ON THE CATEGORY OF PROPS

PHILIP HACKNEY AND MARCY ROBERTSON

Abstract. The category of (colored) props is an enhancement of the category of colored operads, and thus of the category of small categories. The titular category has nice formal properties: it is bicomplete and is a symmetric monoidal category, with monoidal product closely related to the Boardman-Vogt tensor product of operads. Tools developed in this article, which is the first part of a larger work, include a generalized version of multilinearity of functors, a free prop construction defined on certain "generalized" graphs, and the relationship between the category of props and the categories of permutative categories and of operads.

1. Introduction

This paper lays the foundation for a multistage project developing the notion of "higher prop." Here we establish the formal properties necessary to do homotopy theory: the category of props (enriched over a suitable symmetric monoidal category) is complete, cocomplete (Theorem 15), and closed symmetric monoidal (Theorem 39). These are the basic properties required for the sequel [14], where we construct a cofibrantly generated model structure on the category of props enriched in simplicial sets. Later papers will develop combinatorial models for up-to-homotopy props, following the dendroidal approach [4, 23, 24] as well as comparisons between these models (as in [5, 6] in the dendroidal setting). For motivation for the project as a whole, see the introduction to [14].

Operads are a tool used to model (co)algebraic structures, i.e associative, associative and commutative, co-associative co-commutative, Lie, Poisson, etc. They were first introduced in the 70's in algebraic topology [3, 21], experienced a renaissance in the 90's [13], and now are ubiquitous throughout various areas of mathematics (see [20] for a survey). Operads can model structures which have operations with multiple inputs and a single output. A basic example is the operad

Key words and phrases. colored operad, colored prop, multicategory.

Ass, whose algebras are the monoids. Coalgebras over **Ass** are precisely comonoids, so we see that operads can also be used to model structures with coöperations [20, 3.71].

Operads, however, cannot model all algebraic structures of interest. For example, it is well known that there is no operad which models groups. A shadow of this fact occurs when we work over k-modules: algebras over **Ass** are k-algebras, coalgebras over **Ass** are k-coalgebras, but there is no operad which models Hopf algebras, which possess both multiplication and comultiplication. In order to study families of algebras of this type, one must pass to the strictly richer category of props, which was introduced by MacLane [17] long before the invention of operads. Simply put, a prop can control mixed algebraic and coalgebraic structures, like Hopf algebras. Another important example are the various cobordism categories, which may be minimally described by a prop. Thus certain varieties of field theories are algebras over a prop whose morphisms are cobordisms.

Both of these examples are monochrome props, so why might one consider props with more general color sets? Let us justify this with an example, a subcategory, and an area of application. The example is the existence of a 2-colored prop, whose algebras consist of two Hopf algebras together with a morphism from the first to the second. More generally, given a prop T there is a prop S so that algebras over S are maps of T-algebras, although this comes at the cost of a doubling of colors. Secondly, we allow general color sets because we may then consider props as generalized categories. The category of small categories embeds in the category of props (see section 1.5). Finally, props are an ideal way to study certain problems in computer science. We could consider a prop whose morphisms are functions with multiple inputs and outputs, e.g. a function which takes a full customer record and returns the customer's phone number and name. The colors of this prop are the various data types of the language (`int`, `float`, `string`, etc.) and user-defined types. Propic composition can describe piping inputs of some functions into outputs of others, and can also describe parallel execution of functions. We will discuss several other examples in detail in section 1.3.

1.1. **Acknowledgments.** We would like to thank the referee for pointing out an oversight in a previous version of our free prop construction.

1.2. **Definition of prop.** Intuitively, a prop is a generalization of a category. We still have sets of objects, but arrows $x \to y$ are replaced by multilinear operations which may have n-inputs and m-outputs, i.e. maps are multilinear maps $x_1 \otimes \cdots \otimes x_n \to y_1 \otimes \cdots \otimes y_m$. In the monochrome case, a prop may be defined as a symmetric monoidal category freely generated by a single object. The intuition is that this is essentially a generalization of Lawvere theories that work in non-Cartesian contexts. More explicitly, a *prop* $\mathcal{T}$ consists of the following data:

- A *set* of colors[1] $C = \mathrm{Col}(\mathcal{T})$,
- for every (ordered) list of colors $a_1, \dots, a_n, b_1, \dots, b_k \in C$ (where $n, m \geq 0$), a set of operations
$$\mathcal{T}(a_1, \dots, a_n; b_1, \dots, b_m) = \mathcal{T}(\langle a_i \rangle_{i=1}^n ; \langle b_k \rangle_{k=1}^m),$$
- a specified element $\mathrm{id}_c \in \mathcal{T}(c; c)$ for each $c \in C$,
- an associative *vertical composition*
$$\begin{aligned}\mathcal{T}(\langle a_i \rangle_{i=1}^n ; \langle b_k \rangle_{k=1}^m) \times \mathcal{T}(\langle c_j \rangle_{j=1}^p ; \langle a_i \rangle_{i=1}^n) &\to \mathcal{T}(\langle c_j \rangle_{j=1}^p ; \langle b_k \rangle_{k=1}^m)\\ (f, g) &\mapsto f \circ_v g,\end{aligned}$$
- an associative *horizontal composition*
$$\begin{aligned}\mathcal{T}(\langle a_i \rangle_{i=1}^n ; \langle b_k \rangle_{k=1}^m) \times \mathcal{T}(\langle a_i \rangle_{i=n+1}^{n+p} ; \langle b_k \rangle_{k=m+1}^{m+q}) &\to \mathcal{T}(\langle a_i \rangle_{i=1}^{n+p} ; \langle b_k \rangle_{k=1}^{m+q})\\ (f, g) &\mapsto f \circ_h g,\end{aligned}$$
- a map $\sigma^* : \mathcal{T}(\langle a_i \rangle_{i=1}^n ; \langle b_k \rangle_{k=1}^m) \to \mathcal{T}(\langle a_{\sigma(i)} \rangle_{i=1}^n ; \langle b_k \rangle_{k=1}^m)$ for every element $\sigma \in \Sigma_n$, and
- a map $\tau_* : \mathcal{T}(\langle a_i \rangle_{i=1}^n ; \langle b_k \rangle_{k=1}^m) \to \mathcal{T}(\langle a_i \rangle_{i=1}^n ; \langle b_{\tau^{-1}(k)} \rangle_{k=1}^m)$ for every $\tau \in \Sigma_m$.

We typically utilize the notation $\langle a_i \rangle_{i=1}^n$ or $\langle a_1, \dots, a_n \rangle$ for a (possibly empty) list of elements $a_1, \dots, a_n$, omitting the brackets where appropriate (e.g. when $n = 1$). We will frequently denote elements of the set $\mathcal{T}(\langle a_i \rangle_{i=1}^n ; \langle b_k \rangle_{k=1}^m)$ by
$$f : \langle a_i \rangle_{i=1}^n \to \langle b_k \rangle_{k=1}^m .$$
The above data are required to satisfy the following axioms:

- The elements id_c are identities for the *vertical* composition, i.e.
$$\begin{aligned}f \circ_v (\mathrm{id}_{a_1} \circ_h \cdots \circ_h \mathrm{id}_{a_n}) &= f\\ (\mathrm{id}_{b_1} \circ_h \cdots \circ_h \mathrm{id}_{b_m}) \circ_v f &= f.\end{aligned} \tag{1}$$
- The horizontal and vertical compositions satisfy an interchange rule
$$(f \circ_v g) \circ_h (f' \circ_v g') = (f \circ_h f') \circ_v (g \circ_h g') \tag{2}$$

[1] aka 'objects'

whenever the vertical compositions on the left are well-defined.

- The vertical composition is compatible with the symmetric group actions in the sense that

$$f \circ_v (\sigma_* g) = (\sigma^* f) \circ_v g$$
$$\sigma^*(f \circ_v g) = f \circ_v (\sigma^* g) \tag{3}$$
$$\tau_*(f \circ_v g) = (\tau_* f) \circ_v g,$$

  where τ and σ are permutations on the appropriate number of letters.

- Suppose that f has n inputs and m outputs and g has p inputs and q outputs. If $\sigma \in \Sigma_n$, $\bar{\sigma} \in \Sigma_p$, $\tau \in \Sigma_m$, and $\bar{\tau} \in \Sigma_q$ and we write $\sigma \times \bar{\sigma} \in \Sigma_n \times \Sigma_p \hookrightarrow \Sigma_{n+p}$ then the horizontal composition satisfies

$$(\sigma^* f) \circ_h (\bar{\sigma}^* g) = (\sigma \times \bar{\sigma})^*(f \circ_h g) \tag{4}$$
$$(\tau_* f) \circ_h (\bar{\tau}_* g) = (\tau \times \bar{\tau})_*(f \circ_h g).$$

  Furthermore, if $\sigma_{xy} \in \Sigma_{x+y}$ is the permutation whose restrictions are increasing bijections

$$\sigma_{xy} : [1, y] \xrightarrow{\cong} [x+1, x+y]$$
$$\sigma_{xy} : [1+y, x+y] \xrightarrow{\cong} [1, x]$$

  then

$$(\sigma_{p,n})^*(\sigma_{m,q})_*(f \circ_h g) = g \circ_h f. \tag{5}$$

- The maps σ^* and τ_* satisfy the interchange rule $\sigma^*\tau_* = \tau_*\sigma^*$ and are *actions*:

$$\sigma^*\bar{\sigma}^* = (\bar{\sigma}\sigma)^* \qquad \tau_*\bar{\tau}_* = (\tau\bar{\tau})_*.$$

Remark 6. There are several variations on the term "prop" in the literature. Boardman and Vogt use the name 'colored PROP' for a prop which is completely determined by operations with n-inputs and only one output [3, Definition 2.44]. With this definition colored PROPs are the same thing as colored operads or multicategories (see [16, 2.3.1]). The definition we have given here is in line with the original due to MacLane, see, for instance, [10, 11, 15, 18, 19].

Definition 7. A homomorphism of props $f : \mathcal{R} \to \mathcal{T}$ consists of a map $\mathrm{Col}(\mathcal{R}) \to \mathrm{Col}(\mathcal{T})$ and for each input-output profile $a_1, \dots, a_n; b_1, \dots, b_m$ in $\mathrm{Col}(\mathcal{R})$ a map $\mathcal{R}(\langle a_i \rangle_{i=1}^n ; \langle b_j \rangle_{j=1}^m) \longrightarrow \mathcal{T}(\langle f a_i \rangle_{i=1}^n ; \langle f b_j \rangle_{j=1}^m)$ which commutes with all composition, identity, and symmetry operations. The category of props and prop homomorphisms is denoted **Prop**.

It is straightforward to generalize the definitions from this section to a *prop enriched in a symmetric monoidal category* $(\mathcal{E}, \boxtimes, I)$. In the data, one replaces the sets of operations $\mathcal{T}(\langle a_i \rangle_{i=1}^n ; \langle b_k \rangle_{k=1}^m)$ with objects of $\mathcal{E}$, the specified elements id_c by maps $I \to \mathcal{T}(c;c)$, and all products $\times$ by $\boxtimes$. In the axioms, all equalities on elements should be expressed instead by requiring that the relevant diagrams commute.

1.3. **Examples of props.**

Example 8. The Segal prop (see [26]) is a prop of infinite dimensional complex orbifolds. The space of morphisms is defined as the moduli space $P_{m,n}$ of complex Riemann surfaces bounding $m+n$ labeled nonoverlapping holomorphic holes. The surfaces should be understood as compact smooth complex curves, not necessarily connected, along with $m+n$ biholomorphic maps of the closed unit disk to the surface. The precise nonoverlapping condition is that the closed disks in the inputs (outputs) do not intersect pairwise and an input disk may intersect an output disk only along the boundary. This technicality brings in the symmetric group morphisms, including the identity, to the prop, but does not create singular Riemann surfaces by composition. The moduli space means that we consider isomorphism classes of such objects. The composition of morphisms in this prop is given by sewing the Riemann surfaces along the boundaries, using the equation $zw = 1$ in the holomorphic parameters coming from the standard one on the unit disk. The tensor product of morphisms is the disjoint union. This prop plays a crucial role in conformal field theory.

Example 9. Suppose that C is a set and we have a family $\mathbf{X} = \{X_c\}_{c\in C}$ of objects in some symmetric monoidal category $(\mathcal{E}, \boxtimes, I)$. This data determines an prop[2] $\mathcal{E}nd_{\mathbf{X}}$ with color set C, called the *endomorphism prop of* $\mathbf{X}$. It is defined by

$$\mathcal{E}nd_{\mathbf{X}}(c_1, \dots, c_n; d_1, \dots, d_m) = \mathcal{E}(X_{c_1} \boxtimes \cdots \boxtimes X_{c_n}, X_{d_1} \boxtimes \cdots \boxtimes X_{d_m}),$$

together with the Σ-actions and compositions coming from the monoidal structure. As in the case of operad algebras, if $\mathcal{T}$ is a prop, then a $\mathcal{T}$-*algebra* is a prop map $\mathcal{T} \to \mathcal{E}nd_{\mathbf{X}}$.

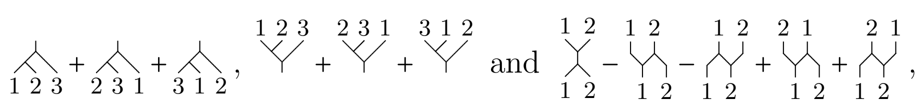
Example 10. [7, 2.1.4]A more geometric example, due to Sullivan, is the Lie bialgebra prop. A Lie *bialgebra* is a Lie algebra $\mathfrak{g}$ with the structure of a Lie coalgebra given by a one-cocycle $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ on $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $\mathfrak{g} \wedge \mathfrak{g}$, i.e., the linear map $\delta(g)$ satisfies the cocycle condition:

$$\delta([g_1, g_2]) = g_1\delta(g_2) - g_2\delta(g_1)$$

[2]We can think of this as an $\mathcal{E}$-prop if $\mathcal{E}$ is *closed* symmetric monoidal.

for all $g_1, g_2 \in \mathfrak{g}$. Lie bialgebras are so-called quasi-classical limits of quantum groups (more precisely, quantum universal enveloping algebras) and they play a key role in deformation theory (see, for example [10, 25]). One can construct a monochrome prop which has (n,m)-ary operations defined as quotient spaces of vector spaces spanned by graphs of a certain type. Explicitly, when m or n is 0, we define $L(n,m) := 0$. For $m, n \geq 1$, the space $L(n,m)$ may be defined as follows.

Consider the vector space spanned freely by the (isomorphism classes of) directed oriented trivalent graphs Γ with $n+m$ legs labeled as inputs $1, \dots, n$ and outputs $1, \dots, m$. The graphs need not be connected, but must be finite. A leg is either an edge whose one end is free, that is, not a vertex, while the other end is a vertex, or a half-edge of an edge with two free ends. The adjective directed refers to the choice of directions on each edge, so that the legs are directed from the inputs and toward the outputs and the directions define a partial order on the set of vertices. Trivalent here means that all vertices must have one incoming and two outgoing edges or two incoming and one outgoing edges. Graphs with no vertices, i.e., disjoint unions of edges each of which connects an input with an output, are allowed. An orientation on a graph means the choice of an ordering on the set of edges, up to the sign of a permutation. We define $L(n,m)$ to be the quotient of this space of graphs by relations generated by



with labels indicating, in the obvious way, the corresponding permutations of the inputs and outputs.

