R}$$

We specify also the equality between terms of A/R
eq-quotient

$$\frac{a \in A \quad b \in A \quad d \in R(a,b)}{[a] = [b] \in A/R}$$

E-quotient

$$\frac{s \in A/R \quad l(x) \in L([x])[x \in A] \quad l(x) = l(y) \in L([x])[x \in A, y \in A, d \in R(x,y)]}{\mathsf{Q}(l,s) \in L(s)}$$

C-quotient

$$\frac{a \in A \quad l(x) \in L([x])[x \in A] \quad l(x) = l(y) \in L([x])[x \in A, y \in A, d \in R(x,y)]}{\mathsf{Q}(l,[a]) = l(a) \in L([a])}$$

We also want to make equivalence relations effective:
Effectiveness condition

$$\frac{a \in A \quad b \in A \quad [a] = [b] \in A/R}{R(a,b)\ true}$$

We also add the codes of quotient sets in the introduction rules for the first two universes and their corresponding conversion rules, as in section 2.2. Moreover, like for the intensional propositional equality set, the introduction of equality on quotient sets yields the admissibility of the following rule:

$$\frac{a \in A \quad b \in A \quad R(a,b)\ true}{[a] = [b] \in A/R}$$

This extension of eTT, called eTT^Q, is consistent, because there exists an interpretation in classical set theory (ZFC) with two strongly inaccessible cardinals, by interpreting the quotient sets in classical quotient sets and the first two universes respectively in the set of small sets and in the set of large sets.

In the presence of extensional propositional equality type, the rule for intensional quotient sets become equivalent to those of extensional quotient sets and the same holds with respect to effectiveness conditions. So, we can reproduce in eTT^Q the proof of the previous section, and we conclude

$$\Pi_{a \in U_0} T_0(a) \ \vee \ \neg T_0(a) \ \mathit{true}$$

which is expected to fail for small sets.

Chapter 3

The type theory of Heyting pretoposes

Summary We present a type theory, based on dependent types and proof-terms, which is valid and complete with respect to the class of Heyting pretoposes with a natural numbers object. The type theory of Heyting pretoposes turns out to be extensional in the presence of the extensional propositional equality type and of the extensional quotient type. Subobjects are characterized as "mono" types.

3.1 Introduction

An elementary topos can be viewed as a generalized universe of sets to develop mathematics. From a logical point of view, topos theory corresponds to an intuitionistic higher order logic with typed variables [LS86]. Suitable toposes provide models for restricted Zermelo set theory [MM92]. Recently, Joyal and Moerdijk built a model of the whole intuitionistic set theory by using the notion of small map and by taking a Heyting pretopos with a natural numbers object as the categorical universe [JM95]. The search for a type theory of Heyting pretoposes with a natural numbers object arises from the purpose of giving a pure type-theoretic description of the models for intuitionistic set theory in [JM95], after analyzing the notion of "Small map" from a type-theoretic point of view. From now on, we shall refer to a Heyting pretopos with a natural numbers object as H-pretopos.

With respect to a topos, a Heyting pretopos lacks exponentials and the subobject classifier. Indeed, a pretopos is a category equipped with finite limits, stable finite disjoint sums and stable effective quotients of equivalence relations. A Heyting pretopos is a pretopos, where the pullback functor on subobjects has a right adjoint.

Makkay and Reyes found that pretoposes can be characterized with respect to the logical categories, which are the necessary structures to interpret the many-sorted coherent logic [MR77]. Here, we want to find a type theory complete with respect to H-pretoposes, where there is no syntactic distinction between formulas and sorts.

The type theory of H-pretoposes, called HP, is a calculus of dependent types, with a formation rule for every type and introduction, elimination and conversion rules for terms of the same type according to the style of Martin-Löf's type theory. In order to interpret the dependencies, we use the fact that any H-pretopos is locally a H-pretopos, i.e. for every object $A \in Ob\mathcal{P}$ of the H-pretopos $\mathcal{P}$, $\mathcal{P}/A$ is a H-pretopos.

The main difficulty in finding a type theory of H-pretoposes is exactly to describe the relation between the codomain fibration of a H-pretopos and the fibration of its subobjects. Indeed, we have to express the fact that the subobjects form a Heyting algebra and are sufficiently complete to interpret quantifiers.

It is possible to characterize proof-theoretically the subobjects as "mono" types: we say that a dependent type $B(x)[x \in A]$ is *mono*, when

$$y = z \in B(x)\ [x \in A, y \in B(x), z \in B(x)]$$

is derivable. Indeed, in the categorical semantics (see chapter 5) the interpretation of a mono type will turn out to be in correspondence with a monomorphism.

The mono dependent type is the crucial concept for the proof-theoretical characterization of the right adjoint to the "pullback functor" on subobjects. As a matter of fact, in order to represent this right adjoint, we introduce a dependent product type restricted to a mono type, called the forall type.

Moreover, we consider the indexed sum type instead of the simple product type, since for its interpretation it is sufficient to have pullbacks. In the syntactic H-pretopos, by the indexed sum types we establish a correspondence between subobjects and mono dependent types.

In conclusion, in the type theory of H-pretoposes there are the following types: the terminal type, the indexed sum type, the extensional equality type corresponding to finite limits, the quotient type based only on mono equivalence relations corresponding to the quotient of an equivalence relation, the disjoint sum type together with the false type corresponding to finite disjoint coproducts and finally, the forall type corresponding to the right adjoint on subobjects. The presence of the extensional propositional equality is crucial in order to prove that the syntactic category of closed types and suitable terms of HP is an H-pretopos.

To prove the completeness theorem of HP with respect to the class of H-pretoposes, we need to add two axioms, which do not follow the schema of all the other rules: effectiveness for quotients of mono equivalence relations and disjointness of sums.

3.2 The type theory HP

We start with the description of the dependent type theory HP, valid and complete with respect to H-pretoposes, as we will see in the next chapters. This typed system is equipped with types, which should be thought of as sets or data types, and with typed terms which represent proofs of the types to which they belong. In the following we present the formation rules for types and the introduction and elimination rules for terms. In the style of Martin-Löf's type theory, we have four kinds of judgements [NPS90]:

$$A\ type \qquad A = B \qquad a \in A \qquad a = b \in A$$

that is the type judgement, the equality between types, the term judgement and the equality between terms of the same type. The contexts of these judgements are telescopic [dB91], since types are allowed to depend on variables of other types. The contexts are generated by the following rules

$$1C)\quad \emptyset \quad cont \qquad 2C)\quad \frac{\Gamma \quad cont \qquad A\ type\ [\Gamma]}{\Gamma, x \in A \quad cont}\ (x \in A \notin \Gamma)$$

plus the rules of equality between contexts [Str91], [Pit95]. In the following, we present the inference rules to construct type judgements and term judgements with their equality judgements by recursion. One should also add all the inference rules that express reflexivity, symmetry and transitivity of the equality between types and terms and the set equality rule

$$conv)\quad \frac{a \in A\ [\Gamma] \quad A = B\ [\Gamma]}{a \in B\ [\Gamma]}$$

Moreover, by the following rule we assume typed variables

$$var)\quad \frac{\Gamma, x \in A, \Delta \quad cont}{x \in A\ [\Gamma, x \in A, \Delta]}$$

We can derive then the structural rules of weakening, substitution and of a suitable exchange.
Now, we give the formation rules for types specific to HP and then the introduction, elimination and conversion rules of its terms.
We adopt the usual definitions of bound and free occurrences of variables and we identify two terms under α-conversion.

Remark 3.2.1 In the following, the context common to all judgements involved in a rule will be omitted. The typed variable appearing in a context is meant to be added to the implicit context as the last one.

Terminal type

$$\text{Tr)}\ \top\ type \qquad \text{I-Tr)}\ \star \in \top \qquad \text{C-Tr)}\ \frac{t \in \top}{t = \star \in \top}$$

False type

$$\text{Fs)}\ \bot\ type \qquad \text{E-Fs)}\ \frac{a \in \bot \quad A\ type}{\mathsf{r_o}(a) \in A}$$

Indexed Sum type

$$\Sigma)\ \frac{C(x)\ type\ \ [x \in B]}{\Sigma_{x \in B} C(x)\ type} \qquad \text{I-}\Sigma)\ \frac{b \in B \quad c \in C(b)}{\langle b, c\rangle \in \Sigma_{x \in B} C(x)}$$

$$\text{E-}\Sigma)\ \frac{d \in \Sigma_{x \in B} C(x) \quad m(x,y) \in M(\langle x,y\rangle)\ [x \in B, y \in C(x)]}{\mathsf{split}(d,m) \in M(d)}$$

$$\text{C-}\Sigma)\ \frac{b \in B \quad c \in C(b) \quad m(x,y) \in M(\langle x,y\rangle)\ [x \in B, y \in C(x)]}{\mathsf{split}(\langle b,c\rangle, m) = m(b,c) \in M(\langle b,c\rangle)}$$

Equality type

$$\text{Eq)}\ \frac{C\ type \quad c \in C \quad d \in C}{\mathsf{Eq}(C,c,d)\ type} \qquad \text{I-Eq)}\ \frac{c \in C}{\mathsf{eq_C}(c) \in \mathsf{Eq}(C,c,c)}$$

$$\text{E-Eq)}\ \frac{p \in \mathsf{Eq}(C,c,d)}{c = d \in C} \qquad \text{C-Eq)}\ \frac{p \in \mathsf{Eq}(C,c,d)}{p = \mathsf{eq_C} \in \mathsf{Eq}(C,c,d)}$$

Disjoint Sum type

$$+)\ \frac{C\ type \quad D\ type}{C + D\ type} \qquad \text{I}_1\text{-+)}\ \frac{c \in C}{\mathsf{inl}(c) \in C + D} \qquad \text{I}_2\text{-+)}\ \frac{d \in D}{\mathsf{inr}(d) \in C + D}$$

$$\text{E-+)}\ \frac{w \in C + D \quad a_C(x) \in A(\mathsf{inl}(x))\ [x \in C] \quad a_D(y) \in A(\mathsf{inr}(y))\ [y \in D]}{\mathcal{D}(w, a_C, a_D) \in A(w)}$$

$$\text{C}_1\text{-+)}\ \frac{c \in C \quad a_C(x) \in A(\mathsf{inl}(x))\ [x \in C] \quad a_D(y) \in A(\mathsf{inr}(y))\ [y \in D]}{\mathcal{D}(\mathsf{inl}(c), a_C, a_D) = a_C(c) \in A(\mathsf{inl}(c))}$$

$$\text{C}_2\text{-+)}\ \frac{d \in D \quad a_C(x) \in A(\mathsf{inl}(x))\ [x \in C] \quad a_D(y) \in A(\mathsf{inr}(y))\ [y \in D]}{\mathcal{D}(\mathsf{inr}(d), a_C, a_D) = a_D(d) \in A(\mathsf{inr}(d))}$$

Disjointness

$$\frac{c \in C \quad d \in D \quad \mathsf{inl}(c) = \mathsf{inr}(d) \in C + D}{\mathsf{m}(c,d) \in \bot}$$

Forall type

$$\forall)\ \frac{C(x)\ type[x \in B] \quad y = z \in C(x)\ [x \in B, y \in C(x), z \in C(x)]}{\forall_{x \in B} C(x)\ type}$$

$$\text{I-}\forall)\ \frac{c \in C(x)[x \in B] \quad y = z \in C(x)\ [x \in B, y \in C(x), z \in C(x)]}{\lambda x^B.c \in \forall_{x \in B} C(x)}$$

$$\text{E-}\forall)\ \frac{b \in B \quad f \in \forall_{x \in B} C(x)}{\mathsf{Ap}(f,b) \in C(b)}$$

$$\beta\text{C-}\forall)\ \frac{b \in B \quad c \in C(x)[x \in B] \quad y = z \in C(x)\ [x \in B, y \in C(x), z \in C(x)]}{\mathsf{Ap}(\lambda x^B.c, b) = c(b) \in C(b)}$$

$$\eta\text{C-}\forall)\ \frac{f \in \forall_{x \in B} C(x)}{\lambda x^B.\mathsf{Ap}(f,x) = f \in \forall_{x \in B} C(x)}$$

Quotient type

$$\text{Q)}\ \frac{\begin{array}{ll} R(x,y)\ type\ [x \in A, y \in A], & z = w \in R(x,y)[x \in A, y \in A, z \in R(x,y), w \in R(x,y)] \\ c_1 \in R(x,x)[x \in A], & c_2 \in R(y,x)[x \in A, y \in A, z \in R(x,y)] \\ \multicolumn{2}{l}{c_3 \in R(x,z)[x \in A, y \in A, z \in A, w \in R(x,y), w' \in R(y,z)]} \end{array}}{A/R\ type}$$

$$\text{I-Q)}\ \frac{a \in A}{[a] \in A/R} \qquad \text{eq-}Q)\ \frac{a \in A \quad b \in A \quad d \in R(a,b)}{[a] = [b] \in A/R}$$

$$\text{E-Q)}\ \frac{p \in A/R \quad l(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x,y)]}{\mathsf{Q}(l,p) \in L(p)}$$

$$\text{C-Q)}\ \frac{a \in A \quad l(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x,y)]}{\mathsf{Q}(l,[a]) = l(a) \in L([a])}$$

Effectiveness

$$\frac{a \in A \quad b \in A \quad [a] = [b] \in A/R}{\mathsf{f}(a,b) \in R(a,b)}$$

Natural Numbers type

$$\text{nat)}\ N\ type \qquad \text{I}_1\text{-nat)}\ 0 \in N \qquad \text{I}_2\text{-nat)}\ \frac{n \in N}{\mathsf{s}(n) \in N}$$

$$\text{E-nat)}\ \frac{n \in N \quad a \in L(0) \quad l(x,y) \in L(\mathsf{s}(x))\ [x \in N, y \in L(x)]}{\mathsf{Rec}(a,l,n) \in L(n)}$$

$$\text{C}_1\text{-nat)}\ \frac{a \in L(0) \quad l(x,y) \in L(\mathsf{s}(x))\ [x \in N, y \in L(x)]}{\mathsf{Rec}(a,l,0) = a \in L(0)}$$

$$\text{C}_2\text{-nat)}\ \frac{n \in N \quad a \in L(0) \quad l(x,y) \in L(\mathsf{s}(x))\ [x \in N, y \in L(x)]}{\mathsf{Rec}(a,l,\mathsf{s}(n)) = l(n, \mathsf{Rec}(a,l,n)) \in L(\mathsf{s}(n))}$$

Thus, we have finished with the presentation of our calculus. In the ηC-$\forall$ rule the variable x does not appear in f, since we can abstract only on the last variable of the context by the introduction rule of the forall type. From now on we shall often omit the word *type* in the type judgements.
Note that the disjointness axiom is not derivable from the other rules. Indeed, we can obtain a model for the calculus, which falsifies disjointness by using a domain with only one element (see [Smi88]), where the quotient type A/R is interpreted as A.

Actually, from now on, we will refer to an equivalent formulation of the calculus HP, where the elimination and conversion rules for the indexed sum type are replaced by the following rules:

$$\text{E}_{\mathbf{1}}\text{-}\Sigma)\ \frac{d \in \Sigma_{x \in B} C(x)}{\pi_1^B(d) \in B} \qquad \text{E}_{\mathbf{2}}\text{-}\Sigma)\ \frac{d \in \Sigma_{x \in B} C(x)}{\pi_2^{C(\pi_1(d))}(d) \in C(\pi_1(d))}$$

$$\beta_1\mathbf{C\text{-}\Sigma)}\ \frac{b \in B \quad c \in C(b)}{\pi_1^B(\langle b,c \rangle) = b \in B} \qquad \beta_2\mathbf{C\text{-}\Sigma)}\ \frac{b \in B \quad c \in C(b)}{\pi_2^{C(b)}(\langle b,c \rangle) = c \in C(b)}$$

$$\eta\mathbf{C\text{-}\Sigma)}\ \frac{d \in \Sigma_{x \in B} C(x)}{\langle \pi_1^B(d), \pi_2^{C(\pi_1(d))}(d) \rangle = d \in \Sigma_{x \in B} C(x)}$$

Every type for which we can prove

$$\frac{B(x)\ type\ [x \in A]}{y = z \in B(x)\ [x \in A, y \in B(x), z \in B(x)]}$$

is called a mono type, that is a proof-irrelevant type.
In particular, the $forall$ type is mono.
In the following, given a judgement $b(x) \in B(x)[x \in A]$ by the expression

$$(x)b(x)$$

we mean the equivalence class of $b(x) \in B(x)[x \in A]$ under the following relation:

$$b(x) \in B(x)[x \in A] \sim b(y) \in B(y)[y \in A]$$

Moreover, we write b for $(x)b(x)$. Actually, in order to have such expressions we should pass to the type theory with higher ariety [NPS90], with the warning that what it is a type here, it is a set in [NPS90]. By adding the so called function type, given $b(x) \in B(x)[x \in A]$ we have the abstraction, that is $(x)b \in (x \in A)B(x)$, the application, the β-conversion and the η-conversion, that is $(x)b(x) = b \in (x \in A)B(x)$.

Note that the E-quotient and C-quotient rules of the quotient type are derivable, by using the indexed sum type, from the following restricted elimination rule of the quotient type for types not depending on A/R,

$\mathbf{E_s}$-quotient

$$\frac{M\ type \quad m(x) \in M\ [x \in A] \quad m(x) = m(y) \in M\ [x \in A, y \in A, d \in R(x,y)]}{\mathsf{Q_s}(m, z) \in M\ [z \in A/R]}$$

together with the following two conversion rules, also derivable in HP: one is the β-conversion

$\boldsymbol{\beta_s}$C-quotient

$$\frac{a \in A \quad m(x) \in M\ [x \in A] \quad m(x) = m(y) \in M\ [x \in A, y \in A, d \in R(x,y)]}{\mathsf{Q_s}(m, [a]) = m(a) \in M}$$

and the other one is the η-conversion stating the uniqueness of $\mathsf{Q_s}$:

$\boldsymbol{\eta_s}$C-quotient

$$\frac{t(z) \in M\ [z \in A/R]}{\mathsf{Q_s}((x)t([x]), z) = t(z) \in M\ [z \in A/R]}$$

Indeed, given the judgements $l(x) \in L([x])[x \in A]$ and $l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x,y)]$, we use the E_s-quotient rule on $\langle [x], l(x) \rangle \in \Sigma_{z \in A/R} L(z)\ [x \in A]$. So, by the second projection of the indexed sum type, we can define $\mathsf{Q}(l,p) \in L(p)$ for $p \in A/R$, which turns out to be well defined by β_s and η_s conversion rules. Indeed, given the following two judgements

$$l(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x,y)]$$

we get

$$\langle [x], l(x) \rangle \in \Sigma_{z \in A/R} L(z)\ [x \in A]$$

and

$$\langle [x], l(x) \rangle = \langle [y], l(y) \rangle \in \Sigma_{z \in A/R} L(z)\ [x \in A, y \in A, d \in R(x,y)]$$

Hence, by E_s-quotient rule

$$\mathsf{Q_s}((x)\langle [x], l(x) \rangle, z) \in \Sigma_{z \in A/R} L(z)\ [z \in A/R]$$

and we define

$$\mathsf{Q}(l, z) \equiv \pi_2(\mathsf{Q_s}((x)\langle [x], l(x) \rangle, z))$$

where

$$\pi_2(\mathsf{Q_s}((x)\langle [x], l(x) \rangle, z)) \in L(\pi_1(\mathsf{Q_s}((x)\langle [x], l(x) \rangle,\ z)))\ [z \in A/R]$$

$\mathsf{Q}(l, z)$ is well defined, because by η_sC-quotient rule

$$\pi_1(\mathsf{Q_s}(\ (x)\langle [x], l(x) \rangle,\ z) = \mathsf{Q_s}(\ (x')\pi_1(\mathsf{Q_s}((x)\langle [x], l(x) \rangle,\ [x']),\ z) \in A/R$$

but we derive

$$\pi_1(\mathsf{Q_s}(\ (x)\langle [x], l(x) \rangle,\ [x']) = [x'] \in A/R\ [x' \in A]$$

hence

$$\mathsf{Q_s}(\ (x')\pi_1(\mathsf{Q_s}(\ (x)\langle [x], l(x) \rangle, [x']\ ), z) = \mathsf{Q_s}((x')[x'],\ z) \in A/R\ [z \in A/R]$$

and again by η_sC-quotient rule

$$\mathsf{Q_s}(\ (x')[x'],\ z) = z \in A/R\ [z \in A/R]$$

so we conclude

$$\pi_1(\mathsf{Q_s}((x)\langle[x], l(x)\rangle, z) = z \in A/R\ [z \in A/R]$$

Also, the E-Nat rule and the conversion rules of the natural numbers type are derivable, by using the indexed sum type, from the following restricted elimination rule of the natural numbers type for types not depending on N

$\mathbf{E_s}$-Nat

$$\frac{L\ type \quad a \in L \quad l(y) \in L\ [y \in L]}{\mathsf{Rec}_s(a, l, n) \in L\ [n \in N]}$$

together with the following three conversion rules, also derivable in HP: two are the β-conversions

$\boldsymbol{\beta_s}\mathbf{C_1}$-Nat

$$\frac{a \in L \quad l(y) \in L\ [y \in L]}{\mathsf{Rec}_s(a, l, 0) = a \in L}$$

$\boldsymbol{\beta_s}\mathbf{C_2}$-Nat

$$\frac{a \in L \quad l(y) \in L\ [y \in L]}{\mathsf{Rec}_s(a, l, \mathsf{s}(n)) = l(\mathsf{Rec}_s(a, l, n)) \in L\ [n \in N]}$$

and the other one is the η-conversion stating the uniqueness of Rec_s:

$\boldsymbol{\eta_s}$C-Nat

$$\frac{\begin{array}{lll} a \in L & l(y) \in L\ [y \in L] & t(n) \in L\ [n \in N] \\ t(0) = a \in L & t(\mathsf{s}(n)) = l(t(n)) \in L & \end{array}}{\mathsf{Rec}_s(a, l, n) = t(n) \in L\ [n \in N]}$$

Indeed, given the judgements $a \in L(0)$ and $l(x, y) \in L(\mathsf{s}(x))\ [x \in N, y \in L(x)]$, we use the E_s-Nat elimination rule on $\langle 0, a\rangle \in \Sigma_{n\in N}L(n)$ and $\langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle \in \Sigma_{n\in N}L(n)\ [z \in \Sigma_{n\in N}L(n)]$. So, by the second projection of the indexed sum type, we can define the recursion term, which turns out to be well defined by β_s and η_s conversion rules.
Indeed, given the following two judgements

$$a \in L(0) \text{ and } l(x, y) \in L(\mathsf{s}(x))\ [x \in N, y \in L(x)]$$

we get

$$\langle 0, a\rangle \in \Sigma_{n\in N}L(n)$$

and

$$\langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle \in \Sigma_{n\in N}L(n)\ [z \in \Sigma_{n\in N}L(n)]$$

hence, by E_s-Nat rule we obtain

$$\mathsf{Rec}_s(\langle 0, a\rangle, (z)\langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle, n) \in \Sigma_{n\in N}L(n)$$

and we define

$$\mathsf{Rec}(a, l, n) \equiv \pi_2(\mathsf{Rec}_s(\langle 0, a\rangle,\ (z)\langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle\ , n))$$

where

$$\begin{gathered} \pi_2(\mathsf{Rec}_s(\langle 0, a\rangle,\ \langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle\ , n)) \\ \in L(\pi_1(\mathsf{Rec}_s(\langle 0, a\rangle,\ (z)\langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle\ , n))) \end{gathered}$$

$\mathsf{Rec}(a, l, n)$ is well defined because by η_sC-Nat rule we can derive

$$\pi_1(\mathsf{Rec}_s(\langle 0, a\rangle,\ (z)\langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle\ , n)) = n \in N\ [n \in N]$$

Indeed, by $\beta_s\mathrm{C}_1$-Nat rule on the zero

$$\pi_1(\mathsf{Rec}_s(\langle 0, a\rangle,\ \langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle\ , 0)) = \pi_1(\langle 0, a\rangle) = 0$$

and by $\beta_s sC_2$-Nat rule on the successor

$$\pi_1(\mathsf{Rec}_s(\langle 0, a\rangle,\ \langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle\ , \mathsf{s}(n))) = \pi_1(\langle \mathsf{s}(u), l(u, w)\rangle)$$

where
$u = \pi_1(\mathsf{Rec}_s(\langle 0, a\rangle,\ \langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle\ , n)))$
$w = \pi_2(\mathsf{Rec}_s(\langle 0, a\rangle,\ \langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle\ , n)))$
and then

$$\pi_1(\mathsf{Rec}_s(\langle 0, a\rangle,\ \langle \mathsf{s}(\pi_2(z)), l(\pi_1(z), \pi_2(z))\rangle\ , \mathsf{s}(n))) = \mathsf{s}(\pi_1(\mathsf{Rec}_s(\langle 0, a\rangle,\ \langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle\ , n)))$$

So, by η_sC-Nat rule we obtain

$$\pi_1(\mathsf{Rec}_s(\langle 0, a\rangle,\ \langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle\ , n)) = \mathsf{Rec}_s(0, \mathsf{s}(n), n)$$

and again by η_sC-Nat rule

$$\mathsf{Rec}_s(0, \mathsf{s}(n), n) = n \in N$$

Finally, we conclude

$$\pi_1(\mathsf{Rec}_s(\langle 0, a\rangle,\ \langle \mathsf{s}(\pi_1(z)), l(\pi_1(z), \pi_2(z))\rangle\ , n)) = n \in N\ [n \in N]$$

Remark 3.2.2 Note that the extensional propositional equality type is crucial to derive the conversion rules stating the uniqueness of the elimination constants for the quotient type and the natural numbers type.

3.2.1 The signature of the calculus HP.

In order to give a more rigorous presentation of the type theory HP, we assigne to it a signature $Sg(HP)$, as in [Pit95]. We write the signature in the typed lambda calculus with β and η equalities based on the following types, that we call sorts, to avoid confusion with the types of $\mathcal{T}$:

- $TYPES$, $TERMS$ are ground sorts;
- $\alpha \to \beta$ is a sort, provided that α and β are sorts.

Therefore, the signature consists of a collections of meta-constants, given by type-valued function symbols

$$C : \alpha \to TYPES$$

and term-valued function symbols

$$s : \alpha \to TERMS$$

where α is a sort.

Remark 3.2.3 We could also describe the type-valued function symbol as in [NPS90], where a unique ground sort O is considered.

Def. 3.2.4 We call raw types the expressions of sort $TYPES$ and raw terms the expressions of sort $TERMS$, which are built up from the function symbols of the signature and a fixed countably number of variables $Var = \{x_1, x_2, \ldots\}$ of sort $TERMS$.

We give the definition of $Sg(HP)$ in correspondence with the type formation rules and the terms introduced in the introduction, elimination, conversion rules and in the axioms of HP. Notice that in giving the signature, we consider a variant of the formulation of the type theory HP, where in the case of elimination rule for the quotient type and the natural numbers type we have restricted elimination rules with the corresponding conversion rules.

1. With respect to the terminal type

$$\top : TYPES$$

$$\star : TERMS$$

2. With respect to the false type we define

$$\bot : TYPES$$

$$\mathsf{r_o} : TERMS \to TERMS$$

3. With respect to the equality type we define

$$\mathsf{Eq} : TYPES \to TERMS \to TERMS \to TYPES$$

$$\mathsf{eq} : TYPES \to TERMS \to TERMS$$

and we put $\mathsf{eq}_X(x) \equiv \mathsf{eq}(X, x)$.

4. With respect to the indexed sum type, we define

$$\Sigma : TYPES \to (TERMS \to TYPES) \to TYPES$$

and we put $\Sigma_{x \in X} Y(x) \equiv \Sigma(X, Y)$,

$$\langle\ \rangle : TYPES \to (TERMS \to TYPES) \to TERMS \to TERMS \to TERMS$$

and we put $\langle x, y\rangle_{X,Y} \equiv \langle\ \rangle(X, Y, x, y)$,

$$\pi_1 : TYPES \to (TERMS \to TYPES) \to TERMS \to TERMS$$

$$\pi_2 : TYPES \to (TERMS \to TYPES) \to TERMS \to TERMS$$

and we put ${\pi_1}^X(x) \equiv \pi_1(X, Y, x)$ and ${\pi_2}^Y(x) \equiv \pi_2(X, Y, x)$.

5. With respect to the disjoint sum type we define

$$+ : TYPES \to TYPES \to TYPES$$

and we put $X + Y \equiv +(X, Y)$,

$$\mathsf{inl} : TYPES \to TYPES \to TERMS \to TERMS$$

$$\mathsf{inr} : TYPES \to TYPES \to TERMS \to TERMS$$

and we put $\mathsf{inl}_{X,Y}(x) \equiv \mathsf{inl}(X, Y, x)$ and $\mathsf{inr}_{X,Y}(x) \equiv \mathsf{inr}(X, Y, x)$,

$$D : TYPES \to TYPES \to (TERMS \to TYPES) \to$$

$$\to TERMS \to (TERMS \to TERMS) \to (TERMS \to TERMS) \to TERMS$$

and we put $\mathcal{D}_{X,Y,Z}(x, y, z) \equiv \mathcal{D}(X, Y, Z, x, y, z)$,

$$\mathsf{m}_\bot : TYPES \to TYPES \to TERMS \to TERMS \to TERMS$$

and we put $\mathsf{m}_{\bot,X,Y}(x, y) \equiv \mathsf{m}_\bot(X, Y, x, y)$.

6. With respect to the forall type we define

$$\forall : TYPES \to (TERMS \to TYPES) \to TYPES$$

and we put $\forall_{x \in X} Y(x) \equiv \forall(X, Y)$,

$$\lambda : TYPES \to (TERMS \to TYPES) \to (TERMS \to TERMS) \to TERMS$$

and we put $\lambda(X, Y, y) \equiv \lambda_{X,Y} x^X.y(x)$,

$$\mathsf{Ap} : TYPES \to (TERMS \to TYPES) \to TERMS \to TERMS \to TERMS$$

and we put $\mathsf{Ap}_{X,Y}(x, y) \equiv \mathsf{Ap}(X, Y, x, y)$.

7. With respect to the quotient type we define

$$/ : TYPES \to (TERMS \to TERMS \to TYPES) \to TYPES$$

and we put $X/Y \equiv /(X,Y)$,

$$[\ ] : TYPES \to (TERMS \to TERMS \to TYPES) \to TERMS \to TERMS$$

and we put $[x]_{X/Y} \equiv [\ ](X,Y,x)$,

$$\mathsf{Q_s} : TYPES \to (TERMS \to TERMS \to TYPES) \to TYPES \to$$

$$\to (TERMS \to TERMS) \to TERMS \to TERMS$$

and we put ${\mathsf{Q_s}}_{X,Y,Z}(x,y) \equiv \mathsf{Q_s}(X,Y,Z,x,y)$, where $\mathsf{Q_s}$ corresponds to the signature introduced in the restricted elimination rule E$_s$-quotient,

$$\mathsf{f} : TYPES \to (TERMS \to TERMS \to TYPES) \to TERMS \to TERMS \to TERMS$$

and we put $\mathsf{f}_{X,Y}(x,y) \equiv \mathsf{f}(X,Y,x,y)$.

8. With respect to the natural numbers type we define

$$N : TYPES$$

$$0_N : TERMS$$

$$\mathsf{s}_N : TERMS \to TERMS$$

$$\mathsf{Rec}_s : TYPES \to TERMS \to (TERMS \to TERMS) \to TERMS \to TERMS$$

and we put ${\mathsf{Rec}_s}_X(x,y,z) \equiv \mathsf{Rec}_s(X,x,y,z)$, where Rec_s corresponds to the signature introduced in the restricted elimination rule E$_s$-Nat.

3.3 The syntactic H-pretopos

We recall the categorical definition of a Heyting pretopos [MR77], [JM95].

Def. 3.3.1 A *pretopos* is a category equipped with finite limits, stable finite disjoint sums and stable effective quotients of equivalence relations. A *Heyting pretopos* is a pretopos where the pullback functor on subobjects has a right adjoint.

We recall that with a H-pretopos we mean a Heyting pretopos with a natural numbers object (see the appendix in [JM95]).
Now, we show how to build a H-pretopos with the type theory in order to prove the completeness theorem w.r.t. H-pretopoi. We define the syntactic category $\mathcal{P}_T$ as follows.

Def. 3.3.2 The objects of $\mathcal{P}_T$ are the closed types of HP, $A, B, C\dots$ and the morphisms between two types, A and B, are the expressions $(x)b(x)$ (see [NPS90]) corresponding to

$$b(x) \in B[x \in A]$$

where the type B does not depend on A. The composition in $\mathcal{P}_T$ is defined by substitution, that is given $(x)b(x) \in \mathcal{P}_T(A,B)$ and $(y)c(y) \in \mathcal{P}_T(B,C)$ their composition is $(x)c(b(x))$. We state that $(x)b(x) \in P(A,B)$ and $(x)b'(x) \in P(A,B)$ are equal iff we can derive

$$b(x) = b'(x) \in B[x \in A]$$

The identity is $(x)x \in P(A,A)$ obtained by $x \in A[x \in A]$.

In this section we are going to prove that

Proposition 3.3.3 *The category $\mathcal{P}_T$ is a H-pretopos.*

First of all we prove that $\mathcal{P}_T$ has *finite limits*.
The *terminal object* is $\top$ and from any object A the morphism towards $\top$ is

$$(x)\star \in \mathcal{P}_T(A, \top)$$

which is unique by the conversion rule for $\top$.
Given $c \in \mathcal{P}_T(A, C)$ and $d \in \mathcal{P}_T(B, C)$ the *pullback* is given by

$$\Sigma_{x\in A}\Sigma_{y\in B}\mathsf{Eq}(C, c(x), d(y))$$

where the first projection to A is

$$(z)\pi_1^A(z) \in \mathcal{P}_T(\Sigma_{x\in A}\Sigma_{y\in B}\mathsf{Eq}(C, c(x), d(y)), A)$$

and the second projection to B is

$$(z)\pi_1^B(\pi_2^A(z)) \in \mathcal{P}_T(\Sigma_{x\in A}\Sigma_{y\in B}\mathsf{Eq}(C, c(x), d(y)), B)$$

From now on, we simply write $a =_A b$ to mean $\mathsf{Eq}(A, a, b)$ and often we will simply write eq_C, instead of $\mathsf{eq}_C(c)$.

