# Logic in 2D, Metalogic in 3D: The Language of Category Theory

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**Abstract**

The fundamental concepts of category theory are systematized in the language of a bifibrant double category. We define the language of all bifibrant double categories: the *co/descent calculus* is a system for double weighted co/limits, and much more; and we can explore it visually in three dimensions: colors, strings, beads, and flows.

[**Note.** This is a first draft, with basic definitions and theorems, but limited exposition.]

[This program is too big for one person! If you're interested, send me a message.]

# Contents

# 0 Introduction

Category Theory is known as a unifying language of mathematics [14]. In recent years, Applied Category Theory has begun to explore it as a language for all kinds of science [4]. I propose that a category theory is a *logic*, as in a *language of thinking*.

In this context, my doctoral thesis creates *metalogic*: the three-dimensional category of bifibrant double categories. The language is vastly powerful, yet intuitive and practical in the forms of *string diagrams* and the *co/descent calculus*.

Based on the visual language of thinking, I propose an education and research program.

## 0.1 Category theory is Logic

The basic concepts of category theory

type and process, relation and transformation
identity and composition, adjunction and representation

are systematized in the language of a *bifibrant double category*, a concept presently known as "proarrow equipment" or "framed bicategory" [18]. Such a language can be understood simply as a *logic*, i.e. a system of *thoughts of a world*:

A world is a category of types of things, and processes between types.

A thought of the world is a relation of types (a judgement), and
a process of thinking is a transformation of relations (an inference).

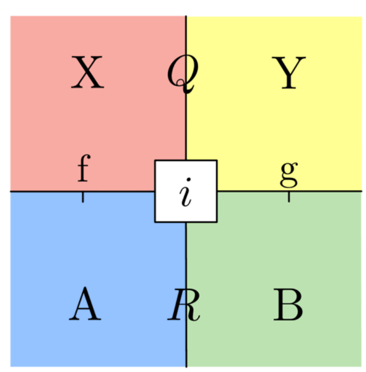
Relations and transformations form a category, and these "thoughts" form a bifibration from the world to the world, with operations of parallel composition and identity.

In this view, category theory is the realization that *thought is connection*, the dimension beyond the world in which type relates to type, and process transforms to process.

Yet a process is a special kind of connection, and so thought encompasses the world: each process forms a dual pair of relations. By composition, thoughts are pushed forward or pulled backward along processes; this is the "bifibrance" of a logic.

The language exists in two dual forms: *syntax* and *imagery*, a.k.a. string diagrams [16]: dual to object, arrow, square is color, string, bead. We distinguish processes from relations by a downward pointer, and their action on relations is drawn as bending.

| **bif. dbl. cat.** | dim. | **logic** |
|---:|:---:|:---|
| object | 0 | type |
| tight morphism | V | process |
| loose morphism | H | relation |
| square | 2 | transformation |

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The simplest kind of logic is *binary logic*: sets and functions, relations and entailments; i.e. the predicate logic of sets. Type theory has realized that relations have content beyond truth values, and in a few decades we have made a multiverse of logics to explore.

So how do we make logics? This is summarized in the motto:

a category is a matrix with composition and identity.

A category is a type of objects, indexing a matrix of morphisms, with the structure of composition and identity. In [18], Shulman presented the two ways we construct logics:

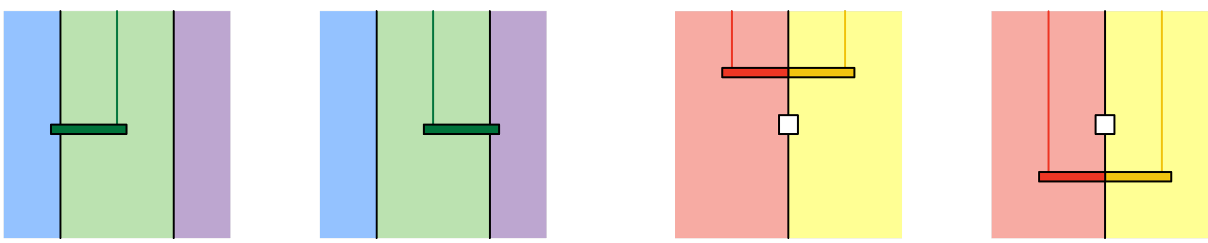
1. A *bifibered monoidal category* $\mathcal{R} \to \mathbb{A}$ forms a logic, in which a relation $R : \mathrm{A} \,|\, \mathrm{B}$ is an object $R$ over $\mathrm{A} \times \mathrm{B}$; this is a matrix, i.e. two-variable type $\mathrm{a} : \mathrm{A}, \mathrm{b} : \mathrm{B} \vdash R(\mathrm{a}, \mathrm{b}) : \mathbb{V}$.

2. *Monads* in a logic, self-relations with composition and identity, form a richer logic. A monad in a logic of matrices is a category, “enriched in” or “internal to” that logic.

The two constructions define the language of *co/ends* [15]: a bimodule of monads is a matrix with composition actions; these compose by *coend*, a coequalizer of a coproduct, and “divide” or transform by *end*, an equalizer of a product.

$$\begin{aligned} R \circ S &= \Sigma \mathrm{b} \quad R(-, \mathrm{b}) \otimes S(\mathrm{b}, -) \\ [P, Q] &= \Pi \mathrm{x}\ \Pi \mathrm{y} \quad P(\mathrm{x}, \mathrm{y}) \to Q(\mathrm{x}, \mathrm{y}) \end{aligned}$$

Categories are self-relations, which *act* on relations of categories, defining “active logic”: coend is the *bilinear* existential, and end is the *natural* universal. [13]



$(r \cdot \mathrm{b}, s) \equiv (r, \mathrm{b} \cdot s) \qquad i(\mathrm{x} \cdot p \cdot y) \equiv \mathrm{x} \cdot i(p) \cdot \mathrm{y}$

Category theory is presently seen as a network of concepts, without a central ground. While it is true that generality begets interdefinability, the “fundamentality” of concepts must be understood by how we *construct* the universe of categories, and this leads directly to the language of coends and ends. In this way, category theory is generalized logic.