1.4. **Relationship with colored operads.** Recall that a colored *operad*[3] $\mathcal{O}$ is a structure with a color set $\mathrm{Col}\,\mathcal{O}$ and hom sets $\mathcal{O}(c_1, \dots, c_n; c)$ for each list of colors $c_1, \dots, c_n, c$, together with appropriate composition operations. The precise definition is a bit more involved than that of prop (see [2]) since the operadic composition mixes together horizontal and vertical propic compositions. However, we can regard colored operads as a special type of prop, namely those which are completely determined by the 'one-output part'. Let **Operad** be the category of colored operads, which we shall now refer to simply as *operads* (see [22]). There is a forgetful functor

$$U : \mathbf{Prop} \to \mathbf{Operad}$$

[3]which is variously called 'symmetric multicategory' or simply 'operad'

which takes $\mathcal{T}$ to an operad $U(\mathcal{T})$ with $\operatorname{Col}\mathcal{T} = \operatorname{Col}U(\mathcal{T})$. The morphism sets are defined by

$$U(\mathcal{T})(a_1,\dots,a_n;b) = \mathcal{T}(a_1,\dots,a_n;b).$$

Operadic composition

$$\gamma : U(\mathcal{T})(\langle a_i\rangle_{i=1}^n;b) \times \prod_{i=1}^n U(\mathcal{T})(\langle c_{i,j}\rangle_{j=1}^{p_i};a_i) \to U(\mathcal{T})\left(\left\langle\langle c_{i,j}\rangle_{j=1}^{p_i}\right\rangle_{i=1}^n;b\right)$$

is then given by

$$\gamma(g,\langle f_i\rangle_{i=1}^n) = g \circ_v (f_1 \circ_h \cdots \circ_h f_n).$$

Proposition 11. *The forgetful functor* $U : \mathbf{Prop} \longrightarrow \mathbf{Operad}$ *has a left adjoint* $F : \mathbf{Operad} \longrightarrow \mathbf{Prop}$ *with* $\operatorname{Col}F(\mathcal{O}) = \operatorname{Col}\mathcal{O}$*. An element of* $F\mathcal{O}(a_1,\dots,a_n;b_1,\dots b_m)$ *is given by* $(\theta,\langle f_j\rangle_{j=1}^m)$ *where* $\theta\colon\{1,\dots,n\}\to\{1,\dots,m\}$ *is a function and* $f_j : \langle a_i\rangle_{\theta(i)=j} \to b_j$ *is in* $\mathcal{O}$*.*

Proof. To show that we have a prop we need to first define the vertical and horizonal composition relations. Vertical composition of $(\theta,\langle f_j\rangle)$ and $(\phi,\langle g_k\rangle)$ is defined to be

$$\psi = (\phi\circ\theta,\langle h_k\rangle)$$
$$h_k = \gamma(g_k,\langle f_j\rangle_{\phi(j)=k})$$

where γ is the operadic composition. Consider $(\theta_1,\langle f_{1j}\rangle),(\theta_2,\langle f_{2j}\rangle)$, where

$$\theta_1 : \{1,\dots,n_1\} \to \{1,\dots,m_1\}$$
$$\theta_2 : \{1,\dots,n_2\} \to \{1,\dots,m_2\}$$

and $f_{\ell j} : \langle a_{\ell i}\rangle_{\theta_\ell(i)=j} \to b_{\ell j}$. The horizontal composition of these is (θ,f_k), where

$$\theta : \{1,\dots,n_1+n_2\} \to \{1,\dots,m_1+m_2\}$$
$$i \mapsto \begin{cases}\theta_1(i) & i\le n_1\\ \theta_2(i-n_1)+m_1 & i>n_1\end{cases}$$

and

$$f_k = \begin{cases} f_{1k} & 1\le k\le m_1\\ f_{2(k-m_1)} & m_1+1\le k\le m_1+m_2.\end{cases}$$

If τ is an element of Σ_m, we define the left action

$$\tau_* : F(\mathcal{O})(\langle a_i\rangle_{i=1}^n;\langle b_k\rangle_{k=1}^m) \to F(\mathcal{O})(\langle a_i\rangle_{i=1}^n;\left\langle b_{\tau^{-1}(k)}\right\rangle_{k=1}^m)$$

by $(\theta, \langle f_j \rangle_{j=1}^m) \mapsto (\tau \circ \theta, \left\langle f_{\tau^{-1}(j)} \right\rangle_{j=1}^m)$. If σ is an element of Σ_n then we define the right action

$$\sigma^* : \mathcal{T}(\langle a_i \rangle_{i=1}^n ; \langle b_k \rangle_{k=1}^m) \to \mathcal{T}(\left\langle a_{\sigma(i)} \right\rangle_{i=1}^n ; \langle b_k \rangle_{k=1}^m)$$

by $(\theta, \langle f_j \rangle_{j=1}^m) \mapsto (\theta \circ \sigma, \left\langle \gamma_j^* f_j \right\rangle_{j=1}^m)$. Here, γ_j is the composition

$$\gamma_j : \theta^{-1}(j) \to \sigma^{-1}\theta^{-1}(j) \xrightarrow{\sigma} \theta^{-1}(j)$$

where the first map is the order preserving bijection. It is now left as an exercise to verify that the axioms of a prop are satisfied.

Let $* : \{1, \ldots, n\} \to \{1\}$ denote the unique map. We see that a map of operads $q : \mathcal{O} \to U(\mathcal{T})$ uniquely determines a map of props $q' : F(\mathcal{O}) \to \mathcal{T}$ so that $(U(q'))(*, f) = q(f)$. Thus, F is left adjoint to U. □

Since there is only one map $\theta : \{1, \ldots, n\} \to \{1\}$, we have

Proposition 12. *If $\mathcal{O}$ is an operad then*

$$F(\mathcal{O})(a_1, \ldots, a_n; b) \cong \mathcal{O}(a_1, \ldots, a_n; b).$$

Consequently, $UF \cong \mathrm{id}_{\mathbf{Operad}}$. □

1.5. **Relationship with categories.** Informally, we can say that inside every operad lies a category which makes up the linear part (i.e. the operations with one input and one output) of that operad. In fact, we have an "enrichment" of the category **Prop** over **Cat**.[4] We can assign to each operad $\mathcal{O}$ a genuine category $U_0(\mathcal{O})$ whose object set is the color set of $\mathcal{O}$ and has morphisms given by $U_0(\mathcal{O})(a, b) := \mathcal{O}(a; b)$ for any two colors a, b in $\mathcal{O}$. Composition and identity operations are induced by those of $\mathcal{O}$. This relationship with category theory is useful in making sense of ideas which do not have obvious meaning in the setting of operads or props.

The functor U_0 admits a left adjoint, denoted by F_0, which takes a category $\mathcal{C}$ to an operad $F_0(\mathcal{C})$ with $\mathrm{Ob}(F_0\mathcal{C}) := \mathrm{Ob}(\mathcal{C})$. The linear operations are just the composition maps of $\mathcal{C}$, i.e. $F_0(\mathcal{C})(a; b) := \mathcal{C}(a, b)$, and the higher operations are all trivial, i.e. $F_0(\mathcal{C})(a_1, ..., a_n; b) = \varnothing$ for $n \neq 1$. Composition and units are induced from $\mathcal{C}$ in the obvious way, and it is an easy exercise to check the necessary axioms of an operad are satisfied.

[4]Coming from the fact that **Prop** is enriched over itself; cf. 3.1.

1.6. **Graphs and megagraphs.** We now fix our notion of (directed) graph, which is essentially the same as that in [11, A.1]. The graphs in this paper have a finite set of vertices V, a finite set of edges E, and functions

$$s : E \to V_+ = V \sqcup \{*\}$$
$$t : E \to V_+$$

which take an edge e to its tail $s(e)$ and its head $t(e)$. Notice that we allow for either of these to be trivial, i.e. we allow half-edges and edges that are incident to no vertices. A *cycle* is a list of edges $e_1, \dots, e_n$ such that $t(e_i) = s(e_{i+1}) \in V$ and $t(e_n) = s(e_1) \in V$. We will want to work with graphs which do not have cycles; in particular, we have no loops (cycles with $n = 1$). We will denote all the data of a graph by $G := (E, V, s, t)$ and will write

$$\mathrm{in}(v) = t^{-1}(v) \qquad \mathrm{out}(v) = s^{-1}(v)$$

for the sets of input and output edges of a vertex.

A morphism of graphs $f : G \to G'$ consists of functions $f_E : E \to E'$ and $f_V : V_+ \to V'_+$ with $f_V(*) = *$, $f_V(v) \neq *$, $sf_E(e) = f_V s(e)$, and $tf_E(e) = f_V t(e)$. The first two conditions ensure that f_E preserves the *type* of edges, i.e. edges go to edges, half-edges go to half-edges pointing in the same direction, and non-incident edges go to non-incident edges.

To define the underlying "graphs" of the category **Prop**, consider the free monoid monad acting on a set S

$$\mathbb{M} : \mathbf{Set} \to \mathbf{Set}$$
$$S \mapsto \coprod_{k \geq 0} S^{\times k}.$$

Elements of $\mathbb{M}S$ are just (finite) ordered lists of elements of S. There are right and left actions of the symmetric groups on the components of $\mathbb{M}S$. More compactly we could say that there are both right and left actions of the symmetric *groupoid* $\Sigma = \coprod_{n \geq 0} \Sigma_n$ on $\mathbb{M}S$. A Σ-bimodule is a set with compatible left and right Σ-actions.

We now describe an extension of the notion of graph, namely one in which edges are permitted to have multiple inputs and outputs.[5] See Figure 1.

Definition 13. A *megagraph* $\mathcal{X}$ consists of a set of objects X_0, a set of arrows X_1, two functions $s : X_1 \to \mathbb{M}X_0$ and $t : X_1 \to \mathbb{M}X_0$, which we will write as the span

$$\mathbb{M}X_0 \xleftarrow{s} X_1 \xrightarrow{t} \mathbb{M}X_0.$$

[5] Unlike multigraphs, it makes sense to consider undirected megagraphs.

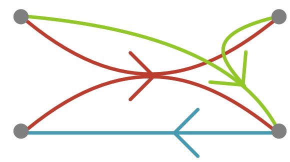


FIGURE 1. An example megagraph with four vertices and three megaedges, each having a different color.

Furthermore, X_1 should possess both right and left Σ actions. These actions should have an interchange property $\tau \cdot (x \cdot \sigma) = (\tau \cdot x) \cdot \sigma$ and should be compatible with those on $\mathbb{M}X_0$, so $t(\tau \cdot x) = \tau \cdot t(x)$ and $s(x \cdot \sigma) = s(x) \cdot \sigma$.

A map of megagraphs $f : \mathcal{X} \to \mathcal{Y}$ is determined by maps $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$ so that the diagram

$$\begin{array}{ccccc} \mathbb{M}X_0 & \xleftarrow{s} & X_1 & \xrightarrow{t} & \mathbb{M}X_0 \\ \downarrow{\scriptstyle \mathbb{M}f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \mathbb{M}f_0} \\ \mathbb{M}Y_0 & \xleftarrow{s} & Y_1 & \xrightarrow{t} & \mathbb{M}Y_0 \end{array}$$

commutes. The collection of megagraphs determines a category which we call **Mega**.

Notice that a megagraph $\mathcal{X}$ would be called a X_0-colored Σ-bimodule in [10]. We deal so frequently with color change that the current viewpoint seems appropriate. We also would like to point out that every megagraph has an underlying *directed hypergraph* (see [12, 27]) obtained by forgetting the symmetric group actions.

There is a forgetful functor U from **Prop** to **Mega**, defined by

$$\begin{aligned} (U\mathcal{T})_0 &= \operatorname{Col}\mathcal{T} \\ (U\mathcal{T})_1 &= \coprod_{\substack{\langle a_i\rangle_{i=1}^n, \langle b_j\rangle_{j=1}^m \\ \in \mathbb{M}(\operatorname{Col}\mathcal{T})}} \mathcal{T}(a_1, \dots, a_n; b_1, \dots, b_m) \end{aligned}$$

with the induced source and target maps.

Theorem 14. *The functor* U : **Prop** → **Mega** *has a left adjoint* F : **Mega** → **Prop**.

The proof of this theorem is contained in appendix A. We would like to note that our construction of $F(\mathcal{X})$ is necessarily isomorphic to that in the fixed color setting given in [10], but we still need to show adjointness in the case where the color sets may vary and maps need not preserve color. For this purpose we prefer to have a very explicit description of $F(\mathcal{X})$.

2. The category of props is complete and cocomplete

Limits in **Prop** are obtained by taking the corresponding limits on colors and morphisms in **Set**. The goal for this section is thus to show

Theorem 15. *The category* **Prop** *is cocomplete.*

Our proof is a minor adaptation of that in [9, §4] for the category of multicategories.

Recall that a *permutative category* is a symmetric monoidal category $\mathcal{C}$ with a strictly associative product $\oplus$, a strict unit 0, a swap map $\gamma : a \oplus b \cong b \oplus a$ which has the *equalities* $\gamma\gamma = \mathrm{id}$, $\gamma = (\gamma \oplus 1)(1 \oplus \gamma)$, and $(a \oplus 0 \overset{\gamma}{\to} 0 \oplus a \overset{=}{\to} a) = (a \oplus 0 \overset{=}{\to} a)$; see [8, 3.1]. A *strict map* $f : \mathcal{C} \to \mathcal{D}$ of permutative categories is a functor with $f(a \oplus b) = fa \oplus fb$, $f(0) = 0$, and

$$[f(a \oplus b) \overset{=}{\to} fa \oplus fb \overset{\gamma}{\to} fb \oplus fa] = [f(a \oplus b) \overset{f\gamma}{\to} f(b \oplus a) \overset{=}{\to} fb \oplus fa].$$

Let **Perm** be the category of permutative categories and strict morphisms, which is cocomplete by [9, 4.1]. There is a functor $U : \mathbf{Perm} \to \mathbf{Prop}$ which is given on objects by

$$\begin{aligned} \operatorname{Col} U(\mathcal{C}) &= \operatorname{Ob} \mathcal{C} \\ U(\mathcal{C})(c_1, \ldots, c_n; d_1, \ldots, d_m) &= \mathcal{C}(c_1 \oplus \cdots \oplus c_n, d_1 \oplus \cdots \oplus d_m). \end{aligned}$$

If $f : \mathcal{C} \to \mathcal{D}$ is a homomorphism of permutative categories, then there is an evident homomorphism of props $U(f) : U(\mathcal{C}) \to U(\mathcal{D})$ given by

$$\operatorname{Col} U(\mathcal{C}) = \operatorname{Ob} \mathcal{C} \to \operatorname{Ob} \mathcal{D} = \operatorname{Col} U(\mathcal{D})$$

$$\begin{aligned} U(\mathcal{C})(\langle c_i \rangle_{i=1}^n ; \langle d_j \rangle_{j=1}^m) &= \mathcal{C}(c_1 \oplus \cdots \oplus c_n, d_1 \oplus \cdots \oplus d_m) \\ &\overset{f}{\to} \mathcal{D}(fc_1 \oplus \cdots \oplus fc_n, fd_1 \oplus \cdots \oplus fd_m) = U(\mathcal{D})(\langle fc_i \rangle_{i=1}^n ; \langle fd_j \rangle_{j=1}^m). \end{aligned}$$

Remark 16. Notice that *all* props arise in this way – a prop is the same thing as a permutative category $\mathcal{C}$ which has a set of indecomposable objects S and $\operatorname{Ob} \mathcal{C} = \mathbb{M}S$ with $\oplus$ given by concatenation.[6]

Proposition 17. *The functor U has a left adjoint.*

Proof. The left adjoint L is constructed as follows. If $\mathcal{T}$ is a prop, then the objects of $L(\mathcal{T})$ are finite lists of colors of $\mathcal{T}$:

$$\operatorname{Ob} L(\mathcal{T}) = \mathbb{M} \operatorname{Col}(\mathcal{T}) = \coprod_{k \geq 0} \operatorname{Col}(\mathcal{T})^{\times k}.$$

[6] This unraveling of the definition in the monochrome case is pointed out in [25].