3.3.1 The disjoint coproduct

The *coproduct* of A and B is defined by $A + B$, where the injections are

$$(x)\ \mathsf{inl}(x) \in \mathcal{P}_T(A, A+B) \quad \text{and} \quad (y)\ \mathsf{inr}(y) \in \mathcal{P}_T(B, A+B)$$

Given $c \in \mathcal{P}_T(A, C)$ and $d \in \mathcal{P}_T(B, C)$ the mediating morphism $c \oplus d$ from $A + B$ to C is $(w)\mathcal{D}(w, c, d)$. Coproducts are disjoint by the rule of disjointness. Moreover, coproducts are stable under pullback. For this purpose, we prove that

Lemma 3.3.4 *$A + B$ is isomorphic in $\mathcal{P}_T$ to*

$$\Sigma_{w\in A+B}(\Sigma_{x\in A}\ \mathsf{inl}(x) =_{A+B} w) + (\Sigma_{y\in B}\ \mathsf{inr}(y) =_{A+B} w)$$

Proof.
We put the following abbreviations

$$A +_P B \equiv \Sigma_{w\in A+B}(\Sigma_{x\in A}\ \mathsf{inl}(x) =_{A+B} w) + (\Sigma_{y\in B}\ \mathsf{inr}(y) =_{A+B} w)$$

and for any $w \in A + B$

$$\tilde{A}(w) \equiv \Sigma_{x\in A}\ \mathsf{inl}(x) =_{A+B} w \qquad \tilde{B}(w) \equiv \Sigma_{y\in B}\ \mathsf{inr}(y) =_{A+B} w$$

where the injections of $\tilde{A}(w)+\tilde{B}(w)$ are $\mathsf{inl}^P(z) \in \tilde{A}(w)+\tilde{B}(w)\ [z \in \tilde{A}(w)]$ and $\mathsf{inr}^P(z) \in \tilde{A}(w)+\tilde{B}(w)\ [z \in \tilde{B}(w)]$. We consider $(z)\pi_1(z) \in \mathcal{P}_T(A +_P B, A + B)$ and we define its inverse δ as

$$(w)\langle w, \mathcal{D}(w, d_1, d_2)\rangle \in \mathcal{P}_T(A + B, A +_P B)$$

where d_1 corresponds to

$$\mathsf{inl}^P(\langle x, \mathsf{eq}_{A+B}\rangle) \in \tilde{A}(\mathsf{inl}(x)) + \tilde{B}(\mathsf{inl}(x))\ [x \in A]$$

and d_2 corresponds to

$$\mathsf{inr}^P(\langle y, \mathsf{eq}_{A+B}\rangle)\tilde{A}(\mathsf{inr}(y)) + \tilde{B}(\mathsf{inr}(y))\ [y \in B]$$

We can easily see that π_1 is the inverse morphism of δ. Indeed, $\pi_1 \cdot \delta = id$ follows from the elimination rule for the disjoint sum type. In order to prove that $\delta \cdot \pi_1 = id$, that is to find a proof of

$$\langle \pi_1(z), \mathcal{D}(\pi_1(z), d_1, d_2) \rangle = z \in A +_P B \ [z \in A +_P B]$$

it is sufficient to derive a proof of

$$\pi_2(z) = \mathcal{D}(\pi_1(z), d_1, d_2) \in \tilde{A}(\pi_1(z)) + \tilde{B}(\pi_1(z)) \ [z \in A +_P B]$$

So, we show how by the elimination rule for the disjoint sum type we derive a proof of

$$\mathcal{D}(\pi_1(z), d_1, d_2) =_{A+_P B} z_2 \ [z \in A +_P B, z_2 \in \tilde{A}(\pi_1(z)) + \tilde{B}(\pi_1(z))]$$

Indeed, suppose $z \in A +_P B$ and $w_1 \in \tilde{A}(\pi_1(z))$, from which we get $\mathsf{inl}(\pi_1^A(w_1)) = \pi_1(z) \in A + B$ and then

$$\mathcal{D}(\pi_1(z), d_1, d_2) = \mathcal{D}(\mathsf{inl}(\pi_1^A(w_1)), d_1, d_2) = \mathsf{inl}^P(\langle \pi_1^A(w_1), \mathsf{eq}_{A+B} \rangle) = \mathsf{inl}^P(w_1)$$

that is we get a proof of $\mathcal{D}(\pi_1(z), d_1, d_2) =_{A+_P B} \mathsf{inl}^P(w_1) \ [z \in A +_P B, w_1 \in \tilde{A}(\pi_1(z))]$. Analogously, we derive a proof of $\mathcal{D}(\pi_1(z), d_1, d_2) =_{A+_P B} \mathsf{inr}^P(w_2) \ [z \in A +_P B, w_2 \in \tilde{B}(\pi_1(z))]$. So, given $z \in A +_P B$, by elimination rule with respect to $\tilde{A}(\pi_1(z)) + \tilde{B}(\pi_1(z))$ we get a proof of

$$\mathcal{D}(\pi_1(z), d_1, d_2) =_{A+_P B} z_2 \ [z \in A +_P B, z_2 \in \tilde{A}(\pi_1(z)) + \tilde{B}(\pi_1(z))]$$

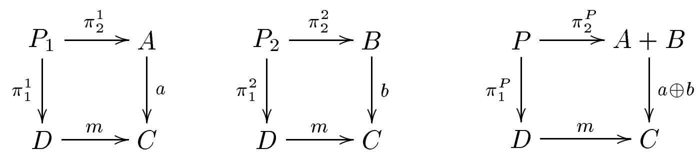
Now, suppose $z \in A +_P B$, since $\pi_2(z) \in \tilde{A}(\pi_1(z)) + \tilde{B}(\pi_1(z))$ by substitution and by elimination of the extensional equality type we conclude

$$\pi_2(z) = \mathcal{D}(\pi_1(z), d_1, d_2) \in \tilde{A}(\pi_1(z)) + \tilde{B}(\pi_1(z))$$

■

Proposition 3.3.5 *In $\mathcal{P}_T$ coproducts are stable under pullbacks.*

Proof.
Given the following pullbacks



we have to show that in $\mathcal{P}_T/D$

$$\pi_1^1 \oplus \pi_1^2 \simeq \pi_1^P$$

For this purpose we define

$$\gamma : P_1 + P_2 \to P$$

as $\gamma \equiv (w)\mathcal{D}(w, d_1, d_2)$ where d_1 corresponds to

$$\langle \pi_1(w_1), \langle \mathsf{inl}(\pi_1(\pi_2(w_1))), \mathsf{eq}_C \rangle \rangle \in P \ [w_1 \in P_1]$$

and d_2 corresponds to

$$\langle \pi_1(w_2), \langle \mathsf{inr}(\pi_1(\pi_2(w_2))), \mathsf{eq}_C \rangle \rangle \in P \ [w_2 \in P_2]$$

We can notice that $\pi_1^P \cdot \gamma = \pi_1^1 \oplus \pi_1^2$ and that $\pi_2^P \cdot \gamma = (\mathsf{inl} \cdot \pi_2^1) \oplus (\mathsf{inr} \cdot \pi_2^2)$.
Moreover, we want to define

$$\gamma^{-1} : P \to P_1 + P_2$$

First of all, we consider that, given $w \in P$, we get $\pi_1(\pi_2(w)) \in A + B$, hence, by δ defined in the above lemma we deduce

$$\pi_2(\delta(\pi_1(\pi_2(w)))) \in \tilde{A}(\pi_1(\pi_2(w))) + \tilde{B}(\pi_1(\pi_2(w)))$$

Now, we use the elimination rule with respect to $\tilde{A}(\pi_1(\pi_2(w))) + \tilde{B}(\pi_1(\pi_2(w)))$ and we define

$$\gamma^{-1} \equiv (w)\mathcal{D}(\pi_2(\delta(\pi_1(\pi_2(w)))), d'_1, d'_2)$$

where d'_1 corresponds to

$$\mathsf{inl}(\langle \pi_1(w), \langle \pi_1(x'), \mathsf{eq}_C \rangle\rangle) \in P_1 + P_2 \ [w \in P, x' \in \tilde{A}(\pi_1(\pi_2(w)))]$$

Indeed, from $w \in P$ and $x' \in \tilde{A}(\pi_1(\pi_2(w)))$ we get

$$m(\pi_1(w)) = (a \oplus b)(\pi_1(\pi_2(w))) \text{ and } \pi_1(\pi_2(w)) = \mathsf{inl}(\pi_1(x'))$$

therefore $m(\pi_1(w)) = a(\pi_1(x'))$. In an analogous way, we define d'_2 as

$$\mathsf{inr}(\langle \pi_1(w), \langle \pi_1(y'), \mathsf{eq}_C \rangle\rangle) \in P_1 + P_2 \ [w \in P, y' \in \tilde{B}(\pi_1(\pi_2(w)))]$$

We can prove that γ^{-1} is the inverse morphism of γ by the elimination rule of the disjoint sum type.
■

3.3.2 The quotient of an equivalence relation

Given an equivalence relation

$$R \overset{g}{\rightarrowtail} A \times A$$

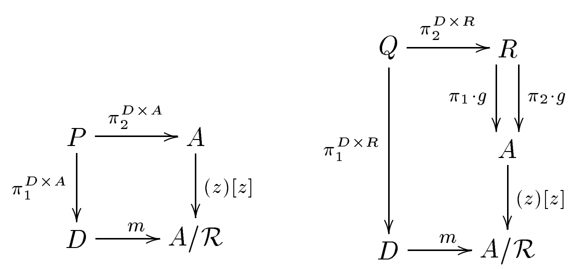
in the syntactic category $\mathcal{P}_T$, we consider the following mono type:

$$\mathcal{R}(x, x') \equiv \Sigma_{y \in R}\, g(y) =_{A \times A} \langle x, x' \rangle [x \in A, x' \in A]$$

It is easy to check that the categorical definition of equivalence relation implies that $\mathcal{R}(x, x')[x \in A, x' \in A]$ is an equivalence relation from the type-theoretical point of view. Let $A/\mathcal{R}$ be the quotient with respect to $\mathcal{R}(x, x')[x \in A, x' \in A]$. We can prove that $(z)[z] \in \mathcal{P}_T(A, A/\mathcal{R})$ is the coequalizer of $\pi_2 \cdot g \in \mathcal{P}_T(R, A)$ and $\pi_2 \cdot g \in \mathcal{P}_T(R, A)$, by the elimination and conversion rules of the quotient type. The uniqueness property of the coequalizer follows from the η_sC-quotient rule.
In $\mathcal{P}_T$ the categorical equivalence relations are effective, by the rule of effectiveness and by the fact that equivalence relations are monic. Moreover, we prove stability of quotients for equivalence relations.

Proposition 3.3.6 *In $\mathcal{P}_T$ equivalence relations are stable effective.*

Proof.
In the following we write $\pi_i^{A \times D}$ for the i'th projection from the vertex of the pullback of unspecified arrows $A \to \cdot \leftarrow D$. We will omit to label the projections, when their domains and codomains are clear from the context.
Given $m \in \mathcal{P}_T(D, A/\mathcal{R})$ let us consider the following pullbacks:



where

$$P \equiv \Sigma_{w \in D} \Sigma_{x \in A} \mathsf{Eq}(A/\mathcal{R}, m(w), [x]) \quad \text{and} \quad Q \equiv \Sigma_{w \in D} \Sigma_{y \in R} \mathsf{Eq}(A/\mathcal{R}, m(w), [(\pi_1 \cdot g)(y)])$$

Moreover, let us consider these two pullbacks:

$$\begin{array}{ccc} Q & \xrightarrow{\pi_2^{D\times R}} & R \\ {\scriptstyle (\pi_2^{D\times A})^*(\pi_1\cdot g)}\downarrow & & \downarrow{\scriptstyle \pi_1\cdot g} \\ P & \xrightarrow[\pi_2^{D\times A}]{} & A \end{array} \qquad \begin{array}{ccc} Q & \xrightarrow{\pi_2^{D\times R}} & R \\ {\scriptstyle (\pi_2^{D\times A})^*(\pi_2\cdot g)}\downarrow & & \downarrow{\scriptstyle \pi_2\cdot g} \\ P & \xrightarrow[\pi_2^{D\times A}]{} & A \end{array}$$

where

$$(\pi_2^{D\times A})^*(\pi_1 \cdot g) \equiv (w)\langle \pi_1(w), \langle \pi_1(g(\pi_1(\pi_2(w)))), \mathsf{eq}\rangle\rangle$$

and

$$(\pi_2^{D\times A})^*(\pi_2 \cdot g) \equiv (w)\langle \pi_1(w), \langle \pi_2(g(\pi_1(\pi_2(w)))), \mathsf{eq}\rangle\rangle$$

We must show that in $P/\mathcal{P}_T$

$$\pi_1^{D\times A} \simeq \mathsf{coeq}((\pi_2^A)^*(\pi_1 \cdot g), (\pi_2^A)^*(\pi_2 \cdot g))$$

We recall that the objects of the category $P/\mathcal{P}_T$ are the morphisms $b : P \to B$ of $\mathcal{P}$, and the morphisms of $P/\mathcal{P}_T$ from $b : P \to B$ to $b' : P \to B'$ are the morphisms $t : B \to B'$ of $\mathcal{P}$ such that $t \cdot b = b'$. We can observe that the pullback given by the effectiveness

$$\begin{array}{ccc} R & \xrightarrow{\pi_2\cdot g} & A \\ {\scriptstyle \pi_1\cdot g}\downarrow & & \downarrow{\scriptstyle (z)[z]} \\ A & \xrightarrow{(z)[z]} & A/\mathcal{R} \\ & \nearrow{\scriptstyle m} & \\ D & & \end{array}$$

can be completed in a cube of pullbacks, therefore

$$\begin{array}{ccc} Q & \xrightarrow{(\pi_2^A)^*(\pi_2\cdot g)} & P \\ {\scriptstyle (\pi_2^A)^*(\pi_1\cdot g)}\downarrow & & \downarrow{\scriptstyle \pi_1^{D\times A}} \\ P & \xrightarrow[\pi_1^{D\times A}]{} & D \end{array}$$

is a pullback and hence $\langle \pi_2^*(\pi_1 \cdot g), \pi_2^*(\pi_2 \cdot g)\rangle$ is an equivalence relation as kernel pair of $\pi_1^{D\times A}$.
Hence, let us consider the coequalizer of $\pi_2^*(\pi_1 \cdot g)$ and $\pi_2^*(\pi_2 \cdot g)$

$$[-] : P \to P/m^*(\mathcal{R})$$

where $P/m^*(\mathcal{R})$ is the quotient type concerning the equivalence relation $\langle \pi_2^*(\pi_1 \cdot g), \pi_2^*(\pi_2 \cdot g)\rangle$.
Since $\pi_1^{D\times A} \cdot \pi_2^*(\pi_1 \cdot g) = \pi_1^{D\times A} \cdot \pi_2^*(\pi_2 \cdot g)$ and $[-]$ is the coequalizer of $\pi_2^*(\pi_1 \cdot g)$ and $\pi_2^*(\pi_2 \cdot g)$, there exists a map

$$Q^P : P/m^*(\mathcal{R}) \to D$$

such that $Q^P \cdot [-] = \pi_1^{D\times A}$. In order to prove that Q^P is an isomorphism, we need the following lemma:

Lemma 3.3.7 *The arrow Q^P is a monomorphism.*

Proof.
We show that we can derive in HP

$$\forall z \in P/m^*(\mathcal{R})\ \forall z' \in P/m^*(\mathcal{R})\ (Q^P(z) =_D Q^P(z') \to z =_{P/m^*(\mathcal{R})} z')$$

In order to find a proof-term of this type, we use the elimination rule for the quotient type. Suppose we have a proof of

$$Q^P([w]) =_D Q^P([w']) \ [w \in P, w' \in P]$$

then given $w \in P$ and $w' \in P$, by the elimination rule of the equality type and by definition of Q^P, we deduce $Q^P([w]) = \pi_1^{D\times A}(w)$, from which we get $\pi_1^{D\times A}(w) = \pi_1^{D\times A}(w') \in D$. But, since $w \in P$ and $w' \in P$, we also get $m(\pi_1^{D\times A}(w)) = [\pi_1(\pi_2(w))]$ and $m(\pi_1^{D\times A}(w')) = [\pi_1(\pi_2(w'))]$, from which we finally have $[\pi_1(\pi_2(w))] = [\pi_1(\pi_2(w'))]$. By effectiveness we derive $\mathsf{f}(\pi_1(\pi_2(w)), \pi_1(\pi_2(w'))) \in \mathcal{R}(\pi_1(\pi_2(w)), \pi_1(\pi_2(w')))$.
So we get

$$\pi_1(\pi_2(w)) = (\pi_1 \cdot g)(\pi_1(\mathsf{f}(\pi_1(\pi_2(w)), \pi_1(\pi_2(w')))))$$

$$\pi_1(\pi_2(w')) = (\pi_2 \cdot g)(\pi_1(\mathsf{f}(\pi_1(\pi_2(w)), \pi_1(\pi_2(w')))))$$

from which we conclude $[w] = [w'] \in P/m^*(\mathcal{R})$. In this way, we have derived a proof of $Q^P([w]) = Q^P([w']) \to [w] =_{P/m^*(\mathcal{R})} [w'] \ [w \in P, w' \in P]$, since the proof-terms of this type are compatible with respect to $m^*(\mathcal{R})$, from which by the elimination rule of the quotient type and by the introduction rule for the forall type we conclude.
■ (of lemma 3.3.7)
Therefore, we can define the inverse morphism of Q^P.
In order to do that, by the elimination rule for the quotient type $A/\mathcal{R}$ we want to derive a proof-term of

$$\forall z \in A/\mathcal{R}\ \forall d \in D\ (z =_{A/\mathcal{R}} m(d)) \to (\Sigma_{w\in P/m^*(\mathcal{R})} Q^P(w) =_D d)$$

This type is well formed, since Q^P is a monomorphism by the previous lemma.
Given $d \in D$ and $x \in A$, supposed $[x] = m(d)$, then $\langle d, \langle x, \mathsf{eq}_{A/\mathcal{R}}\rangle\rangle \in P$ and we derive

$$\langle [\langle d, \langle x, \mathsf{eq}_{A/\mathcal{R}}\rangle\rangle], \mathsf{eq}_D\rangle \in \Sigma_{w\in P/m^*(\mathcal{R})} Q^P(w) =_D d$$

since $Q^P([\langle d, \langle x, \mathsf{eq}_{A/\mathcal{R}}\rangle\rangle]) = \pi_1(\langle d, \langle x, \mathsf{eq}_{A/\mathcal{R}}\rangle\rangle) = d$.
So we get

$$q([x], d) \in [x] =_{A/\mathcal{R}} m(d) \to \Sigma_{w\in P/m^*(\mathcal{R})} Q^P(w) =_D d$$

where $q([x], d) \equiv \lambda w^{\mathsf{Eq}}.\langle [\langle d, \langle x, \mathsf{eq}_{A/\mathcal{R}}\rangle\rangle], \mathsf{eq}_D\rangle$.
Now by the elimination rule for the quotient type $A/\mathcal{R}$ we get

$$\mathsf{Q}(z, q([x], d)) \in z =_{A/\mathcal{R}} m(d) \to \Sigma_{w\in P/m^*(\mathcal{R})} Q^P(w) =_D d$$

and we conclude by the introduction rule for the forall type.
For short, we call $f \equiv \lambda z.\lambda d.\mathsf{Q}(z, q([x], d))$ and finally, we define

$$\mathcal{T} : D \to P/m^*(\mathcal{R})$$

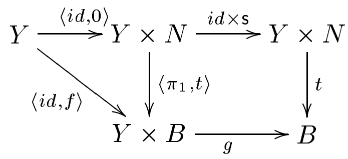
as follows: for every $d \in D$ $\mathcal{T}(d) \equiv \pi_1(\mathsf{Ap}(\mathsf{Ap}(\mathsf{Ap}(f, m(d)), d), \mathsf{eq}_{A/\mathcal{R}}))$. Now, it is easy to show that $Q^P \cdot \mathcal{T} = id$ and since Q^P is a mono, Q^P turns out to be an isomorphism. In conclusion, π_1^D is a coequalizer of $(\pi_2^A)^*(\pi_1 \cdot g)$ and $(\pi_2^A)^*(\pi_2 \cdot g)$.
■ (of proposition 3.3.6)

3.3.3 The natural numbers object

The syntactic H-pretopos is equipped with a natural numbers object. The natural numbers object is the closed type N. Given a closed type Y the zero map is

$$(x)\langle x, 0\rangle \in \mathcal{P}_T(Y, Y \times N)$$

and the successor map corresponds to $\mathsf{s}(n)\ [n \in N]$. We put $id \times \mathsf{s} \equiv (w)\langle \pi_2(w), \mathsf{s}(\pi_2(w))\rangle$. Given the morphisms $\langle id, f\rangle \in \mathcal{P}_T(Y, Y \times B)$ and $g \in \mathcal{P}_T(Y \times B, B)$, we can prove that there exists an unique morphism $t \in \mathcal{P}_T(Y \times N, B)$ such that the following diagram commutes in all its parts:



By hypothesis we get

$$f(y) \in B\ [y \in Y] \quad \text{and} \quad g(\langle y, w\rangle) \in B\ [y \in Y, w \in B]$$

By the elimination rule of the natural numbers type we derive

$$\mathsf{Rec}_s(f(y), g, z) \in B\ [y \in Y, z \in N]$$

So, we put $t \equiv (x)\mathsf{Rec}_s(f(\pi_1(x)), g, \pi_2(x))$, which is the required morphism to make the diagram commute by the conversion rules for the natural numbers type.

3.3.4 About subobjects

In order to show that the subobjects of any object of $\mathcal{P}_T$ form a Heyting algebra and are sufficiently complete to interpret quantifiers, we need to prove that each pullback functor on subobjects has a right adjoint. For this purpose, we show that the pullback functor on subobjects is isomorphic to the functor $Prop(-) : \mathcal{P}_T^{op} \to Cat$ defined in the following.

Def. 3.3.8 For any object $A \in Ob\mathcal{P}_T$, the objects of the category $\boldsymbol{Prop(A)}$ are the equivalence classes of mono types depending on A, $B(x)\ [x \in A]$, under the relation of equiprovability, and the morphisms are the terms $f \in B(x) \to C(x)$ where $B(x) \to C(x) \equiv \forall_{B(x)}(C(x))$, since $C(x)$ is mono. The identity is $\lambda y.y \in B(x) \to B(x)$. The composition of $f \in B(x) \to C(x)$ and $g \in C'(x) \to D(x)$, supposing that $C(x)$ is equivalent to $C'(x)$ and in particular $s \in C(x) \to C'(x)$, is given by $\lambda y.\mathsf{Ap}(g, \mathsf{Ap}(s, \mathsf{Ap}(f, y))) \in B(x) \to D(x)$.

Therefore, we can define the above functor $Prop(-) : \mathcal{P}_T^{op} \to Cat$:

Def. 3.3.9 For any object $A \in Ob\mathcal{P}_T$, $Prop(A)$ is the above defined category and given a morphism $m \in \mathcal{P}_T(D, A)$ we define $Prop(m)$ as the following functor: for any $B(x)\ [x \in A]$

$$Prop(m)(B(x)\ [x \in A]) \equiv B(m(z))\ [z \in D]$$

and for every $t \in B(x) \to C(x)$, given $z \in D$, we define

$$Prop(m)(t) \equiv \lambda w \in B(m(z)).\mathsf{Ap}(t[x := m(z)], w)$$

which is a term of type $B(m(z)) \to C(m(z))$.

We can easily notice that $Prop(-)$ is a well defined functor.
We also consider the functor $Sub(-) : \mathcal{P}_T^{op} \to Cat$, defined as follows. For every $A \in Ob\mathcal{P}_T$, we associate the poset category $Sub(A)$, whose objects are the subobjects on A of $\mathcal{P}_T$ and the morphism, necessarily unique, between subobjects is induced by the morphisms of $\mathcal{P}_T/A$, from any monomorphism representing the domain subobject to any monomorphism representing the codomain subobject. For every $t : A \to B$, $Sub(t)$ is the restriction of pullback functor on subobjects.

Proposition 3.3.10 *The functor* $Sub(-) : \mathcal{P}_T^{op} \to Cat$ *is naturally isomorphic to the functor* $Prop(-) : \mathcal{P}_T^{op} \to Cat$

Proof.
For any $A \in Ob\mathcal{P}_T$ we define the functor

$$\psi_1(A) : Sub(A) \to Prop(A)$$

in this manner: given a mono $B \overset{t}{\rightarrowtail} A$, $\psi_1(A)(t)$ is the equivalence class of $\Sigma_{y\in B}t(y) =_A x\ [x \in A]$, which is a mono type. Indeed, we can notice that a morphism t of $\mathcal{P}_T$ is a mono if and only if $\Sigma_{y\in B}t(y) =_A x\ [x \in A]$ is a mono type. Note that $\psi_1(A)(t)$ is well-defined on subobjects.
Given $\begin{array}{ccc} B & \xrightarrow{m} & B' \\ & {\scriptstyle t}\searrow \quad \swarrow{\scriptstyle t'} & \\ & A & \end{array}$ we define

$$\psi_1(A)(m) \equiv \lambda w.\langle m(\pi_1(w)), \mathsf{eq}_A\rangle$$

of type $\Sigma_{y\in B}t(y) =_A x \to \Sigma_{z\in B'}t'(z) =_A x\ [x \in A]$. It is easy to see that $\psi_1(A)$ is a functor and that $(\psi_1(A))_{A\in Ob\mathcal{P}_T}$ is a natural transformation, that is for every $m(y) \in A\ [y \in D]$ in $\mathcal{P}_T$ the following diagram commutes

$$\begin{array}{ccc} Sub(A) & \xrightarrow{\psi_1(A)} & Prop(A) \\ {\scriptstyle m^*}\downarrow & & \downarrow{\scriptstyle Prop(m)} \\ Sub(D) & \xrightarrow{\psi_1(D)} & Prop(D) \end{array}$$

Moreover, we define

$$\psi_2(A) : Prop(A) \to Sub(A)$$

in this manner: for every mono type $B(x)\ [x \in A]$ we put $\psi_2(A)(B(x)\ [x \in A]) \equiv \pi_1$, where $\pi_1 \in \mathcal{P}_T(\Sigma_{x\in A}B(x), A)$ is the expression that corresponds to the judgement $\pi_1(w) \in A\ [w \in \Sigma_{x\in A}B(x)]$. Note that π_1 is a monomorphism, since $B(x)\ [x \in A]$ is a mono type.
For every $s \in B'(x) \to B(x)$ we define

$$\psi_2(A)(s) \equiv (w)\langle\pi_1(w), \mathsf{Ap}(s, \pi_2(w))\rangle$$

such that the following diagram commutes $\begin{array}{ccc} \Sigma_{x\in A}B'(x) & \xrightarrow{id\times s} & \Sigma_{x\in A}B(x) \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & A & \end{array}$. It is easy to see that $\psi_2(A)$ is the inverse functor of $\psi_1(A)$ and that $(\psi_2(A))_{A\in Ob\mathcal{P}_T}$ is a natural transformation.
■
Now we prove that

Proposition 3.3.11 *For every morphism $m(y) \in A\ [y \in D]$ in $\mathcal{P}_T$, there exists the right adjoint of m^*.*

$$Sub(A) \underset{\forall_m}{\overset{m^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} Sub(D)$$

Proof. By the previous proposition, it is enough to show that $Prop(m)$ has a right adjoint. For every mono type $B(y)\ [y \in D]$ we put

$$\forall_m(B(y)\ [y \in D]) \equiv \forall_{y\in D}(x =_A m(y)) \to B(y)\ [x \in A]$$

whose value at a mono type is indeed a mono type. It is well-defined on subobjects, since it preserves equiprovability. Moreover, we define a bijection

$$Prop(D)(Prop(m)(C(x)), B(y)) \underset{\psi_2}{\overset{\psi_1}{\rightleftarrows}} Prop(A)(C(x), \forall_m(B(y)))$$

as follows: for any $t \in C(m(y)) \to B(y)$ $[y \in D]$ we put for any $x \in A$

$$\psi_1(t) \equiv \lambda z.\lambda y.\lambda w.\mathsf{Ap}(t, z)$$

and for any $s \in C(x) \to \forall_m(B(y))$ $[x \in A]$ and any $y \in D$ we put

$$\psi_2(s) \equiv \lambda z.\mathsf{Ap}(\mathsf{Ap}(\mathsf{Ap}(s[x := m(y)], z), y), \mathsf{eq}_A)$$

It is easy to see that ψ_1 and ψ_2 are inverse to each other and that they are natural on the first variable. This is sufficient to assure that the pullback functor on subobjects has a right adjoint.
■

Remark 3.3.12 Note that the extensional propositional equality type is crucial to get a H-pretopos out of the category $\mathcal{P}_T$, if we consider terms as morphisms and the definitional equality as the equality of morphisms. Indeed, we need the extensional equality type to get equalizers. We also use it to prove stability of the various categorical properties and existence of right adjoints to pullback functors on subobjects. Moreover, we need it to prove uniqueness of the universal properties of the various categorical constructors.

Chapter 4

The type theory of elementary toposes

Summary We propose a type theory, based on dependent types and proof-terms, which is valid and complete with respect to the class of elementary toposes. This theory is obtained from the first order fragment of Martin-Löf's Constructive Type Theory by adding the type corresponding to the subobject classifier. This is the type of closed mono types, whose equality is given by equiprovability. Indeed, this type can be seen as the quotient of the intensional type of propositions under the equivalence relation of equiprovability.

4.1 Introduction

The axiomatization of a Grothendieck topos, free of set-theoretic assumptions, led Lawvere and Tierney to produce the categorical notion of elementary topos. According to Lawvere, an elementary topos can be thought as a generalized universe of sets. The formalization of this idea is expressed by the so called Mitchell-Benabou language, associated with any topos. But, in this language there is a syntactic distinction between the objects of the topos corresponding to types and the subobjects corresponding to formulas, which are terms of the subobject classifier. Moreover, there are no constructors to turn formulas into types.

Here, for toposes we propose the type theory $\mathcal{T}_t$, where the formulas correspond to particular dependent types, as we have already seen in the type theory of Heyting pretoposes. This type theory, which is complete with respect to elementary toposes, is obtained by extending the first order fragment of Martin-Löf's Constructive Type Theory, with the Omega type corresponding to the subobject classifier. In this theory subobjects are represented by dependent types with at most one proof, already called mono types in chapter 3. So, the novelty of this type theory for elementary toposes is that it consists only of dependent types equipped with terms corresponding to their proofs, where the isomorphism "propositions as closed mono types" holds. The mono type is the crucial concept for the proof-theoretical characterization of the subobject classifier of the topos, since in the categorical semantics the interpretation of a mono type turns out to be in correspondence with a monomorphism.

With this type theory, we can build a syntactic topos, whose objects are closed types and whose morphisms are terms. In contrast, in the syntactic topos built up from the Mitchell-Benabou language as in [LS86],[Bel88], the objects are closed terms of powersets and the morphisms are functional relations.

Also, with this type theory, we could compare Martin-Löf's Constructive Type Theory with topos theory, since in both frameworks intuitionistic mathematics can be developed.

4.2 The type theory $\mathcal{T}_t$

The type theory for toposes is obtained by enlarging with the Omega type the first order fragment of the extensional version of Martin-Löf's Intuitionistic Type Theory [Mar84]. This first order fragment

contains the terminal type, indexed sum types, extensional equality types, product types and we call it ML_0.
Therefore, in the style of Martin-Löf's type theory we have four kinds of judgements [NPS90]:

$$A\ type \qquad A = B \qquad a \in A \qquad a = b \in A$$

that is the type judgement, the equality between types, the term judgement and the equality between terms of the same type. The contexts of these judgements are telescopic [dB91], since types are allowed to depend on variables of other types. The contexts are generated by the following rules

$$1C)\quad \emptyset \quad cont \qquad 2C)\quad \frac{\Gamma \quad cont \qquad A\ type\ [\Gamma]}{\Gamma, x \in A \quad cont}\ (x \in A \not\in \Gamma)$$

plus the rules of equality between contexts [Str91], [Pit95]. In the following, we present the inference rules to construct type judgements and term judgements with their equality judgements by recursion. One should also add all the inference rules that express reflexivity, symmetry and transitivity of the equality between types and terms and the set equality rule

$$conv)\quad \frac{a \in A\ [\Gamma] \quad A = B\ [\Gamma]}{a \in B\ [\Gamma]}$$

for all the four kinds of judgements [NPS90]. Moreover, by the following rule we assume typed variables

$$var)\quad \frac{\Gamma, x \in A, \Delta \quad cont}{x \in A\ [\Gamma, x \in A, \Delta]}$$

The structural rules of weakening, substitution and of a suitable exchange can be derived.
We adopt the usual definitions of bound and free occurrences of variables and we identify two terms under α-conversion.

Remark 4.2.1 In the following, the context common to all judgements involved in a rule will be omitted. The typed variable appearing in a context is meant to be added to the implicit context as the last one.

Now, we show the inference rules of $\mathcal{T}_t$ corresponding to the first order fragment ML_o of the extensional version of Martin-Löf's type theory as in [Mar84].
Terminal type

$$\textbf{F-ter)}\quad \top\ type$$

$$\textbf{I-ter)}\quad \star \in \top \qquad \textbf{C-ter)}\quad \frac{t \in \top}{t = \star \in \top}$$

Indexed Sum type

$$\textbf{F-}\Sigma\textbf{)}\quad \frac{C(x)\ type[x \in B]}{\Sigma_{x\in B}C(x)\ type}$$

$$\textbf{I-}\Sigma\textbf{)}\quad \frac{b \in B \quad c \in C(b)}{\langle b, c\rangle \in \Sigma_{x\in B}C(x)}$$

$$\text{E}_{\mathbf{1}}\text{-}\Sigma)\quad \frac{d \in \Sigma_{x\in B}C(x)}{\pi_1(d) \in B} \qquad \text{E}_{\mathbf{2}}\text{-}\Sigma)\quad \frac{d \in \Sigma_{x\in B}C(x)}{\pi_2(d) \in C(\pi_1(d))}$$

$$\beta_1\textbf{C-}\Sigma\textbf{)}\quad \frac{b \in B \quad c \in C(b)}{\pi_1(\langle b, c\rangle) = b \in B} \qquad \beta_2\textbf{C-}\Sigma\textbf{)}\quad \frac{b \in B \quad c \in C(b)}{\pi_2(\langle b, c\rangle) = c \in C(b)}$$

$$\eta\textbf{C-}\Sigma\textbf{)}\quad \frac{d \in \Sigma_{x\in B}C(x)}{\langle \pi_1(d), \pi_2(d)\rangle = d \in \Sigma_{x\in B}C(x)}$$

Equality type

$$\textbf{F-Eq)}\quad \frac{C\ type \quad c \in C \quad d \in C}{\mathsf{Eq}(C, c, d)\ type} \qquad \textbf{I-Eq)}\quad \frac{c \in C}{\mathsf{eq}_C(c) \in \mathsf{Eq}(C, c, d)}$$

$$\textbf{E-Eq)}\ \frac{p \in \mathsf{Eq}(C,c,d)}{c = d \in C} \qquad \textbf{C-Eq)}\ \frac{p \in \mathsf{Eq}(C,c,d)}{p = \mathsf{eq}_C(c) \in \mathsf{Eq}(C,c,d)}$$

Product type

$$\textbf{F-}\Pi)\ \frac{C(x)\ type\ [x \in B]}{\Pi_{x\in B} C(x)\ type}$$

$$\textbf{I-}\Pi)\ \frac{c \in C(x)[x \in B]}{\lambda x^B.c \in \Pi_{x\in B} C(x)} \qquad \textbf{E-}\Pi)\ \frac{b \in B \quad f \in \Pi_{x\in B} C(x)}{\mathsf{Ap}(f,b) \in C(b)}$$

$$\beta\text{C-}\Pi)\ \frac{b \in B \quad c \in C(x)[x \in B]}{\mathsf{Ap}(\lambda x^B.c,b) = c(b) \in C(b)} \qquad \eta\text{C-}\Pi)\ \frac{f \in \Pi_{x\in B} C(x)}{\lambda x^B.\mathsf{Ap}(f,x) = f \in \Pi_{x\in B} C(x)}$$

The novelty of the type theory for toposes is the Omega type, corresponding to the subobject classifier, where propositions correspond to closed mono types, that is closed types with at most one proof.
We recall that a dependent type $B\ type\ [\Gamma]$ is *mono*, if we can derive

$$y = z \in B\ [\Gamma, y \in B, z \in B]$$

The mono types are called proof-irrelevant in the literature, as for example in [Hof95]. In the Omega type there are the codes of the mono types up to equiprovability.
Here, we present the rules that the Omega type should satisfy to represent the subobject classifier.