The monoidal product of two lists is given by concatenation. Given two lists $\langle a_i\rangle_{i=1}^n$ and $\langle b_j\rangle_{j=1}^m$, we define

$$L(\mathcal{T})(\langle a_i\rangle_{i=1}^n, \langle b_j\rangle_{j=1}^m) = \mathcal{T}(\langle a_i\rangle_{i=1}^n, \langle b_j\rangle_{j=1}^m).$$

If $f: \mathcal{T} \to \mathcal{T}'$ is a prop homomorphism, then we get a homomorphism of permutative categories which on objects is

$$\operatorname{Ob} L(\mathcal{T}) = \mathbb{M}\operatorname{Col}(\mathcal{T}) \xrightarrow{\mathbb{M}f} \mathbb{M}\operatorname{Col}(\mathcal{T}') = \operatorname{Ob} L(\mathcal{T}')$$

and on morphisms is induced directly from $\mathcal{T}$. □

Remark 18. The left adjoint **Operad** → **Perm** given in [9] factors through our L. Specifically, the main part of their construction is actually giving the map F : **Operad** → **Prop** from Proposition 11. We have a composition of adjunctions

$$\mathbf{Operad} \overset{F}{\rightleftarrows} \mathbf{Prop} \overset{L}{\rightleftarrows} \mathbf{Perm} \tag{19}$$

which recovers the adjunction [9, 4.2].

Lemma 20. *The left adjoint* L : **Prop** → **Perm** *reflects isomorphisms.*

Proof. Let $\star$ be the terminal object of **Prop**. Specifically, $\operatorname{Col}(\star) = \{1\}$ is a one element set and $\star(\underbrace{1,\dots,1}_{n};\underbrace{1,\dots,1}_{m})$ is a one element set for all $n, m \geq 0$. We identify the objects of the permutative category $L(\star)$ with the set of nonnegative integers $\mathbb{N}$.

Consider the unit $\eta: \mathrm{id}_{\mathbf{Prop}} \Rightarrow UL$. If $\mathcal{T}$ is a prop then $\eta_{\mathcal{T}}: \mathcal{T} \to UL\mathcal{T}$ takes a color $a \in \operatorname{Col}(\mathcal{T})$ to the one-element list $\langle a\rangle \in \operatorname{Col}(UL\mathcal{T}) = \mathbb{M}\operatorname{Col}(\mathcal{T})$. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\eta_{\mathcal{T}}} & UL\mathcal{T} \\ \downarrow & & \downarrow \\ \star & \xrightarrow{\eta_\star} & UL\star \end{array}$$

and suppose that we have

$$\begin{array}{ccc} f: \langle a_i^1\rangle_{i=1}^{j_1}, \dots, \langle a_i^n\rangle_{i=1}^{j_n} & \longrightarrow & \langle b_i^1\rangle_{i=1}^{k_1}, \dots, \langle b_i^m\rangle_{i=1}^{k_m} \\ \| & & \| \\ \Big\langle \langle a_i^x\rangle_{i=1}^{j_x} \Big\rangle_{x=1}^{n} & \longrightarrow & \Big\langle \langle b_i^y\rangle_{i=1}^{k_y} \Big\rangle_{y=1}^{m} \end{array}$$

in $UL\mathcal{T}$. The image of f in $UL\star$ is

$$\bar{f}: \langle j_x\rangle_{x=1}^n \to \langle k_y\rangle_{k=1}^m$$

If all of these list lengths j_x, k_y are 1, then f is in the image of $\eta_{\mathcal{T}}$. On the other hand, if f is in the image of $\eta_{\mathcal{T}}$, then $\bar{f}$ is in the image of

$\eta_\star$, so all of these lengths must be one. In other words, the image of η is the preimage of the subprop[7] of $UL\star$ which is generated by single object $\{1\}$.

Suppose that $\alpha : \mathcal{T} \to \mathcal{T}'$ is a map of props so that $L\alpha$ is an isomorphism. The diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\alpha} & \mathcal{T}' \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\ UL\mathcal{T} & \xrightarrow[\cong]{UL\alpha} & UL\mathcal{T}' \\ & \searrow \quad \swarrow & \\ & UL\star & \end{array}$$

shows that $UL\alpha$ is an isomorphism between the preimages as above. By construction of $L\mathcal{T}$, the unit is injective, so α is an isomorphism. □

Lemma 21. *The functor L preserves equalizers.*

Proof. As noted in the proof of [9, 4.5], equalizers in **Perm** are created in **Cat**. Consider the equalizer diagrams

$$\mathcal{T}' \xrightarrow{\alpha} \mathcal{T} \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \mathcal{T}''$$

$$\mathcal{C} \to L\mathcal{T} \underset{L\gamma}{\overset{L\beta}{\rightrightarrows}} L\mathcal{T}'',$$

where the first is in **Prop** and the second is in **Perm**. We wish to show that $\mathcal{C} = L\mathcal{T}'$; certainly $L\mathcal{T}' \subset \mathcal{C}$. Notice that $\operatorname{Ob}\mathcal{C} \subset \coprod_{k\in\mathbb{N}} (\operatorname{Col}\mathcal{T})^{\times k}$ is the subset consisting of lists $\langle a_i\rangle_{i=1}^n$ such that $L\beta\left(\langle a_i\rangle_{i=1}^n\right) = L\gamma\left(\langle a_i\rangle_{i=1}^n\right)$, i.e. $\beta a_i = \gamma a_i$ for all i. Thus if $\langle a_i\rangle_{i=1}^n \in \operatorname{Ob}\mathcal{C}$ then so is a_i. But then if $a \in \operatorname{Ob}\mathcal{C}$, we have $\beta a = L\beta a = L\gamma a = \gamma a$, so $\operatorname{Ob}\mathcal{C} \subset \operatorname{Ob} L\mathcal{T}'$.

We have shown that $\operatorname{Ob}\mathcal{C} = \operatorname{Ob} L\mathcal{T}'$, and we know that $L\mathcal{T}' \subset \mathcal{C} \subset L\mathcal{T}$. Suppose that $f \in \mathcal{C}(\langle a_i\rangle_{i=1}^n ; \langle b_j\rangle_{j=1}^m)$. Then

$$f \in L\mathcal{T}(\langle a_i\rangle_{i=1}^n ; \langle b_j\rangle_{j=1}^m) = \mathcal{T}(a_1, \dots, a_n; b_1, \dots, b_m)$$

has the property that $\beta(f) = L\beta(f) = L\gamma(f) = \gamma(f)$, so considering f as an element of $\mathcal{T}(a_1, \dots, a_n; b_1, \dots, b_m)$ we see that it is actually an element of $\mathcal{T}'(a_1, \dots, a_n; b_1, \dots, b_m)$. Thus we have shown that $\mathcal{C}(\langle a_i\rangle_{i=1}^n ; \langle b_j\rangle_{j=1}^m) \subset L\mathcal{T}'(\langle a_i\rangle_{i=1}^n ; \langle b_j\rangle_{j=1}^m)$, and conclude that $\mathcal{C} = L\mathcal{T}'$. □

Proof of Theorem 15. We apply the dual of [1, Ch.3, Theorem 3.14] to L, using that L has a right adjoint, L reflects isomorphisms by

[7] cf. Definition 29

Lemma 20, **Prop** has all equalizers, and L preserves equalizers by Lemma 21. The cited theorem then gives that the adjunction

$$L : \mathbf{Prop} \rightleftarrows \mathbf{Perm} : U$$

is comonadic. In other words, **Prop** is equivalent to the category of coalgebras over the comonad LU on **Perm**. Cocompleteness of this category of coalgebras follows from that of **Perm** (see exercise 2 in [18, VI.2]), so **Prop** is cocomplete. □

3. A closed symmetric monoidal structure on Prop

3.1. **Prop is enriched over Prop.** Suppose that $\mathcal{R}$ and $\mathcal{T}$ are two props. We define a mapping prop between them, which we denote by $\mathcal{H}om(\mathcal{R},\mathcal{T})$. The colors of $\mathcal{H}om(\mathcal{R},\mathcal{T})$ are just prop maps $\mathcal{R} \to \mathcal{T}$. We now must define a propic natural transformation; to begin, let us take $p+q$ prop maps $f_1, \dots, f_p, g_1, \dots, g_q$ from $\mathcal{R}$ to $\mathcal{T}$. A (p,q) *natural transformation*

$$\xi : \langle f_1, \dots, f_p \rangle \Rightarrow \langle g_1, \dots, g_q \rangle$$

is a collection of $\mathcal{T}$-morphisms

$$\xi_a \in \mathcal{T}(f_1 a, \dots, f_p a; g_1 a, \dots, g_q a),$$

one for each $a \in \operatorname{Col}\mathcal{R}$. There is, of course, some consistency condition: if $\phi : \langle a_1, \dots, a_n \rangle \to \langle b_1, \dots, b_m \rangle$ is in $\mathcal{R}$, then the following octagon must commute.

(22)
$$\begin{array}{ccc}
& \Big\langle \langle f_j a_i \rangle_{j=1}^{p} \Big\rangle_{i=1}^{n} \xrightarrow{\langle \xi_{a_i} \rangle} \Big\langle \langle g_\ell a_i \rangle_{\ell=1}^{q} \Big\rangle_{i=1}^{n} & \\
\swarrow \cong & & \cong \searrow \\
\Big\langle \langle f_j a_i \rangle_{i=1}^{n} \Big\rangle_{j=1}^{p} & & \Big\langle \langle g_\ell a_i \rangle_{i=1}^{n} \Big\rangle_{\ell=1}^{q} \\
\downarrow \langle f_j \phi \rangle & & \downarrow \langle g_\ell \phi \rangle \\
\Big\langle \langle f_j b_k \rangle_{k=1}^{m} \Big\rangle_{j=1}^{p} & & \Big\langle \langle g_\ell b_k \rangle_{k=1}^{m} \Big\rangle_{\ell=1}^{q} \\
\searrow \cong & & \cong \swarrow \\
& \Big\langle \langle f_j b_k \rangle_{j=1}^{p} \Big\rangle_{k=1}^{m} \xrightarrow[\langle \xi_{b_k} \rangle]{} \Big\langle \langle g_\ell b_k \rangle_{\ell=1}^{q} \Big\rangle_{k=1}^{m} &
\end{array}$$

This is a convenient abuse of notation which we employ frequently. 'Commutativity' of this octagon means precisely that

(23) $$\bar{\tau}_* \big(\langle g_\ell \phi \rangle_{\ell=1}^{q} \big) \circ_v \tau_* \big(\langle \xi_{a_i} \rangle_{i=1}^{n} \big) = \bar{\sigma}^* \big(\langle \xi_{b_k} \rangle_{k=1}^{m} \big) \circ_v \sigma^* \big(\langle f_j \phi \rangle_{j=1}^{p} \big)$$

where $\sigma, \bar{\sigma}, \tau, \bar{\tau}$ are the obvious interchange permutations given by the symbol '$\cong$' in (22) and where angular brackets denote *horizontal compositions*, e.g.

$$\langle \xi_{a_i} \rangle = \xi_{a_1} \circ_h \xi_{a_2} \circ_h \cdots \circ_h \xi_{a_n}.$$

We declare that $\mathcal{H}om(\mathcal{R}, \mathcal{T})(\langle f_j \rangle_{j=1}^p, \langle g_\ell \rangle_{\ell=1}^q)$ be the set of (p,q) natural transformations $\langle f_1, \dots, f_p \rangle \Rightarrow \langle g_1, \dots, g_q \rangle$.

Proposition 24. *The collection of natural transformations $\mathcal{H}om(\mathcal{R}, \mathcal{T})$ is a prop.*

Proof. Let ξ, ξ' be natural transformations, and $f \in \mathrm{Col}(\mathcal{H}om(\mathcal{R}, \mathcal{T})) = \mathrm{Hom}(\mathcal{R}, \mathcal{T})$. We define the prop structure by defining the maps at each $a \in \mathrm{Col}(\mathcal{R})$:

$$(\xi \circ_v \xi')_a = \xi_a \circ_v \xi'_a \qquad (\mathrm{id}_f)_a = \mathrm{id}_{f(a)}$$
$$(\xi \circ_h \xi')_a = \xi_a \circ_h \xi'_a \qquad [\sigma^* \tau_* (\xi)]_a = \sigma^* \tau_* [\xi_a].$$

All of the axioms of the prop then follow directly from the fact that $\mathcal{R}$ and $\mathcal{T}$ are props. One must show that these actually give natural transformations, but verifying that diagram (22) commutes for these various assignments follows by modifying the diagrams for ξ and ξ'. □

If one wishes to show something is a natural transformation, it is often easier to show that the above diagram commutes on a *generating set*. We now prove that this is enough.

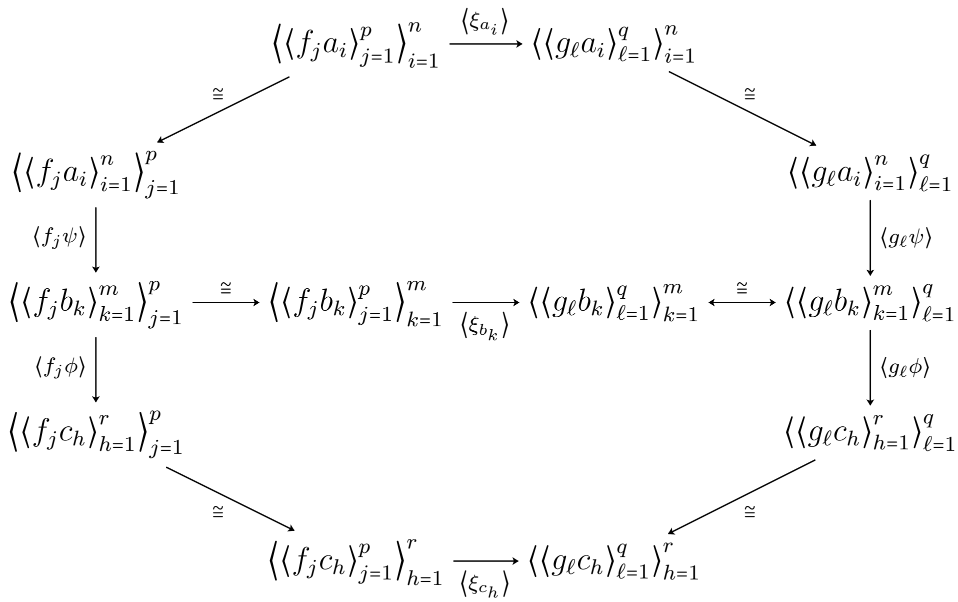
Definition 25. Let $f_1, \dots, f_p, g_1, \dots, g_q \in \mathbf{Prop}(\mathcal{R}, \mathcal{T})$, $\phi : \langle a_i \rangle_{i=1}^n \to \langle b_k \rangle_{k=1}^m$ in $\mathcal{R}$, and let ξ assign, for each $a \in \mathrm{Col}(\mathcal{R})$, a map

$$\xi_a : \langle f_1 a, \dots, f_p a \rangle \to \langle g_1 a, \dots, g_q a \rangle$$

in $\mathcal{T}$. We say that ξ *is natural with respect to* ϕ if the diagram (22) on page 14 commutes for the map ϕ. If S is a set of maps and ξ is natural with respect to each $\phi \in S$, then we say ξ *is natural with respect to* S.

Lemma 26. *Let ξ be as in Definition 25. If ξ is natural with respect to a composable pair ϕ and ψ, then ξ is natural with respect to $\phi \circ_v \psi$.*

Proof.



□

Lemma 27. *Let ξ be as in Definition 25. If ξ is natural with respect to ϕ and ψ, then ξ is natural with respect to $\phi \circ_h \psi$.*

Proof. Let

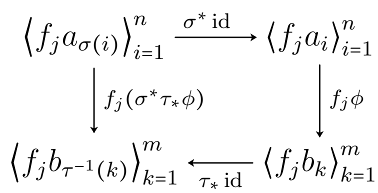
$$\phi : \langle a_1, \dots, a_n \rangle \to \langle b_1, \dots, b_m \rangle$$
$$\psi : \langle a_{n+1}, \dots, a_{n'} \rangle \to \langle b_{m+1}, \dots, b_{m'} \rangle$$

In figure 2, the middle rectangle commutes since it is the horizontal composition of two octagons which commute, using that ξ is natural with respect to ϕ and with respect to ψ. The left and right squares commute by (5). □

Lemma 28. *Let ξ be as in Definition 25. If ξ is natural with respect to ϕ, then ξ is natural with respect to $\sigma^*\phi$ and $\tau_*\phi$.*

Proof. We will show that ξ is natural with respect to $\sigma^*\tau_*\phi$.

Notice that (3) implies that $\sigma^*\phi = \phi \circ_v (\sigma^* \mathrm{id})$ and $(\tau_* \mathrm{id}) \circ_v \phi = \tau_*\phi$. In the current setting we thus have a commutative diagram



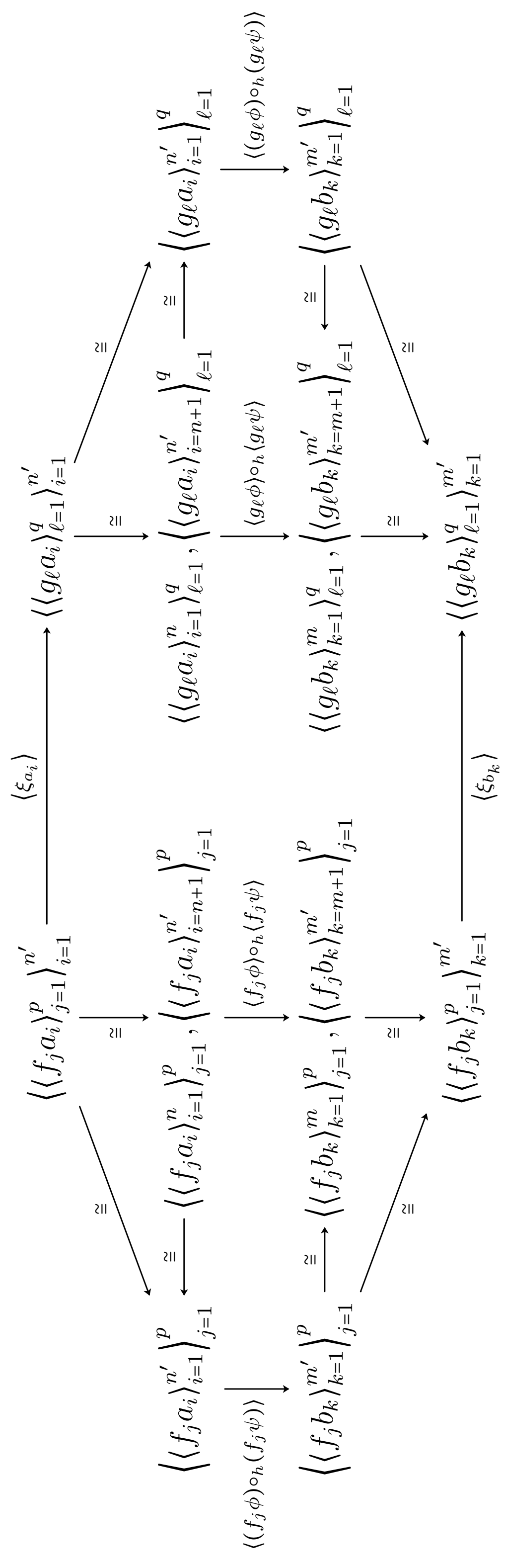


FIGURE 2. Naturality of horizontal composition

for each j, where the top and bottom maps are isomorphisms. This diagram also commutes if we replace f_j by g_ℓ.

We then have a commutative diagram

$$\begin{array}{ccccc}
\left\langle \langle f_j a_{\sigma(i)} \rangle_{j=1}^{p} \right\rangle_{i=1}^{n} & \xrightarrow[\cong]{block} & \left\langle \langle f_j a_i \rangle_{j=1}^{p} \right\rangle_{i=1}^{n} & \xrightarrow{\langle \xi_{a_i} \rangle} & \langle \langle g_\ell a_i \rangle_{\ell=1}^{q} \rangle_{i=1}^{n} \\
\downarrow \cong & & \cong \downarrow & & \\
\left\langle \langle f_j a_{\sigma(i)} \rangle_{i=1}^{n} \right\rangle_{j=1}^{p} & \xrightarrow{\langle \sigma^* \, \mathrm{id} \rangle} & \left\langle \langle f_j a_i \rangle_{i=1}^{n} \right\rangle_{j=1}^{p} & & \\
\langle f_j(\sigma^* \tau_* \phi) \rangle \downarrow & & \downarrow \langle f_j \phi \rangle & & \\
\left\langle \langle f_j b_{\tau^{-1}(k)} \rangle_{k=1}^{m} \right\rangle_{j=1}^{p} & \xleftarrow{\langle \tau_* \, \mathrm{id} \rangle} & \left\langle \langle f_j b_k \rangle_{k=1}^{m} \right\rangle_{j=1}^{p} & & \\
\downarrow \cong & & \cong \downarrow & & \\
\left\langle \langle f_j b_{\tau^{-1}(k)} \rangle_{j=1}^{p} \right\rangle_{k=1}^{m} & \xrightarrow[\cong]{block} & \left\langle \langle f_j b_k \rangle_{j=1}^{p} \right\rangle_{k=1}^{m} & \xrightarrow[\langle \xi_{b_k} \rangle]{} & \langle \langle g_\ell b_k \rangle_{\ell=1}^{q} \rangle_{k=1}^{m}
\end{array}$$

as well as a similar one for the g_ℓ. These glue together along the commutative octagon (22) for ϕ. The resulting large commutative diagram shows that the octagon for $\sigma^* \tau_* \phi$ commutes. □

Definition 29. Suppose that S is a set of morphisms in a prop $\mathcal{T}$. Then the *subprop generated by* S, denoted $\langle S \rangle$, is the smallest subprop of $\mathcal{T}$ containing all elements of S.

This subprop $\langle S \rangle$ must contain all of the identity maps on the colors appearing in the source and target lists of elements of S. It must also contain all morphisms obtained by iterated compositions and symmetric group actions from elements of S and these identity maps. The collection of such morphisms forms a prop, as we saw in the construction of the free prop in appendix A. Therefore we have an alternate characterization of $\langle S \rangle$.

Proposition 30. *Let S be a set of morphisms in $\mathcal{R}$ and let $\langle S \rangle$ be the subprop of $\mathcal{R}$ generated by S. If ξ is natural with respect to S then ξ is natural with respect to $\langle S \rangle$. In particular, if $\mathcal{R}$ is generated by S, then ξ is a natural transformation.*

Proof. This follows from the preceding paragraph and the three preceding lemmas. □

We would hope that this enrichment be compatible with the existing enrichment on the category of operads. We cannot insist that the adjunction be enriched, since the categories are enriched over different things. We do have

Proposition 31. *Given the adjunction* $F\colon \mathbf{Operad} \rightleftarrows \mathbf{Prop}\colon U$*, the isomorphism* $\mathrm{Hom}_{\mathbf{Prop}}(F(\mathcal{O}),\mathcal{T}) \cong \mathrm{Hom}_{\mathbf{Operad}}(\mathcal{O},U(\mathcal{T}))$ *extends to an isomorphism of operads*

$$U(\mathcal{H}om(F(\mathcal{O}),\mathcal{T})) \cong \mathcal{H}om_{\mathbf{Operad}}(\mathcal{O},U(\mathcal{T}))$$

where the right hand side is the internal hom in **Operad***.*

Proof. Suppose $\xi \in \mathcal{H}om_{\mathbf{Operad}}(\mathcal{O},U(\mathcal{T}))$ is a p-natural transformation $\langle f_1,\dots,f_p\rangle \to g$, where $f_1,\dots,f_p,g$ are operad maps $\mathcal{O} \to U(\mathcal{T})$. This means that we have maps $\xi_a : f_1 a,\dots,f_p a \to g(a)$ in $U(\mathcal{T})$ for each $a \in \mathrm{Col}\,\mathcal{O}$ which satisfy the compatibility condition from [9, 2.2].

Let $\bar{f}_1,\dots,\bar{f}_p,\bar{g}$ be the adjoints of the above operad maps and note that $\bar{f}a = fa$ for all $a \in \mathrm{Col}(\mathcal{O}) = \mathrm{Col}(F\mathcal{O})$. Furthermore, the maps

$$\xi_a : \left\langle \bar{f}_1 a,\dots,\bar{f}_p a\right\rangle \to \bar{g}$$

are already in $\mathcal{T}$ itself by the definition of U. We define the adjoint

$$\bar{\xi} : \left\langle \bar{f}_1,\dots,\bar{f}_p\right\rangle \to \bar{g}$$

to be

$$\bar{\xi}_a = \xi_a : \left\langle \bar{f}_1 a,\dots,\bar{f}_p a\right\rangle \to \bar{g}a.$$

Notice that the class of maps $\phi : \langle a_1,\dots,a_n\rangle \to b$ generate $F(\mathcal{O})$, so by Proposition 30 it is enough to show that (22) commutes for $\phi \in F(\mathcal{O})$ with one output. Now we are in the situation with $q = 1$ and $m = 1$, so the octagon becomes

$$\begin{array}{ccc}
\left\langle \left\langle \bar{f}_j a_i \right\rangle_{j=1}^{p} \right\rangle_{i=1}^{n} & \xrightarrow{\langle \bar{\xi}_{a_i}\rangle} & \langle \bar{g} a_i\rangle_{i=1}^{n} \\
\downarrow \cong & & \\
\left\langle \left\langle \bar{f}_j a_i \right\rangle_{i=1}^{n} \right\rangle_{j=1}^{p} & & \downarrow \bar{g}\phi \\
\downarrow \langle \bar{f}_j \phi\rangle & & \\
\langle f_j b\rangle_{j=1}^{p} & \xrightarrow[\bar{\xi}_b]{} & \bar{g} b
\end{array}$$

But since ϕ only has one output and $UF = \mathrm{id}$ by Proposition 12, ϕ is actually in $\mathcal{O}$ so $\bar{f}_j\phi \in \mathcal{T}$ is actually in $U(\mathcal{T})$, and equals $f_j\phi$. Thus the diagram above is a diagram in $U(\mathcal{T})$, and is exactly the diagram from [9, 2.2], so commutes.

Conversely, suppose that we have a $(p,1)$-natural transformation

$$\xi : \langle f_1,\dots,f_p\rangle \to g$$

in $\mathcal{H}om(F(\mathcal{O}),\mathcal{T})$, where $f_1,\dots,f_p,g$ are prop maps from $F(\mathcal{O}) \to \mathcal{T}$. The collection of $(p,1)$-natural transformations constitute the set of arrows of $U(\mathcal{H}om(F(\mathcal{O}),\mathcal{T}))$. Let $\bar{f}_1,\dots,\bar{f}_p,\bar{g} : \mathcal{O} \to U(\mathcal{T})$ be the adjoints of $f_1,\dots,f_p,g$. Since $\xi_a \in \mathcal{T}$ has only one output, it is actually in $U(\mathcal{T})$. We thus define $\bar{\xi}_a : \left\langle \bar{f}_j a\right\rangle_{j=1}^p \to \bar{g}a$ to be $U(\xi_a)$. Let $\phi : \langle a_1,\dots,a_n\rangle \to b$ in $\mathcal{O}$. Then the octagon (22) commutes for $F\phi$ and ξ. Applying U, we get the commutative diagram from [9, 2.2], so $\bar{\xi} = (\bar{\xi}_a)$ is a p-natural transformation. □

3.2. **Bilinear maps of props.** Suppose that $\mathcal{R},\mathcal{S}$, and $\mathcal{T}$ are props. A bilinear map $(\mathcal{R},\mathcal{S}) \to \mathcal{T}$ consists of the following data.

(1) a function $\chi : \operatorname{Col}\mathcal{R} \times \operatorname{Col}\mathcal{S} \to \operatorname{Col}\mathcal{T}$
(2) for each $\phi : \langle a_1,\dots,a_n\rangle \to \langle b_1,\dots,b_m\rangle$ in $\mathcal{R}$ and $c \in \operatorname{Col}\mathcal{S}$, a morphism
$$\chi(\phi,c) : \langle \chi(a_1,c),\dots,\chi(a_n,c)\rangle \to \langle \chi(b_1,c),\dots,\chi(b_m,c)\rangle$$
in $\mathcal{T}$.
(3) for each $\psi : \langle c_1,\dots,c_p\rangle \to \langle d_1,\dots,d_q\rangle$ in $\mathcal{S}$ and $a \in \operatorname{Col}\mathcal{R}$, a morphism
$$\chi(a,\psi) : \langle \chi(a,c_1),\dots,\chi(a,c_p)\rangle \to \langle \chi(a,d_1),\dots,\chi(a,d_q)\rangle$$
in $\mathcal{T}$.

These are required to satisfy the axioms

(1) if $a \in \operatorname{Col}\mathcal{R}$ then $\chi(a,-)$ is a map of props $\mathcal{S} \to \mathcal{T}$,
(2) if $c \in \operatorname{Col}\mathcal{S}$ then $\chi(-,c)$ is a map of props $\mathcal{R} \to \mathcal{T}$, and
(3) the octagon

(32)
$$\begin{array}{ccc}
& \left\langle\langle\chi(a_i,c_j)\rangle_{i=1}^n\right\rangle_{j=1}^p \xrightarrow{\langle\chi(\phi,c_j)\rangle} \left\langle\langle\chi(b_k,c_j)\rangle_{k=1}^m\right\rangle_{j=1}^p & \\
\swarrow\cong & & \cong\searrow \\
\left\langle\langle\chi(a_i,c_j)\rangle_{j=1}^p\right\rangle_{i=1}^n & & \left\langle\langle\chi(b_k,c_j)\rangle_{j=1}^p\right\rangle_{k=1}^m \\
\downarrow{\scriptstyle\langle\chi(a_i,\psi)\rangle} & & \downarrow{\scriptstyle\langle\chi(b_k,\psi)\rangle} \\
\left\langle\langle\chi(a_i,d_\ell)\rangle_{\ell=1}^q\right\rangle_{i=1}^n & & \left\langle\langle\chi(b_k,d_\ell)\rangle_{\ell=1}^q\right\rangle_{k=1}^m \\
\searrow\cong & & \cong\swarrow \\
& \left\langle\langle\chi(a_i,d_\ell)\rangle_{i=1}^n\right\rangle_{\ell=1}^q \xrightarrow[\langle\chi(\phi,d_\ell)\rangle]{} \left\langle\langle\chi(b_k,d_\ell)\rangle_{k=1}^m\right\rangle_{\ell=1}^q &
\end{array}$$

commutes.