Formation

$$\Omega\ type$$

Introduction

$$\frac{B\ type \quad y = z \in B\ [y \in B, z \in B]}{\{B\} \in \Omega}$$

Equality

$$\frac{\begin{array}{ll} B\ type & y = z \in B\ [y \in B, z \in B] \\ C\ type & y = z \in C\ [y \in C, z \in C] \\ f \in B \leftrightarrow C & \end{array}}{\{B\} = \{C\} \in \Omega}$$

Elimination

$$\frac{q \in \Omega}{T(q)\ type} \qquad \frac{q \in \Omega \quad c \in T(q) \quad d \in T(q)}{c = d \in T(q)}$$

β-conversion

$$\frac{B\ type \quad y = z \in B\ [y \in B, z \in B]}{\langle \mathsf{r}_B, \mathsf{r}_B^{-1} \rangle \in T(\{B\}) \leftrightarrow B}$$

η-conversion

$$\frac{q \in \Omega}{\{T(q)\} = q \in \Omega}$$

From these rules we derive

$$\frac{\begin{array}{ll} B\ type & y = z \in B\ [\Gamma_n, y \in B, z \in B] \\ C\ type & y = z \in C\ [\Gamma_n, y \in C, z \in C] \\ \{B\} = \{C\} \in \Omega & \end{array}}{\langle \mathsf{r}_C \cdot \mathsf{r}_B^{-1}, \mathsf{r}_B \cdot \mathsf{r}_C^{-1} \rangle \in B \leftrightarrow C}$$

where $\mathsf{r}_C \cdot \mathsf{r}_B^{-1} \equiv \lambda x.\mathsf{r}_C(\mathsf{r}_B^{-1}(x))$ and $B \leftrightarrow C \equiv \Pi_{x\in B} C$.
We use the notation $\{B\}$ for the subset induced by B. Note that, for every $q \in \Omega$, we can find a proof of

$$T(q) \leftrightarrow \mathsf{Eq}(\Omega, q, \{\top\})$$

So, finally, we propose the following inference rules for the Omega type, as a refinement of the previous ones, where we put $T(q) \equiv \mathsf{Eq}(\Omega, q, \{\top\})$:

The Omega type

Formation

$$\textbf{F-}\boldsymbol{\Omega})\qquad \Omega\ type$$

Introduction

$$\textbf{I-}\boldsymbol{\Omega})\quad \frac{B\ type \quad y = z \in B\ [y \in B, z \in B]}{\{B\} \in \Omega}$$

Equality

$$\textbf{eq-}\boldsymbol{\Omega})\quad \frac{\begin{array}{ll} B\ type & y = z \in B\ [y \in B, z \in B] \\ C\ type & y = z \in C\ [y \in C, z \in C] \\ f \in B \leftrightarrow C & \end{array}}{\{B\} = \{C\} \in \Omega}$$

β**-conversion**

$$\beta\textbf{C-}\boldsymbol{\Omega})\quad \frac{B\ type \quad y = z \in B\ [y \in B, z \in B]}{\langle \mathsf{r}_B, \mathsf{r}_B^{-1} \rangle \in \mathsf{Eq}(\Omega, \{B\}, \{\top\}) \leftrightarrow B}$$

η**-conversion**

$$\eta\textbf{C-}\boldsymbol{\Omega})\quad \frac{q \in \Omega}{\{\mathsf{Eq}(\Omega, q, \{\top\})\} = q \in \Omega}$$

In conclusion, we call $\mathcal{T}_t$ the type theory consisting of the rules of ML_0, together with the rules for the Omega type defined above.
By these rules of $\mathcal{T}_t$, when $d \in \Omega$, we do not introduce a new type $T(d)$, of which we do not know the proofs. But, anyway, we introduce a link between the proofs of the equality type $\mathsf{Eq}(\Omega, \{B\}, \{\top\})$ and the type B, which restores some information that has been forgotten.

Indeed, in the introduction rule of equality on the Omega type we forget the proof of equiprovability, that we want to restore in the β-conversion rule. This fact will be very clear in the next section, where we show that the Omega type is the quotient under equiprovability of the type of mono types. The possibility to restore the information forgotten in the introduction rule of equality on the Omega type is given by effectiveness of the quotient type. This is possible, since equiprovability between mono types is a mono equivalence relation.

Remark 4.2.2 Note that in $\mathcal{T}_t$ we derive that if B is a mono type

$$\mathsf{r}_B^{-1} = \lambda z.\mathsf{eq} \in B \to \mathsf{Eq}(\Omega, \{B\}, \{\top\})$$

Indeed, since, given $z \in B$, we get

$$\langle \lambda w.*, \lambda x.z \rangle \in B \leftrightarrow \top$$

from which we obtain

$$\{B\} = \{\top\} \in \Omega$$

and we conclude by the introduction rule for the extensional equality type.

So, from now on, we consider a variant of $\mathcal{T}_t$ where the β-conversion of the Omega type is the following
β**-conversion**

$$\beta\textbf{C-}\boldsymbol{\Omega})\quad \frac{B\ type \quad y = z \in B\ [y \in B, z \in B]}{\mathsf{r}_B \in \mathsf{Eq}(\Omega, \{B\}, \{\top\}) \to B}$$

With $\mathcal{T}_t$, we see that the impredicativity of toposes is restricted to mono types, but the Omega type is not necessarily itself a mono type.

4.2.1 The signature of the calculus $\mathcal{T}_t$.

We give the definition of the signature for the calculus $\mathcal{T}_t$, in correspondence with the type formation rules and the terms introduced in the introduction, elimination, conversion rules (see 3.2.1 for a definition of signature). Notice that in giving the signature, we consider the variant of the formulation of the type theory $\mathcal{T}_t$, with the restricted β conversion rule for the Omega type.

1. With respect to the terminal type we define

$$\top : TYPES$$

$$\star : TERMS$$

2. With respect to the equality type we define

$$\mathsf{Eq} : TYPES \to TERMS \to TERMS \to TYPES$$

$$\mathsf{eq} : TYPES \to TERMS \to TERMS$$

and we put $\mathsf{eq}_X(x) \equiv \mathsf{eq}(X, x)$.

3. With respect to the indexed sum type, we define

$$\Sigma : TYPES \to (TERMS \to TYPES) \to TYPES$$

and we put $\Sigma_{x \in X} Y(x) \equiv \Sigma(X, Y)$,

$$\langle\ \rangle : TYPES \to (TERMS \to TYPES) \to TERMS \to TERMS \to TERMS$$

and we put $\langle x, y \rangle_{X,Y} \equiv \langle\ \rangle(X, Y, x, y)$,

$$\pi_1 : TYPES \to (TERMS \to TYPES) \to TERMS \to TERMS$$

$$\pi_2 : TYPES \to (TERMS \to TYPES) \to TERMS \to TERMS$$

and we put ${\pi_1}^X(x) \equiv \pi_1(X, Y, x)$ and ${\pi_2}^Y(x) \equiv \pi_2(X, Y, x)$.

4. With respect to the product type, we define

$$\Pi : TYPES \to (TERMS \to TYPES) \to TYPES$$

and we put $\Pi_{x \in X} Y(x) \equiv \Pi(X, Y)$,

$$\lambda : TYPES \to (TERMS \to TYPES) \to (TERMS \to TERMS) \to TERMS$$

and we put $\lambda(X, Y, y) \equiv \lambda_{X,Y} x^X.y(x)$,

$$\mathsf{Ap} : TYPES \to (TERMS \to TYPES) \to TERMS \to TERMS \to TERMS$$

and we put $\mathsf{Ap}_{X,Y}(x, y) \equiv \mathsf{Ap}(X, Y, x, y)$.

5. With respect to the Omega type we define

$$\Omega : TYPES$$

$$\{\ \}_\Omega : TYPES \to TERMS$$

and we put $\{X\}_\Omega \equiv \{\ \}_\Omega(X)$,

$$\mathsf{r} : TYPES \to TERMS$$

and we put $\mathsf{r}_X \equiv \mathsf{r}(X)$.

4.3 A calculus with intensional Omega and restricted quotients.

The rules of $\mathcal{T}_t$ can be derived inside an extension of ML_0 with extensional effective quotients restricted to mono equivalence relations, as in the type theory of Heyting pretoposes (see chapter 3), and with the intensional Omega type, which is the intensional type of propositions. This intensional Omega type resembles the type *Prop* of the Calculus of Constructions, but, here, propositions are only closed mono types. The Omega type is called intensional, since the equality on it is given by the equality of mono types, with the warning that the coding and decoding between propositions and mono types enjoy β and η-conversions. The name of this extension of ML_0 is $\mathcal{T}_q$.
Here, we propose the following rules for the intensional type of propositions:

The intensional Omega type

Formation

$$\Omega^i \ type$$

Introduction

$$\frac{B \ type \quad y = z \in B \ [y \in B, z \in B]}{c(B) \in \Omega^i}$$

Equality

$$\frac{\begin{array}{l} B \ type \quad y = z \in B \ [y \in B, z \in B] \\ C \ type \quad y = z \in C \ [y \in C, z \in C] \\ B = C \end{array}}{c(B) = c(C) \in \Omega^i}$$

Elimination

$$\frac{p \in \Omega^i}{D(p) \ type} \qquad \frac{p \in \Omega^i \quad c \in D(p) \quad d \in D(p)}{c = d \in D(p)}$$

β-conversion

$$\frac{B \ type \quad y = z \in B \ [y \in B, z \in B]}{D(c(B)) = B}$$

η-conversion

$$\frac{p \in \Omega^i}{c(D(p)) = p \in \Omega^i}$$

The rules for the quotient types based only proof-irrelevant relations with the effectiveness axiom are the following ones:
Quotient type

Formation

$$\frac{\begin{array}{ll} R(x,y) \ type \ [x \in A, y \in A], & z = w \in R(x,y)[x \in A, y \in A, z \in R(x,y), w \in R(x,y)] \\ c_1 \in R(x,x)[x \in A], & c_2 \in R(y,x)[x \in A, y \in A, z \in R(x,y)] \\ \multicolumn{2}{l}{c_3 \in R(x,z)[x \in A, y \in A, z \in A, w \in R(x,y), w' \in R(y,z)]} \end{array}}{A/R \ type}$$

I-quotient

$$\frac{a \in A \quad A/R \ type}{[a] \in A/R}$$

eq-quotient

$$\frac{a \in A \quad b \in A \quad d \in R(a,b)}{[a] = [b] \in A/R}$$

E-quotient

$$\frac{s \in A/R \quad l(x) \in L([x])[x \in A] \quad l(x) = l(y) \in L([x])[x \in A, y \in A, d \in R(x,y)]}{\mathsf{Q}(l,s) \in L(s)}$$

C-quotient

$$\frac{a \in A \quad l(x) \in L([x])[x \in A] \quad l(x) = l(y) \in L([x])[x \in A, y \in A, d \in R(x,y)]}{\mathsf{Q}(l, [a]) = l(a) \in L([a])}$$

Effectiveness

$$\frac{a \in A \quad b \in A \quad [a] = [b] \in A/R}{\mathsf{f}(a,b) \in R(a,b)}$$

Therefore we can prove:

Proposition 4.3.1 *In T_q we can derive the rules of the Omega type.*

Proof. In T_q we define the Omega type as follows:

$$\Omega \equiv \Omega^i / \leftrightarrow$$

where $\leftrightarrow \equiv D(x) \leftrightarrow D(y)\ [x \in \Omega^i, y \in \Omega^i]$ is a mono equivalence relation.
Moreover, for every closed mono type B we define

$$\{B\} \equiv [c(B)]$$

and for every $q \in \Omega^i / \leftrightarrow$ we abbreviate $T(q) \equiv \mathsf{Eq}(\Omega^i / \leftrightarrow, q, [c(\top)])$.
Now, we show that the β-conversion for the Omega type holds. Precisely, we define

$$\mathsf{r}_B \equiv \lambda z.\pi_2(\mathsf{f}(c(B), c(\top)))(*)$$

since, given $z \in \mathsf{Eq}(\Omega, [c(B)], [c(\top)])$, by the elimination rule for the extensional equality type we get

$$[c(B)] = [c(\top)] \in \Omega$$

and by effectiveness we conclude that

$$\mathsf{f}(c(B), c(\top)) \in D(c(B)) \leftrightarrow D(c(\top))$$

that is $\mathsf{f}(c(B), c(\top)) \in B \leftrightarrow \top$.
Now, we show that the η-conversion for the Omega type holds by the elimination rule of the quotient type. Indeed, we claim that, for every $p \in \Omega^i$, we get a proof of

$$\mathsf{Eq}(\Omega, \{T([p])\}, [p])$$

from which, since the proof-term assigns equal values to equiprovable elements of Ω^i, by the elimination rule of the quotient type we get a proof of

$$\mathsf{Eq}(\Omega, \{T(q)\}, q)\ [q \in \Omega]$$

Therefore, we conclude that the η-conversion holds by the elimination rule of the extensional equality type. Now, suppose $p \in \Omega^i$, since $\{T([p])\} \equiv [c(\mathsf{Eq}(\Omega, [p], [c(\top)]))]$, by definition of equality w.r.t. Ω and by effectiveness we prove that

$$\{T([p])\} = [p] \in \Omega$$

is derivable if and only if there is a proof of

$$D(c(\mathsf{Eq}(\Omega, [p], [c(\top)]))) \leftrightarrow D(p)$$

that is, by β-conversion w.r.t. Ω^i, if and only if there is a proof of $\mathsf{Eq}(\Omega, [p], [c(\top)]) \leftrightarrow D(p)$. So now, we derive a proof of $\mathsf{Eq}(\Omega, [p], [c(\top)]) \leftrightarrow D(p)$. Indeed, given $z \in \mathsf{Eq}(\Omega, [p], [c(\top)])$, by the elimination rule of the extensional equality type, we get $[p] = [c(\top)]$. By effectiveness we obtain $\mathsf{f}(p, c(\top)) \in D(p) \leftrightarrow D(c(\top))$, that is by β-conversion w.r.t. Ω^i, $\mathsf{f}(p, c(\top)) \in D(p) \leftrightarrow \top$. We conclude $\pi_2(f(p, c(\top)))(*) \in D(p)$ and finally we get

$$\lambda z.\pi_2(\mathsf{f}(p, c(\top)))(*) \in \mathsf{Eq}(\Omega, [p], [c(\top)]) \rightarrow D(p)$$

Moreover, given $z \in D(p)$, we get

$$\langle \lambda w.*, \lambda x.z \rangle \in D(p) \leftrightarrow \top$$

from which, as $\top = D(c(\top))$, we obtain $[p] = [c(\top)]$. Therefore, we derive that $\mathsf{eq}_\Omega \in \mathsf{Eq}(\Omega, [p], [c(\top)])$. Finally, we conclude

$$\lambda z.\mathsf{eq}_\Omega \in D(p) \to \mathsf{Eq}(\Omega, [p], [c(\top)])$$

By introduction rule of the equality type, we get a proof of $\mathsf{Eq}(\Omega, \{T([p])\}, [p])$, as we claimed.
■
The calculus $\mathcal{T}_q$ is consistent, because it can be interpreted in the topos of natural numbers, where the interpretation of the intensional Ω^i coincides with that of Ω. Note that in the topos of natural numbers the monomorphisms on every object form a set.

4.4 The syntactic topos

We recall the categorical definition of a topos [MR77], [MM92].

Def. 4.4.1 *A **topos** is a category equipped with finite limits, exponentials and a subobject classifier.*

Here, we show how to build up a topos with the type theory, in order to prove the completeness theorem with respect to the class of toposes. We define the syntactic category $\mathcal{S}_T$ as follows.

Def. 4.4.2 The objects of $\mathcal{S}_\mathbf{T}$ are the closed types of $\mathcal{T}_t$, $A, B, C \ldots$ and the morphisms from the type A to the type B, are the expressions $(x)b(x)$ (see [NPS90]) corresponding to

$$b(x) \in B\ [x \in A]$$

where the type B does not depend on A. The composition in $\mathcal{S}_T$ is defined by substitution, that is, given $(x)b(x) \in \mathcal{S}_T(A, B)$ and $(y)c(y) \in \mathcal{S}_T(B, C)$, their composition is $(x)c(b(x))$. We state that $(x)b(x) \in P(A, B)$ and $(x)b'(x) \in P(A, B)$ are equal iff we can derive

$$b(x) = b'(x) \in B[x \in A]$$

The identity is $(x)x \in P(A, A)$ obtained by $x \in A\ [x \in A]$.

Along this section we are going to prove that

Proposition 4.4.3 *The category $\mathcal{S}_T$ is a topos.*

Proof. First of all we prove that $\mathcal{S}_T$ has *finite limits.*
The *terminal object* is $\top$ and from any object A the morphism to $\top$ is

$$(x)\star \in \mathcal{S}_T(A, \top)$$

which is unique by the conversion rule for $\top$.
Given $c \in \mathcal{S}_T(A, C)$ and $d \in \mathcal{S}_T(B, C)$ the *pullback* is given by

$$\Sigma_{x \in A}\Sigma_{y \in B}\mathsf{Eq}(C, c(x), d(y))$$

where the first projection to A is

$$(z)\pi_1^A(z) \in \mathcal{S}_T(\Sigma_{x \in A}\Sigma_{y \in B}Eq(C, c(x), d(y)), A)$$

and the second projection to B is

$$(z)\pi_1^B(\pi_2^A(z)) \in \mathcal{S}_T(\Sigma_{x \in A}\Sigma_{y \in B}\mathsf{Eq}(C, c(x), d(y)), B)$$

The *right adjoint to the pullback functor* is descrived as in [See84]. For every morphism $m : D \to A$ of $\mathcal{S}_T$, for every object $b : B \to D$ of $\mathcal{S}_T/D$, we put

$$\forall_m(b) \equiv \pi_1 : \Sigma_{x \in A}C(x) \to A$$

where for $x \in A$

$$C(x) \equiv \forall_{y \in D}(x =_A m(y)) \rightarrow \Sigma_{z \in B} b(z) =_D y$$

In the syntactic category $\mathcal{S}_T$, the *subobject classifier* is Ω.
The *true* map is

$$\{\top\} \in \Omega\ [x \in \top]$$

Moreover, given a monomorphism $B \stackrel{t}{\rightarrowtail} A$ its characteristic map is

$$\{\Sigma_{y \in B} t(y) =_A x\} \in \Omega\ [x \in A]$$

It is easy to prove that the pullback of the characteristic map with the $True$ map is isomorphic to t.

$$\begin{array}{ccc} B & \xrightarrow{\ \simeq\ } & \Sigma_{x \in A}\Sigma_{z \in \top}(\{\Sigma_{y \in B} t(y) =_A x\} =_\Omega \{\top\}) \\ & {\scriptstyle t} \searrow & \swarrow {\scriptstyle \pi_1} \\ & A & \end{array}$$

By the equality on Ω and the η-C conversion rule of Ω, the characteristic map is unique.
Indeed, for every $q(x) \in \Omega[x \in A]$ such that

$$\begin{array}{ccc} B & \xrightarrow{\ \simeq\ } & \Sigma_{x \in A}\Sigma_{z \in \top}(q(x) =_\Omega \{\top\}) \\ & {\scriptstyle t} \searrow & \swarrow {\scriptstyle \pi_1} \\ & A & \end{array}$$

by η-C conversion rule of Ω and by the equality on Ω

$$q(x) = \{\mathsf{Eq}(\Omega, q(x), \{\top\})\} = \{\Sigma_{y \in B} t(y) =_A x\}$$

■

Remark 4.4.4 As we said in 3.3.12, the extensional propositional equality type is crucial to get a topos out of the category $\mathcal{S}_T$, if we consider terms as morphisms and the definitional equality as the equality of morphisms. Indeed, we need the extensional equality type to get equalizers. We also use it to prove existence of right adjoints to pullback functors and in the universal property of the subobject classifier.

Remark 4.4.5 In a topos, it is known that the image functor from the codomain fibration to the subobject fibration is first order logical if and only is the internal axiom of choice holds (see [Joh77] page 145, and [See83] page 528). We say that this functor is first order logical, if it translates the interpretation of connectives of the first order type theory ML_0, as it is given in a locally cartesian closed category in [Law69], [Law70], [See84] through the codomain fibration, into the interpretation of first order many-sorted predicative logic, as it is given through the fibration of subobjects, for example, in [MM92]. Note that, in this case, the provability of a predicate seen as a type in the first order type theory ML_0, namely that there is morphism from the terminal object of the fiber in which the predicate is interpreted, entails the provability of the predicate interpreted as a subobject, namely its image is the identity. The converse, i.e. the provability in the subobject fibration entails the provability in the codomain fibration, is valid, if the external axiom of choice holds (that is every epimorphism has got a retraction). In other words, suppose to translate a type A into a mono type, for example by quotienting it under the terminal type $A/\top$ (this is not a problem, since in a topos there are effective quotients [MM92]). Then, the logic of first order predicative types, following propositions as types, can be translated into the logic of mono types such that the logic of types and mono types become equivalent, if and only if there is a choice operator (that is we should have a proof-term $c \in A/\top \rightarrow A$). But, in the presence of external axiom of choice, we get a boolean topos, that is we fall into the classical logic. Therefore, such a choice operator with a Heyting semantics of connectives can not be added to the type theory of toposes. In other words, if we refuse to fall into classical logic the natural translation of connectives from types to mono types does not preserve provability in both directions.

Chapter 5

The semantics in a categorical universe

Summary We describe the categorical semantics of the dependent type theories for H-pretoposes and for toposes. We show how to build a model out of a H-pretopos for the type theory HP and out of a topos for the type theory $\mathcal{T}_t$. After defining a partial interpretation of each calculus, we prove the validity theorem with respect to the corresponding class of universes.

5.1 Introduction

Our notion of model for the type theories of universes, described in the previous chapters, combines the notion of model given by display maps [HP89], [See84] together with the tools provided by contextual categories to interpret substitution correctly [Car86]. We shall emphasize context formation. Indeed, the judgement $B\ [\Gamma]$, asserting that B is a dependent type under the context Γ, is interpreted as a suitable sequence of morphisms of $\mathcal{P}$ to the terminal object. Moreover, the judgement $b \in B[\Gamma]$, asserting that b is a term of type B under the context Γ, is interpreted as a section of the last morphism of the sequence representing the dependent type B. Since we want to express substitution by means of pullback, which is determined up to isomorphisms, we use fibred functors, as in [Hof94], to interpret substitution correctly. But in our semantics, a type judgement corresponds to a sequence of fibred functors, which represents the type under a context with all its possible substitutions, and a term judgement corresponds to a natural transformation, which also represents the term under a context with all its possible substitutions.

Our models for the two type theories correspond to particular contextual categories, where the category of contexts is equivalent to the universe under consideration. Indeed, our model is a categorical universe, with a choice of its structure, where the interpretation of judgements is defined by taking the reindexing functor of the split fibration equivalent to the codomain fibration of the categorical universe. It is worthwhile to say that is enough to consider a split fibration of the codomain fibration in order to obtain a correct interpretation not only of substitution, but also of the other constructors.

In the appendix, we outline the description of the contextual categories with attributes, suitable to model the type theories of H-pretoposes and of toposes.

5.2 The categorical semantics

Since we intend to model a type theory, we shall assume that, with a universe, a given choice of its categorical constructors is made. More precisely, with a H-pretopos $\mathcal{P}$, we fix choices of

- finite limits: 1 is the terminal object and for every object A of $\mathcal{P}$ the unique morphism to the terminal object is $!_A : A \to 1$; for every $t : D \to A$ and $f : B \to A$ the following diagram is a pullback $\begin{array}{ccc} D_\Sigma & \xrightarrow{t^*(f)} & D \\ {\scriptstyle f^*(t)}\downarrow & & \downarrow{\scriptstyle t} \\ B & \xrightarrow[f]{} & A \end{array}$, so that for every $f : A \to B$, we can define the pullback functor $f^* : \mathcal{P}/B \to \mathcal{P}/A$, which associates $f^*(t)$ to every $t : D \to A$ and to every morphism $b : t \to s$ of $\mathcal{P}/B$ the unique morphism $\langle f^*(t),_B b \cdot t^*(f)\rangle$ to the pullback of s along f; since we have defined pullback, the product of A and B is the vertex of the pullback of $!_A$ and $!_B$ and the two projections are $\pi_1{}^A \equiv !_A{}^*(!_B) : A \times B \to A$ and $\pi_2{}^B \equiv !_B{}^*(!_A) : A \times B \to B$, and finally the equalizer of $a : A \to B$ and $b : A \to B$ is $(\langle a, b\rangle)^*(\langle id_B, id_B\rangle) : E \to A$, where $\langle a, b\rangle$ is the unique morphism to the product $B \times B$ such that $\pi_1{}^B \cdot \langle a, b\rangle = a$ and $\pi_2{}^B \cdot \langle a, b\rangle = b$;

- finite coproducts: O is the initial object and, for every object A of $\mathcal{P}$, the unique morphism from the initial object is $?_A : O \to A$; for every objects A and B in $\mathcal{P}$, $A \oplus B$ is the coproduct together with the injections $\epsilon_1 : A \to A \oplus B$ and $\epsilon_2 : B \to A \oplus B$ and given $a : A \to C$ and $b : B \to C$ $a \oplus b : A \oplus B \to C$ is the unique morphism such that $a \oplus b \cdot \epsilon_1 = a$ and $a \oplus b \cdot \epsilon_2 = b$;
- quotients of equivalence relations: for every equivalence relation $\rho : R \to A \times A$ there is a quotient $c = coeq(\pi_1 \cdot \rho, \pi_2 \cdot \rho)$, where π_i for $i = 1, 2$ are the two projections of the pullback of $!_A : A \to 1$ along itself;
- right adjoints on subobjects of the specified pullback functors: for every morphism $f : A \to B$ the functor $\forall_f(-) : Mon(A) \to Mon(B)$ is the right adjoint to the restricted pullback functor $f^* : Mon(B) \to Mon(A)$, where $Mon(A)$ is the subcategory of $\mathcal{P}/A$, whose objects are monomorphism;
- a natural numbers object $\mathcal{N}$ with the zero map $o : 1 \to \mathcal{N}$ and the successor map $s : \mathcal{N} \to \mathcal{N}$.

With a topos $\mathcal{P}$, we fix choices of

- finite limits (as in the case of a H-pretopos);
- exponentials: for every object A in $\mathcal{P}$, the functor $A \to - : \mathcal{P} \to \mathcal{P}$ is the right adjoint of the functor $- \times A : \mathcal{P} \to \mathcal{P}$, which associates $B \times A$ to every object B of $\mathcal{P}$, and $\langle f \cdot \pi_1, \pi_2\rangle : B \times A \to C \times A$ to every morphism $f : B \to C$ of $\mathcal{P}$;
- a subobject classifier $\mathcal{P}(1)$ with a map $True : 1 \to \mathcal{P}(1)$ such that for every monomorphism $X \xrightarrow{\phi} A$ there is a unique characteristic map $ch(r) : A \to \mathcal{P}(1)$ such that $ch(r)^*(True)$ is isomorphic to r in $\mathcal{P}/A$.

An essential feature for the interpretation of a dependent type theory is the local property of the universe under consideration. Indeed, for every object A of the H-pretopos (topos) $\mathcal{P}$, the comma category $\mathcal{P}/A$ is an H-pretopos (topos, respectively).

The proof for the topos can be found in [MM92] or [Joh77]. The local property of a H-pretopos is derived from the fact that the forgetful functor $U : \mathcal{P}/A \to \mathcal{P}$ creates limits and for every $f : A \to B$ the pullback functor $f^* : \mathcal{P}/B \to \mathcal{P}/A$ preserves coproducts and quotients. Note that given an equivalence relation in $\mathcal{P}/D$

$$\begin{array}{ccc} R & \xrightarrow{\rho} & A \times_D A \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle a \cdot \pi_1} & \\ & D & \end{array}$$

$\langle \pi_1 \cdot \rho, \pi_2 \cdot \rho\rangle : R \to A \times A$ is also an equivalence relation in $\mathcal{P}$, where $\pi_i : A \times_D A \to A$ for $i : 1, 2$ are the projections of the pullback of $a : A \to D$ along itself in $\mathcal{P}$.

In a H-pretopos also Beck-Chevalley conditions for right adjoints are satisfied. It is easy to see that for every object A of the H-pretopos $\mathcal{P}$, a natural numbers object in $\mathcal{P}/A$ is $\pi_1 : A \times \mathcal{N} \to A$, where $\mathcal{N}$ is a natural object of $\mathcal{P}$.

The reason to require the local property of the structure of a universe is that constructing a type, depending on a context Γ, from other types corresponds to a categorical property of $\mathcal{P}/A$, where A is determined by Γ. Moreover, since substitution corresponds to pullback, the various categorical properties must be stable under pullback.

Remark 5.2.1 From now on, we shall mean with the categorical universe $\mathcal{P}$ a H-pretopos, when we refer to the type theory HP for H-pretoposes, and a topos, when we refer to the type theory $\mathcal{T}_t$ for toposes. Indeed, the categorical semantics for the two type theories is the same with regard to the interpretation of type and term judgements. One differs from the other only for the structure suitable to interpret some particular type and term constructors.

The idea is to interpret a dependent type as a sequence of morphisms of a given universe $\mathcal{P}$, ending with the terminal object 1, whereas the terms are sections of the last morphism of the type to which they belong. Thus, we consider the algebraic development of the fibration *cod* of $\mathcal{P}$: it is the category $Pgr(\mathcal{P})$.

Def. 5.2.2 *The objects of the category* $\boldsymbol{Pgr(\mathcal{P})}$ *are finite sequences* $a_1, a_2, ..., a_n$ *of morphisms of* $\mathcal{P}$

$$A_n \xrightarrow{a_n} \cdots\cdots A_2 \xrightarrow{a_2} A_1 \xrightarrow{a_1} 1$$

and a morphism from $a_1, a_2, ..., a_n$ *to* $b_1, b_2, ..., b_m$ *is a morphism* b *of* $\mathcal{P}$ *such that* $b_n \cdot b = a_n$

$$\begin{array}{ccccccc} & & & & A_n & \xrightarrow{\;b\;} & B_n \\ & & & & & {\scriptstyle a_n}\searrow \quad \swarrow{\scriptstyle b_n} & \\ 1 & \xleftarrow{!_{A_1}} & A_1 & \cdots\cdots & \xleftarrow{a_{n-1}} & A_{n-1} & \end{array}$$

provided $m = n$ *and* $a_i = b_i$ *for* $i = 1, ..., n-1$.