We will write $\operatorname{Bilin}(\mathcal{R},\mathcal{S};\mathcal{T})$ for the collection of bilinear maps. Unravelling the definitions gives natural bijections

$$\operatorname{Hom}(\mathcal{R},\mathcal{H}om(\mathcal{S},\mathcal{T})) \cong \operatorname{Bilin}(\mathcal{R},\mathcal{S};\mathcal{T}) \cong \operatorname{Hom}(\mathcal{S},\mathcal{H}om(\mathcal{R},\mathcal{T})). \tag{33}$$

We would like to show that the collection of bilinear maps is the color set for a prop $\mathcal{B}ilin(\mathcal{R},\mathcal{S};\mathcal{T})$. To this end, suppose that we have a list

$$\chi_1,\dots,\chi_v,\varsigma_1,\dots,\varsigma_w$$

of bilinear maps $(\mathcal{R},\mathcal{S})\to\mathcal{T}$. A (v,w)-morphism

$$\xi:\langle\chi_1,\dots,\chi_v\rangle\to\langle\varsigma_1,\dots,\varsigma_w\rangle$$

in $\mathcal{B}ilin(\mathcal{R},\mathcal{S};\mathcal{T})$ consists of a choice of (v,w) morphisms

$$\xi_{(a,c)}:\langle\chi_1(a,c),\dots,\chi_v(a,c)\rangle\to\langle\varsigma_1(a,c),\dots,\varsigma_w(a,c)\rangle$$

for each $a\in\mathrm{Col}(\mathcal{R})$ and $c\in\mathrm{Col}(\mathcal{S})$. These are subject to two compatibility conditions. The first is that if we have

$$\begin{aligned}&\phi\in\mathcal{R}(a_1,\dots,a_n;b_1,\dots,b_m)\\&c\in\mathrm{Col}(\mathcal{S})\end{aligned}$$

then the octagon

$$\begin{array}{ccccc}
& \Big\langle\langle\chi_j(a_i,c)\rangle_{i=1}^n\Big\rangle_{j=1}^v & \xrightarrow{\langle\chi_j(\phi,c)\rangle} & \Big\langle\langle\chi_j(b_\ell,c)\rangle_{\ell=1}^m\Big\rangle_{j=1}^v & \\
\swarrow\cong & & & & \cong\searrow \\
\Big\langle\langle\chi_j(a_i,c)\rangle_{j=1}^v\Big\rangle_{i=1}^n & & & & \Big\langle\langle\chi_j(b_\ell,c)\rangle_{j=1}^v\Big\rangle_{\ell=1}^m \\
\Big\downarrow{\scriptstyle \langle\xi_{(a_i,c)}\rangle} & & & & \Big\downarrow{\scriptstyle \langle\xi_{(b_\ell,c)}\rangle} \\
\langle\langle\varsigma_k(a_i,c)\rangle_{k=1}^w\rangle_{i=1}^n & & & & \langle\langle\varsigma_k(b_\ell,c)\rangle_{k=1}^w\rangle_{\ell=1}^m \\
\cong\searrow & & & & \swarrow\cong \\
& \langle\langle\varsigma_k(a_i,c)\rangle_{i=1}^n\rangle_{k=1}^w & \xrightarrow[\langle\varsigma_k(\phi,c)\rangle]{} & \langle\langle\varsigma_k(b_\ell,c)\rangle_{\ell=1}^m\rangle_{k=1}^w &
\end{array}$$

commutes. Similarly, if

$$\begin{aligned}&\psi\in\mathcal{S}(c_1,\dots,c_p;d_1,\dots,d_q)\\&a\in\mathrm{Col}(\mathcal{R})\end{aligned}$$

then the octagon

$$
\begin{array}{ccccc}
& \Big\langle \langle \chi_j(a,c_i)\rangle_{i=1}^{p} \Big\rangle_{j=1}^{v} & \xrightarrow{\langle \chi_j(a,\psi)\rangle} & \Big\langle \langle \chi_j(a,d_\ell)\rangle_{\ell=1}^{q} \Big\rangle_{j=1}^{v} & \\
\swarrow{\scriptstyle \cong} & & & & \searrow{\scriptstyle \cong} \\
\Big\langle \langle \chi_j(a,c_i)\rangle_{j=1}^{v} \Big\rangle_{i=1}^{p} & & & & \Big\langle \langle \chi_j(a,d_\ell)\rangle_{j=1}^{v} \Big\rangle_{\ell=1}^{q} \\
\Big\langle \xi_{(a,c_i)} \Big\rangle \Big\downarrow & & & & \Big\downarrow \Big\langle \xi_{(a,d_\ell)} \Big\rangle \\
\langle \langle \varsigma_k(a,c_i)\rangle_{k=1}^{w} \rangle_{i=1}^{p} & & & & \langle \langle \varsigma_k(a,d_\ell)\rangle_{k=1}^{w} \rangle_{\ell=1}^{q} \\
\searrow{\scriptstyle \cong} & & & & \swarrow{\scriptstyle \cong} \\
& \langle \langle \varsigma_k(a,c_i)\rangle_{i=1}^{p} \rangle_{k=1}^{w} & \xrightarrow[\langle \varsigma_k(a,\psi)\rangle]{} & \langle \langle \varsigma_k(a,d_\ell)\rangle_{\ell=1}^{q} \rangle_{k=1}^{w} &
\end{array}
$$

commutes.

Proposition 34. *With these morphisms $\mathcal{B}ilin(\mathcal{R},\mathcal{S};\mathcal{T})$ is a prop and*

$$\mathcal{H}om(\mathcal{R},\mathcal{H}om(\mathcal{S},\mathcal{T})) \cong \mathcal{B}ilin(\mathcal{R},\mathcal{S};\mathcal{T}) \cong \mathcal{H}om(\mathcal{S},\mathcal{H}om(\mathcal{R},\mathcal{T})).$$

□

3.3. **The tensor product of props.** If X is an algebra over two props $\mathcal{T}$ and $\mathcal{T}'$, we might ask for some sort of compatibility of the actions of $\mathcal{T}$ and $\mathcal{T}'$. We can interchange the actions in a suitable, reasonable sense if and only if X is an algebra over another prop $\mathcal{T}\otimes\mathcal{T}'$. We describe this prop in this section as a universal bilinear target $(\mathcal{T},\mathcal{T}') \to \mathcal{T}\otimes\mathcal{T}'$.

Let $\mathcal{R}$ and $\mathcal{S}$ be (small) props and consider the coproducts

$$\coprod_{\mathrm{Col}(\mathcal{R})} \mathcal{S} \qquad \& \qquad \coprod_{\mathrm{Col}(\mathcal{S})} \mathcal{R}.$$

Maps from $\coprod_{\mathrm{Col}(\mathcal{R})} \mathcal{S} \to \mathcal{T}$ can be thought of as $\mathcal{R}$-parametrized maps $\mathcal{S} \to \mathcal{T}$. Each consists of a function $\chi : \mathrm{Col}(\mathcal{R}) \times \mathrm{Col}(\mathcal{S}) \to \mathrm{Col}(\mathcal{T})$ together with a function $\mathrm{Col}(\mathcal{R}) \to \mathrm{Hom}(\mathcal{S},\mathcal{T})$ which extends χ's adjoint $\mathrm{Col}(\mathcal{R}) \to \mathrm{Hom}(\mathrm{Col}(\mathcal{S}),\mathrm{Col}(\mathcal{T}))$. Similarly maps $\coprod_{\mathrm{Col}(\mathcal{S})} \mathcal{R} \to \mathcal{T}$ are the same as $\mathcal{S}$-parametrized maps $\mathcal{R} \to \mathcal{T}$.

We observe that by forgetting structure, a bilinear map $(\mathcal{R},\mathcal{S}) \to \mathcal{T}$ may be thought of as either an $\mathcal{S}$-parametrized map or as an $\mathcal{R}$-parametrized map. In addition, these two parametrized maps share the same function $\chi : \mathrm{Col}(\mathcal{R}) \times \mathrm{Col}(\mathcal{S}) \to \mathrm{Col}(\mathcal{T})$.

We take the pushout

$$\begin{array}{ccc} \mathrm{Col}(\mathcal{R}) \times \mathrm{Col}(\mathcal{S}) & \longrightarrow & \coprod_{\mathrm{Col}(\mathcal{R})} \mathcal{S} \\ \downarrow & & \downarrow \\ \coprod_{\mathrm{Col}(\mathcal{S})} \mathcal{R} & \longrightarrow & \mathcal{R} \# \mathcal{S} \end{array}$$

where the upper left corner is the free prop on this set of colors. Thus maps from $\mathcal{R}\#\mathcal{S}$ are the same thing as a pair of parametrized maps which share the same χ.

Fix morphisms $\phi : \langle a_1, \dots, a_n \rangle \to \langle b_1, \dots, b_m \rangle$ in $\mathcal{R}$ and $\psi : \langle c_1, \dots, c_p \rangle \to \langle d_1, \dots, d_q \rangle$ in $\mathcal{S}$, and define megagraphs $\mathcal{X}(\phi, \psi)$ and $\mathcal{Y}(\phi, \psi)$ as follows. We declare

$$X_0 = Y_0 = (\{a_1, \dots, a_n\} \times \{c_1, \dots, c_p\}) \cup (\{b_1, \dots, b_m\} \times \{d_1, \dots, d_q\}),$$

and let X_1 be a two point set $\{*_1, *_2\}$ and $Y_1 = *$. Both source functions evaluate to $\left\langle \langle (a_i, c_j) \rangle_{i=1}^n \right\rangle_{j=1}^p$ and both target functions evaluate to $\langle \langle (b_k, d_\ell) \rangle_{k=1}^m \rangle$. We consider the map of megagraphs $\mathcal{X}(\phi, \psi) \to \mathcal{Y}(\phi, \psi)$ which is the identity map on objects and the map of megagraphs $\mathcal{X}(\phi, \psi) \to U(\mathcal{R}\#\mathcal{S})$ which takes $*_1$ to the composite

$$\begin{aligned} \left\langle \langle (a_i, c_j) \rangle_{i=1}^n \right\rangle_{j=1}^p \xrightarrow{\langle (\phi, c_j) \rangle} \left\langle \langle (b_k, c_j) \rangle_{k=1}^m \right\rangle_{j=1}^p \xrightarrow{\cong} \left\langle \langle (b_k, c_j) \rangle_{j=1}^p \right\rangle_{k=1}^m \\ \xrightarrow{\langle (b_k, \psi) \rangle} \left\langle \langle (b_k, d_\ell) \rangle_{\ell=1}^q \right\rangle_{k=1}^m \xrightarrow{\cong} \left\langle \langle (b_k, d_\ell) \rangle_{k=1}^m \right\rangle_{\ell=1}^q \end{aligned}$$

and $*_2$ to the composite

$$\begin{aligned} \left\langle \langle (a_i, c_j) \rangle_{i=1}^n \right\rangle_{j=1}^p \xrightarrow{\cong} \left\langle \langle (a_i, c_j) \rangle_{j=1}^p \right\rangle_{i=1}^n \xrightarrow{\langle (a_i, \psi) \rangle} \left\langle \langle (a_i, d_\ell) \rangle_{\ell=1}^q \right\rangle_{i=1}^n \\ \xrightarrow{\cong} \left\langle \langle (a_i, d_\ell) \rangle_{i=1}^n \right\rangle_{\ell=1}^q \xrightarrow{\langle (\phi, d_\ell) \rangle} \left\langle \langle (b_k, d_\ell) \rangle_{k=1}^m \right\rangle_{\ell=1}^q . \end{aligned}$$

These are the two paths around the octagon (32) in the definition of bilinear map.

We define $\mathcal{R} \otimes \mathcal{S}$ to be the pushout

$$\begin{array}{ccc} \coprod_{(\phi,\psi)} F\mathcal{X}(\phi, \psi) & \longrightarrow & \mathcal{R}\#\mathcal{S} \\ \downarrow & & \downarrow \\ \coprod_{(\phi,\psi)} F\mathcal{Y}(\phi, \psi) & \longrightarrow & \mathcal{R} \otimes \mathcal{S} \end{array}$$

where $F\mathcal{X}(\phi, \psi) \to \mathcal{R}\#\mathcal{S}$ is the adjoint of the previously defined map.

The next theorem follows from the construction of $\mathcal{R} \otimes \mathcal{S}$.

Theorem 35. *If $\mathcal{R}$ and $\mathcal{S}$ are props then there is a bilinear map $(\mathcal{R},\mathcal{S}) \to \mathcal{R}\otimes\mathcal{S}$ which is universal among bilinear maps from $(\mathcal{R},\mathcal{S})$. In other words, this map induces a natural isomorphism*

$$\mathrm{Bilin}(\mathcal{R},\mathcal{S};\mathcal{T}) \cong \mathrm{Hom}(\mathcal{R}\otimes\mathcal{S},\mathcal{T}).$$

An easy consequence of this and (33) is that

$$\mathrm{Hom}(\mathcal{R}\otimes\mathcal{S},\mathcal{T}) \cong \mathrm{Hom}(\mathcal{R},\mathcal{H}om(\mathcal{S},\mathcal{T})) \cong \mathrm{Hom}(\mathcal{S},\mathcal{H}om(\mathcal{R},\mathcal{T})). \tag{36}$$

Let $\chi : (\mathcal{R},\mathcal{S}) \to \mathcal{R}\otimes\mathcal{S}$ be the universal bilinear map. For ϕ a morphism in $\mathcal{R}$ and c in $\mathrm{Col}(\mathcal{S})$, we write $\phi\otimes c = \chi(\phi,c)$. Similarly, $a\otimes\psi = \chi(a,\psi)$ for $a\in\mathrm{Col}(\mathcal{R})$ and ψ a morphism of $\mathcal{S}$.

Proposition 37. *The set of morphisms*

$$S = \{\,\phi\otimes c, a\otimes\psi \mid a\in\mathrm{Col}(\mathcal{R}), c\in\mathrm{Col}(\mathcal{S}), \phi\in\mathcal{R}, \text{ and } \psi\in\mathcal{S}\,\}$$

generate $\mathcal{R}\otimes\mathcal{S}$. Moreover, the set of (p,q)-natural transformations ξ in $\mathcal{H}om(\mathcal{R}\otimes\mathcal{S},\mathcal{T})$ are precisely the set of those ξ which are natural with respect to S (as in definition 25).

Proof. The subprop generated by S is universal for bilinear maps. The second statement is Proposition 30. □

Proposition 38. *There is a natural isomorphism of props*

$$\mathcal{H}om(\mathcal{R}\otimes\mathcal{S},\mathcal{T}) \cong \mathcal{H}om(\mathcal{R},\mathcal{H}om(\mathcal{S},\mathcal{T}))$$

whose restriction to color sets is (36).

Proof. There are isomorphisms

$$\mathcal{H}om(\mathcal{R}\otimes\mathcal{S},\mathcal{T}) \cong \mathcal{B}ilin(\mathcal{R},\mathcal{S};\mathcal{T}) \cong \mathcal{H}om(\mathcal{R},\mathcal{H}om(\mathcal{S},\mathcal{T}))$$

given by Propositions 37 and 34, respectively. □

Theorem 39. *The tensor product $\otimes$ makes* **Prop** *a closed symmetric monoidal category.*

Proof. Symmetry is clear from construction, the unit axioms are obvious, and the fact that $-\otimes\mathcal{S}$ is adjoint to $\mathcal{H}om(\mathcal{S},-)$ is (36). Here the unit is the monochrome prop and exactly one morphism. As in [9], the associativity isomorphisms come from the isomorphisms

$$\begin{aligned}
\mathrm{Hom}((\mathcal{R}\otimes\mathcal{S})\otimes\mathcal{T},\mathcal{U}) &\underset{(36)}{\cong} \mathrm{Hom}(\mathcal{R}\otimes\mathcal{S},\mathcal{H}om(\mathcal{T},\mathcal{U}))\\
&\underset{(36)}{\cong} \mathrm{Hom}(\mathcal{R},\mathcal{H}om(\mathcal{S},\mathcal{H}om(\mathcal{T},\mathcal{U})))\\
&\underset{38}{\cong} \mathrm{Hom}(\mathcal{R},\mathcal{H}om(\mathcal{S}\otimes\mathcal{T},\mathcal{U}))\\
&\underset{(36)}{\cong} \mathrm{Hom}(\mathcal{R}\otimes(\mathcal{S}\otimes\mathcal{T}),\mathcal{U})
\end{aligned}$$

via the Yoneda lemma. The pentagon relation for this associativity isomorphism follows using the same diagrams one must draw to see the analogous result in the case of operads in [9, §4]. □

A natural question is whether this tensor product is 'compatible' with the usual Boardman-Vogt tensor product on operads. We have the following:

Proposition 40. *If* $\mathcal{O}, \mathcal{P} \in \mathbf{Operad}$*, then*

$$F(\mathcal{O} \otimes_{BV} \mathcal{P}) \cong F(\mathcal{O}) \otimes F(\mathcal{P}).$$

Proof. This is a straightforward computation using several natural isomorphisms. We have, for any prop $\mathcal{T}$,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Prop}}(F(\mathcal{O} \otimes_{BV} \mathcal{P}), \mathcal{T}) &= \mathrm{Hom}_{\mathbf{Operad}}(\mathcal{O} \otimes_{BV} \mathcal{P}, U(\mathcal{T})) \\ &= \mathrm{Hom}_{\mathbf{Operad}}(\mathcal{O}, \mathcal{H}om_{\mathbf{Operad}}(\mathcal{P}, U(\mathcal{T})). \end{aligned}$$

By Proposition 31, we know that $\mathcal{H}om_{\mathbf{Operad}}(\mathcal{P}, U(\mathcal{T}))$ is isomorphic to $U(\mathcal{H}om(F(\mathcal{P}), \mathcal{T}))$, so

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{Operad}}(\mathcal{O}, \mathcal{H}om_{\mathbf{Operad}}(\mathcal{P}, U(\mathcal{T}))) \\ =&\mathrm{Hom}_{\mathbf{Prop}}(\mathcal{O}, U(\mathcal{H}om(F(\mathcal{P}), \mathcal{T}))) \\ =&\mathrm{Hom}_{\mathbf{Prop}}(F(\mathcal{O}), \mathcal{H}om(F(\mathcal{P}), \mathcal{T})) \\ =&\mathrm{Hom}_{\mathbf{Prop}}(F(\mathcal{O}) \otimes F(\mathcal{P}), \mathcal{T}). \end{aligned}$$

□

Appendix A. The free prop on a megagraph

Fix a symmetric megagraph $\mathcal{X}$. Let $G = (E, V, s, t)$ be a graph.