Remark 5.2.3 We recall that given a category $\mathcal{P}$ we can define the category $\mathcal{P}^{\rightarrow}$, whose objects are the morphisms of $\mathcal{P}$

$$\begin{array}{c} X \\ \downarrow{\scriptstyle \phi} \\ A \end{array}$$

and the morphisms are pairs of morphisms of $\mathcal{P}$, $f : X \rightarrow Y$ and $u : A \rightarrow B$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ A & \xrightarrow[\;u\;]{} & B \end{array}$$

Besides, given a universe $\mathcal{P}$ the following functors are fibrations (see [Ben85], [Jac91] for the definition):

- $cod_{\mathcal{P}} : \mathcal{P}^{\rightarrow} \longrightarrow \mathcal{P}$ defined by:

$$\left(\begin{array}{c} X \\ \downarrow{\scriptstyle \phi} \\ A \end{array} \right) \mapsto A$$

  and

$$(f, u) \quad \mapsto \quad u$$

- $dom_{\mathcal{P}} : \mathcal{P}^{\rightarrow} \longrightarrow \mathcal{P}$ defined by:

$$\left(\begin{array}{c} X \\ \downarrow{\scriptstyle \phi} \\ A \end{array} \right) \mapsto X$$

  and

$$(f, u) \quad \mapsto \quad f$$

Remark 5.2.4 We would like to interpret substitution by means of pullback, using the reindexing pseudofunctor, with respect to the fibration $cod_{\mathcal{P}}$, $F : \mathcal{P}^{OP} \longrightarrow Cat$ defined as follows: F associates to every $A \in Ob\mathcal{P}$ the category $\mathcal{P}/A$ and to every morphism $f : B \rightarrow A$ of $\mathcal{P}$ the pullback pseudo-functor $f^* : \mathcal{P}/A \rightarrow \mathcal{P}/B$. But, in general, for an arbitrary choice of pullbacks, F would not be a functor: for instance, even $F(id)$ may not be an identity. Therefore, if substitution were interpreted by F then it would not be well defined. The solution is to replace F by an equivalent pseudofunctor $S : \mathcal{P}^{OP} \longrightarrow Cat$, which is in fact a functor [Ben85], [Jac91]. S is defined as follows. For every object A in $\mathcal{P}$, $S(A) \equiv Fib(\mathcal{P}/A, \mathcal{P}^{\rightarrow})$, where $Fib(\mathcal{P}/A, \mathcal{P}^{\rightarrow})$ is the category of fibred functors $\sigma : \mathcal{P}/A \rightarrow \mathcal{P}^{\rightarrow}$, from the fibration $dom_{\mathcal{P}}$ to the fibration $cod_{\mathcal{P}}$ (they send cartesian morphisms of $dom_{\mathcal{P}}$ to cartesian morphisms of $cod_{\mathcal{P}}$). A fibred functor $\sigma : \mathcal{P}/A \rightarrow \mathcal{P}^{\rightarrow}$ associates to every triangle $\begin{array}{ccc} C & \xrightarrow{t} & B \\ & {\scriptstyle b'}\searrow \;\; \swarrow{\scriptstyle b} & \\ & A & \end{array}$ a pullback diagram $\begin{array}{ccc} C' & \xrightarrow{q(t,\sigma(b))} & B' \\ {\scriptstyle \sigma(b\cdot t)}\downarrow & & \downarrow{\scriptstyle \sigma(b)} \\ C & \xrightarrow[t]{} & B \end{array}$. The morphisms of $Fib(\mathcal{P}/A, \mathcal{P}^{\rightarrow})$ are natural transformations ρ such that for every $b : B \rightarrow A$ the second member of $\rho(b)$ is the identity (recall that $\rho(b)$ is a morphism of $\mathcal{P}^{\rightarrow}$), that is the triangle $\begin{array}{ccc} & \xrightarrow{\rho_1(b)} & \\ & {\scriptstyle \sigma(b)}\searrow \;\; \swarrow{\scriptstyle \tau(b)} & \\ & B & \end{array}$ commutes. Moreover, for a morphism $f : B \rightarrow A$ of $\mathcal{P}$, the functor $S(f) : Fib(\mathcal{P}/A, \mathcal{P}^{\rightarrow}) \rightarrow Fib(\mathcal{P}/B, \mathcal{P}^{\rightarrow})$ associates to every fibred functor σ a fibred functor $\sigma[f]$. $\sigma[f]$ is defined as follows: for every $t : C \rightarrow B$, $\sigma[f](t) \equiv \sigma(f \cdot t)$. Besides, for every natural transformation ρ, $S(f)(\rho) \equiv \rho[f]$, where $\rho[f](t) \equiv \rho(f \cdot t)$ for every $t : C \rightarrow B$. S is the reindexing functor with respect to the fibration $p_{\mathcal{G}(S)} : Fib(\mathcal{P}/-, \mathcal{P}^{\rightarrow}) \rightarrow \mathcal{P}$, which is the Grothendieck construction on the functor S defined as follows. The object of $Fib(\mathcal{P}/-, \mathcal{P}^{\rightarrow})$ are (A, σ) such that $\sigma \in ObFib(\mathcal{P}/A, \mathcal{P}^{\rightarrow})$ and the morphisms from (A, σ) to (B, τ) are (f, ρ), where $f \in \mathcal{P}(A, B)$ and $\rho : \sigma \rightarrow S(f)(\tau)$ is a morphism of $Fib(\mathcal{P}/A, \mathcal{P}^{\rightarrow})$. The fibration $\mathcal{G}(S)$ is a projection: $p_{\mathcal{G}(S)}(A, \sigma) = A$ for every $(A, \sigma) \in ObFib(\mathcal{P}/-, \mathcal{P}^{\rightarrow})$ and $p_{\mathcal{G}(S)}((f, \rho)) = f$ for every $(f, \rho) \in MorFib(\mathcal{P}/-, \mathcal{P}^{\rightarrow})$. Note that the pseudofunctor $F : \mathcal{P}^{OP} \longrightarrow Cat$ is equivalent to the functor S in the appropriate 2-category of pseudofunctors, and also the codomain fibration $cod_{\mathcal{P}}$ is equivalent to $p_{\mathcal{G}(S)}$. The functors establishing such an equivalence can be described as follows. We define a functor

$$(-)(id) : Fib(\mathcal{P}/A, \mathcal{P}^{\rightarrow}) \rightarrow \mathcal{P}/A$$

which associates to every $\sigma \in ObFib(\mathcal{P}/A, \mathcal{P}^{\rightarrow})$ its evaluation $\sigma(id)$ on the identity of the object A, and for every $\sigma, \tau \in ObFib(\mathcal{P}/A, \mathcal{P}^{\rightarrow})$ the morphism part

$$(-)(id)_{\sigma,\tau} : S(A)(\sigma, \tau) \rightarrow \mathcal{P}/A(\sigma(id), \tau(id))$$

associates to every $\rho \in S(A)(\sigma, \tau)$ the morphism $\begin{array}{ccc} D & \xrightarrow{\rho(id)} & B \\ & {\scriptstyle \sigma(id)}\searrow \;\; \swarrow{\scriptstyle \tau(id)} & \\ & A & \end{array}$. Moreover, we define the functor

$$\widehat{(-)} : \mathcal{P}/A \rightarrow Fib(\mathcal{P}/A, \mathcal{P}^{\rightarrow})$$

establishing the equivalence with $(-)(id)$. The object part of $(-)(id)$ associates to every object $b : B \rightarrow A$ of $\mathcal{P}/A$ the fibred functor $\widehat{b}$, defined as $\widehat{b}(t) \equiv t^*(b)$ for every $t : D \rightarrow A$ and extended to morphisms by the universal property of pullback. For every $a : C \rightarrow A$ and $b : B \rightarrow A$, the morphism part of $(-)(id)$

$$\widehat{(-)}_{\widehat{a},\widehat{b}} : \mathcal{P}/A(a, b) \rightarrow S(A)(\widehat{a}, \widehat{b})$$

associates to every $\begin{array}{ccc} C & \xrightarrow{g} & B \\ & {\scriptstyle a}\searrow \;\; \swarrow{\scriptstyle b} & \\ & A & \end{array}$ the natural transformation $\overline{g} : \widehat{a} \rightarrow \widehat{b}$ defined in this way: for every $t : D \rightarrow A$, we put $\overline{g}(t) \equiv \langle \widehat{a}(t),_B g \cdot a^*(t) \rangle$, where $\langle \widehat{a}(t),_B g \cdot a^*(t) \rangle$ is the unique morphism to the pullback of b along t induced by $\widehat{a}(t)$ and $g \cdot a^*(t)$.

We use fibred functors to interpret the dependent types with all its possible substitutions, as in [Hof94]. Moreover, we use natural transformations to represent terms with all its possible substitutions. We call preinterpretation an assignment of fibred functors to type judgements and of natural transformations to term judgements. To this purpose, we consider the category $Pgf(\mathcal{P})$, where the judgements of the type theories HP and $\mathcal{T}_t$ are preinterpreted. We put $I(\sigma) = A$ if $\sigma \in [\mathcal{P}/A, \mathcal{P}^{\rightarrow}]$.

Def. 5.2.5 *The objects of the category* $\boldsymbol{Pgf(\mathcal{P})}$ *are finite sequences* $\sigma_1, \sigma_2, ..., \sigma_n$ *of fibred functors such that* $\sigma_1(id_{A_1}), \sigma_2(id_{A_2}), ..., \sigma_n(id_{A_n})$ *is an object of* $Pgr(\mathcal{P})$, *where* $A_i = I(\sigma_i)$ *for* $i = 1, ..., n$. *The morphisms of* $Pgf(\mathcal{P})$ *from* $\sigma_1, \sigma_2, ..., \sigma_m$ *to* $\tau_1, \tau_2, ..., \tau_n$ *are defined only if* $m = n$ *and* $\sigma_i = \tau_i$ *for* $i = 1, ..., n-1$, *and they are natural transformations from the functor* σ_n *to* τ_n *such that, if* $A_n = I(\sigma_n) = I(\tau_n)$, *then for every* $b : B \to A_n$ *the second member of* $\rho(b)$ *is the identity (recall that* $\rho(b)$ *is a morphism of* $\mathcal{P}^{\to}$*), that is the triangle* $\begin{array}{ccc} \cdot & \xrightarrow{\rho_1(b)} & \cdot \\ {\scriptstyle \sigma_n(b)} \searrow & & \swarrow {\scriptstyle \tau_n(b)} \\ & B & \end{array}$ *commutes.*

In the following, we simply write σ_i instead of $\sigma_i(id_{A_i})$. Moreover, since the second member of $\rho(b)$ is always the identity, we confuse $\rho(b)$ with the first member $\rho_1(b)$.

Besides, notice that by naturality any component $\rho(b)$ of a morphism ρ of $Pgf(\mathcal{P})$ is determined by the properties of pullback from $\rho(id_{A_n})$. Indeed, if we consider $\begin{array}{ccc} B & \xrightarrow{b} & A_n \\ {\scriptstyle b} \searrow & & \swarrow {\scriptstyle id} \\ & A_n & \end{array}$, we get that $\rho(b)$ is equal to $\langle \sigma_n(b),_B \ \rho(id_{A_n}) \cdot q(b, \sigma_n(id)) \rangle$, from $\sigma_n(b)$ to $\tau_n(b)$, and it is the unique morphism to the pullback of $\tau_n(id)$ along b, according to the functorial choice of pullbacks of τ, induced by $\sigma_n(b)$ and $\rho(id_{A_n}) \cdot q(b, \sigma_n(id))$. We conclude that $\rho \equiv (-)(id)_{\sigma,\tau}{}^{-1}(\rho(id))$.

Finally, for every $A \in Ob\mathcal{P}$, we define the fibred functor $i_A : \mathcal{P}/A \to \mathcal{P}^{\to}$ associating to every triangle $\begin{array}{ccc} C & \xrightarrow{t} & B \\ {\scriptstyle b'} \searrow & & \swarrow {\scriptstyle b} \\ & A & \end{array}$ the following pullback diagram $\begin{array}{ccc} C & \xrightarrow{t} & B \\ {\scriptstyle id} \downarrow & & \downarrow {\scriptstyle id} \\ C & \xrightarrow[t]{} & B \end{array}$.

5.3 The interpretation and validity

Given a universe $\mathcal{P}$, before defining the interpretation, we define a preinterpretation $\tilde{\mathcal{I}_{\mathcal{P}}}_{|} : \mathcal{T} \longrightarrow Pgf(\mathcal{P})$ on the type and term judgements derivable in the type theory $\mathcal{T}$. With $\mathcal{T}$ we shall mean the type theory HP or $\mathcal{T}_t$. $\tilde{\mathcal{I}_{\mathcal{P}}}_{|}$ will turn out to be defined as a restriction of an *a priori* partial interpretation $\tilde{\mathcal{I}_{\mathcal{P}}}$ from the pseudo-judgements of $\mathcal{T}$, which are expressions in the form of a judgement with the signature of $\mathcal{T}$.

The preinterpretation says how to interpret a dipendent type and a typed term after any possible substitution. The interpretation of type and term judgements corresponds to evaluate their preinterpretations on the identical substitution.

Moreover, we define a valuation $\mathcal{V} : Pgf(\mathcal{P}) \longrightarrow Pgr(\mathcal{P})$ in this manner: for every object of $Pgf(\mathcal{P})$ $\sigma_1, \sigma_2, ..., \sigma_n$

$$\mathcal{V}(\sigma_1, \sigma_2, ..., \sigma_n) = \sigma_1(id_{A_1}), \sigma_2(id_{A_2}), ..., \sigma_n(id_{A_n})$$

where $A_i = I(\sigma_i)$ for $i = 1, ..., n$, and for every morphism ρ of $Pgf(\mathcal{P})$ between $\sigma_1, \sigma_2, ..., \sigma_n$ and $\tau_1, \tau_2, ..., \tau_n$

$$\mathcal{V}(\rho) = \rho(id_{A_n})$$

Finally, the interpretation $\mathcal{I}_{\mathcal{P}} : \mathcal{T} \longrightarrow Pgr(\mathcal{P})$ is defined as $\mathcal{I}_{\mathcal{P}} \equiv \mathcal{V} \cdot \tilde{\mathcal{I}_{\mathcal{P}}}_{|}$

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\mathcal{I}_{\mathcal{P}}} & Pgr(\mathcal{P}) \\ {\scriptstyle \tilde{\mathcal{I}_{\mathcal{P}}}_{|}} \searrow & & \nearrow {\scriptstyle \mathcal{V}} \\ & Pgf(\mathcal{P}) & \end{array}$$

So, a type judgement of HP or $\mathcal{T}_t$ will be preinterpreted as a sequence of fibred functors, since a fibred functor is used to represent the dependent type with all the possible substitutions in its free variables [Hof94], and it will turn out to be interpreted as an object of $Pgr(\mathcal{P})$, by the evaluation of the fibred functors on the identical substitution. Precisely, a type judgement with empty context, that is a *closed type*, will be simply interpreted as a sequence of only one arrow to the terminal object 1 of $\mathcal{P}$: for example, every judgement of HP or $\mathcal{T}_t$

$$A\,[\,]$$

will be preinterpreted as a fibred functor

$$\alpha_1 : \mathcal{P}/1 \to \mathcal{P}^{\to}$$

and interpreted as

$$A_\Sigma \xrightarrow{!_{A_\Sigma}} 1$$

since $\alpha_1(id_1) = !_{A_\Sigma}$ with $A_\Sigma = dom(\alpha_1(id_1))$.
More generally, a *dependent type* judgement of HP or $\mathcal{T}_t$

$$B(x_1, ..., x_n)\ [x_1 \in A_1, ..., x_n \in A_{n-1}(x_1, ..., x_{n-1})]$$

will be preinterpreted as the object of $Pgf(\mathcal{P})$

$$\alpha_1, \alpha_2, ..., \alpha_n, \beta$$

and hence interpreted as

$$1 \xleftarrow{\alpha_1(id)} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\beta(id)} B_\Sigma$$

In the following, for short, we write $\Gamma_n \equiv x_1 \in A_1, ..., x_n \in A_n(x_1, ..., x_{n-1})$.
The *equality between types* will be preinterpreted as equality between objects of $Pgf(\mathcal{P})$ and hence interpreted as the equality between objects of $Pgr(\mathcal{P})$.
The *typed term* judgements will be interpreted as morphisms of $Pgr(\mathcal{P})$:
given the type judgement $B(x_1, ..., x_n)\ [\Gamma_n]$ interpreted as $1 \xleftarrow{\alpha_1(id)} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\beta(id)} B_\Sigma$ the term judgement

$$b \in B(x_1, ..., x_n)\ [\Gamma_n]$$

will be preinterpreted as a natural transformation b^I from $\alpha_1, \alpha_2, ..., \alpha_n, i_{A_{\Sigma n}}$ to $\alpha_1, \alpha_2, ..., \alpha_n, \beta$, and it will be interpreted as $b^I(id)$, that is a section of $\beta(id)$

$$\begin{array}{ccccccc}
 & & & & A_{\Sigma n} & \xrightarrow{b(id)} & B_\Sigma \\
 & & & & & {\scriptstyle id} \searrow \quad \swarrow {\scriptstyle \beta(id)} & \\
1 & \xleftarrow[!_{A_{\Sigma 1}}]{} & A_{\Sigma 1} & \cdots\cdots & \xleftarrow[\alpha_n(id)]{} & A_{\Sigma n} &
\end{array}$$

The *equality between terms* will be preinterpreted as equality between natural transformations and hence interpreted as the equality between morphisms of $Pgr(\mathcal{S})$. From now on, for short, we simply write b^I to mean $b^I(id)$.

Essentially, we define the preinterpretation

$$\tilde{\mathcal{I}_{\mathcal{P}}} : pseudo(\mathcal{T}) \longrightarrow Pgf(\mathcal{P})$$

as an *a priori* partial function from the pseudo-judgements of $\mathcal{T}$, $pseudo(\mathcal{T})$, about dependent types and terms, by induction on their complexity (see, for example, [Pit95], [Str91]). Indeed, later we will show that the preinterpretation is well defined on the type and term judgements derivable in the theory, by induction on the derivation. But, we will prove this in the validity theorem, because for our purpose we also need the validity of the judgements about equality between types and terms. With regard to this, see, for example, the elimination rule for the extensional propositional equality type, the formation rule for the forall type in the type theory HP and the introduction rule for the Omega type in the type theory $\mathcal{T}_t$.

Remark 5.3.1 As already said, a complication of these dependent type theories is that by the dependency of types from terms, we have to consider the equality between types and between terms. Indeed, the proofs that some types or terms are well formed depend on these equality judgements. So, we give a partial interpretation.

Another difficulty is that in the presence of the set equality rule *conv*), an interpretation defined by induction on the derivations must be checked to be well-defined such that the interpretation of a judgement does not depend on the derivation, if the derivation is not unique.

We think that this could be done by avoiding all the weakening, substitution and set equality rules in the formulation of the calculus. But our formulations of the dependent type theories do not let us to prove the set equality rule *conv*).

5.3.1 The partial interpretation

We define $pseudo(\mathcal{T})$ as the pseudo-judgements of a type theory $\mathcal{T}$, consisting of expressions of these four kinds

$$A(\underline{x})\ type\ [\Gamma] \qquad A(\underline{x}) = B(\underline{x})\ [\Gamma]$$
$$a(\underline{x}) \in A(\underline{x})\ [\Gamma] \qquad a(\underline{x}) = b(\underline{x})\ \in A(\underline{x})\ [\Gamma]$$

where $\underline{x} \equiv x_1, \ldots, x_n$ and

- $\Gamma \equiv [\ ]$ or $\Gamma \equiv x_1 \in A_1, ..., x_n \in A_n(x_1, ..., x_{n-1})$ is a list of distinct typed variables, and $A_i(x_1, ..., x_{i-1})$ for $i = 1, \ldots, n$ is a raw type, that may depend on variables, previously listed; we call this list a pseudo-context;
- $A(\underline{x})$ is a raw type, whose variables occur in the pseudo-context Γ;
- $a(\underline{x})$ is a raw term, whose variables occur in the pseudo-context Γ.

Remark 5.3.2 We will omit to write the type in the signature of a term. Indeed, we assume that whenever a new symbol is introduced in the introduction, elimination and conversion rules of the type theory about a term judgement $a \in A\ type\ [\Gamma]$, the types that should be appear in the signature of the term a are determined by A and by the types of the premisses.

We define an *a priori* partial preinterpretation $\tilde{\mathcal{I}_\mathcal{P}} : pseudo(\mathcal{T}) \longrightarrow Pgf(\mathcal{P})$ of the pseudo-judgements of the type theory HP and $\mathcal{T}_t$

$$A\ type\ [\Gamma] \qquad a \in A\ [\Gamma]$$

by induction on the complexity of pseudo-judgements.
The complexity of pseudo-judgements is defined by recursion, as in [Str91]. We assume to know the complexity of raw types and raw terms. We define the depth of a context. The depth of the empty context is 0. The depth of a pseudo-context $\Gamma, x \in A$ is the complexity of the pseudo-judgement $A\ type\ [\Gamma]$. The complexity of a pseudo-judgement $A\ type\ [\Gamma]$ is the sum of the complexity of the type-valued function symbol A with the depth of the pseudo-context Γ. The complexity of a pseudo-judgement $a \in A\ [\Gamma]$ is the sum of the complexity of the term-valued function symbol a with the complexity of $A\ type\ [\Gamma]$.
Moreover, we preinterpret the pseudo-contexts:

- for the empty context
$$\tilde{\mathcal{I}_\mathcal{P}}([\ ]) \equiv 1$$
- for a generic pseudo-context
$$\tilde{\mathcal{I}_\mathcal{P}}(\Gamma, x \in A) \equiv \tilde{\mathcal{I}_\mathcal{P}}(A\ type\ [\Gamma])$$

As for the judgements, we define the partial preinterpretation of the pseudo-judgements of equality:

- $\tilde{\mathcal{I}_\mathcal{P}}(A = B\ [\Gamma])$ is preinterpreted as
$$\tilde{\mathcal{I}_\mathcal{P}}(A\ type\ [\Gamma]) = \tilde{\mathcal{I}_\mathcal{P}}(B\ type\ [\Gamma])$$
provided that $\tilde{\mathcal{I}_\mathcal{P}}(A\ type\ [\Gamma])$ and $\tilde{\mathcal{I}_\mathcal{P}}(B\ type\ [\Gamma])$ are defined;

- $\tilde{\mathcal{I}_\mathcal{P}}(a = b \in A\ [\Gamma])$ is preinterpreted as

$$\tilde{\mathcal{I}_\mathcal{P}}(a \in A\ [\Gamma]) = \tilde{\mathcal{I}_\mathcal{P}}(b \in A\ [\Gamma])$$

provided that $\tilde{\mathcal{I}_\mathcal{P}}(a \in A\ [\Gamma])$ and $\tilde{\mathcal{I}_\mathcal{P}}(b \in A\ [\Gamma])$ are defined.

Remark 5.3.3 Note that it is sufficient to specify only the interpretation of a term, since the preinterpretation of a term is the corresponding natural transformation of $Pgf(\mathcal{P})$ determined by its evaluation on the identity.

The interpretation of the type theory HP

Now, given a H-pretopos $\mathcal{P}$, we proceed to define the partial preinterpretation $\tilde{\mathcal{I}_\mathcal{P}}$ from the pseudo-judgements of the type theory HP into $Pgf(\mathcal{P})$, and therefore, a partial interpretation into $Pgr(\mathcal{P})$, by induction on the complexity of the pseudo-judgements. In the inductive hypothesis, we will refer to the interpretation of a pseudo-judgement, assuming that also the preinterpretation is given. The interpretation of the *assumption of variable* is the following:
provided that the pseudo-judgement

$$C(x_1, ..., x_n, x)\ [\Gamma_n, x \in B(x_1, ..., x_n)]$$

is interpreted as

$$!_{A_{\sigma 1}}, \alpha_2(id), ..., \alpha_n(id), \beta(id), \gamma(id)$$

and the pseudo-judgement

$$B(x_1, ..., x_n)[x_1 \in A_1, ..., x_n \in A_n(x_1, ..., x_{n-1}), x \in B(x_1, ..., x_n), y \in C(x_1, ..., x_n, x)]$$

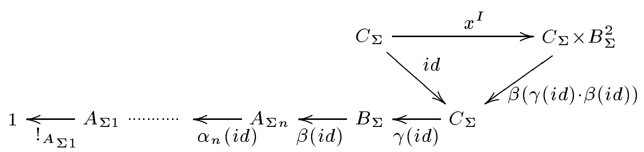
as

$$!_{A_{\sigma 1}}, \alpha_2(id), ..., \alpha_n(id), \beta(id), \gamma(id), \beta(\gamma(id) \cdot \beta(id))$$

then

$$x \in B(x_1, ..., x_n)[x_1 \in A_1, ..., x_n \in A_n(x_1, ..., x_{n-1}), x \in B(x_1, ..., x_n), y \in C(x_1, ..., x_n, x)]$$

is interpreted as



where $x^I(id) = \gamma(id)^*(\triangle_{B_\Sigma})$ and $\triangle_{B_\Sigma} \equiv \langle id_{B_\Sigma}, id_{B_\Sigma}\rangle$. In order to interpret the inference rule about the assumption of variable, when the context Δ is made of more than one typed variable, we repeat the semantic operation of weakening, interpreted as in the lemmas of weakening (we refer to section 5.3.2 for its interpretation).
Now, we go on by defining the interpretation on the signature given by the formation, introduction and elimination rules for types and terms.

Remark 5.3.4 Note that we will write $\widehat{A}$ for the fibred functor $\widehat{!_A} : \mathcal{P}/1 \to \mathcal{P}^{\to}$.

1. The *Terminal type* pseudo-judgement

$$\top\ [\,]$$

is interpreted as

$$1 \xleftarrow{!_{1_\Sigma}} 1_\Sigma$$

where $1_\Sigma \equiv dom((id_1)^*(id_1))$, and we recall that the preinterpretation $\widehat{1} : \mathcal{P}/1 \to \mathcal{P}^{\to}$ is the functor defined in the following manner: for every $D \xrightarrow{!_D} 1$ we put $\widehat{1}(!_D) = (!_D)^*(id_1)$ and on the morphisms it is defined through the pullback. Hence we get $\widehat{1}(id_1) = !_{1_\Sigma}$.

Moreover, if the pseudo-context Γ_n is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n}$$

we interpret

$$\top\ [\Gamma_n]$$

as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\widehat{1}(!_{A_{\Sigma n}})} A_{\Sigma n} \times 1_\Sigma$$

and we interpret

$$\star \in \top\ [\Gamma_n]$$

as

$$\begin{array}{ccccc} & & A_{\Sigma n} & \xrightarrow{\star^I(id)} & A_{\Sigma n} \times 1_\Sigma \\ & & {\scriptstyle id} \searrow & & \swarrow {\scriptstyle \widehat{1}(!_{A_{\Sigma n}})} \\ 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} & \cdots\cdots & \xleftarrow{\alpha_n(id)} & A_{\Sigma n} & \end{array}$$

where $\star^I(id) \equiv \langle id_{A_{\Sigma n}}, (!_{1\Sigma})^{-1} \cdot !_{A_{\Sigma n}} \rangle$.

2. If the pseudo-context Γ_n is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n}$$

the *False type* pseudo-judgement

$$\bot\ [\Gamma_n]$$

is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\widehat{0}(!_{A_{\Sigma n}})} A_{\Sigma n} \times 0_\Sigma$$

where 0 is the initial object of $\mathcal{P}$ and $0_\Sigma \equiv dom((id_1)^*(!_0))$ and the preinterpretation $\widehat{0} : \mathcal{P}/1 \to \mathcal{P}^{\to}$ is the functor defined in the following manner: for every $D \xrightarrow{!_D} 1$ we put $\widehat{0}(!_D) = (!_D)^*(!_0)$. Therefore $\widehat{0}(id_1) = !_{0_\Sigma}$. On the morphisms it is defined through the pullback.

The signature introduced in the *elimination rule* of the false type is interpreted in the following manner:

provided that the pseudo-judgements $a \in \bot\ [\Gamma_n]$ and $A\ [\Gamma_n]$ are interpreted as

$$\begin{array}{ccccc} & & A_{\Sigma n} & \xrightarrow{a^I} & A_{\Sigma n} \times 0_\Sigma \\ & & {\scriptstyle id} \searrow & & \swarrow {\scriptstyle \widehat{0}(!_{A_{\Sigma n}})} \\ 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} & \cdots\cdots & \xleftarrow{\alpha_n(id)} & A_{\Sigma n} & \end{array} \qquad 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\alpha(id)} A_\Sigma$$

we interpret

$$\mathsf{r_o}(a) \in A\ [\Gamma_n]$$

as

$$\begin{array}{ccccc} & & A_{\Sigma n} & \xrightarrow{?_{A_\Sigma} \cdot (a^I(id))} & A_\Sigma \\ & & {\scriptstyle id} \searrow & & \swarrow {\scriptstyle \alpha(id)} \\ 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} & \cdots\cdots & \xleftarrow{\alpha_n(id)} & A_{\Sigma n} & \end{array}$$

where $?_{A_\Sigma}$ is the unique morphism from $A_{\Sigma n} \times 0_\Sigma$ to $A_{\Sigma n}$, because 0 is a strict initial object and then $\widehat{0}(!_D)$ is an initial object in $\mathcal{P}/D$.

3. The *Indexed Sum type*.

   Provided that the pseudo-judgement $C(y)\ [\Gamma_n, y \in B]$ is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\beta(id)} B_{\Sigma} \xleftarrow{\gamma(id)} C_{\Sigma}$$

   we interpret

$$\Sigma_{y \in B} C(y)\ [\Gamma_n]$$

   as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\Sigma_\beta(\gamma)(id)} C_{\Sigma}$$

   where $\Sigma_\beta(\gamma) : \mathcal{P}/A_{\Sigma n} \to \mathcal{P}^{\to}$ is the functor defined in the following manner: for every $D \xrightarrow{t} A_{\Sigma n}$ we put $\Sigma_\beta(\gamma)(t) = \beta(t) \cdot \gamma(q(t, \beta(id)))$ and on the morphisms it is defined through the pullback. It is well defined, since the corresponding Beck-Chevalley conditions hold in any H-pretopos.

   The *pair* term is interpreted in the following manner:

   provided that the pseudo-judgements $b \in B\ [\Gamma_n]$ and $c \in C(b)\ [\Gamma_n]$ are interpreted as

$$\begin{array}{ccc} A_{\Sigma n} & \xrightarrow{b^I} & B_\Sigma \\ & {\scriptstyle id} \searrow \quad \swarrow {\scriptstyle \beta(id)} & \\ 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} & A_{\Sigma n} & \end{array} \qquad \begin{array}{ccc} A_{\Sigma n} & \xrightarrow{c^I} & A_{\Sigma n} \times C_\Sigma \\ & {\scriptstyle id} \searrow \quad \swarrow {\scriptstyle \gamma(b^I)} & \\ 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} & A_{\Sigma n} & \end{array}$$

   we interpret

$$\langle b, c \rangle \in \Sigma_{y \in B} C(y)\ [\Gamma_n]$$

   as

$$\begin{array}{ccc} B_{\Sigma} & \xrightarrow{\langle b,c \rangle^I} & C_\Sigma \\ & {\scriptstyle id} \searrow \quad \swarrow {\scriptstyle \Sigma_\beta(\gamma)(id)} & \\ 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} & A_{\Sigma n} & \end{array}$$

   where $\langle b, c \rangle^I(id) = q(b^I(id), \gamma(id)) \cdot (c^I(id))$.

   The *first projection* of the indexed sum type is interpreted in this manner:

   provided that the pseudo-judgement $d \in \Sigma_{y \in B} C(y)\ [\Gamma_n]$ is interpreted as

$$\begin{array}{ccc} A_{\Sigma n} & \xrightarrow{d^I} & C_\Sigma \\ & {\scriptstyle id} \searrow \quad \swarrow {\scriptstyle \Sigma_\beta(\gamma)(id)} & \\ 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} & A_{\Sigma n} & \end{array}$$

   we interpret

$$\pi_1(d) \in B\ [\Gamma_n]$$

   as

$$\begin{array}{ccc} A_{\Sigma n} & \xrightarrow{(\pi_1(d))^I} & B_\Sigma \\ & {\scriptstyle id} \searrow \quad \swarrow {\scriptstyle \beta(id)} & \\ 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} & A_{\Sigma n} & \end{array}$$

   where $(\pi_1(d))^I(id) \equiv \gamma(id) \cdot (d^I(id))$.

   The *second projection* of the indexed sum type is interpreted in this manner:

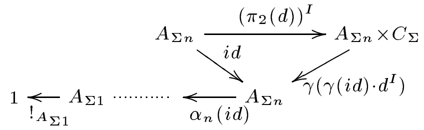
   provided that the pseudo-judgement $d \in \Sigma_{y \in B} C(y)\ [\Gamma_n]$ is interpreted as

$$\begin{array}{ccc} A_{\Sigma n} & \xrightarrow{d^I} & C_\Sigma \\ & {\scriptstyle id} \searrow \quad \swarrow {\scriptstyle \Sigma_\beta(\gamma)(id)} & \\ 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} & A_{\Sigma n} & \end{array}$$

we interpret

$$\pi_2(d) \in C(\pi_1(d))\ [\Gamma_n]$$

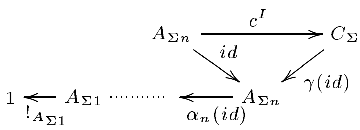
as

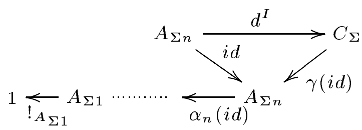


where $(\pi_2(d))^I(id) \equiv \langle id, d^I(id)\rangle$ is the unique morphism to the pullback between $\gamma(id)$ and $\gamma(id) \cdot (d^I(id))$.

4. The *Equality type.*

Provided that the pseudo-judgements $c \in C\ [\Gamma_n]$ and $d \in C\ [\Gamma_n]$ are interpreted as





we interpret

$$\mathsf{Eq}(C, c, d)\ [\Gamma_n]$$

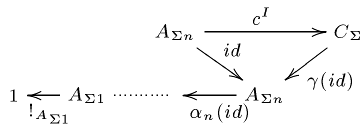
as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{Eq(c^I,d^I)(id)} E_\Sigma$$

where $Eq(c^I, d^I) : \mathcal{P}/A_{\Sigma n} \to \mathcal{P}^{\to}$ is the functor defined in the following manner: for every $D \xrightarrow{t} A_{\Sigma n}$ we put $Eq(c^I, d^I)(t) = (\langle c^I(t), d^I(t)\rangle)^*(\triangle_{D\times C_\Sigma})$ with $\triangle_{D\times C_\Sigma} \equiv \langle id_{D\times C_\Sigma}, id_{D\times C_\Sigma}\rangle$. It is well defined since each pullback functor preserves equalizers, as it has a left adjoint. On the morphisms it is defined through the pullback.

The signature introduced in the *introduction rule* of the equality type is interpreted in the following manner:

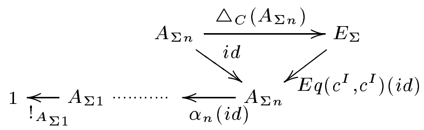
provided that the pseudo-judgement $c \in C\ [\Gamma_n]$ is interpreted as



we interpret

$$\mathsf{eq}_C(c) \in \mathsf{Eq}(C, c, c)\ [\Gamma_n]$$

as



where $\triangle_C(A_{\Sigma n}) \equiv \langle id_{A_{\Sigma n}}, _C\, c^I(id)\rangle$ is the unique morphism to the pullback that defines the equalizer $Eq(c^I, c^I)(id)$ induced by $id_{A_{\Sigma n}}$ and $c^I(id)$.
Therefore, $\mathsf{eq}_C(c)^I \equiv \widehat{\triangle_C(A_{\Sigma n})}$.

5. The *Disjoint Sum type.*

Provided that the pseudo-judgements $C\ [\Gamma_n]$ and $D\ [\Gamma_n]$ are interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\gamma(id)} C_\Sigma \qquad 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\delta(id)} D_\Sigma$$

we interpret

$$C + D\ [\Gamma_n]$$

as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\gamma\oplus\delta(id)} C_\Sigma \oplus D_\Sigma$$

where $\gamma \oplus \delta : \mathcal{P}/A_{\Sigma n} \to \mathcal{P}^{\to}$ is the functor defined in the following manner: for every $D \xrightarrow{t} A_{\Sigma n}$ we put $\gamma \oplus \delta(t) = \gamma(t) \oplus \delta(t)$ and on the morphisms it is defined through the pullback. It is well defined, since coproducts are stable under pullback.

The *first* and *second injections* of the introduction rules of the disjoint sum type are interpreted in this manner:

provided that the pseudo-judgements $c \in C\ [\Gamma_n]$ and $d \in D\ [\Gamma_n]$ are interpreted as

$$\begin{array}{ccc} A_{\Sigma n} & \xrightarrow{c^I} & C_\Sigma \\ & {\scriptstyle id}\searrow \quad \swarrow{\scriptstyle \gamma(id)} & \\ 1 \xleftarrow[!_{A_{\Sigma 1}}]{} A_{\Sigma 1} \cdots\cdots \xleftarrow[\alpha_n(id)]{} & A_{\Sigma n} & \end{array} \qquad \begin{array}{ccc} A_{\Sigma n} & \xrightarrow{d^I} & D_\Sigma \\ & {\scriptstyle id}\searrow \quad \swarrow{\scriptstyle \delta(id)} & \\ 1 \xleftarrow[!_{A_{\Sigma 1}}]{} A_{\Sigma 1} \cdots\cdots \xleftarrow[\alpha_n(id)]{} & A_{\Sigma n} & \end{array}$$

we interpret

$$\mathsf{inl}(c) \in C + D\ [\Gamma_n] \quad \text{and} \quad \mathsf{inr}(d) \in C + D\ [\Gamma_n]$$

as

$$\begin{array}{ccc} A_{\Sigma n} & \xrightarrow{\epsilon_1(c^I)} & C_\Sigma\oplus D_\Sigma \\ & {\scriptstyle id}\searrow \quad \swarrow{\scriptstyle \gamma\oplus\delta(id)} & \\ 1 \xleftarrow[!_{A_{\Sigma 1}}]{} A_{\Sigma 1} \cdots\cdots \xleftarrow[\alpha_n(id)]{} & A_{\Sigma n} & \end{array} \qquad \begin{array}{ccc} A_{\Sigma n} & \xrightarrow{\epsilon_2(d^I)} & C_\Sigma\oplus D_\Sigma \\ & {\scriptstyle id}\searrow \quad \swarrow{\scriptstyle \gamma\oplus\delta(id)} & \\ 1 \xleftarrow[!_{A_{\Sigma 1}}]{} A_{\Sigma 1} \cdots\cdots \xleftarrow[\alpha_n(id)]{} & A_{\Sigma n} & \end{array}$$

where $(\mathsf{inl}(c))^I(id) \equiv \epsilon_1 \cdot (c^I(id))$, $(\mathsf{inr}(d))^I(id) \equiv \epsilon_2 \cdot (d^I(id))$ and ϵ_1, ϵ_2 are the injections of the coproduct $C_\Sigma \oplus D_\Sigma$.