Definition 41. A *decoration of* G *by* $\mathcal{X}$ consists of the following data:

- a function $D_0 : E \to X_0$,
- a function $D_1 : V \to X_1$,
- for each vertex $v \in V$, an ordering on the input edges $\mathrm{in}(v)$ and the output edges $\mathrm{out}(v)$;
- an ordering on both $\mathrm{in}(G)$ and on $\mathrm{out}(G)$,

which are subject to the compatibility conditions

$$\begin{array}{ccc} V & \xrightarrow{D_1} & X_1 \\ {\scriptstyle in}\downarrow & & \downarrow{\scriptstyle s} \\ \mathbb{M}E & \xrightarrow[\mathbb{M}D_0]{} & \mathbb{M}X_0 \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{D_1} & X_1 \\ {\scriptstyle out}\downarrow & & \downarrow{\scriptstyle t} \\ \mathbb{M}E & \xrightarrow[\mathbb{M}D_0]{} & \mathbb{M}X_0. \end{array}$$

The reader will notice that by specifying that there be an ordering on $in(v)$ (respectively, an ordering on: $out(v)$, $\mathrm{in}(G)$, or $\mathrm{out}(G)$) we can consider the set $\mathrm{in}(v)$ (respectively, $\mathrm{out}(v)$, $\mathrm{in}(G)$ or $\mathrm{out}(G)$) as an element of $\mathbb{M}E$. We will use this fact frequently below.

We also remark that in our definition of decoration, the symmetric megagraph $\mathcal{X}$ is fixed. As such, we will usually refer to a decoration of G by $\mathcal{X}$ as a *decoration of* G, or sometimes just a *decoration*. We will denote decorations by variants of '$\mathfrak{g}$'.

Let us choose a single graph from each isomorphism class and let $\widetilde{\Gamma}$ denote the set of all decorations of all chosen graphs by the symmetric megagraph $\mathcal{X}$. We define an equivalence relation on $\widetilde{\Gamma}$ using graph automorphisms as follows. Let G be a graph, let $\mathfrak{g}$, $\widetilde{\mathfrak{g}}$ be two decorations of G, and let $f : G \to G$ be a graph automorphism. We say that f *relates* $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ if:

(1) $\mathbb{M}f_0(\mathrm{in}(G)) = \widetilde{in}(G)$;
(2) $\mathbb{M}f_0(\mathrm{out}(G)) = \widetilde{out}(G)$;
(3) if both
   (a) $f_0(\mathrm{out}(v)) = \sigma \cdot \widetilde{out}(f_1(v))$ and
   (b) $f_0(\mathrm{in}(v)) = \widetilde{in}(f_1(v)) \cdot \tau$,
   then $D_1(v) = \sigma \cdot \widetilde{D}_1(f_1(v)) \cdot \tau$.

The identity automorphism $\mathrm{id} : G \to G$ produces two special relations that we will employ frequently in what follows.

Interior permutations: If $I = \mathrm{in}(v) \cap \mathrm{out}(v')$ and $\gamma \in \Sigma_I$ (considered as a subgroup of both $\Sigma_{\mathrm{in}(v)}$ and $\Sigma_{\mathrm{out}(v')}$), then a decoration $\mathfrak{g}$ is related to a modified decoration $\widetilde{\mathfrak{g}}$ where the only changes are
- $\widetilde{D}_1(v) = (D_1 v) \cdot \gamma$ and $in(v)$ is replaced by $\mathrm{in}(v) \cdot \gamma$
- $\widetilde{D}_1(v') = \gamma^{-1} \cdot (D_1 v')$ and $out(v')$ is replaced by $\gamma^{-1} \cdot \mathrm{out}(v')$

Exterior permutations: Let $\mathfrak{g}$ be a decoration and $I \subset \mathrm{in}(G) \cap \mathrm{in}(v)$ be a subset with the induced ordering from $\mathrm{in}(v)$. Let γ be such that $I \cdot \gamma^{-1} \subset \mathrm{in}(G)$ is an ordered inclusion. Then $\mathfrak{g}$ is related to a decoration $\widetilde{\mathfrak{g}}$ where the only changes are $\widetilde{D}_1(v) \cdot \gamma = D_1(v)$ and $\widetilde{in}(v) \cdot \gamma = \mathrm{in}(v)$. A similar relation holds if one considers outputs and left actions.

We will denote by Γ the quotient of the set $\widetilde{\Gamma}$ by the relation which is generated by all graph automorphisms. The set Γ is the morphism set for the free prop $F(\mathcal{X})$. The decorations of graphs are obtained by formally composing elements of X_1.

To elaborate, notice that we have an inclusion

$$X_1 \hookrightarrow \Gamma.$$

Suppose that $x \in X_1$, $s(x) = \langle a_i \rangle_{i=1}^n$, and $t(x) = \langle b_k \rangle_{k=1}^m$. The n, m corolla $C_{n,m}$ is the graph with one vertex which has n incoming and m outgoing edges. Choose an ordering for the incoming edges $e_1, \dots, e_n$ and an ordering on the outgoing edges $e_{n+1}, \dots, e_{n+m}$. Let $\mathfrak{g}(x)$ be the decoration on $C_{n,m}$ with $D_1(v) = x$, $in(v) = \langle e_i \rangle_{i=1}^n = \mathrm{in}(C_{n,m})$, and $\mathrm{out}(v) = \langle e_i \rangle_{i=n+1}^{n+m} = \mathrm{out}(C_{n,m})$. For this to be a decoration we must assign

$$D_0(e_i) = \begin{cases} a_i & i \in [1, n] \\ b_{i-n} & i \in [n+1, n+m]. \end{cases}$$

We remark that any other choice of order amounts to a graph automorphism given by permuting the edges. The reader should also note that every decoration of the corolla $C_{n,m}$ is related to $\mathfrak{g}(x)$ for some x.

A.1. **Definition of prop structure on the collection Γ.** We will now describe a prop $F(\mathcal{X})$ whose set of morphisms is Γ and whose color set $\mathrm{Col}(F(\mathcal{X}))$ is X_0. Let $\langle a_i \rangle_{i=1}^n$ and $\langle b_k \rangle_{k=1}^m$ be lists of elements of X_0 and then define

$$F(\mathcal{X})(\langle a_i \rangle_{i=1}^n ; \langle b_k \rangle_{k=1}^m)$$

to be the set of all equivalence classes of decorations such that

$$\mathbb{M}D_0(\mathrm{in}(G)) = \langle a_i \rangle_{i=1}^n \qquad \text{and}$$
$$\mathbb{M}D_0(\mathrm{out}(G)) = \langle b_k \rangle_{k=1}^m .$$

For each $c \in \mathrm{Col}(F(\mathcal{X})) = X_0$, we will define the identity elements id_c, as decorations of the graph with $E = \{*\}$ and $V = \varnothing$.

Vertical composition

$$F(\mathcal{X})(\langle a_i \rangle_{i=1}^n ; \langle b_k \rangle_{k=1}^m) \times F(\mathcal{X})(\langle c_j \rangle_{j=1}^p ; \langle a_i \rangle_{i=1}^n) \to F(\mathcal{X})(\langle c_j \rangle_{j=1}^p ; \langle b_k \rangle_{k=1}^m)$$

is defined as follows. Let $\mathfrak{g}^1, \mathfrak{g}^2$ be a pair of decorations we wish to compose. We construct a graph H with vertex set $V^1 \sqcup V^2$. If $\mathrm{in}(G^1) = (e_1^1, \dots, e_n^1)$ and $\mathrm{out}(G^2) = (e_1^2, \dots, e_n^2)$, then the edge set of H is $(E^1 \sqcup E^2)/\sim$ where the $\sim$ is given by $e_i^1 \sim e_i^2$. Before we had $t^2(e_i^2) = *$ and $s^1(e_i^1) = *$, but in the new graph H we define $t^{1+2}(e_i^2) \coloneqq t^1(e_i^1)$ and $s^{1+2}(e_i^1) \coloneqq s^2(e_i^2)$. Let G^{1+2} be the previously chosen representative of the isomorphism class of H, which will be the underlying graph of the composition $\mathfrak{g}^1 \circ_v \mathfrak{g}^2 = \mathfrak{g}^{1+2}$.

Now that we have defined the graph which is to be decorated, we can define the decoration. First, we set $D_1^{1+2} = D_1^1 \sqcup D_1^2$. The function $D_0^1 \sqcup D_0^2$ induces a function $D_0^{1+2} : E^{1+2} \to X_0$ since $\mathbb{M}D_0(\mathrm{in}(G^1)) = \mathbb{M}D_0(\mathrm{out}(G^2))$. The ordering on the input and output edges of vertices are the same as those from $\mathfrak{g}^1$ and $\mathfrak{g}^2$: if, say, $v \in V_1$ and $\mathrm{in}(v) =$

$(a_1, \ldots, a_\ell)$ in $\mathfrak{g}^1$, then in $\mathfrak{g}^{1+2}$ we have $\mathrm{in}(v) = ([a_1], \ldots, [a_\ell])$ where $[-]$ denotes the equivalence relation on $E^1 \sqcup E^2$. We define the order of $\mathrm{in}(G^{1+2})$ to be the same as the order on $\mathrm{in}(G^2)$ and the order of $\mathrm{out}(G^{1+2})$ to be the same as the order on $\mathrm{out}(G^1)$.

We observe that an automorphism of G^1 or G^2 induces an automorphism of G^{1+2}, and thus we see that the definition of $\mathfrak{g}^1 \circ_v \mathfrak{g}^2$ does not depend on the choice of representatives in $\widetilde{\Gamma}$, but only the equivalence classes in Γ.

Proposition 42. *The vertical composition defined in the previous paragraph is associative.*

Proof. We wish to show that $(\mathfrak{g}^1 \circ_v \mathfrak{g}^2) \circ_v \mathfrak{g}^3 = \mathfrak{g}^{1+2} \circ_v \mathfrak{g}^3 = \mathfrak{g}^{(1+2)+3}$ is equal to $\mathfrak{g}^1 \circ_v (\mathfrak{g}^2 \circ_v \mathfrak{g}^3) = \mathfrak{g}^1 \circ_v \mathfrak{g}^{2+3} = \mathfrak{g}^{1+(2+3)}$. It is clear that $V^{(1+2)+3} = V^{1+(2+3)}$ since disjoint union is associative. When forming $E^{(1+2)+3}$ we first identify $\mathrm{out}(G^2)$ and $\mathrm{in}(G^1)$ and then identify $\mathrm{out}(G^3)$ with $\mathrm{in}(G^{1+2}) = \mathrm{in}(G^2)$. This is the same as first identifying $\mathrm{in}(G^2)$ with $\mathrm{out}(G^3)$ and then identifying $\mathrm{out}(G^{2+3}) = \mathrm{out}(G^2)$ with $\mathrm{in}(G^1)$, so $E^{(1+2)+3} = E^{1+(2+3)}$. At this point we see that $G^{(1+2)+3}$ and $G^{1+(2+3)}$ as defined above have the same source and target maps.

The decorations $\mathfrak{g}^{(1+2)+3}$ and $\mathfrak{g}^{1+(2+3)}$ are identical given the way they are induced from $\mathfrak{g}^1$, $\mathfrak{g}^2$, and $\mathfrak{g}^3$. □

The horizontal composition $\mathfrak{g}^1 \circ_h \mathfrak{g}^2 = \mathfrak{g}^{1+2}$ of two decorations $\mathfrak{g}^1$ and $\mathfrak{g}^2$ is given by disjoint union of the underlying graphs with decoration given by disjoint union. The only thing we must declare is that $\mathrm{in}(G^{1+2})$ is ordered so that $\mathrm{in}(G^1) < \mathrm{in}(G^2)$ and $\mathrm{out}(G^{1+2})$ is ordered so that $\mathrm{out}(G^1) < \mathrm{out}(G^2)$. This is clearly associative since (ordered) disjoint union is associative.

The symmetric action is given by the action on (ordered) input and output edges of the graph. If $\mathfrak{g}$ has input edges $\mathrm{in}(G) \in \mathbb{M}E$ then $\sigma^*\mathfrak{g}$ has exactly the same structure as $\mathfrak{g}$ except that we give the input vertices of G the order $\mathrm{in}(G) \cdot \sigma$. Similarly, $\tau_*\mathfrak{g}$ just has the modified order $\tau \cdot \mathrm{out}(G)$ on output vertices.

Lemma 43. *The vertical composition is compatible with the symmetric group actions in the sense that*

$$\mathfrak{g}^1 \circ_v (\sigma_* \mathfrak{g}^2) = (\sigma^* \mathfrak{g}^1) \circ_v \mathfrak{g}^2 \tag{44}$$

$$\sigma^*(\mathfrak{g}^1 \circ_v \mathfrak{g}^2) = \mathfrak{g}^1 \circ_v (\sigma^* \mathfrak{g}^2) \tag{45}$$

$$\tau_*(\mathfrak{g}^1 \circ_v \mathfrak{g}^2) = (\tau_* \mathfrak{g}^1) \circ_v \mathfrak{g}^2. \tag{46}$$

Proof. If $\mathfrak{g}$ is a decoration of G, then we will write $\mathrm{in}(\mathfrak{g})$ for $\mathrm{in}(G)$ and $\mathrm{out}(\mathfrak{g})$ for $\mathrm{out}(G)$ since we will be dealing with so many orders in this proof.

For (44) it is enough to check that the underlying graphs are the same. But we form graph for the left-hand decoration by identifying the ordered sets $\mathrm{in}(\mathfrak{g}^1) = (e^1_1, \ldots, e^1_n)$ and

$$\mathrm{out}(\sigma_* \mathfrak{g}^2) = \sigma \cdot \mathrm{out}(\mathfrak{g}^2) = (e^2_{\sigma^{-1}(1)}, \ldots, e^2_{\sigma^{-1}(n)})$$

and we form the graph for the right-hand decoration by identifying the ordered sets $\mathrm{out}(\mathfrak{g}^2) = (e^2_1, \ldots, e^2_n)$ and

$$\mathrm{in}(\sigma^* \mathfrak{g}^1) = \mathrm{in}(\mathfrak{g}^1) \cdot \sigma = (e^1_{\sigma(1)}, \ldots, e^1_{\sigma(n)}).$$

In the first case we are identifying $e^1_i \sim e^2_{\sigma^{-1}(i)}$ while in the second we are identifying $e^2_i \sim e^1_{\sigma(i)}$; since this is the same identification we see that the underlying graph of both compositions is the same.

The only part of a decoration that σ^* changes is the order on $\mathrm{in}(G)$, so

$$\mathrm{in}(\sigma^*(\mathfrak{g}^1 \circ_v \mathfrak{g}^2)) = \mathrm{in}(\mathfrak{g}^1 \circ_v \mathfrak{g}^2) \cdot \sigma = \mathrm{in}(\mathfrak{g}^2) \cdot \sigma = \mathrm{in}(\sigma^* \mathfrak{g}^2) = in(\mathfrak{g}^1 \circ_v (\sigma^* \mathfrak{g}^2))$$

gives (45). Equation (46) follows similarly. □

Proposition 47. *The definitions above make $F(\mathcal{X})$ into a prop.*

Proof. It is immediate from construction of $\circ_v$ that (1) holds.