The signature introduced in the *elimination rule* for the disjoint sum type is interpreted in this manner:

provided that the pseudo-judgements $a_C \in A(\mathsf{inl}(y))\ [\Gamma_n, y \in C]$ and $a_D \in A(\mathsf{inr}(z))\ [\Gamma_n, z \in D]$ are interpreted as

$$\begin{array}{ccc} C_\Sigma & \xrightarrow{a_C^I} & A_\Sigma\times C_\Sigma \\ & {\scriptstyle id}\searrow \quad \swarrow{\scriptstyle \alpha(\epsilon_1)} & \\ 1 \xleftarrow[!_{A_{\Sigma 1}}]{} A_{\Sigma 1} \cdots\cdots \xleftarrow[\alpha_n(id)]{} A_{\Sigma n} \xleftarrow[\gamma(id)]{} & C_\Sigma & \end{array} \qquad \begin{array}{ccc} D_\Sigma & \xrightarrow{a_D^I} & A_\Sigma\times D_\Sigma \\ & {\scriptstyle id}\searrow \quad \swarrow{\scriptstyle \alpha(\epsilon_2)} & \\ 1 \xleftarrow[!_{A_{\Sigma 1}}]{} A_{\Sigma 1} \cdots\cdots \xleftarrow[\alpha_n(id)]{} A_{\Sigma n} \xleftarrow[\delta(id)]{} & D_\Sigma & \end{array}$$

we interpret

$$\mathcal{D}(w, a_C, a_D) \in A(w)\ [\Gamma_n, w \in C + D]$$

as

$$\begin{array}{ccc} C_\Sigma\oplus D_\Sigma & \xrightarrow{D^I} & A_\Sigma \\ & {\scriptstyle id}\searrow \quad \swarrow{\scriptstyle \alpha(id)} & \\ 1 \xleftarrow[!_{A_{\Sigma 1}}]{} A_{\Sigma 1} \cdots\cdots \xleftarrow[\gamma\oplus\delta(id)]{} & C_\Sigma\oplus D_\Sigma & \end{array}$$

where $\mathcal{D}^I(id) \equiv (q(\epsilon_1, \alpha(id)) \cdot a_C^I(id)) \oplus (q(\epsilon_2, \alpha(id)) \cdot a_D^I(id))$. We prove that $\mathcal{D}^I(id)$ is a section of $\alpha(id)$. Indeed, we get $\alpha(id) \cdot (\mathcal{D}^I(id) \cdot \epsilon_1) = \alpha(id) \cdot (q(\epsilon_1, \alpha(id)) \cdot a_C^I(id)) = (\epsilon_1 \cdot \alpha(\epsilon_1)) \cdot a_C^I(id) = \epsilon_1$, by the definitions of $\mathcal{D}^I$ and α and by the hypothesis on a_C^I, and, hence, analogously $\alpha(id) \cdot (\mathcal{D}^I(id) \cdot \epsilon_2) = \epsilon_2$. Since $\epsilon_1 \oplus \epsilon_2 = id$, we conclude that $\mathcal{D}^I(id)$ is a section of $\alpha(id)$.

Now, we interpret the signature introduced in the axiom of *Disjointness*

$$\mathsf{m}(c, d) \in \bot\ [\Gamma_n] \quad \text{as} \quad \begin{array}{ccc} A_{\Sigma n} & \xrightarrow{\langle id, t\rangle} & A_{\Sigma n}\times 0_\Sigma \\ & {\scriptstyle id}\searrow \quad \swarrow{\scriptstyle \widehat{0}(!_{A_{\Sigma n}})} & \\ 1 \xleftarrow[!_{A_{\Sigma 1}}]{} A_{\Sigma 1} \cdots\cdots \xleftarrow[\alpha_n(id)]{} & A_{\Sigma n} & \end{array}$$

provided that $\epsilon_1(c^I(id)) = \epsilon_2(d^I(id))$, where $\langle id, t\rangle$ is the unique morphism to the pullback of $!_0$ and $!_{A_{\Sigma n}}$ and t is defined as follows. Since the coproduct is disjoint, the unique morphisms $p_1 : O \to C_\Sigma$ and $p_2 : O \to D_\Sigma$ are the projections of a pullback square between ϵ_1 and ϵ_2 (though this is not the specified one from the structure of $\mathcal{P}$). So, by the hypothesis $\epsilon_1(c^I(id)) = \epsilon_2(d^I(id))$, there exists a unique $t : A_{\Sigma n} \to 0$ such that $p_1 \cdot t = c^I(id)$ and $p_2 \cdot t = d^I(id)$.

6. The *Forall type*.

Provided that the pseudo-judgement $C(y)\ [\Gamma_n, y \in B]$ is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\beta(id)} B_\Sigma \xleftarrow{\gamma(id)} C_\Sigma$$

and $\gamma(id)$ is a monomorphism, we interpret

$$\forall_{y\in B} C(y)\ [\Gamma_n]$$

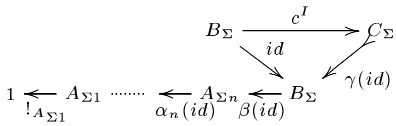
as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\forall_\beta \gamma(id)} \forall_\beta(C_\Sigma)$$

where $\forall_\beta\gamma : \mathcal{P}/A_{\Sigma n} \to \mathcal{P}^{\to}$ is the functor defined in the following manner: for every $D \xrightarrow{t} A_{\Sigma n}$ we put $\forall_\beta\gamma(t) = \forall_{\beta(t)}\gamma(q(t, \beta(id)))$. On the morphisms it is defined through the pullback. It is well defined, since $\gamma(id)$ is a monomorphism and on the morphism part $\forall_\beta\gamma(-)$ sends a morphism of $\mathcal{P}/A_{\Sigma n}$ to a pullback square by the corresponding Beck-Chevalley conditions.

The *abstraction* of the forall type is interpreted in the following manner:

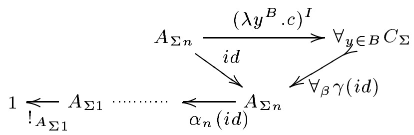
provided that the pseudo-judgement $c \in C(y)\ [\Gamma_n, y \in B]$ is interpreted as



we interpret

$$\lambda y^B.c \in \forall_{y\in B} C(y)\ [\Gamma_n]$$

as



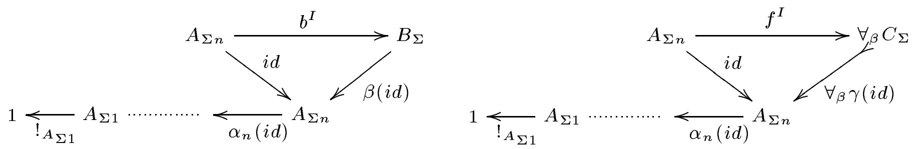
where $(\lambda y^B.c)^I(id) \equiv \psi(c^I(id))$ and ψ is the bijection

$$\psi : \mathcal{P}/B_\Sigma(id_{B_{\Sigma n}}), \gamma(id)) \to \mathcal{P}/A_{\Sigma n}(id_{A_{\Sigma n}}, \forall_{\beta(id)}(\gamma(id)))$$

since $\mathcal{P}/B_\Sigma(id_{B_{\Sigma n}}), \gamma(id)) \simeq \mathcal{P}/B_\Sigma(\beta(id)^*(id_{A_{\Sigma n}}), \gamma(id)) \simeq \mathcal{P}/A_{\Sigma n}(id_{A_{\Sigma n}}, \forall_{\beta(id)}(\gamma(id)))$, where the latter isomorphism is obtained by the bijection of the adjunction $\beta(id)^* \dashv \forall_\beta$.

The *application* of the forall type is interpreted in the following manner:

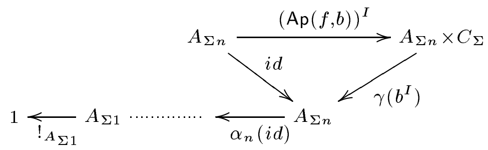
provided that the pseudo-judgements $b \in B\ [\Gamma_n]$ and $f \in \forall_{y\in B} C(y)\ [\Gamma_n]$ are interpreted as



we interpret

$$\mathsf{Ap}(f, b) \in C(b)\ [\Gamma_n]$$

as



where $(\mathsf{Ap}(f,b))^I(id) \equiv \langle id,_{B_\Sigma} \psi^{-1}(f^I(id)) \cdot b^I(id)\rangle$ is the morphism to the pullback of $\gamma(id)$ along $b^I(id)$ and ψ^{-1} is the inverse of ψ.

7. The *Quotient type*.

Suppose that the pseudo-judgement

$$R(x,y) \; type \; [x \in A, y \in A]$$

is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\alpha(id)\cdot\pi_1} A_\Sigma \times A_\Sigma \xleftarrow{\rho(id)} R_\Sigma$$

where $\rho(id)$ is an equivalence relation in $\mathcal{P}/A_{\Sigma n}$ and

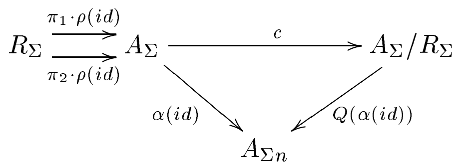
$$A \; [\Gamma_n]$$

is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\alpha(id)} A_\Sigma$$

with $\pi_1 \equiv \alpha(\alpha(id))$ and $\pi_2 \equiv q(\alpha(id), \alpha(id))$.

Since $\rho(id)$ is an equivalence relation in $\mathcal{P}/A_{\Sigma n}$, there exists the coequalizer $c : A_\Sigma \to A_\Sigma/R_\Sigma$ of $\pi_1 \cdot \rho(id)$ and $\pi_2 \cdot \rho(id)$ and we get $Q(\alpha(id))$ such that the following triangle diagram commutes



Therefore, we interpret

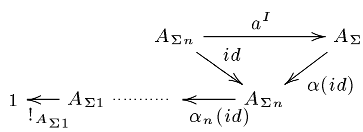
$$A/R \; [\Gamma_n]$$

as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{Q(\alpha)(id)} A_\Sigma/R_\Sigma$$

where $Q(\alpha) : \mathcal{P}/A_{\Sigma n} \to \mathcal{P}^{\to}$ is the functor defined in the following manner: for every $D \xrightarrow{t} A_{\Sigma n}$ we put $Q(\alpha)(t) \equiv Q(\alpha(t))$, where $Q(\alpha(t))$ is the unique morphism such that $\alpha(t) = Q(\alpha(t)) \cdot c(t)$ and $c(t)$ is the quotient of the equivalence relation $\rho(q(q(t, \alpha(id)), \pi_1)$. $Q(\alpha)$ is well defined because the quotient is stable under pullbacks. On the morphisms it is defined through the pullback.

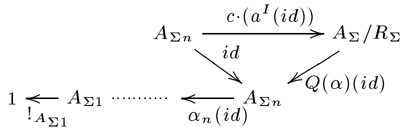
The signature of the *introduction rule* of the quotient type is interpreted in the following manner: provided that the pseudo-judgement $a \in A \; [\Gamma_n]$ is interpreted as



we interpret

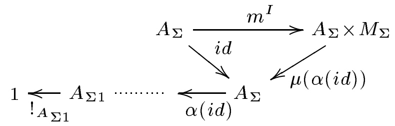
$$[a] \in A/R \; [\Gamma_n]$$

as



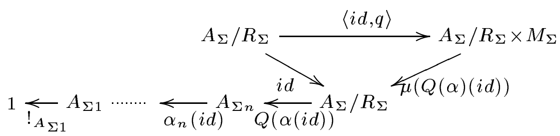
Now, we interpret the signature introduced in the restricted *elimination rule*, E$_s$-quotient.

Suppose that the pseudo-judgement $m(x) \in M\ [\Gamma_n, x \in A]$ is interpreted as



and that $m^I(id) \cdot (\pi_1 \cdot \rho(id)) = m^I(id) \cdot (\pi_2 \cdot \rho(id))$. Therefore, as c is the coequalizer of $\pi_1 \cdot \rho(id)$ and $\pi_2 \cdot \rho(id)$, there exists a morphism q in $\mathcal{P}/A_{\Sigma n}$ such that $q \cdot c = q(\alpha(id), \mu(id)) \cdot m^I(id)$. Since by hypothesis $\mu(id) \cdot (q(\alpha(id), \mu(id)) \cdot m^I(id)) = \alpha(id)$, we also have that by uniqueness $\mu(id) \cdot q = Q(\alpha(id))$.

We finally define the interpretation of $\mathsf{Q_s}(m, z) \in M\ [\Gamma_n, z \in A/R]$ as



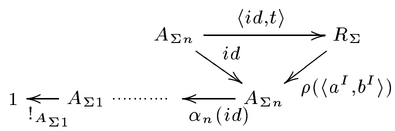
Now, we interpret the signature introduced in the axiom of *Effectiveness*.
Suppose that the pseudo-context Γ_n is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n}$$

We interpret

$$\mathsf{f}(a, b) \in R(a, b)\ [\Gamma_n]$$

as



provided that $c \cdot a^I(id) = c \cdot b^I(id)$, where t is defined as follows. Since the quotient is effective in $\mathcal{P}/A_{\Sigma n}$, then there exists a morphism $t : A_{\Sigma n} \to R_\Sigma$ such that $(\pi_1 \cdot \rho(id)) \cdot t = a^I(id)$ and $(\pi_2 \cdot \rho(id)) \cdot t = b^I(id)$.

8. The *Natural Numbers type*.
Suppose that the pseudo-context Γ_n is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n}$$

then we interpret

$$N\ [\Gamma_n]$$

as

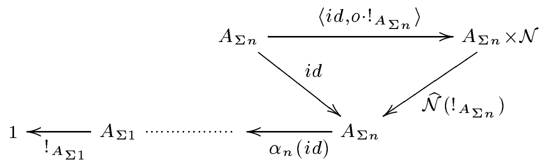
$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\widehat{\mathcal{N}}(!_{A_{\Sigma n}})} A_{\Sigma n} \times \mathcal{N}$$

where $\mathcal{N}$ is a natural numbers object of $\mathcal{P}$ and we recall that $\widehat{\mathcal{N}} : \mathcal{P}/1 \to \mathcal{P}^{\to}$ is the functor defined in the following manner: for every $D \xrightarrow{!_D} 1$ we put $\widehat{\mathcal{N}}(!_D) = (!_D)^*(!_{\mathcal{N}})$ and on the morphisms it is defined through the pullback.

Moreover, under the same assumption about Γ_n, the *zero*

$$0 \in N\ [\Gamma_n]$$

is interpreted as

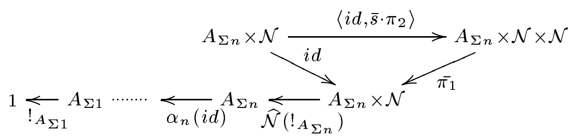


where $o : 1 \to \mathcal{N}$ is the zero map in the H-pretopos $\mathcal{P}$;

and the *successor*

$$\mathsf{s}(x) \in N\ [\Gamma_n, x \in N]$$

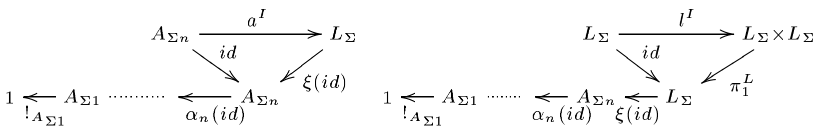
is interpreted as



where $s : \mathcal{N} \to \mathcal{N}$ is the successor map in the H-pretopos $\mathcal{P}$, $\bar{s} \equiv id_1^*(s)$ considering $s \in \mathcal{P}/1(!_{\mathcal{N}}, !_{\mathcal{N}})$, $\bar{\pi_1} \equiv \widehat{\mathcal{N}}(!_{A_{\Sigma n}\times\mathcal{N}})$ and $\pi_2 \equiv q(!_{A_{\Sigma n}}, \widehat{\mathcal{N}}(id))$ and finally $\langle id, \bar{s}\cdot\pi_2\rangle$ is the unique morphism to the pullback of $!_{A_{\Sigma n}\times\mathcal{N}}$ and $\widehat{\mathcal{N}}(!_1)$,

Now, we interpret the signature introduced in the weaker *elimination rule* E_s-Nat.

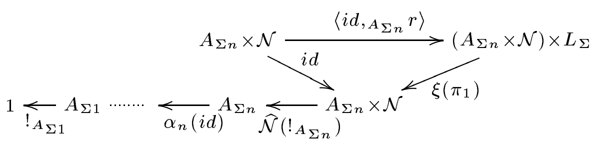
Provided that the pseudo-judgements $a \in L\ [\Gamma_n]$ and $l(y) \in L\ [\Gamma_n, y \in L]$ are interpreted as



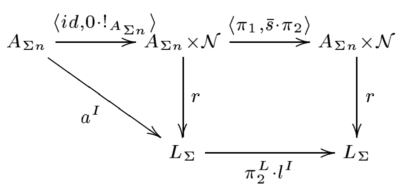
where $\pi_1^L \equiv \xi(\xi(id))$, then we interpret

$$\mathsf{Rec}_s(a, l, n) \in L\ [\Gamma_n, n \in N]$$

as



where r is the unique morphism that makes the following diagram commute by the property of natural numbers object in $\mathcal{P}/A_{\Sigma n}$



with $\pi_1 \equiv \widehat{\mathcal{N}}(!_{A_{\Sigma n}})$ and $\pi_2^L \equiv q(\xi(id), \xi(id))$.

5.3.2 The validity of the type theory HP

In order to prove the validity theorem we need to know how the rules for weakening and substitution are interpreted.
Weakening and substitution of variables in types and terms are expressed by pullback:

Lemma 5.3.5 *The weakening of a variable in type and term judgements is interpreted as follows: given the pseudo-judgements*

$$B(x_1, ..., x_n)\ [\Gamma_n] \quad and \quad D(x_1, ..., x_j)\ [\Gamma_j]$$

where $n \geq j$, interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\beta(id)} B_\Sigma \qquad 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_j(id)} A_{\Sigma j} \xleftarrow{\delta(id)} D_\Sigma$$

then the pseudo-judgement

$$B(x_1, ..., x_n)\ [\Gamma_j, y \in D, x_{j+1} \in A_{j+1}, \ldots, x_n \in A_n]$$

is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots A_{\Sigma j} \xleftarrow{\delta(id)} D_\Sigma \xleftarrow{\alpha_{j+1}(t_j)} D_\Sigma \times A_{\Sigma j+1} \cdots\cdots \xleftarrow{\alpha_n(t_{n-1})} D_\Sigma \times A_{\Sigma n} \xleftarrow{\beta(t_n)} D_\Sigma \times B_\Sigma$$

where $t_j \equiv \delta(id)$ and if $n \geq j+1$, $t_i \equiv q(t_{i-1}, \alpha_i(id))$ for $i = j+1, \ldots, n$
and given the pseudo-judgement

$$b \in B(x_1, ..., x_n)\ [\Gamma_n] \quad and \quad D(x_1, ..., x_j)\ [\Gamma_j]$$

where $n \geq j$, interpreted as

$$\begin{array}{ccccc} & & A_{\Sigma n} & \xrightarrow{b^I(id)} & B_\Sigma \\ & & \searrow^{id} & \swarrow_{\beta(id)} & \\ 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} & \cdots\cdots & \xleftarrow{\alpha_n(id)} A_{\Sigma n} & & \end{array} \qquad 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_j(id)} A_{\Sigma j} \xleftarrow{\delta(id)} D_\Sigma$$

then

$$b \in B(x_1, ..., x_n)\ [\Gamma_j, y \in D, x_{j+1} \in A_{j+1}, \ldots, x_n \in A_n]$$

is interpreted as

$$\begin{array}{ccccc} & & D_\Sigma \times A_{\Sigma n} & \xrightarrow{b^I(t_n)} & D_\Sigma \times B_\Sigma \\ & & \searrow^{id} & \swarrow_{\beta(t_n)} & \\ 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots A_{\Sigma j} \xleftarrow{\xi(id)} D_\Sigma & \cdots\cdots & \xleftarrow{\alpha_n(t_{n-1})} D_\Sigma \times A_{\Sigma n} & & \end{array}$$

Lemma 5.3.6 *The substitution of variables in type and term judgements is interpreted as follows: given the pseudo-judgements*

$$B(x_1, ..., x_n)\ [\Gamma_n] \quad and \quad a_j \in A_j\ [\Gamma_{j-1}]$$

where $n \geq j$, interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\beta(id)} B_\Sigma \qquad \begin{array}{ccccc} & & A_{\Sigma j-1} & \xrightarrow{a_j^I(id)} & A_{\Sigma j} \\ & & \searrow^{id} & \swarrow_{\alpha_j(id)} & \\ 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} & \cdots\cdots & \xleftarrow{\alpha_{j-1}(id)} A_{\Sigma j-1} & & \end{array}$$

then the pseudo-judgement

$$B(x_1,\ldots,x_j,a_j,x'_{j+1},\ldots,x'_n)[\Gamma_{j-1},x'_{j+1}\in A'_{j+1},\ldots,x'_n\in A'_n]$$

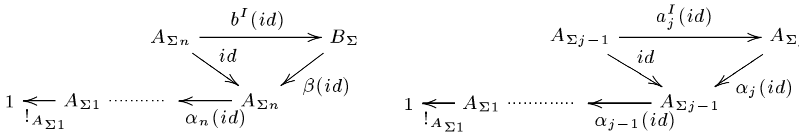
where if $n\geq j+1$, $A'_{j+k}\equiv A_{j+k}[x_j/a_j][x_i/x'_i]_{i=j+1,\ldots,j+k-1}$ *for* $k=1,\ldots n-j$,
is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots A_{\Sigma j-1} \xleftarrow{\alpha_{j+1}(q_j)} A'_{\Sigma j+1} \cdots\cdots \xleftarrow{\alpha_n(q_{n-1})} A'_{\Sigma n} \xleftarrow{\beta(q_n)} B'_{\Sigma}$$

where $q_j\equiv a_j^I(id)$ *and if* $n\geq j+1$, $q_i\equiv q(q_{i-1},\alpha_i(id))$ *for* $i=j+1,\ldots,n$
and given a pseudo-term judgement

$$b\in B(x_1,...,x_n)\ [\Gamma_n] \quad \textit{and} \quad a_j\in A_j\ [\Gamma_{j-1}]$$

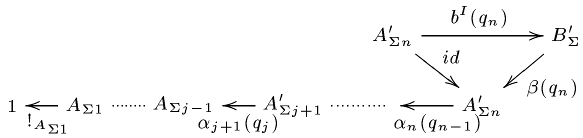
where $n\geq j$, *interpreted as*



then the pseudo-judgement

$$b[x_j/a_j][x_i/x'_i]_{i=j,\ldots,n}\in B(x_1,\ldots,x_j,a_j,x'_{j+1},\ldots,x'_n)\ [\Gamma_{j-1},x'_{j+1}\in A'_{j+1},\ldots,x'_n\in A'_n]$$

is interpreted as



The proofs can be done by induction on the signature of the judgements. For the assumption of variable, the lemma of weakening holds by definition and the lemma of substitution holds because terms are interpreted as sections. In the case of constant type, like $\top$ or N, we use the inductive hypothesis, since the preinterpretation of a context is reduced to that of a type pseudo-judgement. The definition of partial preinterpretation of type and term judgements assures that all these lemmas for weakening and substitution hold for the other type and term constructors.
Now, we are ready to prove the validity theorem:

Theorem 5.3.7 (Validity) *Given a H-pretopos* $\mathcal{P}$, *if A type* $[\Gamma_n]$ *is derivable in HP then* $\mathcal{I}_{\mathcal{P}}(A\ type\ [\Gamma_n])$ *is well defined. If* $a\in A\ [\Gamma_n]$ *is derivable then* $\mathcal{I}_{\mathcal{P}}(a\in A\ [\Gamma_n])$ *is well defined.*
Suppose that A type $[\Gamma_n]$ *and B type* $[\Gamma_n]$ *are derivable in HP, if* $A=B\ [\Gamma_n]$ *is derivable in HP then* $\mathcal{I}_{\mathcal{P}}(A\ type\ [\Gamma_n])=\mathcal{I}_{\mathcal{P}}(B\ type\ [\Gamma_n])$.
Suppose that $a\in A\ [\Gamma_n]$ *and* $b\in A\ [\Gamma_n]$ *are derivable in HP, if* $a=b\in A\ [\Gamma_n]$ *is derivable in HP, then* $\mathcal{I}_{\mathcal{P}}(a\in A\ [\Gamma_n])=\mathcal{I}_{\mathcal{P}}(b\in A\ [\Gamma_n])$.

Proof. The proof can be done by induction on the derivation of the judgement.

Remark 5.3.8 A judgement is valid, if its interpretation is well-defined and hence, in particular an equality judgement for types or terms is valid, if the interpretations of the corresponding type or term judgements are equal. We say that an inference rule holds or is valid, if it preserves the validity of judgements.

Surely, the set rule *conv*) preserves validity of the judgements. Hence, we proceed by proving the validity of the formation, introduction and elimination and conversion rules for the various types with their terms. The lemma of weakening and substitution are crucial for the dependent types, whose rules refer to substitution or weakening such as, for example, the forall type.

1. The formation, introduction and elimination and conversion rules for the *Terminal type* hold because for every object D in $\mathcal{P}$, $\widehat{1}(!_D)$ is a terminal object in $\mathcal{P}/D$.

2. The formation and elimination rules of the *False type* are valid.

3. The formation, introduction and elimination rules of the *Indexed Sum type* are valid. Moreover, the β and η conversion rules for the indexed sum type hold by the properties of pullback. Indeed, the β_1-C conversion rule for the indexed sum type

$$\frac{b \in B \quad c \in C(b)}{\pi_1(\langle b, c\rangle) = b \in B}$$

holds because $\gamma(id) \cdot (q(b^I(id), \gamma(id)) \cdot c^I(id)) = b^I(id) \cdot \gamma(b^I(id)) \cdot c^I(id)$, from which as $c^I(id)$ is a section of $\gamma(b^I(id))$ we get $\gamma(id) \cdot (q(b^I(id), \gamma(id)) \cdot c^I(id) = b^I(id)$.
The β_2-C conversion rule for the indexed sum type

$$\frac{b \in B \quad c \in C(b)}{\pi_2(\langle b, c\rangle) = c \in C(b)}$$

holds because $\langle id, q(b^I(id), \gamma(id)) \cdot c^I(id)\rangle = c^I(id)$, since $c^I(id)$ is a section of $\gamma(b^I(id))$. Finally, the η-C conversion rule for the indexed sum type

$$\frac{d \in \Sigma_{x\in B} C(x)}{\langle \pi_1(d), \pi_2(d)\rangle = d \in \Sigma_{x\in B} C(x)}$$

holds since $q(\gamma(id) \cdot d^I(id), \gamma(id)) \cdot \langle id, d^I(id)\rangle = d^I(id)$.

4. The formation, introduction rules of the *Equality type* hold.

   The E-equality elimination rule
$$\frac{p \in \mathsf{Eq}(C, c, d)}{c = d \in C}$$
   holds because for every $t : D \to A_n$, $p^I(t)$ is a section of $Eq(c^I, d^I)(t)$, which is the equalizer of $c^I(t)$ and $d^I(t)$, so we conclude that $c^I(t) = d^I(t)$. Moreover, the C-equality conversion rule

$$\frac{p \in Eq(C, c, d)}{p = \mathsf{eq}_C \in \mathsf{Eq}(C, c, d)}$$

   holds, because the equalizer is a monomorphism.

5. The formation, introduction and elimination rules of the *Disjoint Sum type* hold.

   The C_1 conversion rule for the disjoint sum type

$$\frac{a_C(x) \in A(\mathsf{inl}(x))\ [x \in C] \quad a_D(y) \in A(\mathsf{inr}(y))\ [y \in D]}{\mathcal{D}(\mathsf{inl}(x), a_C, a_D) = a^I_C(x) \in A(\mathsf{inl}(x))\ [x \in C]}$$

   holds since $\mathcal{D}(\mathsf{inl}(x), a_C, a_D)^I(id) = \langle id\ ,_{C_\Sigma \oplus D_\Sigma}\ (q(\epsilon_1, \alpha(id)) \cdot a^I_C(id) \oplus q(\epsilon_2, \alpha(id)) \cdot a^I_D(id))) \cdot \epsilon_1\rangle$ and by uniqueness of the morphism to a pullback $\mathcal{D}(\mathsf{inl}(x), a_C, a_D)^I(id) = a^I_C(id)$.

   The C_2 conversion rule for the disjoint sum type holds for an analogous reason.

   The axiom of *Disjointness* is valid, since by hypothesis $\epsilon_1(c^I(id)) = \epsilon_2(d^I(id))$.

6. The formation rule of the *Forall type* hold. Indeed, given $C(y)\ [\Gamma_n, y \in B]$ interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\beta(id)} B_\Sigma \xleftarrow{\gamma(id)} C_\Sigma$$

   from the validity of the judgement expressing that $C(y)$ is a mono type, $\gamma(id)$ is a monomorphism as follows. In fact, the interpretations of $z \in C\ [\Gamma_n, y \in B, z \in C, w \in C]$ and $w \in C\ [\Gamma_n, y \in$

$B, z \in C, w \in C]$ are isomorphic with the same isomorphism respectively to the first projection and to the second projection of the product $\gamma(id) \times \gamma(id)$. Therefore, by the interpretation of $z = w \in C\ [\Gamma_n, y \in B, z \in C, w \in C]$, we get that the two projections of $\gamma(id)$, that are $\gamma(\gamma(id))$ and $q(\gamma(id), \gamma(id))$, are equal and we conclude that $\gamma(id)$ is a monomorphism.

The introduction and elimination rules of the *Forall type* hold by the lemma of substitution.

The β-C conversion rule

$$\frac{b \in B\ [\Gamma_n] \quad c \in C\ [\Gamma_n, y \in B] \quad y = z \in C\ [\Gamma_n, x \in B, y \in C, z \in C]}{\mathsf{Ap}(\lambda x^B.c, b) = c(b) \in C\ [\Gamma_n]}$$

holds since $\langle id,_{B_{\Sigma n}} \psi^{-1}(\psi(c^I(id))) \cdot b^I(id)\rangle = c^I(b^I(id))$.

The η-C conversion rule

$$\frac{f \in \forall_{x \in B} C(x)}{\lambda x^A.\mathsf{Ap}(f, x) = f \in \forall_{x \in B} C(x)}$$

holds because

$(\lambda x^A.\mathsf{Ap}(f, x))^I(id) = \psi(\langle id,_{B_\Sigma \times B_\Sigma} \psi^{-1}(\langle id,_{A_{\Sigma n}} f^I(id) \cdot \beta(id)\rangle) \cdot \triangle_{B_\Sigma}\rangle)$
$= \psi(\langle id,_{B_\Sigma \times B_\Sigma} \langle id,_{B_\Sigma} \psi^{-1}(f^I(id)) \cdot q(\beta(id), \beta(id))\rangle \cdot \triangle_{B_\Sigma}\rangle\rangle) = \psi(\psi^{-1}(f^I(id))) = f^I(id)$,
since $\psi^{-1}(\langle id,_{A_{\Sigma n}} f^I(id) \cdot \beta(id)\rangle) = \langle id,_{B_\Sigma} \psi^{-1}(f^I(id)) \cdot q(\beta(id), \beta(id))\rangle$ by Beck-Chevalley conditions for the bijections of the adjunction and the lemma of weakening.

7. The formation rule of the *Quotient type* holds.
Indeed, given the following judgements (we omit Γ_n in the context)

$$\begin{array}{ll} R(x,y)\ type\ [x \in A, y \in A], & z = w \in R(x,y)\ [x \in A, y \in A, z \in R(x,y), w \in R(x,y)], \\ c_1 \in R(x,x)[x \in A], & c_2 \in R(y,x)[x \in A, y \in A, z \in R(x,y)], \\ c_3 \in R(x,z)[x \in A, y \in A, z \in A, w \in R(x,y), w' \in R(y,z)] & \end{array}$$

suppose that $R(x,y)\ type\ [\Gamma_n, x \in A, y \in A]$ is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\alpha(id)\cdot\pi_1} A_\Sigma \times A_\Sigma \xleftarrow{\rho(id)} R_\Sigma$$

with $\pi_1 \equiv \alpha(\alpha(id))$ and $\pi_2 \equiv q(\alpha(id), \alpha(id))$.

From the case of the forall type, we already know that a mono type is interpreted as a monomorphism and here we can prove that $\rho(id)$ turns out to be an equivalence relation in $\mathcal{P}/A_{\Sigma n}$.
Indeed, for *reflexivity*, since $c_1 \in R(x,x)[x \in A]$, there exists a section c_1^I of $\rho(\triangle_{A_\Sigma})$ and $q(\triangle_{A_\Sigma}, \rho(id)) \cdot c_1^I$ is the required morphism that factorizes $\triangle_{A_\Sigma}$ through $\rho(id)$ in $\mathcal{P}/A_{\Sigma n}$.

For *symmetry*, from $c_2 \in R(y,x)[x \in A, y \in A, z \in R(x,y)]$ we get a section c_2^I of $\rho(s \cdot \rho(id))$, where s is the exchange morphism $\langle \pi_2, \pi_1 \rangle$ in $\mathcal{P}/A_{\Sigma n}$. Therefore, $s \cdot \rho(id) = \rho(id) \cdot (q(s \cdot \rho(id), \rho(id)) \cdot c_2^I)$, satisfies the categorical condition for symmetry.

For *transitivity*, from $c_3 \in R(x,z)[x \in A, y \in A, z \in A, w \in R(x,y), w' \in R(y,z)]$ we get a section c_3^I of $\rho(\langle \bar{\pi_1}, \bar{\pi_3}\rangle \cdot \rho(1) \times \rho(2))$, where we call $\bar{\pi_1} \equiv \pi_1 \cdot \alpha(\alpha(id) \cdot \pi_2)$, $\bar{\pi_2} \equiv \pi_2 \cdot \alpha(\alpha(id) \cdot \pi_2)$ and $\bar{\pi_3} \equiv \pi_2 \cdot q(\pi_2, \pi_1)$ and we also abbreviate $\rho(1) \equiv \rho(\langle \bar{\pi_1}, \bar{\pi_2}\rangle)$, $\rho(2) \equiv \rho(\langle \bar{\pi_2}, \bar{\pi_3}\rangle)$ and $\rho(3) \equiv \rho(\langle \bar{\pi_1}, \bar{\pi_3}\rangle)$, and finally $\rho(1) \times \rho(2) \equiv \rho(1) \cdot \rho(\langle \bar{\pi_2}, \bar{\pi_3}\rangle \cdot \rho(1))$. Let us consider the following pullback

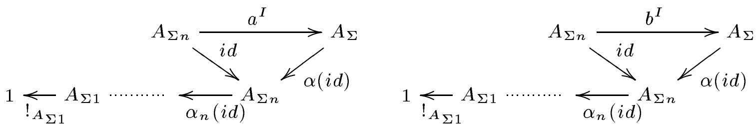
$$\begin{array}{ccc} P & \xrightarrow{p_2} & R \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_2 \cdot \rho(id)} \\ R & \xrightarrow[\pi_1 \cdot \rho(id)]{} & A \end{array}$$

where $p_1 \equiv q(\langle \bar{\pi_2}, \bar{\pi_3}\rangle \cdot \rho(1), \rho(id))$ and $p_2 \equiv q(\langle \bar{\pi_1}, \bar{\pi_2}\rangle, \rho(id)) \cdot \rho(\langle \bar{\pi_2}, \bar{\pi_3}\rangle \cdot \rho(1))$.

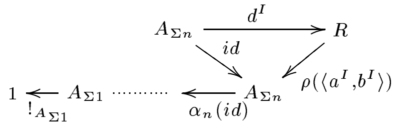
Therefore, we conclude that

$(\pi_1 \cdot \rho(id)) \cdot p_2 =$
$= \pi_1 \cdot \langle \bar{\pi_1}, \bar{\pi_2} \rangle \cdot \rho(1) \times \rho(2)$
$= \pi_1 \cdot \langle \bar{\pi_1}, \bar{\pi_3} \rangle \cdot \rho(1) \times \rho(2)$
$= (\pi_1 \cdot \rho(id)) \cdot q(\langle \bar{\pi_1}, \bar{\pi_3} \rangle \cdot \rho(1) \times \rho(2), \rho(id)) \cdot c_3$
and analogously
$(\pi_2 \cdot \rho(id)) \cdot p_1 =$
$= \pi_2 \cdot \langle \bar{\pi_2}, \bar{\pi_3} \rangle \cdot \rho(1) \times \rho(2)$
$= \pi_2 \cdot \langle \bar{\pi_1}, \bar{\pi_3} \rangle \cdot \rho(1) \times \rho(2)$
$= (\pi_2 \cdot \rho(id)) \cdot q(\langle \bar{\pi_1}, \bar{\pi_3} \rangle \cdot \rho(1) \times \rho(2), \rho(id)) \cdot c_3$
hence, $\langle \pi_1 \cdot \rho(id) \cdot p_2, \pi_2 \cdot \rho(id) \cdot p_1 \rangle$ factorizes through $\rho(id)$, that is the categorical condition for transitivity holds.

The equality rule for the quotient type holds. Indeed, suppose that $a \in A\ [\Gamma_n]$ and $b \in A\ [\Gamma_n]$ are interpreted as



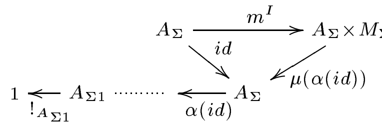
and $d \in R(a, b)\ [\Gamma_n]$ is interpreted as



Then we get that $(\pi_1 \cdot \rho(id)) \cdot (q(\langle a^I, b^I \rangle, \rho(id)) \cdot d^I) = a^I(id)$ and $(\pi_2 \cdot \rho(id)) \cdot (q(\langle a^I, b^I \rangle, \rho(id)) \cdot d^I) = b^I(id)$. We conclude that $c \cdot a^I(id) = c \cdot b^I(id)$, that gives the validity of

$$[a] = [b] \in A/R\ [\Gamma_n]$$

The elimination rule for the quotient type holds, since given $m(x) \in M\ [\Gamma_n, x \in A]$ interpreted as



and given $m(x) = m(y) \in M\ [\Gamma_n, x \in A, y \in A, d \in R(x, y)]$ interpreted as the equality between morphisms, we get that

$$\langle id, m^I(id) \cdot (\pi_1 \cdot \rho(id)) \rangle = \langle id, m^I(id) \cdot (\pi_2 \cdot \rho(id)) \rangle$$

(we recall that $\pi_1 \equiv \alpha(\alpha(id))$ and $\pi_2 \equiv q(\alpha(id), \alpha(id))$). From this, we obtain $m^I(id) \cdot (\pi_1 \cdot \rho(id)) = m^I(id) \cdot (\pi_2 \cdot \rho(id))$.

The β_sC-quotient conversion rule

$$\frac{a \in A \quad m(x) \in M\ [x \in A] \quad m(x) = m(y) \in M\ [x \in A, y \in A, d \in R(x, y)]}{\mathsf{Q_s}(m, [a]) = m(a) \in M}$$

holds since
$(\mathsf{Q_s}(m, [a]))^I(id) = \langle id\ ,_{A_\Sigma / R_\Sigma} \langle id\ ,_{A_{\Sigma n}} q \rangle \cdot c \cdot (a^I(id)) \rangle = \langle id\ ,_{A_{\Sigma n}} q \cdot (c \cdot a^I(id)) \rangle =$
$= \langle id ,_{A_{\Sigma n}} (q(\alpha(id), \mu(id)) \cdot m^I(id)) \cdot a^I(id) \rangle = \langle id\ ,_{A_\Sigma} m^I(id) \cdot a^I(id) \rangle = m(a)^I(id).$

The η_sC-quotient conversion rule

$$\frac{t(z) \in M\ [z \in A/R]}{\mathsf{Q_s}((x)t([x]), z) = t(z) \in M\ [z \in A/R]}$$

holds since

$\mathsf{Q_s}((x)t([x]), z)^I(id) = \langle id\ ,_{A_{\Sigma n}}\ q\rangle$, where $q \cdot c = q(Q(\alpha)(id), \mu(id)) \cdot (t^I(id) \cdot c)$. So by uniqueness of the morphism from a coequalizer we get $q = q(Q(\alpha)(id), \mu(id)) \cdot t^I(id)$ and then $\langle id,_{A_{\Sigma n}}\ q\rangle = t^I(id)$.

The axiom of *Effectiveness* holds

$$\frac{a \in A \quad b \in A \quad [a] = [b] \in A/R}{f(a, b) \in R(a, b)}$$

since, by validity of the hypothesis, $c \cdot a^I(id) = c \cdot b^I(id)$.

8. The formation, introduction and elimination rule for the *Natural Numbers type* hold, since $\widehat{\mathcal{N}}(!_D)$ is a natural numbers object of $\mathcal{P}/D$.

   The conversion rules for the natural numbers type follow from the properties of the natural numbers object. The β_sC$_1$-Nat conversion rule for the natural numbers type

$$\frac{a \in L \quad l(y) \in L\ [y \in L]}{\mathsf{Rec}_s(a, l, 0) = a \in L}$$

   holds since

   $\mathsf{Rec}_s(a, l, 0)^I(id) = \langle id\ ,_{A_{\Sigma n}}\ r \cdot (\langle id, o{\cdot}!_{A_{\Sigma n}}\rangle)\rangle = a^I(id)$ by definition of the natural numbers object $A_{\Sigma n} \times \mathcal{N}$ in $\mathcal{P}/A_{\Sigma n}$.

   The β_sC$_2$-Nat conversion rule for the natural numbers type

$$\frac{a \in L \quad l(y) \in L\ [y \in L]}{\mathsf{Rec}_s(a, l, \mathsf{s}(n)) = l(\mathsf{Rec}_s(a, l, n)) \in L\ [n \in N]}$$

   holds since

   $\mathsf{Rec}_s(a, l, \mathsf{s}(n))^I(id) = \langle id\ ,_{A_{\Sigma n} \times \mathcal{N}}\ (\langle id,_{A_{\Sigma n}}\ r\rangle \cdot q(\widehat{\mathcal{N}}(!_{A_{\Sigma n}}), !_{A_{\Sigma n}})) \cdot \langle id\ , \bar{s} \cdot \pi_2\rangle\rangle =$
   $= \langle id\ ,_{A_{\Sigma n}}\ r \cdot \langle \pi_1\ , \bar{s} \cdot \pi_2\rangle\rangle = \langle id\ ,_{A_{\Sigma n}}\ (\pi_2^L \cdot l^I(id)) \cdot r\rangle =$
   $= \langle id\ ,_{A_{\Sigma n}}\ (\pi_2^L \cdot l^I(id)) \cdot (q(\pi_1, \xi(id)) \cdot \langle id,_{A_{\Sigma n}}\ r\rangle) = l(\mathsf{Rec}_s(a, l, n))^I(id)$
   where we recall that $\pi_1 \equiv \widehat{\mathcal{N}}(!_{A_{\Sigma n}})$ and $\pi_2 \equiv q(!_{A_{\Sigma n}}, \widehat{\mathcal{N}}(id))$.

   The η_sC-Nat conversion rule for the natural numbers type also holds by uniqueness of the morphism that makes the diagram of the natural numbers object commute w.r.t. l^I.

∎

The interpretation of the type theory $\mathcal{T}_t$

Now, given a topos $\mathcal{S}$, we proceed to define the partial preinterpretation $\widetilde{\mathcal{I}_{\mathcal{S}}}$ from the pseudo-judgements of $\mathcal{T}_t$ into $Pgf(\mathcal{S})$, and hence a partial interpretation into $Pgr(\mathcal{S})$, by induction on the complexity of pseudo-judgements. In the inductive hypothesis, we will refer to the interpretation of a pseudo-judgement, assuming that also the preinterpretation is given.
We assume all the definitions, already given in the introduction of this section about the categorical semantics in a universe, replacing $\mathcal{P}$ with the topos $\mathcal{S}$.
The interpretation of the *assumption of variable* and the pseudo-judgements with the signature introduced in the formation, introduction, elimination and conversion rules for the *Terminal type*, the *Indexed Sum type*, the *Equality type* are the same as for the type theory HP of H-pretoposes. It remains to interpret the signature introduced in the formation, introduction, elimination and conversion rules for the Product type and the Omega type.

1. The *Product type*.

Provided that the pseudo-judgement $C(y)\ [\Gamma_n, y \in B]$ is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\beta(id)} B_\Sigma \xleftarrow{\gamma(id)} C_\Sigma$$

we interpret

$$\Pi_{y \in B} C(y)\ [\Gamma_n]$$

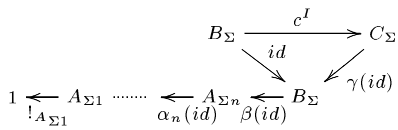
as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\Pi_\beta \gamma(id)} \Pi_\beta(C_\Sigma)$$

where $\Pi_\beta \gamma : \mathcal{S}/A_{\Sigma n} \to \mathcal{S}^{\to}$ is the functor defined in the following manner: for every $D \xrightarrow{t} A_{\Sigma n}$ we put $\Pi_\beta \gamma(t) = \Pi_{\beta(t)} \gamma(q(t, \beta(id)))$. On the morphisms it is defined through the pullback. It is well defined, since the corresponding Beck-Chevalley conditions hold.

The *abstraction* of the product type is interpreted as follows:

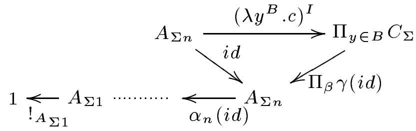
provided that the pseudo-judgement $c \in C(y)\ [\Gamma_n, y \in B]$ is interpreted as



we interpret

$$\lambda y^B.c \in \Pi_{y \in B} C(y)\ [\Gamma_n]$$

as



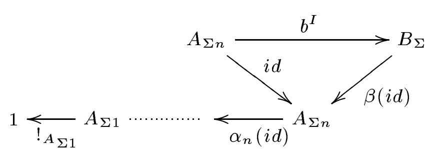
where $(\lambda y^B.c)^I(id) \equiv \psi(c^I(id))$ and and ψ is the bijection

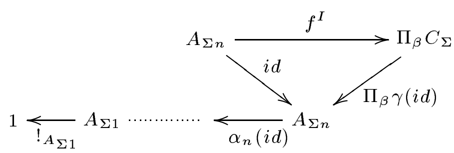
$$\psi : \mathcal{S}/B_\Sigma(id_{B_{\Sigma n}}), \gamma(id)) \to \mathcal{S}/A_{\Sigma n}(id_{A_{\Sigma n}}, \Pi_{\beta(id)}(\gamma(id)))$$

since $\mathcal{S}/B_\Sigma(id_{B_{\Sigma n}}), \gamma(id)) \simeq \mathcal{S}/B_\Sigma(\beta(id)^*(id_{A_{\Sigma n}}), \gamma(id)) \simeq \mathcal{S}/A_{\Sigma n}(id_{A_{\Sigma n}}, \Pi_{\beta(id)}(\gamma(id)))$, where the latter isomorphism is obtained by the bijection of the adjunction $\beta(id)^* \dashv \Pi_\beta$.

The *application* of the product type is interpreted in the following manner:

provided that the pseudo-judgements $b \in B\ [\Gamma_n]$ and $f \in \Pi_{y \in B} C(y)\ [\Gamma_n]$ are interpreted as

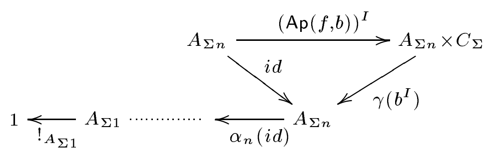




we interpret

$$\mathsf{Ap}(f, b) \in C(b)\ [\Gamma_n]$$

as



where $(\mathsf{Ap}(f, b))^I(id) \equiv \langle id,_{B_\Sigma} \psi^{-1}(f^I(id)) \cdot b^I(id) \rangle$ is the morphism to the pullback of $\gamma(id)$ along $b^I(id)$ and ψ^{-1} is the inverse of ψ.

2. The *Omega type.*
   Provided that the pseudo-context Γ_n is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n}$$

we interpret

$$\Omega\ type\ [\Gamma_n]$$

as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\Omega^I(!_{A_{\Sigma n}})} A_{\Sigma n} \times \mathcal{P}(1)$$

where, for every $D \in Ob\mathcal{P}$, we put $\Omega^I(!_D) \equiv !^*_D(!_{\mathcal{P}(1)})$ and $\mathcal{P}(1)$ is the subobject classifier.

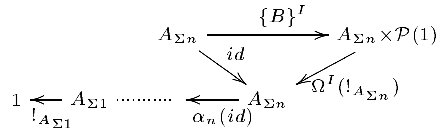
Morever, in the *introduction rule* for the Omega type, provided that the pseudo-judgement $B\ type\ [\Gamma_n]$ is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\beta(id)} B_\Sigma$$

where $\beta(id)$ is a monomorphism, we interpret

$$\{B\} \in \Omega\ [\Gamma_n]$$

as



where $\{B\}^I(id) \equiv ch(\beta(id))$, that is the characteristic morphism of the monomorphism $\beta(id)$ with respect to $A_{\Sigma n} \times \mathcal{P}(1)$.

Now, we interpret the signature introduced in the β-conversion rule for the Omega type.

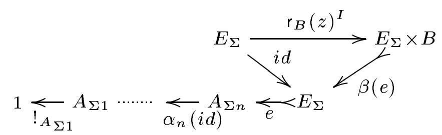
Provided that the pseudo-judgement $B\ type\ [\Gamma_n]$ is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\beta(id)} B_\Sigma$$

and $\beta(id)$ is a monomorphism then, we interpret

$$\mathsf{r}_B(z) \in\in B[\Gamma_n, w \in \mathsf{Eq}(\Omega, \{B\}, \{\top\})[\Gamma_n]$$

as



where $e \equiv Eq(\{B\}^I, \{\top\}^I)(id)$ and $\mathsf{r}_B(z)^I(id) \equiv \langle id, \rho_{B_\Sigma}\rangle$, and ρ_{B_Σ} is the isomorphism in $\mathcal{C}/A_{\Sigma n}$ from the equalizer of $\{B\}^I(id)$ and $True \cdot !_{B_\Sigma}$ to the monomorphism $\beta(id)$.

∎

5.3.3 The validity of the type theory $\mathcal{T}_t$

In order to prove the validity theorem, we need to know how the rules of weakening and substitution are interpreted. The lemmas about the interpretation of the rules of weakening and substitution and their proofs are the same as for the type theory HP (see section 5.3.2).

Theorem 5.3.9 (Validity) *Given a topos $\mathcal{S}$, if A type $[\Gamma_n]$ is derivable in $\mathcal{T}_t$ then $\mathcal{I}_{\mathcal{S}}(A\ type\ [\Gamma_n])$ is well defined. If $a \in A\ [\Gamma_n]$ is derivable in $\mathcal{T}_t$, then $\mathcal{I}_{\mathcal{S}}(a \in A\ [\Gamma_n])$ is well defined.*
Suppose that A type $[\Gamma_n]$ and B type $[\Gamma_n]$ are derivable in $\mathcal{T}_t$, if $A = B\ [\Gamma_n]$ is derivable in $\mathcal{T}_t$ then $\mathcal{I}_{\mathcal{S}}(A\ type\ [\Gamma_n]) = \mathcal{I}_{\mathcal{S}}(B\ type\ [\Gamma_n])$.
Suppose that $a \in A\ [\Gamma_n]$ and $b \in A\ [\Gamma_n]$ are derivable in $\mathcal{T}_t$, if $a = b \in A\ [\Gamma_n]$ is derivable in $\mathcal{T}_t$ then $\mathcal{I}_{\mathcal{S}}(a \in A\ [\Gamma_n]) = \mathcal{I}_{\mathcal{S}}(b \in A\ [\Gamma_n])$.

Proof. The proof can be done by induction on the derivation of the judgement.
We adopt the same definition of validity of an inference rule, as in the proof of the validity theorem for the type theory HP. Surely, the set rule preserves validity of the judgements. For the validity of the formation, introduction, elimination and conversion rules for the *Terminal type*, the *Indexed Sum type*, the *Equality type* and also the *Product type* we refer to the same proof of the type theory HP in section 5.3.2. It remains to prove the validity of formation, introduction, elimination and conversion rules for the Omega type. The formation rule for the Omega type is valid, since the subobject classifier is stable under pullbacks (see the appendix A).
The *introduction rule* for the Omega type is valid, because, provided that the pseudo-judgement B *type* $[\Gamma_n]$ is interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\beta(id)} B_{\Sigma}$$

then $\beta(id)$ turns out to be a monomorphism by the validity of the judgement in the hypothesis of the introduction rule

$$y = z \in B\ [\Gamma_n, y \in B, z \in B]$$

as we have already seen in the case of the interpretation of the forall type in the type theory HP in section 5.3.2. Therefore

$$\{B\} \in \Omega\ [\Gamma_n]$$

turns out to be well interpreted as

$$\begin{array}{ccccccc} & & & A_{\Sigma n} & \xrightarrow{\{B\}^I} & A_{\Sigma n} \times \mathcal{P}(1) \\ & & & & {}_{id}\searrow & \swarrow{}_{\Omega^I(!_{A_{\Sigma n}})} \\ 1 & \xleftarrow{!_{A_{\Sigma 1}}} & A_{\Sigma 1} & \cdots\cdots & \xleftarrow{\alpha_n(id)} & A_{\Sigma n} \end{array}$$

where $\{B\}^I(id) \equiv ch(\beta(id))$, that is the characteristic morphism of the monomorphism $\beta(id)$ with respect to $A_{\Sigma n} \times \mathcal{P}(1)$.
We can show that the equality rule on Ω is valid. Indeed, given

$$\begin{array}{ll} B\ type & y = z \in B\ [\Gamma_n, y \in B, z \in B] \\ C\ type & y = z \in C\ [\Gamma_n, y \in C, z \in C] \end{array}$$

interpreted as

$$1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\beta(id)} B_{\Sigma} \qquad 1 \xleftarrow{!_{A_{\Sigma 1}}} A_{\Sigma 1} \cdots\cdots \xleftarrow{\alpha_n(id)} A_{\Sigma n} \xleftarrow{\gamma(id)} C_{\Sigma}$$

and given $f \in B \leftrightarrow C\ [\Gamma_n]$, by validity of the rules for the product type, $\beta(id)$ and $\gamma(id)$ turn out to be isomorphic, since they are monomorphisms. Hence, by the property of subobject classifier, we get $\{B\}^I(id) = \{C\}^I(id)$.
Moreover, for the same reason as the introduction rule, the β-conversion rule for the Omega type is valid.
Finally, we show that the ηC conversion rule holds. Given $q \in \Omega\ [\Gamma_n]$ interpreted as

$$\begin{array}{ccccccc} & & & A_{\Sigma n} & \xrightarrow{q^I} & A_{\Sigma n} \times \mathcal{P}(1) \\ & & & & {}_{id}\searrow & \swarrow{}_{\Omega^I(!_{A_{\Sigma n}})} \\ 1 & \xleftarrow{!_{A_{\Sigma 1}}} & A_{\Sigma 1} & \cdots\cdots & \xleftarrow{\alpha_n(id)} & A_{\Sigma n} \end{array}$$

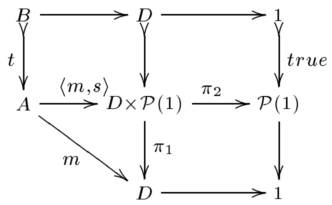
we get that

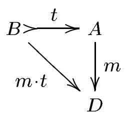
$$\{Eq(\Omega, q, \{\top\})\} = q \in \Omega$$

is satisfied. Indeed, $\{Eq(\Omega, q, \top)\}^I(id)$ is the characteristic map of the equalizer between $q^I(id)$ and the $True$ map pulled back by a suitable ! morphism. This equalizer is isomorphic to the pullback of the $True$ map along $q(!_{A_{\Sigma n}}, \Omega^I(id)) \cdot q^I(id)$, so that we get $\{Eq(\Omega, q, \{\top\})\}^I(id) = q^I(id)$.
■

5.4 Appendix A: about the subobject classifier

We show that the subobject classifier is stable under pullbacks. Consider the pullback of $!_D$ and $!_{\mathcal{P}(1)}$, then the pullback of the $true$ map along $D\times\mathcal{P}(1) \xrightarrow{\pi_2} \mathcal{P}(1)$ in the following diagram



Note that in $\mathcal{S}/D$ for every monomorphism 

, there exists in $\mathcal{S}$ its characteristic morphism $A \xrightarrow{s} \mathcal{P}(1)$ and $\langle m, s\rangle$ turns out to be its characteristic morphism in $\mathcal{S}/D$ by properties of composition of pullbacks.

5.5 Appendix B: our semantics as category with attributes

As we have seen in this chapter, our notion of model for the type theories HP and $\mathcal{T}_t$ consists of a categorical universe, namely respectively a H-pretopos and a topos, with a fixed choice of its structure, where the interpretation is given by the reindexing functor of the split fibration equivalent to the codomain fibration. The fact that these interpretations provide models for the theories is assured by the theorems of validity of HP and $\mathcal{T}_t$ with respect to their corresponding categorical universes. We describe the notion of model for the type theory of Heyting pretoposes and that for the type theory of toposes, in terms of contextual category with attributes.

5.5.1 The contextual category with attributes for HP

The contextual category with attributes for the type theory HP is a contextual category $\mathcal{C}$ as in [Pit95] with attributes to interpret the various type constructors.

Def. 5.5.1 A *contextual category* $\mathcal{C}$ is a category possessing a terminal object, 1, and equipped with the following structure:

- for each object X in $\mathcal{C}$, a collection of $Type_{\mathcal{C}}(X)$, whose elements will be called *X-indexed types* in $\mathcal{C}$;
- for each object X in $\mathcal{C}$, operations assuming to each X-indexed type A an object $X \propto A$, called the *total object* of A, together with a morphism

$$\pi_A : X \propto A \to X$$

called the *project morphism* of A;

- for each morphism $f : Y \to X$ in $\mathcal{C}$, an operation assigning to each X-indexed type A, a Y-indexed type f^*A, called the *pullback of A along X*, together with a morphism $f \propto A : Y \propto f^*A \to X \propto A$, making the following diagram a pullback in $\mathcal{C}$

$$\begin{array}{ccc} Y \propto f^*A & \xrightarrow{f \propto A} & X \propto A \\ {\scriptstyle \pi_{f^*A}} \downarrow & & \downarrow {\scriptstyle \pi_A} \\ Y & \xrightarrow[f]{} & X \end{array}$$

and such that the following *strictness conditions* hold:

$$id_X{}^*A = A \qquad id_X \propto A = id_{X \propto A}$$

$$g^*(f^*A) = (f \cdot g)^*A \qquad (f \propto A) \cdot (g \propto f^*A) = (f \cdot g) \propto A$$

for every $f : Y \to X$ and $g : Z \to Y$ morphism in $\mathcal{C}$.

For each object X, the *global sections* of an X-indexed type A are the morphisms in $\mathcal{C}$

$$a : X \to X \propto A \qquad \text{such that} \qquad \pi_A \cdot a = id_X$$

and for each morphism $f : Y \to X$, using the universal property of pullback, we get the unique morphism

$$f^*a : Y \to Y \propto f^*A$$

such that $\pi_{f^*A} \cdot f^*a = id_Y$ and $(f \propto A) \cdot f^*a = a \cdot f$.

Remark 5.5.2 We can endow the collection $Type_{\mathcal{C}}(X)$ with the category structure, by defining a morphism of $Type_{\mathcal{C}}(X)$ from the indexed type A to B as a morphism in $\mathcal{C}/X$ from π_A to π_B.

Def. 5.5.3 For each morphism $f : Y \to X$ in $\mathcal{C}$ we define the pullback functor along f

$$f^* : Type_{\mathcal{C}}(X) \to Type_{\mathcal{C}}(Y)$$

such that

$$f^*(\ A \xrightarrow{g} B\ ) = \ f^*A \xrightarrow{\langle \pi_{f^*A}, g \cdot (f \propto A) \rangle} f^*B$$

Remark 5.5.4 The attributes, that we are going to define for the various constructors of HP and $\mathcal{T}_t$ are not necessarily the minimal attributes to model each type constructor with its terms, considered separately from the others.

Def. 5.5.5 A category with attributes supports the *false type*, if for each object X in $\mathcal{C}$ there is an indexed type

$$O_X \in Type_{\mathcal{C}}(X)$$

satisfying:

- *Universal Property.* For every $C \in Type_{\mathcal{C}}(X \propto O_X)$, there is a unique global section $\tilde{r_{o}}_C : id_{X \propto O_X} \to \pi_C$
- *Strictness Property.* For each morphism $f : Y \to X$ in $\mathcal{C}$

$$f^*O_X \quad = \quad O_Y$$

Note that in the universal property we could have simply required a global section $\tilde{r_{o}}_C$, stable under pullback, since in the presence of extensional propositional equality type uniqueness follows.

Def. 5.5.6 A category with attributes supports *disjoint sum types*, if for each object X in $\mathcal{C}$ and for every $A \in Type_{\mathcal{C}}(X)$, $B \in Type_{\mathcal{C}}(X)$, there is an indexed type

$$A \oplus B \in Type_{\mathcal{C}}(X)$$

and there are two morphisms in $Type_{\mathcal{C}}(X)$

$$\epsilon_A : A \to A \oplus B$$

$$\epsilon_B : B \to A \oplus B$$

satisfying:

- *Universal Property.* For every $C \in Type_{\mathcal{C}}(X \propto A+B)$, for every global sections $a : id_{X \propto A} \to \pi_{\epsilon_A^* C}$ and $b : id_{X \propto B} \to \pi_{\epsilon_B^* C}$, there is a unique global section $a \oplus b : id_{A \oplus B} \to \pi_C$ such that

$$\epsilon_A^*(a \oplus b) = a \qquad \epsilon_B^*(a \oplus b) = b$$

- *Strictness Property.* For each morphism $f : Y \to X$ in $\mathcal{C}$

$$\begin{array}{rcl} f^*(A \oplus B) & = & f^*A \oplus f^*B \\ f^*(\epsilon_A) & = & \epsilon_{f^*A} \end{array}$$

- *Disjointness.* For every global sections $a : id_X \to \pi_A$ and $b : id_X \to \pi_B$ such that $\epsilon_A \cdot a = \epsilon_B \cdot b$, there exists a unique global section $m : id_X \to \pi_0$, where $0_X \in Type_{\mathcal{C}}(X)$ is the indexed type corresponding to the attribute supporting the false type.

Note that in the universal property we could have simply required a global section $a \oplus b$, stable under pullback.

Remark 5.5.7 In the presence of indexed sum type in HP, we could only define the attribute to interpret a restricted elimination rule for the disjoint sum type, where the type in consideration does not depend on the disjoint sum, by adding the suitable conversion rules, among which that stating the uniqueness of the term introduced in the restricted elimination rule, as in the case of the quotient type and of the natural numbers type (see section 5.3.1).

Def. 5.5.8 A category with attributes supports *extensional equality types*, if for each object X in $\mathcal{C}$, $A \in Type_{\mathcal{C}}(X)$ and for every global sections $a : id_X \to \pi_A$ and $b : id_X \to \pi_A$ in $\mathcal{C}/X$, there is an indexed type

$$\widetilde{Eq}(A, a, b) \in Type_{\mathcal{C}}(X)$$

such that the following conditions are satisfied:

- its projection

$$X \propto \widetilde{Eq}(A,a,b) \xrightarrow{\pi_{\widetilde{Eq}}} X$$

  is the equalizer of a and b in $\mathcal{C}/X$.

- *Strictness Property.* For each morphism $f : Y \to X$ in $\mathcal{C}$

$$f^*(\widetilde{Eq}(A, a, b)) = \widetilde{Eq}(f^*A, f^*(a), f^*(b))$$

Therefore, for every global sections $a : id_X \to \pi_A$, we define $\widetilde{eq}_A(a)$ as the isomorphism such that $\pi_{\widetilde{Eq}(A,a,a)} \cdot \widetilde{eq}_A(a) = id_X$.

Note that we could have also defined this attribute, by simply saying that there is a type $\widetilde{Eq}(A, a, b)$, stable under pullback, such that, whenever there is a global section towards it, then $a = b$ and that there is a global section $\widetilde{eq}_A(a) : id_X \to \pi_{\widetilde{Eq}(A,a,a)}$, stable under pullback such that, for every global section $p : id_X \to \pi_{\widetilde{Eq}(A,a,b)}$, we get $p = \widetilde{eq}_A(a)$.

Def. 5.5.9 A category with attributes supports *forall types*, if for each object X in $\mathcal{C}$, $A \in Type_{\mathcal{C}}(X)$ and $B \in Type_{\mathcal{C}}(X \propto A)$ such that its projection

$$X \propto A \propto B \overset{\pi_B}{\rightarrowtail} X \propto A$$

is a mono in $\mathcal{C}$, there is an indexed type

$$\forall(A, B) \in Type_{\mathcal{C}}(X)$$

whose projection

$$X \propto \forall(A,B) \overset{\pi_{\forall(A,B)}}{\rightarrowtail} X$$

is a mono in $\mathcal{C}$, and there is a morphism in $Type_{\mathcal{C}}(X \propto A)$

$$ap_{A,B} : {\pi_A}^* \forall(A, B) \to B$$

satisfying:

- *Adjointness Property.* $\pi_{\forall(A,B)} : X \propto \forall(A, B) \to X$ is the value of the right adjoint to the pullback functor at π_B with counit $ap_{A,B}$. In other words, for each morphism $f : Y \to X$ in $\mathcal{C}$, and $g : \pi_A^*(f) \to \pi_B$ in $\mathcal{C}/X \propto A$, there is a unique morphism in $\mathcal{C}/X$

$$\psi(g) : f \to \pi_{\forall(A,B)}$$

  satisfying $ap_{A,B} \cdot \pi_A^*(\psi(g)) = g$;

- *Strictness Property.* For each morphism $f : Y \to X$ in $\mathcal{C}$

$$\begin{array}{rcl} f^* \forall(A, B) & = & \forall(f^* A, (f \propto A)^* B) \\ (f \propto A)^*(ap_{A,B}) & = & ap_{f^* A, (f \propto A)^* B} \end{array}$$

Remark 5.5.10 In the adjointness property, we could restrict the requirement of having $\psi(g)$, when g is a global section of B.

Def. 5.5.11 A category with attributes supports *effective quotient types*, with an elimination rule for types not depending on the quotient type, if for each object X in $\mathcal{C}$, $A \in Type_{\mathcal{C}}(X)$ and $R \in Type_{\mathcal{C}}(X \propto A \propto \pi_A^* A)$ such that its projection

$$X \propto A \propto \pi_A^* A \propto R \overset{\pi_R}{\rightarrowtail} X \propto A \propto \pi_A^* A$$

is a mono and $\langle \pi_{\pi_A^* A} \cdot \pi_R, (\pi_A \propto A) \cdot \pi_R \rangle$ is an equivalence relation in $\mathcal{C}/X$, there is an indexed type

$$A/R \in Type_{\mathcal{C}}(X)$$

and there is a morphism in $Type_{\mathcal{C}}(X)$

$$\widetilde{[-]}_{A/R} : \pi_A \to \pi_{A/R}$$

satisfying:

- *Universal Property.* For each $C \in Type_{\mathcal{C}}(X)$ and each morphism $\pi_A \xrightarrow{d} \pi_C$ such that $d \cdot \pi_{\pi_A^* A} \cdot \pi_R = d \cdot (\pi_A \propto A) \cdot \pi_R$, there exists a unique morphism

$$\mathcal{Q}(d) : \pi_{A/R} \to \pi_C$$

  such that $\mathcal{Q}(d) \cdot \widetilde{[-]}_{A/R} = d$.

- *Strictness Property.* For each morphism $f : Y \to X$ in $\mathcal{C}$

$$\begin{array}{rcl} f^*A/R & = & f^*A/((f \propto A) \propto \pi_A^* A)^* R \\ f^*(\widetilde{[-]}_{A/R}) & = & \widetilde{[-]}_{f^*A/((f\propto A)\propto \pi_A^* A)^* R} \end{array}$$

- *Effectiveness.* For every global sections $a : id_X \to \pi_A$ and $b : id_X \to \pi_A$ in $\mathcal{C}/X$ such that

$$\widetilde{[-]}_{A/R} \cdot a = \widetilde{[-]}_{A/R} \cdot b$$

  there exists a unique morphism $\widetilde{f(a,b)} : id_X \to \pi_{\langle a,b\rangle^* R}$ such that $(\pi_R \cdot \langle a,b\rangle \propto R) \cdot \widetilde{f(a,b)} = \langle a,b\rangle$, where $\langle a,b\rangle$ is the morphism induced by a and b towards the pullback of π_A along π_A.

As for the category with attributes supporting the terminal type, indexed sum types, the natural numbers type we refer to [Hof95].

5.5.2 The contextual category with attributes for $\mathcal{T}_t$

We describe the notion of model for the type theory of elementary toposes, in terms of contextual category with attributes. We refer to the definition of contextual category of the section 5.5.1.