It remains to show that the interchange of the horizontal and vertical compositions holds

$$(\mathfrak{g}^1 \circ_v \mathfrak{g}^2) \circ_h (\mathfrak{g}^3 \circ_v \mathfrak{g}^4) = (\mathfrak{g}^1 \circ_h \mathfrak{g}^3) \circ_v (\mathfrak{g}^2 \circ_h \mathfrak{g}^4).$$

We examine the underlying graphs on each side. They have the same vertex set $V_1 \sqcup V_2 \sqcup V_3 \sqcup V_4$. The edge set on each side is $E_1 \sqcup E_2 \sqcup E_3 \sqcup E_4$ with some of the edges identified. On the left, we identify the ordered sets $\mathrm{in}(G^1) \sim \mathrm{out}(G^2)$ as well as $\mathrm{in}(G^3) \sim \mathrm{out}(G^4)$. On the right we identify $(\mathrm{in}(G^1), \mathrm{in}(G^3)) \sim (\mathrm{out}(G^2), \mathrm{out}(G^4))$. These identifications are the same, and we see that both sides have the same underlying graph. As the structure of the compositions is just induced from that on the individual decorations, these decorations are the same as well.

We showed that the vertical composition is compatible with the symmetric group actions in the previous lemma. The compatibility of horizontal composition with symmetric group actions as in (4) and (5) is easy to see from the definition of $\circ_h$. Finally, the interchange rule $\sigma^* \tau_* = \tau_* \sigma^*$ is obvious since σ^* only modifies the order on $\mathrm{in}(G)$ and τ_* only modifies the order on $\mathrm{out}(G)$.

□

Proposition 48. *The assignment $F : \mathbf{Mega} \to \mathbf{Prop}$ is a functor.*

Proof. Suppose that $f = (f_1, f_0) : \mathcal{X} \to \mathcal{Y}$ is a morphism of megagraphs, where $f_i : X_i \to Y_i$; we wish to describe $F(f)$. Let $\mathfrak{g}$ be in $F(\mathcal{X})$, i.e. $\mathfrak{g}$ is a decoration of a graph G by $\mathcal{X}$. We then have a decoration of G by $\mathcal{Y}$, which we denote $\mathfrak{g}^{\mathfrak{f}}$, with $D_0^f = f_0 \circ D_0$ and $D_1^f = f_1 \circ D_1$, and the order structures are the same. We define $F(f)\mathfrak{g} := \mathfrak{g}^{\mathfrak{f}}$. Since composition of set functions is associative, $F(f \circ g) = F(f) \circ F(g)$. □

A.2. **The functors F and U are an adjoint pair.** We now turn to adjointness. We will show that

$$\mathrm{Hom}_{\mathbf{Prop}}(F(\mathcal{X}), \mathcal{T}) = \mathrm{Hom}_{\mathbf{Mega}}(\mathcal{X}, U(\mathcal{T})),$$

where U is the forgetful functor. As mentioned above, we have inclusions

$$X_0 \hookrightarrow \Gamma$$
$$X_1 \hookrightarrow \Gamma$$

so we must show that given $f : \mathcal{X} \to U(\mathcal{T})$ there *exists* a *unique* map of props $K : F(\mathcal{X}) \to \mathcal{T}$ such that $K|_{X_0} = f_0$ and $K|_{X_1} = f_1$. Existence and uniqueness will be shown at the same time through an inductive process. Since elements of Γ have an underlying graph, we can define a filtration $\Gamma_0 \subset \Gamma_1 \subset \dots$ based on the order of the underlying graph; we define K as a limit of partially defined functors $K_p : \Gamma_p \to \mathcal{T}$. Since we must choose an order on inputs and outputs in order to define $\mathfrak{g}(x)$, we fix this choice for all corollas G in the process of defining K_p. In what follows, we will show that each K_p can then be defined in exactly one way.

Here is what we require from $K_p : \Gamma_p \to \mathcal{T}$:

- Compatibility with f:
  - If $c \in X_0 = \mathrm{Col}(F(\mathcal{X}))$, then $K_p(\mathrm{id}_c) = \mathrm{id}_{f_0(c)}$.
  - If $x \in X_1$ then $K_p(\mathfrak{g}(x)) = f_1(x)$.
- The K_p constitute a filtration: $K_p|_{\Gamma_{p-1}} = K_{p-1}$.
- Partial functoriality:
  - If $\mathfrak{g} \in \Gamma_p$ and $\mathfrak{g} = \mathfrak{g}^1 \circ_h \mathfrak{g}^2$, then $K_p(\mathfrak{g}) = K_p(\mathfrak{g}^1) \circ_h K_p(\mathfrak{g}^2)$.
  - If $\mathfrak{g} \in \Gamma_p$ and $\mathfrak{g} = \mathfrak{g}^1 \circ_v \mathfrak{g}^2$, then $K_p(\mathfrak{g}) = K_p(\mathfrak{g}^1) \circ_v K_p(\mathfrak{g}^2)$.
  - If $\mathfrak{g} \in \Gamma_p$ then $K_p(\sigma^* \tau_* \mathfrak{g}) = \sigma^* \tau_* K_p(\mathfrak{g})$.

The graphs with zero vertices are just a collection of non-incident edges; we define

$$K_0(\sigma^* \tau_* (\mathrm{id}_{c_1} \circ_h \cdots \circ_h \mathrm{id}_{c_k})) = \sigma^* \tau_* (\mathrm{id}_{f_0(c_1)} \circ_h \cdots \circ_h \mathrm{id}_{f_0(c_k)}).$$

We define $K_1|_{\Gamma_0} = K_0$, and

$$\begin{aligned} &K_1(\sigma^*\tau_*(\mathrm{id}_{c_1} \circ_h \cdots \circ_h \mathrm{id}_{c_k} \circ_h \mathfrak{g}(x) \circ_h \mathrm{id}_{c_{k+1}} \circ_h \cdots \circ_h \mathrm{id}_{c_\ell})) \\ &\quad = \sigma^*\tau_*(\mathrm{id}_{f_0(c_1)} \circ_h \cdots \circ_h \mathrm{id}_{f_0(c_k)} \circ_h f_1(x) \circ_h \mathrm{id}_{f_0(c_{k+1})} \circ_h \cdots \circ_h \mathrm{id}_{f_0(c_\ell)}), \end{aligned}$$

which covers all decorations on order 1 graphs. Note that K_0 and K_1 are well-defined and satisfy the above conditions.

We now build K_p using K_{p-1}. For this, we need a suitable collection of subgraphs. By a *subgraph* of $G = (E, V)$ we mean a pair of subsets $E^0 \subset E$ and $V^0 \subset V$. These determine a graph with source and target maps induced by those from G, i.e. they are defined by

$$s^0, t^0 : E^0 \hookrightarrow E \xrightarrow{s,t} V_+ \twoheadrightarrow V^0_+$$

so that $s^0(e) = s(e)$ and $t^0(e) = t(e)$ whenever possible (the second arrow is given by $(V \setminus V^0) \mapsto *$). An *admissible subgraph* is a subgraph satisfying the condition that if $v \in V^0$ then any edge incident to v is in E^0.

Notice that if G^0 is an admissible subgraph of G, a decoration $\mathfrak{g}$ on G nearly induces a decoration on G^0. The only thing that is missing is an order on the input and output edges.

Definition 49. A *decomposition* of a graph G is a collection of admissible subgraphs $G^1, \dots, G^n$ so that $V^i \cap V^j = \varnothing$ when $i \neq j$, $V = \bigcup V^i$, and $E = \bigcup E^i$. A *proper* decomposition is one in which each V^i is nonempty.

Notice that the intersection of two decompositions is again a decomposition, where by intersection of $G^1, \dots, G^n$ and $\bar{G}^1, \dots, \bar{G}^m$ we mean

$$G^1 \cap \bar{G}^1, G^1 \cap \bar{G}^2, \dots, G^n \cap \bar{G}^{m-1}, G^n \cap \bar{G}^m.$$

We now isolate a particularly interesting type of decomposition. A *vertical decomposition* of G is a decomposition $G^1, \dots, G^n$ so that

$$\begin{aligned} \mathrm{out}(G^1) &= \mathrm{out}(G) \\ \mathrm{in}(G^n) &= \mathrm{in}(G) \\ \text{and } \mathrm{out}(G^i) &= \mathrm{in}(G^{i-1}) \text{ for } 2 \le i \le n. \end{aligned}$$

If $\mathfrak{g}$ is a decoration on G, then a *vertical decomposition of* $\mathfrak{g}$ is a vertical decomposition of G together with a choice of orders on $\mathrm{out}(G^i)$ and $\mathrm{in}(G^i)$ for $i \in [1, n]$ so that the above equalities hold *as ordered sets.* The data of a vertical decomposition thus gives decorations $\mathfrak{g}^{\mathfrak{i}}$ on G^i, and, moreover, $\mathfrak{g} = \mathfrak{g}^{1} \circ_v \cdots \circ_v \mathfrak{g}^{\mathfrak{n}}$.

Lemma 50. *Suppose that G^0 is an admissible subgraph of G and G^1, G^2 is a vertical decomposition of G. Define subgraphs G^{01} and G^{02} by*

$$V^{01} = V^0 \cap V^1 \qquad V^{02} = V^0 \cap V^2$$
$$E^{01} = (E^0 \cap E^1) \cup (\mathrm{out}(G^0)) \qquad E^{02} = (E^0 \cap E^2) \cup (\mathrm{in}(G^0)).$$

Then G^{01}, G^{02} is a vertical decomposition of G^0.

Proof. To show that G^{0i} is admissible, suppose that $v \in V^{0i} = V^0 \cap V^i$. If e is incident to v in G^0, then e is incident to v in G, so by admissibility of G^i and G^0 we have $v \in E^0 \cap E^i \subset E^{0i}$.

We now show that G^{01}, G^{02} constitutes a decomposition of G^0. A vertex $v \in V^0$ must either be in V^1 or V^2 since G^1, G^2 is a decomposition of G, so $v \in V^0 \cap V^1 = V^{01}$ or $V^0 \cap V^2 = V^{02}$. The same argument holds for edges. We also need to check disjointness of V^{01} and V^{02}, but $V^{01} \cap V^{02} = V^0 \cap V^1 \cap V^2 = V^0 \cap \varnothing$.

Showing that this is a *vertical* decomposition is a little bit more work. Remember that we are trying to show that $\mathrm{out}(G^{01}) = \mathrm{out}(G^0)$, and $\mathrm{in}(G^{01}) = \mathrm{out}(G^{02})$, and $\mathrm{in}(G^{02}) = \mathrm{in}(G^0)$. We will just show that $\mathrm{out}(G^{01}) = \mathrm{out}(G^0)$, and $\mathrm{out}(G^{02}) \subset \mathrm{in}(G^{01})$ since the proofs of the other equality other inclusion are dual.

The inclusion $\mathrm{out}(G^0) \subset \mathrm{out}(G^{01})$ is part of the definition of G^{01}. In order for $e \in \mathrm{out}(G^{01})$, we must have $t^0(e) = *$ or $t^1(e) = *$. If $t^1(e) = *$, then $e \in \mathrm{out}(G^1) = \mathrm{out}(G)$, so $t(e) = *$. Thus $t^0(e) = *$ since G^0 is a subgraph of G. So $e \in \mathrm{out}(G^{01})$ implies that $t^0(e) = *$, so $e \in \mathrm{out}(G^0)$.

We now wish to show that $\mathrm{out}(G^{02}) \subset \mathrm{in}(G^{01})$. If $e \in \mathrm{out}(G^{02})$, then $t^{02}(e) = *$ which means either $t^0(e) = *$ or $t^2(e) = *$. Let us first check that such an e is actually in E^{01}. If $t^0(e) = *$, then $e \in \mathrm{out}(G^0) \subset E^{01}$. If $t^2(e) = *$, then $e \in \mathrm{out}(G^2) = \mathrm{in}(G^1) \subset E^1$. Therefore $t^2(e) = *$ implies $e \in E^0 \cap E^1 \subset E^{01}$.

Now that we know that $\mathrm{out}(G^{02}) \subset E^{01}$, we must show that $s^{01}(e) = *$ for $e \in \mathrm{out}(G^{02})$. Towards this end, we examine

$$\begin{aligned}\mathrm{out}(G^{02}) &= [E^0 \cap \mathrm{out}(G^2)] \cup [\mathrm{in}(G^0) \cap \mathrm{out}(G^0)] \cup [E^2 \cap \mathrm{out}(G^0)] \\ &= [E^0 \cap \mathrm{in}(G^1)] \cup [\mathrm{in}(G^0) \cap \mathrm{out}(G^0)] \cup [E^2 \cap \mathrm{out}(G^0)],\end{aligned}$$

which we now prove by showing the top equality; the bottom follows since G^1, G^2 is a vertical decomposition. The inclusion from right to left is easy, since any element e in the right hand set has $t^0(e) = *$ or $t^2(e) = *$, so $t^{02}(e) = *$. If $e \in \mathrm{out}(G^{02})$ and $t^2(e) = *$ then $e \in \mathrm{out}(G^2)$ so we are in the first set on the right. If $t^0(e) = *$ then $e \in \mathrm{out}(G^0)$. Since $e \in E^{02}$ we either have $e \in E^2$ or $e \in \mathrm{in}(G^0)$, so e is contained in one of the two rightmost sets. Thus the left hand side is contained in the right hand side, and we have shown equality.

It is clear that $E^0 \cap \mathrm{in}(G^1)$ and $\mathrm{in}(G^0) \cap \mathrm{out}(G^0)$ are contained in $\mathrm{in}(G^{01})$. Our remaining work then is to show that $E^2 \cap \mathrm{out}(G^0) \subset \mathrm{in}(G^{01})$.

We first make an observation. If $e \in E^2$ then $s(e) \notin V^1$. If it were, then $s^2(e) = *$ by disjointness of V^1 and V^2, so $s(e) = *$ since $\mathrm{in}(G^2) = \mathrm{in}(G)$. But we cannot have both $s(e) = *$ and $s(e) \in V^1$.

Now consider $e \in E^2 \cap \mathrm{out}(G^0)$. We either have $s(e) \in V^2$ or $s(e) = *$. In both cases we have $s^1(e) = *$, so $e \in \mathrm{in}(G^{01})$. Thus we have shown $\mathrm{out}(G^{02}) \subset \mathrm{in}(G^{01})$, which completes the proof. □

Let us now consider two vertical decompositions G^1, G^2 and G^3, G^4 of the same graph G. Using the same notation from the previous proposition, we have decompositions G^{13}, G^{14} and G^{23}, G^{24} of G^1 and G^2 respectively, and decompositions G^{31}, G^{32} and G^{41}, G^{42} of G^3 and G^4. Notice that $V^{ij} = V^{ji}$ for $i = 1, 2$ and $j = 3, 4$.

As for the edge sets, we have

$$\begin{aligned} E^{13} &= (E^1 \cap E^3) \cup \mathrm{out}(G^1) \\ &= (E^1 \cap E^3) \cup \mathrm{out}(G) \\ &= (E^3 \cap E^1) \cup \mathrm{out}(G^3) = E^{31} \end{aligned}$$

and similarly $E^{24} = E^{42}$. Furthermore,

$$E^{14} \cup E^{23} = (E^1 \cap E^4) \cup (E^2 \cap E^3) = E^{41} \cup E^{32}; \tag{51}$$

to see this, it is enough to check that E^{14}, E^{23}, E^{41}, and E^{32} are subsets of $(E^1 \cap E^4) \cup (E^2 \cap E^3)$. To show, for example, that $(E^1 \cap E^4) \cup \mathrm{in}(G^1) = E^{14} \subset (E^1 \cap E^4) \cup (E^2 \cap E^3)$, take an edge $e \in \mathrm{in}(G^1) = \mathrm{out}(G^2) \subset E^1 \cap E^2$. Since e is in G and G^3, G^4 is a decomposition, either $e \in E^3$ or $e \in E^4$, which implies that

$$e \in (E^1 \cap E^2) \cap E^3 \subset E^2 \cap E^3 \text{ or } e \in (E^1 \cap E^2) \cap E^4 \subset (E^1 \cap E^4),$$

hence $e \in (E^1 \cap E^4) \cup (E^2 \cap E^3)$. The proofs that $E^{23}, E^{41}, E^{32} \subset (E^1 \cap E^4) \cup (E^2 \cap E^3)$ are similar, so (51) holds and we thus have identical vertical compositions

$$\begin{gathered} G^{13}, (G^{14} \cup G^{23}), G^{24} \\ G^{31}, (G^{32} \cup G^{41}), G^{42}. \end{gathered} \tag{52}$$

Note that there are no edges of G between vertices in V^{14} and V^{23}, hence, if G is *connected* then $G^{14} \cup G^{23}$ has strictly fewer vertices than G.

Proposition 53. *Suppose that G is a connected graph of order p, $\mathfrak{g}$ is a decoration on G, and K_{p-1} is defined and satisfies the required* *properties. If* $\mathfrak{g}^1 \circ_v \mathfrak{g}^2$ *and* $\mathfrak{g}^3 \circ_v \mathfrak{g}^4$ *are two proper vertical decompositions of* $\mathfrak{g}$*, then*

$$K_{p-1}(\mathfrak{g}^1) \circ_v K_{p-1}(\mathfrak{g}^2) = K_{p-1}(\mathfrak{g}^3) \circ_v K_{p-1}(\mathfrak{g}^4).$$

Definition 54. If G is connected, we define

$$K_p(\mathfrak{g}) := K_{p-1}(\mathfrak{g}^1) \circ_v K_{p-1}(\mathfrak{g}^2)$$

for any proper vertical decomposition $\mathfrak{g}^1, \mathfrak{g}^2$ of $\mathfrak{g}$.

Remark 55. In the setting of the previous definition, we have

$$\begin{aligned} K_p(\sigma^* \tau_* \mathfrak{g}) = K_{p-1}(\sigma^* \mathfrak{g}^1) \circ_v K_{p-1}(\tau_* \mathfrak{g}^2) &= \sigma^* K_{p-1}(\mathfrak{g}^1) \circ_v \tau_* K_{p-1}(\mathfrak{g}^2) \\ &= \sigma^* \tau_* \left(K_{p-1}(\mathfrak{g}^1) \circ_v K_{p-1}(\mathfrak{g}^2)\right) = \sigma^* \tau_* K_p(\mathfrak{g}), \end{aligned}$$

so we see that compatibility with symmetric group actions follows from the same property on K_{p-1}.

Proof of Proposition 53. We will write $G^\dagger$ for $G^{14} \cup G^{23}$. Choose an order on $\mathrm{in}(G^{13})$ and on $\mathrm{out}(G^{42})$, and use the appropriate orders on everywhere else:

$$\begin{aligned} \mathrm{out}(G^{13}) &= \mathrm{out}(G) & \mathrm{in}(G^{42}) &= \mathrm{in}(G) \\ \mathrm{in}(G^{14}) &= \mathrm{in}(G^1) & &= \mathrm{out}(G^{23}) = \mathrm{out}(G^2) \\ \mathrm{in}(G^{32}) &= \mathrm{in}(G^3) & &= \mathrm{out}(G^{41}) = \mathrm{out}(G^4) \\ \mathrm{in}(G^\dagger) &= \mathrm{out}(G^{42}) & \mathrm{out}(G^\dagger) &= \mathrm{in}(G^{13}). \end{aligned}$$

Using (52), we now have an additional vertical decomposition

$$\mathfrak{g} = \mathfrak{g}^{13} \circ_v \mathfrak{g}^\dagger \circ_v \mathfrak{g}^{24}$$

of $\mathfrak{g}$, as well decompositions

$$\begin{aligned} \mathfrak{g}^1 &= \mathfrak{g}^{13} \circ_v \mathfrak{g}^{14} & \mathfrak{g}^2 &= \mathfrak{g}^{23} \circ_v \mathfrak{g}^{24} \\ \mathfrak{g}^3 &= \mathfrak{g}^{31} \circ_v \mathfrak{g}^{32} & \mathfrak{g}^4 &= \mathfrak{g}^{41} \circ_v \mathfrak{g}^{42} \\ \mathfrak{g}^\dagger &= \mathfrak{g}^{14} \circ_v \mathfrak{g}^{23} = \mathfrak{g}^{32} \circ_v \mathfrak{g}^{41}. \end{aligned}$$

The decorations $\mathfrak{g}^\dagger$ and each $\mathfrak{g}^i$ have fewer than p vertices using the fact that G is connected and properness. From now on we will only write down decorations with fewer than p vertices, so we can safely write K instead of K_{p-1}.

$$\begin{aligned} K(\mathfrak{g}^{13}) \circ_v K(\mathfrak{g}^\dagger) &= K(\mathfrak{g}^{13}) \circ_v K(\mathfrak{g}^{14}) \circ_v K(\mathfrak{g}^{23}) \\ &= K(\mathfrak{g}^{1}) \circ_v K(\mathfrak{g}^{23}) \\ K(\mathfrak{g}^{31}) \circ_v K(\mathfrak{g}^\dagger) &= K(\mathfrak{g}^{31}) \circ_v K(\mathfrak{g}^{32}) \circ_v K(\mathfrak{g}^{41}) \\ &= K(\mathfrak{g}^{3}) \circ_v K(\mathfrak{g}^{41}) \end{aligned}$$

These are equal, so we compose with $K(\mathfrak{g}^{24})$ and find

$$\begin{aligned} K(\mathfrak{g}^{1}) \circ_v K(\mathfrak{g}^{23}) \circ_v K(\mathfrak{g}^{24}) &= K(\mathfrak{g}^{1}) \circ_v K(\mathfrak{g}^{2}) \\ K(\mathfrak{g}^{3}) \circ_v K(\mathfrak{g}^{41}) \circ_v K(\mathfrak{g}^{42}) &= K(\mathfrak{g}^{3}) \circ_v K(\mathfrak{g}^{4}) \end{aligned}$$

are equal as well. □

We now move on to arbitrary decompositions. At the moment, we have only defined K_p for *connected* decompositions on p vertices and for arbitrary decompositions on *fewer* than p vertices.

Let $\mathfrak{g}^1, \dots, \mathfrak{g}^{\mathfrak{k}}$ be decorations whose underlying graphs are connected, so that $\sum |V^i| = p$. We define

$$K_p(\mathfrak{g}^1 \circ_h \cdots \circ_h \mathfrak{g}^{\mathfrak{k}}) = K_p(\mathfrak{g}^1) \circ_h \cdots \circ_h K_p(\mathfrak{g}^{\mathfrak{k}}),$$

which is well-defined since K_p is well-defined whenever the underlying graph is connected. In this same situation we assign

$$K_p(\sigma^* \tau_* (\mathfrak{g}^1 \circ_h \cdots \circ_h \mathfrak{g}^{\mathfrak{k}})) = \sigma^* \tau_* K_p(\mathfrak{g}^1 \circ_h \cdots \circ_h \mathfrak{g}^{\mathfrak{k}}).$$

Let us see that this is well-defined. Suppose that

$$\sigma^* \tau_* (\mathfrak{g}^1 \circ_h \cdots \circ_h \mathfrak{g}^{\mathfrak{k}}) = \bar\sigma^* \bar\tau_* (\bar{\mathfrak{g}}^1 \circ_h \cdots \circ_h \bar{\mathfrak{g}}^{\mathfrak{k}}),$$

and move the symmetric group actions to the other side. We then see

$$\begin{aligned} \mathfrak{g}^1 \circ_h \cdots \circ_h \mathfrak{g}^{\mathfrak{k}} &= (\bar\sigma \sigma^{-1})^* (\tau^{-1} \bar\tau)_* (\bar{\mathfrak{g}}^1 \circ_h \cdots \circ_h \bar{\mathfrak{g}}^{\mathfrak{k}}) \\ &= \sigma_1^* \tau_*^1 \bar{\mathfrak{g}}^{\mathfrak{a}_1} \circ_h \cdots \circ_h \sigma_k^* \tau_*^k \bar{\mathfrak{g}}^{\mathfrak{a}_{\mathfrak{k}}}, \end{aligned}$$

whence

$$\begin{aligned} K_p(\mathfrak{g}^1 \circ_h \cdots \circ_h \mathfrak{g}^{\mathfrak{k}}) &= K_p(\sigma_1^* \tau_*^1 \bar{\mathfrak{g}}^{\mathfrak{a}_1} \circ_h \cdots \circ_h \sigma_k^* \tau_*^k \bar{\mathfrak{g}}^{\mathfrak{a}_{\mathfrak{k}}}) \\ &= K_p(\sigma_1^* \tau_*^1 \bar{\mathfrak{g}}^{\mathfrak{a}_1}) \circ_h \cdots \circ_h K_p(\sigma_k^* \tau_*^k \bar{\mathfrak{g}}^{\mathfrak{a}_{\mathfrak{k}}}) \\ &= \sigma_1^* \tau_*^1 K_p(\bar{\mathfrak{g}}^{\mathfrak{a}_1}) \circ_h \cdots \circ_h \sigma_k^* \tau_*^k K_p(\bar{\mathfrak{g}}^{\mathfrak{a}_{\mathfrak{k}}}) \\ &= (\bar\sigma \sigma^{-1})^* (\tau^{-1} \bar\tau)_* (K_p(\bar{\mathfrak{g}}^1) \circ_h \cdots \circ_h K_p(\bar{\mathfrak{g}}^{\mathfrak{k}})). \end{aligned}$$

Applying $\sigma^* \tau_*$ to both sides gives

$$\sigma^* \tau_* K_p(\mathfrak{g}^1 \circ_h \cdots \circ_h \mathfrak{g}^{\mathfrak{k}}) = \bar\sigma^* \bar\tau_* K_p(\bar{\mathfrak{g}}^1 \circ_h \cdots \circ_h \bar{\mathfrak{g}}^{\mathfrak{k}}).$$

Summary of Proof of Theorem 14. We wished to show

$$\mathrm{Hom}_{\mathbf{Prop}}(F(\mathcal{X}), \mathcal{T}) = \mathrm{Hom}_{\mathbf{Mega}}(\mathcal{X}, U(\mathcal{T})),$$

which amounted to showing that $f : \mathcal{X} \to U(\mathcal{T})$ induces a unique prop map $K : F(\mathcal{X}) \to \mathcal{T}$ with $K|_{X_0} = f_0$ and $K|_{X_1} = f_1$. We filtered the set of all morphisms, Γ, by the number of vertices of the underlying graph of the decoration. We then built the prop map K inductively, with partially defined prop maps $K_p : \Gamma_p \to \mathcal{T}$.

Since K_p needed to agree with K_{p-1} whenever possible, we could *define* $K_p(\mathfrak{g})$ to be $K_{p-1}(\mathfrak{g}^1) \circ_v K_{p-1}(\mathfrak{g}^2)$ or $K_{p-1}(\mathfrak{g}^1) \circ_h K_{p-1}(\mathfrak{g}^2)$ wherever this was possible and made sense. This entire section was devoted to showing that such an assignment was well-defined. The properties essentially came for free. □

References

1. Michael Barr and Charles Wells, *Toposes, triples and theories*, Repr. Theory Appl. Categ. (2005), no. 12, x+288, Corrected reprint of the 1985 original [MR0771116]. MR 2178101
2. Clemens Berger and Ieke Moerdijk, *Resolution of coloured operads and rectification of homotopy algebras*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 31–58. MR 2342815 (2008k:18008)
3. J. M. Boardman and R. M. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin, 1973. MR 0420609 (54 #8623a)
4. Denis-Charles Cisinski and Ieke Moerdijk, *Dendroidal sets as models for homotopy operads*, J. Topol. **4** (2011), no. 2, 257–299, Available at `http://dx.doi.org/10.1112/jtopol/jtq039`. MR 2805991
5. ———, *Dendroidal Segal spaces and ∞-operads*, J. Topol. **6** (2013), no. 3, 675–704, Available at `http://dx.doi.org/10.1112/jtopol/jtt004`.
6. ———, *Dendroidal sets and simplicial operads*, J. Topol. **6** (2013), no. 3, 705–756, Available at `http://dx.doi.org/10.1112/jtopol/jtt006`.
7. Hess K. Voronov A. Cohen, R.L, *String topology and cyclic homology*, Adv. Courses in Math CRM Barcelona., Birkhäuser, Basel, 2006, Translated from the Japanese by M. Reid.
8. A. D. Elmendorf and M. A. Mandell, *Rings, modules, and algebras in infinite loop space theory*, Adv. Math. **205** (2006), no. 1, 163–228. MR 2254311 (2007g:19001)
9. ———, *Permutative categories, multicategories and algebraic K-theory*, Algebr. Geom. Topol. **9** (2009), no. 4, 2391–2441. MR 2558315 (2011a:19002)
10. Yael Frégier, Martin Markl, and Donald Yau, *The L_∞-deformation complex of diagrams of algebras*, New York J. Math. **15** (2009), 353–392. MR 2530153 (2011b:16039)
11. Benoit Fresse, *Props in model categories and homotopy invariance of structures*, Georgian Math. J. **17** (2010), no. 1, 79–160. MR 2640648 (2011h:18011)

12. Giorgio Gallo, Giustino Longo, Stefano Pallottino, and Sang Nguyen, *Directed hypergraphs and applications*, Discrete Appl. Math. **42** (1993), no. 2-3, 177–201, Combinatorial structures and algorithms. MR 1217096 (94e:05187)
13. Victor Ginzburg and Mikhail Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1994), no. 1, 203–272. MR 1301191 (96a:18004)
14. Philip Hackney and Marcy Robertson, *The homotopy theory of props*.
15. Mark W. Johnson and Donald Yau, *On homotopy invariance for algebras over colored PROPs*, J. Homotopy Relat. Struct. **4** (2009), no. 1, 275–315. MR 2559644 (2010j:18014)
16. Tom Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004. MR 2094071 (2005h:18030)
17. Saunders Mac Lane, *Categorical algebra*, Bull. Amer. Math. Soc. **71** (1965), 40–106. MR 0171826 (30 #2053)
18. ———, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR 1712872 (2001j:18001)
19. Martin Markl, *Operads and PROPs*, Handbook of algebra. Vol. 5, Handb. Algebr., vol. 5, Elsevier/North-Holland, Amsterdam, 2008, pp. 87–140. MR 2523450 (2010j:18015)
20. Martin Markl, Steve Shnider, and Jim Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002. MR 1898414 (2003f:18011)
21. J. Peter May, *The Geometry of Iterated Loop Spaces*, Springer-Verlag, Berlin, 1972, Lectures Notes in Mathematics, Vol. 271. MR 0420610 (54 #8623b)
22. Ieke Moerdijk, *Lectures on dendroidal sets*, Simplicial methods for operads and algebraic geometry, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer Basel AG, Basel, 2010, Notes written by Javier J. Gutiérrez, pp. 1–118. MR 2778589
23. Ieke Moerdijk and Ittay Weiss, *Dendroidal sets*, Algebr. Geom. Topol. **7** (2007), 1441–1470, Available at `http://dx.doi.org/10.2140/agt.2007.7.1441`. MR 2366165 (2009d:55014)
24. ———, *On inner Kan complexes in the category of dendroidal sets*, Adv. Math. **221** (2009), no. 2, 343–389, Available at `http://dx.doi.org/10.1016/j.aim.2008.12.015`. MR 2508925 (2010a:55021)
25. Teimuraz Pirashvili, *On the PROP corresponding to bialgebras*, Cah. Topol. Géom. Différ. Catég. **43** (2002), no. 3, 221–239. MR 1928233 (2003i:18012)
26. G. B. Segal, *The definition of conformal field theory*, Differential geometrical methods in theoretical physics (Como, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 250, Kluwer Acad. Publ., Dordrecht, 1988, pp. 165–171. MR 981378 (90d:58026)
27. Alberto Torres F. and Julián Aráoz D., *Combinatorial models for searching in knowledge bases*, Acta Cient. Venezolana **39** (1988), no. 5-6, 387–394. MR 1005374 (90k:68162)

Department of Mathematics, University of California-Riverside
E-mail address: hackney@math.ucr.edu

Department of Mathematics, University of Western Ontario, Canada
E-mail address: mrober97@uwo.ca