Def. 5.5.12 A category with attributes supports *product types*, if for each object X in $\mathcal{C}$, $A \in Type_{\mathcal{C}}(X)$ and $B \in Type_{\mathcal{C}}(X \propto A)$, there is an indexed type

$$\Pi(A,B) \in Type_{\mathcal{C}}(X)$$

and a morphism in $Type_{\mathcal{C}}(X \propto A)$

$$ap_{A,B} : \pi_A^* \Pi(A,B) \to B$$

satisfying:

- *Adjointness Property.* $\pi_{\Pi(A,B)} : X \propto \Pi(A,B) \to X$ is the value of the right adjoint to the pullback functor at $\pi_B : X \propto A \propto B \to X \propto A$ with counit $ap_{A,B}$. In other words, for each morphism $f : Y \to X$ in $\mathcal{C}$, and $g : \pi_A^*(f) \to \pi_B$ in $\mathcal{C}/X \propto A$, there is a unique morphism in $\mathcal{C}/X$

$$cur(g) : f \to \pi_{\Pi(A,B)}$$

  satisfying $ap_{A,B} \cdot \pi_A^*(cur(g)) = g$;

- *Strictness Property.* For each morphism $f : Y \to X$ in $\mathcal{C}$

$$\begin{array}{rcl} f^*\Pi(A,B) & = & \Pi(f^*A, (f \propto A)^* B) \\ (f \propto A)^*(ap_{A,B}) & = & ap_{f^*A,(f\propto A)^*B} \end{array}$$

Remark 5.5.13 In the adjointness property, we could restrict the requirement of having $cur(g)$, when g is a global section of B (see [Hof95]).

Def. 5.5.14 A category with attributes supports the *Omega type*, if for each object X in $\mathcal{C}$, there is a type $\mathcal{P}(1)_X \in Type_{\mathcal{C}}(X)$ and a global section

$$true_X : id_X \to \pi_{\mathcal{P}(1)_X}$$

such that for each $A \in Type_{\mathcal{C}}(X)$, whose projection

$$X \propto A \overset{\pi_A}{\rightarrowtail} X$$

is a mono in $\mathcal{C}$, there is a unique global section

$$ch(A) : id_X \to \pi_{\mathcal{P}(1)_X}$$

in $\mathcal{C}/X$ satisfying the following conditions:

- *Universal Property.* The following diagram is a pullback in $\mathcal{C}/X$

$$\begin{array}{ccc} X \propto A & \longrightarrow & X \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle true_X} \\ X & \xrightarrow[ch(A)]{} & X \propto \mathcal{P}(1)_X \end{array}$$

- *Strictness Property.* For each morphism $f : Y \to X$

$$f^*(\mathcal{P}(1)_X) = \mathcal{P}(1)_Y$$
$$f^*(ch(A)) = ch(f^*A)$$

  where $f^*(ch(A))$ is defined as the unique morphism in $\mathcal{C}/Y$ towards the pullback of π_A along f, induced by id_Y and $ch(A) \cdot f$.

Note that the projection of A is isomorphic to the equalizer of $ch(A)$ and $true_X$. Let us call

$$\rho : eq(ch(A), true_X) \to \pi_A$$

this isomorphism.
For every $A \in Type_{\mathcal{C}}(X)$ such that its projection is a mono, $\tilde{r_A}\ cur(\langle id_{eq(..)}, \rho\rangle)$ in $\mathcal{C}/X$. By the way, we recall that every morphism of $\mathcal{C}$, whose domain is X, becomes a global section of $\mathcal{C}/X$ by taking its graph.

In this definition we assume that the attribute for the extensional equality type is defined as in 5.5.1. About the attributes for the terminal type and the indexed sum type we also refer to section 5.5.1.

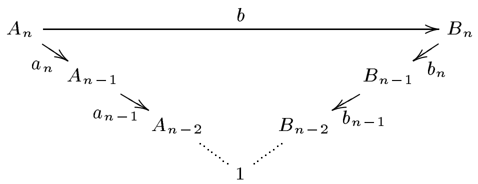
5.5.3 Our model out of a universe as a category with attributes

Now, we see how our notion of model for the type theories HP and $\mathcal{T}_t$, described in this chapter, corresponds to a particular contextual category. This contextual category is given by the reindexing functor of the split fibration equivalent to the codomain fibration, as in the remark 5.2.4, and it is described in the following. Given a H-pretopos or a topos $\mathcal{P}$, we consider, as $\mathcal{C}$, the category of contexts $Cont(\mathcal{P})$ defined as follows:

Def. 5.5.15 *The objects of the category* $\boldsymbol{Cont(\mathcal{P})}$ *are finite sequences* $a_1, a_2, ..., a_n$ *of morphisms of* $\mathcal{P}$

$$A_n \xrightarrow{a_n} \cdots\cdots A_2 \xrightarrow{a_2} A_1 \xrightarrow{a_1} 1$$

and a morphism from $a_1, a_2, ..., a_n$ *to* $b_1, b_2, ..., b_m$ *is simply a morphism* b *of* $\mathcal{P}$



provided that $m = n$ *and* $a_i = b_i$ *for* $i = 1, ..., n-1$.

Moreover, for each object of $Cont(\mathcal{P})$

$$A_n \xrightarrow{a_n} \cdots\cdots A_2 \xrightarrow{a_2} A_1 \xrightarrow{a_1} 1$$

we define

$$Type_{\mathcal{C}}(a_1, a_2, ..., a_n) \equiv Fib(\mathcal{P}/A_n, \mathcal{P}^{\to})$$

Therefore, $Cont(\mathcal{P})$ is equivalent to $\mathcal{P}$ and to $Type_{\mathcal{C}}(1)$ (see the remark 5.2.4). In the case of the model for the type theory HP, the category $\mathcal{P}$ is required to be a H-pretopos and the attributes are defined similarly to the interpretation for the type theory HP in section 5.3.1.
In the case of the model for the type theory $\mathcal{T}_t$, the category $\mathcal{P}$ is required to be a topos and the attributes are defined as in the interpretation for the type theory $\mathcal{T}_t$ in section 5.3.2.

Remark 5.5.16 The class of contextual categories with attributes for the type theory HP ($\mathcal{T}_t$), captured by our semantics, is smaller than the class of contextual categories with attributes for the HP ($\mathcal{T}_t$) calculi. Indeed, not for any contextual category $\mathcal{C}$, we have that $\mathcal{C}$ is equivalent to $Type_{\mathcal{C}}(1)$. By the way, from a contextual category for the type theory HP ($\mathcal{T}_t$) we should get a H-pretopos (a topos) out of the category $Type_{\mathcal{C}}(1)$, as shown in the sections 3.3 and 4.4.

Chapter 6

The completeness theorems

Summary We prove the completeness theorem for the type theory HP with respect to H-pretoposes and the completeness theorem for the type theory $\mathcal{T}_t$ with respect to toposes.

6.1 The proof of completeness

Completeness theorems for the type theories HP and $\mathcal{T}_t$ are proved with respect to a particular class of contextual categories, namely those related to the split fibration equivalent to the codomain fibration.

We know that the completeness theorem with respect to general contextual categories with attributes is quite straightforward (see, for example, [Pit95], [Str91]). Indeed, the interpretation in the syntactic contextual category is faithful, since it turns out to correspond to an identity modulo provable equality between types and between terms. But, since our models are particular contextual categories with attributes, and the interpretation of the indexed sum type is the composition of fibred functors, the interpretation in the syntactic category is no more exactly an identity modulo provable equality. Anyway, this interpretation is isomorphic to a canonical comprehension structure, which does not require new data or choices, useful to prove completeness.

Having seen the validity of the type theory HP with respect to $Pgr(\mathcal{P})$ for every H-pretopos $\mathcal{P}$, we prove the completeness theorem w.r.t. the class of H-pretoposes with a fixed choice of their structure.

In a similar way, having seen the validity of the type theory $\mathcal{T}_t$ with respect to $Pgr(\mathcal{S})$ for every topos $\mathcal{S}$, we prove the completeness theorem w.r.t. the class of toposes with a fixed choice of their structure. For this purpose, given a H-pretopos or a topos $\mathcal{P}$, we define the following category $\mathcal{P}^{\rightarrow_n}$ whose objects are the objects of $Pgr(\mathcal{P})$ and whose morphisms between $\alpha_1, \alpha_2, ..., \alpha_n$ and $\beta_1, \beta_2, ..., \beta_n$ are sequences of morphisms of $\mathcal{P}$ $\phi_1, ..., \phi_n$ such that all the following squares commute

$$\begin{array}{ccc}
A_n & \xrightarrow{\phi_n} & B_n \\
\alpha_n \downarrow & & \downarrow \beta_n \\
A_{n-1} & \xrightarrow{\phi_{n-1}} & B_{n-1} \\
\downarrow \alpha_{n-1} & & \downarrow \beta_{n-1} \\
\vdots & & \vdots \\
A_2 & \xrightarrow{\phi_2} & B_2 \\
\downarrow \alpha_2 & & \downarrow \beta_2 \\
A_1 & \xrightarrow{\phi_1} & B_1 \\
\downarrow \alpha_1 & & \downarrow \beta_1 \\
1 & \xrightarrow{id_1} & 1
\end{array}$$

The proof of the completeness theorem with respect to a class of universes is based on the investigation of the interpretation $\mathcal{I}_{\mathcal{P}_T}$ in the syntactic H-pretopos $\mathcal{P}_T$ and on the investigation of the interpretation $\mathcal{I}_{\mathcal{S}_T}$ in the syntactic topos $\mathcal{S}_T$. These interpretations do not resemble the identity interpretation. Anyway, we will prove that there is a kind of isomorphism between the interpretation $\mathcal{I}_{\mathcal{P}_T}$ and another interpretation of judgements of HP, which we call $\mathcal{J}_{\mathcal{P}_T}$. Analogously, there is also a kind of isomorphism between the interpretation $\mathcal{I}_{\mathcal{S}_T}$ and another interpretation of judgements of $\mathcal{T}_t$, which we call $\mathcal{J}_{\mathcal{S}_T}$. The interpretations $\mathcal{J}_{\mathcal{P}_T}$ and $\mathcal{J}_{\mathcal{S}_T}$ resemble the identity interpretation modulo the equality and are faithful.

6.1.1 The completeness with respect to H-pretoposes

We prove the completeness theorem of the type theory HP with respect to H-pretoposes. In order to do this, we take the syntactic H-pretopos $\mathcal{P}_T$, and we define the interpretation

$$\mathcal{J}_{\mathcal{P}_T} : HP \to Pgr(\mathcal{P}_T)$$

by induction on the number of assumptions in the context in this manner:
$\mathcal{J}_{\mathcal{P}_T}(B\ type\ [x_1 \in A_1, ..., x_n \in A_n(x_1, ..., x_{n-1})])$ is

$$\Sigma_{z_n \in \widetilde{A_n}} \overline{B} \xrightarrow{\pi_1^B} \Sigma_{z_{n-1} \in \widetilde{A_{n-1}}} \overline{A_n} \xrightarrow{\pi_1^n} \cdots \Sigma_{z \in \top} \overline{A_1} \xrightarrow{\pi_1^1} \top$$

where $\widetilde{A_n}$ is the domain of the last morphism of $\mathcal{J}_{\mathcal{P}_T}(A_n\ type[x_1 \in A_1, ..., x_{n-1} \in A_{n-1}(x_1, ..., x_{n-2})])$ and

$$\overline{B} \equiv B[x_1 := \pi_2^1 \cdot \pi_1^2 \cdots \pi_1^n(z_n)] \dots [x_{n-1} := \pi_2^{n-1}(\pi_1^n(z_n))][x_n := \pi_2^n(z_n)]$$

with $\pi_i^n \equiv \lambda x.\pi_i^n(x)$ for $i = 1, 2$, where $\pi_1^n(x)$ and $\pi_2^n(x)$ are the two projections of $\Sigma_{z_{n-1} \in \widetilde{A_{n-1}}} \overline{A_n}$ and π_1^B and π_2^B are the two projections of $\Sigma_{z_n \in \widetilde{A_n}} \overline{B}$. For $i = 1, \dots, n$ $\overline{A_i}$ is defined in the same manner as $\overline{B}$, where $\widetilde{A_1} \equiv \Sigma_{z \in \top} \overline{A_1}$ and $\overline{A_1} \equiv A_1$.
If $b \in B\ [\Gamma_n]$ is a judgement of HP, we put

$$\overline{b} \equiv b\ [x_1 := \pi_2^1 \cdot \pi_1^2 \cdots \pi_1^n(z_n)] \dots [x_{n-1} := \pi_2^{n-1}(\pi_1^n(z_n))][x_n := \pi_2^n(z_n)]$$

and we can derive

$$\overline{b} \in \overline{B}\ [z_n \in \widetilde{A_n}]$$

From now on, we call $B^I = dom(\mathcal{I}_{\mathcal{P}_T}(B\ type\ [\Gamma_n]))$.
In order to prove the completeness theorem, we want to show that

Proposition 6.1.1 *For every judgement*

$$B\ type\ [\Gamma_n]$$

derivable in HP, (which we suppose to be interpreted as $\alpha_1(id), \alpha_2(id), ..., \alpha_n(id), \beta(id)$) there is an isomorphism of $\mathcal{P}_T^{\rightarrow^n}$

$$\phi_{A_1}, ..., \phi_{A_n}, \phi_B$$

between $\mathcal{I}_{\mathcal{P}_T}(B\ type\ [\Gamma_n])$ and $\mathcal{J}_{\mathcal{P}_T}(B\ type\ [\Gamma_n])$ such that for every judgement $b \in B\ [\Gamma_n]$

$$\phi_B \cdot b^I(id) = \langle id, \overline{b} \rangle \cdot \phi_{A_n}$$

and for weakening, for every judgement with $n \geq j$

$$M\ type\ [\Gamma_j]$$

(which we suppose to be interpreted as $\alpha_1(id), \alpha_2(id), ..., \alpha_j(id), \mu(id)$)

$$\phi_B \cdot t_n = p_n \times id \cdot \phi_{M\times B}$$

where $\phi_{M\times B} : (M \times B)^I \to \Sigma_{x\in\widetilde{M}}\overline{B}$, t_i is defined as in the lemma of weakening, $p_j \equiv \pi_1^M \times id$ and if $n \geq j+1$, $p_i \equiv p_{i-1} \times id$ for $i = j+1, \ldots, n$,
and for substitution, for every $a_j \in A_j\ [\Gamma_{j-1}]$ with $n \geq j$

$$\phi_B \cdot q_n = s_n \times id \cdot \phi_{B(a_n)}$$

where q_i is defined as in the lemma of substitution, $s_j \equiv \langle id, \overline{a_n}\rangle \times id$ and if $n \geq j+1$, $s_i \equiv s_{i-1} \times id$ for $i = j+1, \ldots, n$.

Proof. The definition of the isomorphisms is given by induction on the derivation of type and term judgements of HP. In general, given a type judgement

$$B\ type\ [x_1 \in A_1, ..., x_n \in A_n(x_1, ..., x_{n-1})]$$

by inductive hypothesis we know that $\phi_{A_1}, ..., \phi_{A_n}$ is the isomorphism between $\mathcal{I}_{\mathcal{P}_T}(A_n\ type\ [x_1 \in A_1, ..., x_{n-1} \in A_{n-1}(x_1, ..., x_{n-2})])$ and $\mathcal{J}_{\mathcal{P}_T}(A_n\ type\ [x_1 \in A_1, ..., x_{n-1} \in A_{n-1}(x_1, ..., x_{n-2})])$, so we only define the isomorphism $\phi_B : B^I \to \Sigma_{z_n\in\widetilde{A_n}}\overline{B}$ in order to prove that $\phi_{A_1}, ..., \phi_{A_n}, \phi_B$ is the isomorphism in $\mathcal{P}_T^{\to n}$ between $\mathcal{I}_{\mathcal{P}_T}(B\ type\ [\Gamma_n])$ and $\mathcal{J}_{\mathcal{P}_T}(B\ type\ [\Gamma_n])$.
We can define the isomorphisms by induction on the derivation of type and term judgements, since by the validity theorem these are well defined. For the terms it is crucial that isomorphisms commute with the second projections of pullbacks related to weakening and substitution.
For example, we show the inductive steps for the terminal type and for the indexed sum type.

1. Given the *Terminal type* judgement $\top\ [\Gamma_n]$ we define

$$\phi_{A_n\times\top} : \top^I \to \Sigma_{z_n\in\widetilde{A_n}}\top$$

as $\phi_{A_n\times\top}(z) = \langle\phi_{A_n}(\pi_1^\top(z)), \pi_1(\pi_2^\top(z))\rangle$ for $z \in \top^I$, since by definition $\top^I \equiv \Sigma_{z_n\in\widetilde{A_n}}\Sigma_{y\in\top} \star =_\top \star$.
We can easily prove that this isomorphism satisfies all the equations of the proposition.

2. Given the *Indexed Sum type* judgement $\Sigma_{y\in B}C(x_1, ..., x_n, y)[\Gamma_n]$ we define

$$\phi_{\Sigma_{y\in B}C} : (\Sigma_{y\in B}C)^I \to \Sigma_{z_n\in\widetilde{A_n}}\overline{\Sigma_{y\in B}C}$$

as $\phi_{\Sigma_{y\in B}C} \equiv \zeta \cdot \phi_C$ where

$$\zeta : \Sigma_{y\in\widetilde{B}}\overline{C} \to \Sigma_{z_n\in\widetilde{A_n}}\overline{\Sigma_{y\in B}C}$$

is defined in this manner: for every $z \in \Sigma_{y\in\widetilde{B}}\overline{C}$

$$\zeta(z) = \langle\pi_1^B(\pi_1^C(z)), \langle\pi_2^B(\pi_1^C(z)), \pi_2^C(z)\rangle\rangle$$

For short we write Σ for $\Sigma_{y\in B}C$.

We can easily prove that ζ is an isomorphism and in order to check the weakening equation

$$\phi_\Sigma \cdot t_n = p_n \times id \cdot \phi_{M\times\Sigma}$$

it is sufficient to show that

$$p_n \times id \cdot \zeta_{M\times C} = \zeta_C \cdot (p_n \times id) \times id$$

Finally, in analogous way we can prove the substitution equations.

Given the *pair* term $\langle b, c\rangle \in \Sigma_{y\in B}C(x_1, ..., x_n, y)[\Gamma_n]$ we want to prove that

$$\phi_{\Sigma_{y\in B}C} \cdot (\langle b, c\rangle)^I = \langle id, \overline{\langle b, c\rangle}\rangle \cdot \phi_{A_n}$$

Indeed by inductive hypotheses on pullback projections and on substitution
$(\zeta \cdot \phi_C)) \cdot (q(b^I, \gamma(id)) \cdot c^I) =$
$= \zeta \cdot (\langle id, \overline{b}\rangle \times id \cdot \phi_{C(b)}) \cdot c^I =$
$= (\zeta \cdot \langle id, \overline{b}\rangle \times id) \cdot (\langle id, \overline{c}\rangle \cdot \phi_{A_n})$
$= \langle id, \overline{\langle b, c\rangle}\rangle \cdot \phi_{A_n}$.
Given the *first projection* $\pi_1(d) \in B(x_1, ..., x_n)[\Gamma_n]$ we want to prove that

$$\phi_B \cdot (\pi_1(d))^I = \langle id, \overline{\pi_1(d)}\rangle \cdot \phi_{A_n}$$

Indeed, by inductive hypotheses on type judgements and on d^I
$\phi_B \cdot (\gamma(id) \cdot d^I) =$
$\pi_1^{\Sigma_{y \in B} C} \cdot (\phi_C \cdot d^I) =$
$= \pi_1^{\Sigma_{y \in B} C} \cdot (\zeta^{-1} \cdot \langle id, \overline{d}\rangle \cdot \phi_{A_n})$
$= \langle id, \overline{\pi_1(d)}\rangle \cdot \phi_{A_n}$.
Given the *second projection* $\pi_2(d) \in B(x_1, ..., x_n)[\Gamma_n]$ we want to prove that

$$\phi_{C(\pi_1(d))} \cdot (\pi_2(d))^I = \langle id, \overline{\pi_2(d)}\rangle \cdot \phi_{A_n}$$

We start to consider $\phi^{-1}_{C(\pi_1(d))} \cdot (\langle id, \overline{\pi_2(d)}\rangle \cdot \phi_{A_n})$ and note that by induction hypotheses
$\gamma(\pi_1(d)^I) \cdot \phi^{-1}_{C(\pi_1(d))} \cdot (\langle id, \overline{\pi_2(d)}\rangle \cdot \phi_{A_n})) =$
$= (\phi^{-1}_{A_n} \cdot \pi_1^{C(b)}) \cdot (\langle id, \overline{\pi_2(d)}\rangle \cdot \phi_{A_n})) =$
$= id$
and moreover, by induction hypotheses on pullback projections and on substitution
$q(\pi_1(d)^I, \gamma(id)) \cdot (\phi^{-1}_{C(\pi_1(d))} \cdot (\langle id, \overline{\pi_2(d)}\rangle \cdot \phi_{A_n})) =$
$= (\phi_C^{-1} \cdot \langle id, \overline{\pi_1(d)}\rangle \times id) \cdot (\langle id, \overline{\pi_2(d)}\rangle \cdot \phi_{A_n}) =$
$= \phi_C^{-1} \cdot (\zeta^{-1} \cdot \langle id, \overline{d}\rangle) \cdot \phi_{A_n}) =$
$= d^I$.
Therefore, by uniqueness of a morphism to a pullback we conclude that

$$\pi_2(d)^I = \langle id, d^I\rangle = \phi^{-1}_{C(\pi_1(d))} \cdot (\langle id, \overline{\pi_2(d)}\rangle \cdot \phi_{A_n})$$

For all the other types, we can go on defining the isomorphisms that satisfy the various equations by using their term constructors and the inductive hypotheses. For the equality type and the forall type the equations hold directly by the inductive hypotheses, since the last morphism is a mono.
∎
Now we are ready to prove:

Theorem 6.1.2 (completeness) *Suppose that $a \in A\ [\Gamma_n]$ and $b \in A\ [\Gamma_n]$ are derivable in HP, if for every H-pretopos $\mathcal{P}$ $\mathcal{I}_{\mathcal{P}}(a \in A\ [\Gamma_n]) = \mathcal{I}_{\mathcal{P}}(b \in A\ [\Gamma_n])$ then $a = b \in A\ [\Gamma_n]$ is derivable in HP.*
Suppose that A type $[\Gamma_n]$ and B type $[\Gamma_n]$ are derivable in HP, if for every H-pretopos $\mathcal{P}$ $\mathcal{I}_{\mathcal{P}}(A\ type\ [\Gamma_n]) = \mathcal{I}_{\mathcal{P}}(B\ type\ [\Gamma_n])$ then $A = B\ [\Gamma_n]$ is derivable in HP.

Proof.
If $\mathcal{I}_{\mathcal{P}_T}(a \in A\ [\Gamma_n]) = \mathcal{I}_{\mathcal{P}_T}(b \in A\ [\Gamma_n])$ then by the above proposition

$$\phi_B^{-1} \cdot \langle id, \overline{a}\rangle \cdot \phi_{A_n} = a^I = b^I = \phi_B^{-1} \cdot \langle id, \overline{b}\rangle \cdot \phi_{A_n}$$

from which we conclude $a = b \in A\ [\Gamma_n]$. The proof for the judgements about equality between types can be done by double induction on the derivation, considering the interpretation $\mathcal{I}_{\mathcal{P}_T}$ in the syntactic category $\mathcal{P}_T$. When the equality type step occurs in the induction, we can conclude by the completeness for judgements about equality between terms.
∎

6.1.2 The completeness with respect to elementary toposes

We prove the completeness theorem of $\mathcal{T}_t$ with respect to elementary toposes, in the same way as we have done for HP. In order to do this, we consider the syntactic topos $\mathcal{S}_T$, and we define the interpretation

$$\mathcal{J}_{\mathcal{S}_T} : \mathcal{T}_t \to Pgr(\mathcal{S}_T)$$

by induction on the number of assumptions in the context in this manner:
$\mathcal{J}_{\mathcal{S}_T}(B\ type\ [x_1 \in A_1, ..., x_n \in A_n(x_1, ..., x_{n-1})])$ is

$$\Sigma_{z_n \in \widetilde{A_n}} \overline{B} \xrightarrow{\pi_1^B} \Sigma_{z_{n-1} \in \widetilde{A_{n-1}}} \overline{A_n} \xrightarrow{\pi_1^n} \cdots\cdots \Sigma_{z \in \top} \overline{A_1} \xrightarrow{\pi_1^1} \top$$

where $\widetilde{A_n}$ is the domain of the last morphism of $\mathcal{J}_{\mathcal{S}_T}(A_n\ type\ [x_1 \in A_1, ..., x_{n-1} \in A_{n-1}(x_1, ..., x_{n-2})])$ and

$$\overline{B} \equiv B[x_1 := \pi_2^1 \cdot \pi_1^2 \cdots \pi_1^n(z_n)] \ldots [x_{n-1} := \pi_2^{n-1}(\pi_1^n(z_n))][x_n := \pi_2^n(z_n)]$$

with $\pi_i^n \equiv \lambda x.\pi_i^n(x)$ for $i = 1, 2$, where $\pi_1^n(x)$ and $\pi_2^n(x)$ are the two projections of $\Sigma_{z_{n-1} \in \widetilde{A_{n-1}}} \overline{A_n}$ and π_1^B and π_2^B are the two projections of $\Sigma_{z_n \in \widetilde{A_n}} \overline{B}$. For $i = 1, \ldots, n$ $\overline{A_i}$ is defined in the same manner as $\overline{B}$, where $\widetilde{A_1} \equiv \Sigma_{z \in \top} \overline{A_1}$ and $\overline{A_1} \equiv A_1$.
If $b \in B\ [\Gamma_n]$ is a judgement of $\mathcal{T}_t$, we put

$$\overline{b} \equiv b[x_1 := \pi_2^1 \cdot \pi_1^2 \cdots \pi_1^n(z_n)] \ldots [x_{n-1} := \pi_2^{n-1}(\pi_1^n(z_n))][x_n := \pi_2^n(z_n)]$$

and we can derive

$$\overline{b} \in \overline{B}\ [z_n \in \widetilde{A_n}]$$

From now on, we call $B^I = dom(\mathcal{I}_{\mathcal{S}_T}(B\ type\ [\Gamma_n]))$.
In order to prove the completeness theorem we want to show that

Proposition 6.1.3 *For every judgement*

$$B\ type\ [\Gamma_n]$$

derivable in $\mathcal{T}_t$, (which we suppose to be interpreted as $\alpha_1(id), \alpha_2(id), ..., \alpha_n(id), \beta(id)$) there is an isomorphism of $\mathcal{S}_T^{\rightarrow^n}$

$$\phi_{A_1}, ..., \phi_{A_n}, \phi_B$$

from $\mathcal{I}_{\mathcal{S}_T}(B\ type\ [\Gamma_n])$ to $\mathcal{J}_{\mathcal{S}_T}(B\ type\ [\Gamma_n])$ such that for every judgement $b \in B\ [\Gamma_n]$

$$\phi_B \cdot b^I(id) = \langle id, \overline{b} \rangle \cdot \phi_{A_n}$$

and about weakening for every judgement with $n \geq j$

$$M\ type\ [\Gamma_j]$$

(which we suppose to be interpreted as $\alpha_1(id), \alpha_2(id), ..., \alpha_j(id), \mu(id)$)

$$\phi_B \cdot t_n = p_n \times id \cdot \phi_{M \times B}$$

where $\phi_{M \times B} : (M \times B)^I \to \Sigma_{x \in \widetilde{M}} \overline{B}$, t_i is defined as in the lemma of weakening, $p_j \equiv \pi_1^M \times id$ and if $n \geq j+1$, $\pi_i \equiv \pi_{i-1} \times id$ for $i = j+1, \ldots, n$, *and about substitution for every* $a_j \in A_j\ [\Gamma_{j-1}]$ *with* $n \geq j$

$$\phi_B \cdot q_n = s_n \times id \cdot \phi_{B(a_n)}$$

where q_i *is defined as in the lemma of substitution,* $s_j \equiv \langle id, \overline{a_n} \rangle \times id$ *and if* $n \geq j+1$*,* $s_i \equiv s_{i-1} \times id$ *for* $i = j+1, \ldots, n$*.*

Proof. The definition of the isomorphisms is given by induction on the derivation of type and term judgements of $\mathcal{T}_t$. In general, given a type judgement

$$B\ type\ [x_1 \in A_1, ..., x_n \in A_n(x_1, ..., x_{n-1})]$$

by inductive hypothesis we know that $\phi_{A_1}, ..., \phi_{A_n}$ is an isomorphism from $\mathcal{I}_{\mathcal{S}_T}(A_n\ type\ [x_1 \in A_1, ..., x_{n-1} \in A_{n-1}(x_1, ..., x_{n-2})])$ to $\mathcal{J}_{\mathcal{S}_T}(A_n\ type\ [x_1 \in A_1, ..., x_{n-1} \in A_{n-1}(x_1, ..., x_{n-2})])$, so, we only define an isomorphism $\phi_B : B^I \to \Sigma_{z_n \in \widetilde{A_n}} \overline{B}$ in order to prove that $\phi_{A_1}, ..., \phi_{A_n}, \phi_B$ is an isomorphism in $\mathcal{S}_T^{\to n}$ from $\mathcal{I}_{\mathcal{S}_T}(B\ type\ [\Gamma_n])$ to $\mathcal{J}_{\mathcal{S}_T}(B\ type\ [\Gamma_n])$.
We can define the isomorphisms by induction on the derivation of type judgement and term judgement, since by the validity theorem these are well defined. For the terms it is crucial that isomorphisms commute with the second projections of pullbacks related to weakening and substitution. See, for the terminal type, the indexed sum type and the equality type the analogous proposition 6.1.1 for the type theory HP. About the product type, the isomorphism is defined in a similar way as for the indexed sum type, looking at the description of the right adjoint in section 4.4. About the Omega type, see the terminal type.
■
Now, we are ready to prove:

Theorem 6.1.4 (completeness) *Suppose that* $a \in A\ [\Gamma_n]$ *and* $b \in A\ [\Gamma_n]$ *are derivable in* $\mathcal{T}_t$*, if for every topos* $\mathcal{S}$ $\mathcal{I}_{\mathcal{S}}(a \in A\ [\Gamma_n]) = \mathcal{I}_{\mathcal{S}}(b \in A\ [\Gamma_n])$ *then* $a = b \in A\ [\Gamma_n]$ *is derivable in* $\mathcal{T}_t$*.*
Suppose that $A\ type\ [\Gamma_n]$ *and* $B\ type\ [\Gamma_n]$ *are derivable in* $\mathcal{T}_t$*, if for every topos* $\mathcal{S}$ $\mathcal{I}_{\mathcal{S}}(A\ type\ [\Gamma_n]) = \mathcal{I}_{\mathcal{S}}(B\ type\ [\Gamma_n])$ *then* $A = B\ [\Gamma_n]$ *is derivable in* $\mathcal{T}_t$*.*

Proof.
If $\mathcal{I}_{\mathcal{S}_T}(a \in A\ [\Gamma_n]) = \mathcal{I}_{\mathcal{S}_T}(b \in A\ [\Gamma_n])$, then by the above proposition

$$\phi_B^{-1} \cdot \langle id, \overline{a} \rangle \cdot \phi_{A_n} = a^I = b^I = \phi_B^{-1} \cdot \langle id, \overline{b} \rangle \cdot \phi_{A_n}$$

from which we conclude $a = b \in A\ [\Gamma_n]$.
The proof for the judgements about equality between types can be done by double induction on the derivation, considering the interpretation $\mathcal{I}_{\mathcal{S}_T}$ in the syntactic category $\mathcal{S}_T$. When the equality type step occurs in the induction, we can conclude by the completeness for judgements about equality between terms.
■

Chapter 7

The internal type theory of a universe

Summary We present the internal type theory of a Heyting pretopos with a natural numbers object and of a topos. The resulting theories are based respectively on the initial type theories HP and $\mathcal{T}_t$. We prove that there is a sort of equivalence between the type theories and the corresponding category of universes. By using the type theory we also build the free Heyting pretopos and the free topos generated by a category.

7.1 The internal type theory of a Heyting pretopos

Given a H-pretopos $\mathcal{P}$ we want to describe its internal dependent type theory $T(\mathcal{P})$. The type theory is based on the initial type theory HP for H-pretoposes (see section 3.2), augmented with the specific type and term judgements of $\mathcal{P}$. As in the non-dependent case, we give a dependent formulation of the internal language of a universe saying what a type judgement is, what a term judgement is, in a clear order, without considering before raw types and raw terms and then well-formed types and terms, as it is usually done in the dependent case. Indeed, it is meaningless in a dependent theory to consider a type or a term in isolation from the corresponding type or term judgement and its derivation. As in the categorical semantics for the dependent typed calculi in section 5.2, the idea is to consider a dependent type as a sequence of morphisms of $\mathcal{P}$, ending with the terminal object 1, whereas the terms are sections of the last morphism of the type to which they belong. Therefore, the type theory $T(\mathcal{P})$ is formulated in the style of Martin-Löf's type theory with the four kinds of judgements [NPS90] and telescopic contexts. We assume all the inference rules about the formation of contexts, declarations of typed variables, about reflexivity, symmetry and transitivity of the equality between types and terms [NPS90] and the set rule *conv*) as in section 3.2. As in the semantics in section 5.2, a type judgement arises from a object of $Pgf(\mathcal{P})$, which represents a dependent type with all its possible substitutions. More precisely, a type judgement corresponds to the evaluation of a finite sequence of fibred functors on the identity. Indeed, for a sequence of fibred functors $\alpha_1, \alpha_2, ..., \alpha_n, \beta$ of $Pgf(\mathcal{P})$, we define

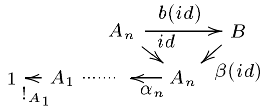
$$\beta^{-1}(x_1, ..., x_n)[x_1 \in \alpha_1^{-1}, ..., x_n \in \alpha_n^{-1}(x_1, ..., x_{n-1})]$$

as the type judgement corresponding to

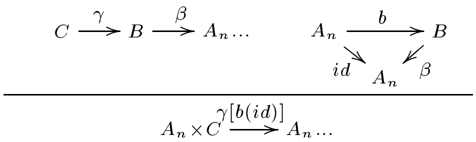
$$B \xrightarrow{\beta(id)} A_n \xrightarrow{\alpha_n(id)} \cdots\cdots A_1 \xrightarrow{\alpha_1(id)} 1$$

by thinking of the fibers of the morphism $\beta(id)$. This notation turns out to be very clear when we look at the category of paths built on any syntactic H-pretopos. The equality between types corresponds to the equality between objects of $Pgf(\mathcal{P})$, which implies the equality between objects of $Pgr(\mathcal{P})$. For short, we use the abbreviation $\Gamma_n \equiv x_1 \in \alpha_1^{-1}, ..., x_n \in \alpha_n^{-1}(x_1, ..., x_{n-1})$ in the contexts. On the other hand, a term judgement arises from a morphism of $Pgf(\mathcal{P})$, which is a natural transformation representing a term with all its possible substitutions. The evaluation of a natural transformation on the identical substitution is a term judgement. Indeed, for a suitable morphism b of $Pgf(\mathcal{P})$ from $\alpha_1, \alpha_1, ..., \alpha_n, i_{A_n}$ to $\alpha_1, \alpha_2, ..., \alpha_n, \beta$, the term judgement

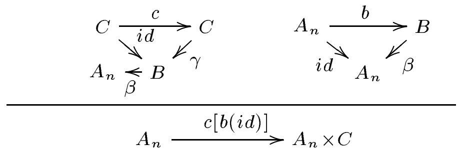
$$b \in \beta^{-1}(x_1, ..., x_n)[\Gamma_n]$$

corresponds to a section of $\beta(id)$  by choosing the identity as the terminal object in $\mathcal{P}/A_n$. The equality between terms corresponds to the equality between morphisms of $Pgf(\mathcal{P})$. So, we add as axioms all the equality judgements that correspond to actual equations holding in $Pgf(\mathcal{P})$. In the following, to make formulas more readable in type judgements, we will write $\beta[\Gamma_n]$ instead of $\beta^{-1}[\Gamma_n]$. In the diagrams we will often write σ_i instead of $\sigma_i(id_{A_i})$ for fibred functors and b instead of $b(id)$ for natural transformations.

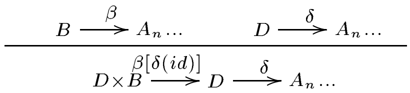
The rules for *substitution* of variables in a type and in a term and for *weakening* of a variable w.r.t type and term judgements are the usual ones and they are defined as their interpretation in the semantics in section 5.2. We only show how they work in these particular cases:

$$sT \ \frac{\gamma[\Gamma_n, y \in \beta] \quad b \in \beta[\Gamma_n]}{\gamma[b(id)][\Gamma_n]}$$ is 

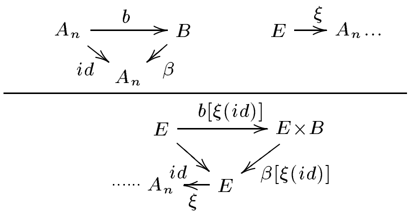
where we put $\gamma[b(id)](id) \equiv \gamma(b(id))$

$$st \ \frac{c \in \gamma[\Gamma_n, y \in \beta] \quad b \in \beta[\Gamma_n]}{c[b(id)] \in \gamma[b(id)][\Gamma_n]}$$ is 

where we put $c[b(id)](id) \equiv c(b(id))$

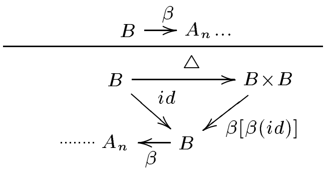
$$wT \ \frac{\beta[\Gamma_n] \quad \delta[\Gamma_n]}{\beta[\Gamma_n, y \in \delta]}$$ is 

where we put $\beta[\delta(id)](id) \equiv \beta(\delta(id))$

$$wt \ \frac{b \in \beta[\Gamma_n] \quad \xi[\Gamma_n]}{b \in \beta[\Gamma_n, w \in \xi]}$$ is 

where we put $b[\xi(id)](id) \equiv (\xi(id))^*(b(id))$, that is the unique morphism of $\mathcal{P}/E$ from $i_{A_n}(\xi(id))$ to $\beta(\xi(id))$, obtained from $b(id)$ by the properties of pullback.

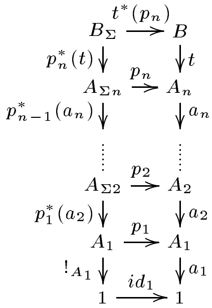
The rule expressing the *assumption of variable* is the following:

$$var \ \frac{\beta \ [\Gamma_n]}{x \in \beta[\Gamma_n, x \in \beta]}$$ is 

where $x(id) \equiv \triangle_B \equiv \langle id_B, id_B \rangle$.

Now, we show the formation rules for types and then the introduction, elimination and conversion rules for their terms.

The *proper types* and *terms* of $T(\mathcal{P})$ are described as follows. Proper type judgements arise from objects of $Pgr(\mathcal{P})$ and proper term judgements arise from morphisms of $Pgr(\mathcal{P})$. For every object of $Pgr(\mathcal{P})$ $\boldsymbol{a_1, a_2, ..., a_n, t}$ we consider the sequence obtained by making the pullback of a_1 along the identity, then by making the pullback of a_2 along the second projection p_1 of the previous pullback, and so on, that is we obtain the following sequence of pullbacks:



where p_i is the second projection of the pullback of a_i and p_{i-1}, for $i = 1, \ldots, n$. Finally, we consider the associate sequence of fibred functors

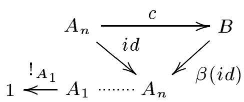
$$\widehat{A_1},\ \widehat{a_2}[p_1],\ \widehat{a_3}[p_2],\ ...,\widehat{a_n}[p_{n-1}],\ \widehat{t}[p_n]$$

where $\widehat{A_1} \equiv \widehat{a_1}$, hence we introduce a new dependent type t^{-1} and finally we state that

$$t^{-1}[x_1 \in A_1, ..., x_n \in a_n^{-1}] \quad \text{is} \quad B_\Sigma \xrightarrow{\widehat{t}[p_n]} A_{\Sigma n} \xrightarrow{\widehat{a_n}[p_{n-1}]} \cdots\cdots A_{\Sigma 1} \xrightarrow{\widehat{A_1}} 1$$

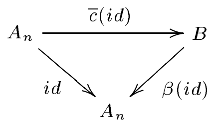
where the Σ subscript is used for the interpretation of the series of judgements of proper types introduced by an object of $Pgr(\mathcal{P})$.

Moreover, given a sequence of fibred functors $\alpha_1, \alpha_2, ..., \alpha_n, \beta$ of $Pgf(\mathcal{P})$, for every morphism $\boldsymbol{c}$ of $Pgr(\mathcal{P})$



we introduce a new term c and we state that

$c \in \beta(id)[x_1 \in A_1, ..., x_n \in \alpha_n]$ is



where $\overline{c}(id) \equiv c$.

Finally, we add all the types and terms of the type theory HP, defined as for the interpretation in section 5.3.1.

Remark 7.1.1 Our definition of internal language of a category follows [LS86], for instance, and it is different from that in [Tay97].

7.2 The relation between the HP type theories and H-pretoposes

There is a sort of equivalence between the internal type theories of H-pretoposes described in section 7.1 and the category of H-pretoposes. As a consequence of this, we can state that the type theory $T(\mathcal{P})$ is the internal language of the H-pretopos $\mathcal{P}$. First of all, we define the following categories:

1. $Th(HP)$ whose objects are the type theories of H-pretoposes, whose initial type theory is HP and whose morphisms are translations: they send types to types so as to preserve the type and term constructors, closed terms to closed terms and variables to variables; we call $Th(HP)^*$ the category whose objects are those of $Th(HP)$, but whose morphisms are translations preserving type and term constructors up to isomorphisms;

2. $HPretop_o$ whose objects are H-pretoposes with a fixed choice of H-pretopos structure and whose morphisms are strict logical functors, that is functors preserving the H-pretopos structure w.r.t. the fixed choices; we call $HPretop$ the category whose objects are those of $HPretop_o$, but whose morphisms are functors preserving the H-pretopos structure up to isomorphisms.

Now, we define a functor from H-pretoposes to type theories

$$T : HPretop_o \longrightarrow Th(HP)$$

that associates to every H-pretopos $\mathcal{P}$ the internal type theory $T(\mathcal{P})$ described in the previous section. The functor T associates to every morphism $F : \mathcal{P} \to \mathcal{D}$ of $HPretop_o$ the translation $T(F) : T(\mathcal{P}) \to T(\mathcal{D})$ defined as follows. Given a fibred functor $\sigma : \mathcal{P}/A \to \mathcal{P}^{\to}$, corresponding to a type judgement, and a natural transformation c, corresponding to a term judgement, we define $T(F)(\sigma)$ and $T(F)(c)$ in the same way as we have defined the interpretation of a type theory in section 5.3.1. If $\sigma = \widehat{b}$ for any $b : B \to A$ of $\mathcal{P}$, then we put $F(\sigma) = \widehat{F(b)}$, since the chosen pullbacks of $\mathcal{P}$ are sent into the chosen pullbacks of $\mathcal{D}$ by F. If σ is introduced by an inference rule of HP, then we simply define $F(\sigma)$ such that $F(\sigma)(id) = F(\sigma(id))$, in order to make $T(F)$ be a translation. For example, we put $F(\Sigma_\beta(\gamma)) \equiv \Sigma_{F(\beta)}(F(\gamma))$. This definition of $T(F)$ is good, since the functor F preserves the H-pretopos structure w.r.t. the fixed choices used in the internal type theories of $\mathcal{P}$ and $\mathcal{D}$.

Moreover, we define a functor from type theories to H-pretoposes

$$P : Th(HP) \longrightarrow HPretop_o$$

that associates to every type theory $\mathcal{T}$ the category $P(\mathcal{T})$, whose objects are closed types $A, B, C, ...$ and whose morphisms are the expressions $(x)b(x)$ corresponding to $b(x) \in B[x \in A]$, where the type B does not depend on A. We can prove that $P(\mathcal{T})$ is a H-pretopos by fixing a choice of its structure as in section 3.3. The functor P associates to every morphism of $Th(HP)$ $L : \mathcal{T} \to \mathcal{T}'$ the functor $P(L) : P(\mathcal{T}) \to P(\mathcal{T}')$ defined as follows. For every closed type A, we put $P(L)(A) \equiv L(A)$, which is well defined since a translation sends closed types to closed types. For every morphism $b(x) \in B[x \in A]$ of $P(\mathcal{T})$ we put

$$P(L)(b(x) \in B[x \in A]) \equiv L(b(x)) \in L(B)[x \in L(A)]$$

Since L is a translation, then $P(L)$ is a functor preserving the H-pretopos structure. In order to describe the relation between type theories and H-pretoposes, we have to consider a type theory $\mathcal{T}$ as a category. We think of $\mathcal{T}$ as the category whose objects correspond to those of $Pgr(P(\mathcal{T}))$, but whose morphisms are sequences of morphisms by which we built a series of commutative squares. More precisely, the objects of $\mathcal{T}$ are the dependent types under a context $B(x_1, ..., x_n)[x_1 \in A_1, ..., x_n \in A_n]$. The morphisms of $\mathcal{T}$ exist only from $B[x_1 \in A_1, ..., x_n \in A_n]$ to $B'[x'_1 \in A'_1, ..., x'_n \in A'_n]$ and they are[1]

$$b' \in B'(a'_1, ..., a'_n)[x_1 \in A_1, ..., x_n \in A_n, y \in B(x_1, ..., x_n)]$$

such that $a_1 \in A'_1[x_1 \in A_1]$ and $a'_i \in A'_i(a'_1, ..., a'_{i-1})[x_1 \in A_1, ..., x_i \in A_i]$ for $i = 1, ..., n$. The composition is the substitution and the identity is $y \in B(x_1, ..., x_n)[x_1 \in A_1, ..., x_n \in A_n, y \in B]$. Therefore, we can consider equivalences of type theories. In the following we mean by ID the identity functor.

Proposition 7.2.1 *Let $T : HPretop_o \to Th(HP)$ and $P : Th(HP) \to HPretop_o$ be the functors defined above. There are two natural transformations: η from ID to $T \cdot P$, thought as functors from $Th(HP)$ to $Th(HP)^*$, and ϵ from $P \cdot T$ to ID, thought as functors from $HPretop_o$ to $HPretop$, such that for every type theory $\mathcal{T}$ and for every H-pretopos $\mathcal{P}$, $\eta_{\mathcal{T}} : \mathcal{T} \to T(P(\mathcal{T}))$ and $\epsilon_{\mathcal{P}} : P(T(\mathcal{P})) \to \mathcal{P}$ are equivalences.*

[1] One could also consider the usual morphisms of contexts.

Proof. In order to obtain the natural transformation η, for every type theory $\mathcal{T}$ we define

$$\eta_{\mathcal{T}} : \mathcal{T} \to T(P(\mathcal{T}))$$

as follows. For any closed type $\eta_{\mathcal{T}}(A[\ ]) \equiv \widehat{A}(id) : A_\Sigma \to 1$. For dependent type judgements, $\eta_{\mathcal{T}}(C(x,y)[x \in A, y \in B(x)])$ is the type judgement of $T(P(\mathcal{T}))$ corresponding to the sequence

$$\Sigma_{z\in\tilde{B}}C(x)_\Sigma \xrightarrow{q_3(id)} \Sigma_{x\in A}B(x)_\Sigma \xrightarrow{q_2(id)} A_\Sigma \xrightarrow{\widehat{A}(id)} 1$$

where $\tilde{B} \equiv \Sigma_{x\in A}B(x)$ and $q_i \equiv \widehat{\pi_1}[p_{i-1}]$ for $i = 2, 3$. This is the dependent type judgement arising from the following sequence

$$\Sigma_{z\in\Sigma_{x\in A}B(x)}C(x) \xrightarrow{\pi_1} \Sigma_{x\in A}B(x) \xrightarrow{\pi_1} A \xrightarrow{*} 1$$

in the internal type theory $T(P(\mathcal{T}))$, as it is described in the previous section. For term judgements, $\eta_{\mathcal{T}}(c \in C(x,y)[x \in A, y \in B(x)])$ is

$$\begin{array}{ccccc}
 & & \Sigma_{x\in A}B(x)_\Sigma & \xrightarrow{\overline{\langle z,\tilde{c}\rangle}[p_2](id)} & \Sigma_{z\in\Sigma_{x\in A}B(x)}\overline{C}(z)_\Sigma \\
 & & \quad\searrow^{id} & & \swarrow_{q_3(id)} \\
1 & \xleftarrow{\widehat{A}(id)} A_\Sigma & \xleftarrow{q_2(id)} & \Sigma_{x\in A}B(x)_\Sigma &
\end{array}$$

where $\tilde{c} \equiv c[x/\pi_1(z), y/\pi_2(z)][z \in \Sigma_{x\in A}B(x)]$. This is the term judgement arising from $\langle z, \tilde{c}\rangle$ in the internal type theory $T(P(\mathcal{T}))$, as it is described in the section 7.1. We can obviously imagine how $\eta_{\mathcal{T}}$ is defined in the case of having a generic context of n types. We can see that η is a natural transformation, since translations preserve indexed sum types and projections. η_T is a translation up to isomorphisms and it is an equivalence of categories since the functor is faithful, full and essentially surjective. Indeed, we can define a natural transformation η^{-1} such that, given a type theory $\mathcal{T}$, the component $\eta^{-1}_{\mathcal{T}} : T(P(\mathcal{T})) \to \mathcal{T}$ is defined as follows. Given a type judgement $B \xrightarrow{\beta(id)} A \xrightarrow{\alpha(id)} 1$ of $T(P(\mathcal{T}))$ we define

$$\eta^{-1}_{\mathcal{T}}(\alpha(id), \beta(id)) \equiv \beta(id)^{-1}(x)[x \in A]$$

where $\beta(id)^{-1}(x) \equiv \Sigma_{z\in B}Eq(A, \beta(id)(z), x)$, that is the fibers of $\beta(id)$. Given the term judgement $\begin{array}{ccc} A & \xrightarrow{c(id)} & B \\ & \searrow^{id} \quad \swarrow_{\beta(id)} & \\ 1 \xleftarrow{\alpha(id)} & A & \end{array}$ of $T(P(\mathcal{T}))$, provided that $c(id)$ is $c(x) \in B[x \in A]$, $\eta^{-1}_{\mathcal{T}}$ associates to it the term judgement of $\mathcal{T}$

$$\langle c(x), \mathsf{eq}\rangle \in \Sigma_{z\in B}Eq(A, \beta(id)(z), x)[x \in A]$$

We can see that η^{-1} is a natural transformation, since translations preserve indexed sum types, projections and equality types. We can prove that, for every type theory $\mathcal{T}$, $\eta_{\mathcal{T}}$ and $\eta^{-1}_{\mathcal{T}}$ give rise to an equivalence of categories (also see [See84]).

Moreover, we define a natural transformation ϵ such that for every H-pretopos $\mathcal{P}$ the component

$$\epsilon_{\mathcal{P}} : P(T(\mathcal{P})) \to \mathcal{P}$$

is defined as follows. $\epsilon_{\mathcal{P}}$ associates to every object $A \xrightarrow{\sigma(id)} 1$ of $P(T(\mathcal{P}))$ the object A and it associates to the morphism $\begin{array}{ccc} A & \xrightarrow{b(id)} & A\times B \\ & \searrow^{id} \quad \swarrow_{\beta(!_A)} & \\ 1 \xleftarrow{\sigma(id)} & A & \end{array}$ the morphism $q(!_A, \beta(id)) \cdot b(id) : A \to B$. We can easily prove that $\epsilon_{\mathcal{P}}$ is a functor preserving the H-pretopos structure up to isomorphisms[2]. We have that $\epsilon_{\mathcal{P}}$ gives rise to

[2]This due to the fact that the split fibration selects a choice of structure different from the choice given with a H-pretopos: see, for instance, the terminal object.

a natural transformation, since the functors preserve the H-pretopos structure w.r.t. the fixed choices. Moreover, $\epsilon_{\mathcal{P}}$ is an equivalence of categories, since it is faithful by uniqueness of morphisms towards pullbacks, full because every section of a fibred functor has got a name in the language, and essentially surjective. Indeed, we can define a natural transformation ϵ^{-1} such that for every H-pretopos $\mathcal{P}$ the component $\epsilon_{\mathcal{P}}^{-1} : \mathcal{P} \to P(T(\mathcal{P}))$ is defined as follows. For every object A of $\mathcal{P}$, $\epsilon_{\mathcal{P}}^{-1}(A)$ is the closed type corresponding to $A_\Sigma \xrightarrow{\widehat{A}(id)} 1$. For every morphism $b : A \to B$ of $\mathcal{P}$, $\epsilon_{\mathcal{P}}^{-1}(b)$ is the term corresponding to

$$\begin{array}{ccc} A_\Sigma & \xrightarrow{\langle id,b'\rangle(id)} & A_\Sigma \times B_\Sigma \\ & {\scriptstyle id}\searrow \quad \swarrow{\scriptstyle \widehat{B}(!_{A_\Sigma})} & \\ 1 \xleftarrow{\widehat{A}(id)} & A_\Sigma & \end{array}$$

where $b' = \pi_B^{-1} \cdot b \cdot \pi_A$ and where π_B and π_A are the second projections of the pullbacks of $!_A$ and $!_B$ along the identity. We conclude that for every H-pretopos $\mathcal{P}$, $\epsilon_{\mathcal{P}}$ and $\epsilon_{\mathcal{P}}^{-1}$ give rise to an equivalence of categories.

■

7.3 The internal type theory of a topos

Given a topos $\mathcal{S}$ we describe its internal dependent type theory $L(\mathcal{S})$, exactly in the same way as for a Heyting pretopos $\mathcal{P}$ in section 7.1. The type theory is based on the initial type theory $\mathcal{T}_t$ for toposes (see section 4.2), augmented with the specific type and term judgements of $\mathcal{S}$. Therefore, also the internal type theory of a topos is formulated in the style of Martin-Löf's type theory with the four kinds of judgements [NPS90], where the contexts are telescopic. We assume all the inference rules about the formation of contexts, declarations of typed variables, about reflexivity, symmetry and transitivity of the equality between types and terms [NPS90] and the *conv*) rule. We only repeat how type and term judgements are defined. For a sequence of fibred functors $\alpha_1, \alpha_2, ..., \alpha_n, \beta$ of $Pgf(\mathcal{S})$, we define

$$\beta^{-1}(x_1, ..., x_n)[x_1 \in \alpha_1^{-1}, ..., x_n \in \alpha_n^{-1}(x_1, ..., x_{n-1})]$$

as the type judgement corresponding to

$$B \xrightarrow{\beta(id)} A_n \xrightarrow{\alpha_n(id)} \cdots\cdots A_1 \xrightarrow{\alpha_1(id)} 1$$

by thinking of the fibers of the morphism $\beta(id)$. The equality between types corresponds to the equality between objects of $Pgf(\mathcal{S})$, which implies the equality between objects of $Pgr(\mathcal{S})$. For short, we use the abbreviation $\Gamma_n \equiv x_1 \in \alpha_1^{-1}, ..., x_n \in \alpha_n^{-1}(x_1, ..., x_{n-1})$ in the contexts. For a morphism b of $Pgf(\mathcal{S})$ from $\alpha_1, \alpha_1, ..., \alpha_n, i_{A_n}$ to $\alpha_1, \alpha_2, ..., \alpha_n, \beta$, the term judgement

$$b \in \beta^{-1}(x_1, ..., x_n)[\Gamma_n]$$

corresponds to the section of $\beta(id)$

$$\begin{array}{ccc} & A_n \xrightarrow{b(id)} B & \\ & {\scriptstyle id}\searrow \quad \swarrow{\scriptstyle \beta(id)} & \\ 1 \xleftarrow{!_{A_1}} A_1 \cdots\cdots & \xleftarrow{\alpha_n} A_n & \end{array}$$

by choosing the identity as the terminal object in $\mathcal{S}/A_n$.

The equality between terms corresponds to the equality between morphisms of $Pgf(\mathcal{S})$.
For the formulation of the structural rules of weakening and substitution and of the type and term judgements that are specific to $\mathcal{S}$, we refer to section 7.1, where $\mathcal{P}$ should be replaced with $\mathcal{S}$.

7.4 The relation between the $\mathcal{T}_t$ type theories and toposes

There is a sort of equivalence between the internal type theories of toposes described in the section 7.3 and the category of toposes. As a consequence of this, we can state that the type theory $L(\mathcal{S})$ is the internal language of the topos $\mathcal{S}$.
First of all, we define the following categories:

1. $Th(\mathcal{T}_t)$ whose objects are the type theories of toposes, whose initial type theory is $\mathcal{T}_t$, and whose morphisms are translations: they send types to types so as to preserve the type and term constructors, closed terms to closed terms and variables to variables; we call $Th(\mathcal{T}_t)^*$ the category whose objects are those of $Th(\mathcal{T}_t)$, but whose morphisms are translations preserving type and term constructors up to isomorphisms;

2. Top_o whose objects are toposes with a fixed choice of topos structure and whose morphisms are strict logical functors, that is functors preserving the topos structure w.r.t. the fixed choices; we call Top the category whose objects are those of Top_o, but whose morphisms are functors preserving the topos structure up to isomorphisms.

We define the functors

$$L : Top_o \to Th(\mathcal{T}_t) \qquad S : Th(\mathcal{T}_t) \to Top_o$$

in the same way we have defined $T : HPretop_o \longrightarrow Th(HP)$ and $P : Th(HP) \longrightarrow HPretop_o$. In an analogous way we can prove:

Proposition 7.4.1 *Let $L : Top_o \to Th(\mathcal{T}_t)$ and $S : Th(\mathcal{T}_t) \to Top_o$ be the functors defined above. There are two natural transformations: η from ID to $L \cdot S$, thought as functors from $Th(\mathcal{T}_t)$ to $Th(\mathcal{T}_t)^*$, and ϵ from $S \cdot L$ to ID, thought as functors from Top_o to Top, such that for every type theory $\mathcal{T}$ and for every topos $\mathcal{S}$, $\eta_{\mathcal{T}} : \mathcal{T} \to L(S(\mathcal{T}))$ and $\epsilon_{\mathcal{S}} : S(L(\mathcal{S})) \to \mathcal{S}$ are equivalences.*

Proof. The only difference with the proof for H-pretoposes is that here, we have to check that the translations preserve the type and term constructors of $\mathcal{T}_t$ and that the functors preserve the topos structure.
∎

7.5 The free H-pretopos

The main idea is to generate a H-pretopos from a given category $\mathcal{C}$ by considering its objects as closed types and its morphisms as terms with a free variable. We can prove the universal property by the construction of the category of paths, which represents the dependent types in a categorical way.

Given a category $\mathcal{C}$, we consider the dependent type theory $T(\mathcal{C})$ generated by the inference rules as follows:

1. For every object $\boldsymbol{A}$ of $Ob\mathcal{C}$ we introduce a new type A and we state the closed type judgement $A\ [\ ]$.

   Given $A \in Ob\mathcal{C}$ and $B \in Ob\mathcal{C}$ we state $A = B\ [\ ]$, if they are the same object in $Ob\mathcal{C}$.

2. For every morphism $\boldsymbol{b : A \to B}$ in $\mathcal{C}$, we introduce a new term $b(x)$ and we state $b(x) \in B\ [x \in A]$, where A and B are closed types.

   Given $b : A \to B$ and $d : A \to B$ in $\mathcal{C}$, we state $b(x) = d(x) \in B\ [x \in A]$, provided that b and d are the same morphism in $\mathcal{C}$.

   Given $b : A \to B$ and $a : D \to A$ in $\mathcal{C}$, we state about composition $b(x)[x := a(y)] = (b \cdot a)(y) \in B\ [y \in D]$.

3. There are all the inference rules of the type theory HP. for H-pretoposes.

Therefore, $T(\mathcal{C})$ is a type theory of H-pretoposes.

Now, we can prove:

Proposition 7.5.1 *Let $P : Th(HP) \longrightarrow HPretop_o$ be the functor described in section 3. The category $P(T(\mathcal{C}))$ is the free H-pretopos generated by the category $\mathcal{C}$ in $P(Th(HP))$.*

Proof. We know that $P(T(\mathcal{C}))$ is a H-pretopos from the definition of P. Given a functor $G : \mathcal{C} \to \mathcal{P}$, from the category $\mathcal{C}$ to the H-pretopos $\mathcal{P}$, we claim that there exists a unique functor $\tilde{G} : P(T(\mathcal{C})) \to \mathcal{P}$ in $HPretop_o$ such that the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{I} & P(T(\mathcal{C})) \\ & {\scriptstyle G}\searrow \quad \swarrow{\scriptstyle \tilde{G}} & \\ & \mathcal{P} & \end{array}$$

commutes, where $I : \mathcal{C} \to P(T(\mathcal{C}))$ is the following functor: for every object $A \in Ob\mathcal{C}$ we put $I(A) \equiv A\,[\ ]$ and for every morphism $b : A \to B$ we put $I(b) \equiv b(x) \in B[x \in A]$.

In order to define $\tilde{G}$ on $P(T(\mathcal{C}))$, we define an interpretation $\mathcal{J} : T(\mathcal{C}) \to Pgr(\mathcal{P})$, by passing to $Pgf(\mathcal{P})$, with the warning that we have to normalize the interpretation. This is done by adding the value of every fibred functor $\sigma \in Fib(\mathcal{P}/1, \mathcal{P}^{\to})$ on the empty context,[3] such that a type judgement will be interpreted by a sequence of $Pgr(\mathcal{P})$ like

$$\alpha_1(\emptyset), \alpha_2(id_{A_1}), ..., \alpha_n(id_{A_{n-1}})$$

The interpretation is given in the same way as for the type theory HP in section 5.3.1, except for closed types and terms, which are interpreted in fibred functors evaluated on $\emptyset$. The reason is that we want to put $\tilde{G}(A[\ ]) \equiv \mathrm{dom}\mathcal{J}(A[\ ])$ and $\tilde{G}(b \in B[x \in A]) \equiv q(\mathcal{J}(A[\ ]), \mathcal{J}(B[\ ])) \cdot \mathcal{J}(b \in B[x \in A])$, but if we adopt for $\mathcal{J}$ the semantics defined in section 5.3.1, then $\tilde{G}$ would commute with G up to isomorphisms. So, for every object A of $Ob\mathcal{P}$, we extend the functor $\widehat{A}$ by adding $\widehat{A}(\emptyset) \equiv !_A$ and for every object B, $q(!_B, \widehat{A}(\emptyset))$ is the second projection of the pullback of $!_B$ and $\widehat{A}(\emptyset)$. For example, for the natural numbers $\mathcal{J}(N[\ ]) \equiv \widehat{\mathcal{N}}(\emptyset) = !_{\mathcal{N}}$, instead of being interpreted as $!_{1\times\mathcal{N}}$ like in the semantics defined in 5.3.1. Moreover, $\mathcal{J}(0 \in N[\ ])$ is

$$\begin{array}{ccc} 1 & \xrightarrow{\widehat{o}(\emptyset)} & \mathcal{N} \\ & {\scriptstyle id_1}\searrow \quad \swarrow{\scriptstyle \widehat{\mathcal{N}}(\emptyset)} & \\ & 1 & \end{array}$$

where $\widehat{o}(\emptyset) \equiv o$ and $o : 1 \to \mathcal{N}$ is the zero map in $\mathcal{P}$.

Finally, given a proper type arising from an object $A \in Ob\mathcal{C}$, we put $\mathcal{J}(A[\ ]) \equiv \widehat{G(A)}(\emptyset)$ and given a proper term arising from a morphism $b : A \to B$ of $\mathcal{C}$, we put $\mathcal{J}(b \in B[x \in A]) \equiv \langle id_{G(A)}, G(b)\rangle$ section of $\widehat{G(B)}(\widehat{G(A)}(\emptyset)) : G(A) \times G(B) \to G(A)$. By definition $\tilde{G}$ preserves the H-pretopos structure and we get $\tilde{G} \cdot I = G$. Moreover, $\tilde{G}$ is obviously unique for fixed choices of the H-pretopos structure, which are required to interpret the type theory $T(\mathcal{C})$ into $Pgr(\mathcal{P})$.

■

The free structure gives rise to a monad. It would be interesting to investigate if the category $HPretop_o$ is monadic on Cat and $Graph$. Or at least, if we prove that $HPretop_o$ is essentially algebraic, as for the categorical models of ITT in [Obt89], we would get a representation theorem of $HPretop_o$ into a category of presheaves [AR94].

7.6 The free topos

As for the free H-pretopos, we generate a topos from a given category $\mathcal{C}$ by considering its objects as closed types and its morphisms as terms with a free variable. We can prove the universal property by the construction of the category of paths, which represents the dependent types in a categorical way. To this purpose we consider the dependent type theory $L(\mathcal{C})$ generated by the same inference rules as in section 7.5, replacing in the last point the rules of the type theory HP, with the rules of the type theory $\mathcal{T}_t$.

Therefore, we can prove:

[3]In a rigorous way, we consider the free category $\mathcal{P}/1^{\top}$ with terminal object $\emptyset$ generated from $\mathcal{P}/1$. So a type and a term with empty context are interpreted respectively as functor $\sigma^{\top}$ and natural transformation $\rho^{\top}$ of $[\mathcal{P}/1^{\top}, \mathcal{P}^{\to}]$ such that $\sigma^{\top}$ and $\rho^{\top}$ restricted to $\mathcal{P}/1$ are in $Fib(\mathcal{P}/1, \mathcal{P}^{\to})$. We extend the fibred functors as described above. For example, $\widehat{\mathcal{N}}^{\top}(\emptyset) \equiv !_{\mathcal{N}}$ and $\widehat{\mathcal{N}}^{\top}(!_B \to \emptyset) \equiv$

$$\begin{array}{ccc} \cdot & \xrightarrow{q(!_B, \widehat{\mathcal{N}}^{\top}(\emptyset))} & \mathcal{N} \\ {\scriptstyle \widehat{\mathcal{N}}^{\top}(!_B)}\downarrow & & \downarrow{\scriptstyle \widehat{\mathcal{N}}^{\top}(\emptyset)} \\ B & \xrightarrow[!_B]{} & 1 \end{array}$$

with $q(!_B, \widehat{\mathcal{N}}^{\top}(\emptyset)) \equiv (\widehat{\mathcal{N}}^{\top}(\emptyset))^{*}(!_B)$. For short we still write σ for $\sigma^{\top}$.

Proposition 7.6.1 *Let $S : Th(\mathcal{T}_t) \longrightarrow Top_o$ be the functor described in the section 7.4. The category $S(L(\mathcal{C}))$ is the free H-pretopos generated by the category $\mathcal{C}$ in $P(Th(\mathcal{T}_t))$.*

Proof. We know that $S(L(\mathcal{C}))$ is a topos from the definition of S.

Exactly, as for the free H-pretopos, given a functor $G : \mathcal{C} \to \mathcal{S}$, from the category $\mathcal{C}$ to the topos $\mathcal{S}$, we claim that there exists a unique functor $\tilde{G} : S(L(\mathcal{C})) \to \mathcal{S}$ in Top_o such that the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ I\ } & S(L(\mathcal{C})) \\ & {\scriptstyle G}\searrow \quad \swarrow{\scriptstyle \tilde{G}} & \\ & \mathcal{S} & \end{array}$$

commutes, where $I : \mathcal{C} \to S(L(\mathcal{C}))$ is the following functor: for every object $A \in Ob\mathcal{C}$ we put $I(A) \equiv A\ [\ ]$ and for every morphism $b : A \to B$ we put $I(b) \equiv b(x) \in B[x \in A]$.

In order to define $\tilde{G}$ on $S(L(\mathcal{C}))$, we define an interpretation $\mathcal{J} : L(\mathcal{C}) \to Pgr(\mathcal{S})$, by passing to $Pgf(\mathcal{S})$, with the warning that we have to normalize the interpretation. This is done in the same way as for the free H-pretopos, except that we have the product type and the subobject classifier. We normalize the interpretation of the subobject classifier as that of the natural numbers object.

∎

By the free topos generated by an arbitrary category, we get a presentation of a monad on Cat, with respect to which the category of toposes is monadic on Cat [DK83].

7.7 Some other free structures: the *Lex* and LCC^+ categories

A similar correspondence to that one between type theories and H-pretoposes can be established for the category *Lex* and LCC^+. The category *Lex*, whose objects are the categories with finite limits and whose morphisms are functors strictly preserving finite limits, provides a valid and complete semantics for the type theory with terminal type, extensional equality types and indexed sum types. In the same way, the LCC^+ category, whose objects are the locally cartesian closed categories with finite coproducts and a natural numbers object and whose morphisms are functors strictly preserving the ITT structure, provides a valid and complete semantics for the fragment of Martin-Löf's type theory with extensional equality and without universes and well-orders [Mar84]. These validity and completeness theorems can be proved in a similar way to that for H-pretoposes and for toposes. We can easily notice that these dependent type theories enable us to build the free structure for *Lex* and LCC^+ over Cat, in the same way we proved for the category $HPretop_o$ and Top_o. The free structures give a presentation of two monads, whose algebras correspond respectively to *Lex* and LCC^+, since *Lex* and ITT are monadic over $Graph$ [Bur81] and admit an equational presentation.

Conclusions and further research

The type theories of Heyting pretoposes and of elementary toposes can be used to give translations of categorical proofs from topos theory into type theory and vice versa.

These typed calculi make clear that topos and H-pretopos theory are governed by the isomorphism *propositions as mono types.*

On the contrary, in Martin-Löf's Constructive Type Theory logic is captured via the Curry-Howard isomorphism *proposition as types.*

So, it seems more natural to consider a type theory, where the notion of Proposition is distinct from the notion of Type or Set. After establishing the various isomorphisms between propositions and other types, then we can analize the various frameworks to develop intuitionistic mathematics, like Topos Theory presented as a type theory, Martin-Löf's Constructive Type Theory and also the Calculus of Constructions.

Moreover, since the type theories of Heyting pretoposes and of toposes are extensional, while Martin-Löf's Constructive Type Theory is intensional, it should be analyzed how much of such type theories can be saved in a more intensional setting.

Another direction of application of the type theory of Heyting pretoposes with a natural numbers object is to describe the notion of small maps via type theory and then to get a type-theoretic description of the models for the whole intuitionistic set theory as in [JM95].

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