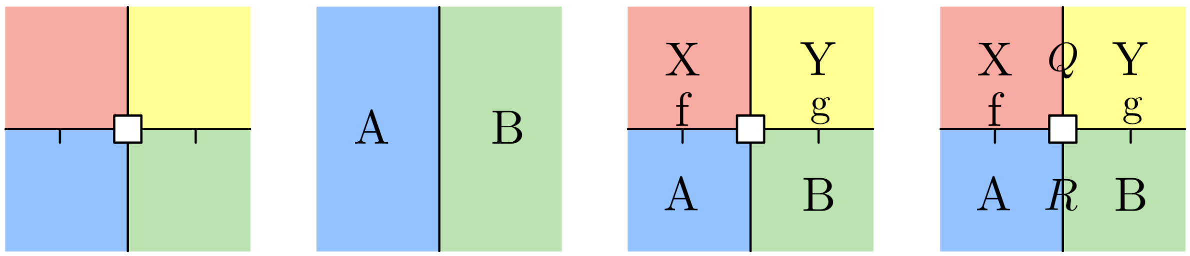
Universal constructions are systematically derived in the language: composition and transformation form a tensor-hom adjunction, giving formulae for lifts and extensions, and weighted limits and colimits are representations thereof.

$$\begin{aligned} [R \circ S, T] &= \Pi \mathrm{a}\ \Pi \mathrm{c} \quad (\Sigma \mathrm{b}\ R(\mathrm{a}, \mathrm{b}) \otimes S(\mathrm{b}, \mathrm{c})) \ \to\ T(\mathrm{a}, \mathrm{c}) \\ &\cong \\ [R, S \to T] &= \Pi \mathrm{a}\ \Pi \mathrm{b} \quad R(\mathrm{a}, \mathrm{b}) \ \to\ (\Pi \mathrm{c}\ S(\mathrm{b}, \mathrm{c}) \to T(\mathrm{a}, \mathrm{c})) \end{aligned}$$

The coYoneda lemma is the fact that the hom of a category is its identity relation, and the Yoneda lemma is the curried form of this fact.

Fundamental ideas are made simple and clear in the language. Presenting CT as logic provides not only a central ground of category theory, but also a systematic and direct exposition of the full power of the language of categories.

Moreover, string diagrams provide an intuitive and systematic presentation of these fundamental ideas. Because imagery is *dual* to syntax, no exclusionary choice is needed: the two combine to form the visual formal language of "color syntax".




$\mathbb{C}$ $\mathbb{C}(\mathrm{A}, \mathrm{B}) : \mathrm{Cat}$ $\mathbb{C}(\mathrm{f}, \mathrm{g}) : \mathrm{Prof}$ $\mathbb{C}(\mathrm{f}, \mathrm{g})(Q, R) : \mathrm{Set}$

A string diagram is the general form of a concept, and writing syntax in the diagram determines a specific instance, i.e. substitution into a dependent type. Reasoning can smoothly transition levels of generality, from an entire logic to a specific transformation. In color syntax, the language of categories is both intuitive and practical.

## 0.2 Metalogic

Now, the central insight of the thesis: for each pair of types in a logic, there is a *category* of relations, and the structure of a logic is composition and identity of relations.

A logic is a *matrix of categories* with composition and identity.

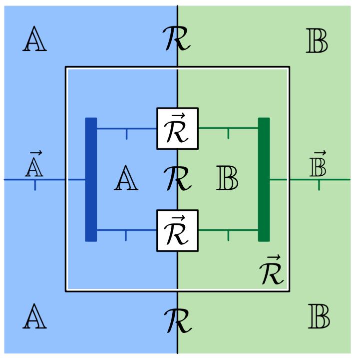
The language of logics is the higher-dimensional co/end calculus: the *co/descent calculus*. Because bifibrant double categories unify the fundamental concepts of category theory, we propose the co/descent calculus to be the unified metalanguage of category theory.

We develop the notion of a "matrix of categories", and its three-dimensional language, as follows.

**Chapter 1: Spans of categories.**

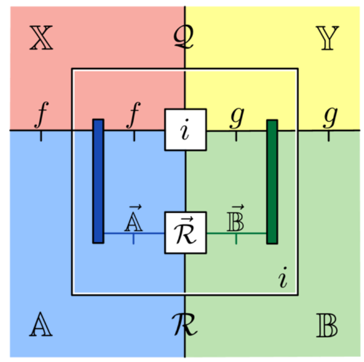
A *span of categories* $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ is a equivalent to a matrix of categories $\mathcal{R}(\mathrm{A}, \mathrm{B})$ and profunctors $\vec{\mathcal{R}}(\mathrm{a}, \mathrm{b})$, with *sequential composition* and identity. In the same way, a *span of profunctors* $i \leftarrow f \rightarrow g$ is equivalent to a matrix of profunctors $i(\mathrm{f}, \mathrm{g}) : \mathcal{Q}(\mathrm{X}, \mathrm{Y}) \mid \mathcal{R}(\mathrm{A}, \mathrm{B})$ with composition and identity.

We introduce *three-dimensional* string diagrams: spans of categories are horizontal strings, profunctors are vertical bars, and functors are drawn as a closed loop or "bead within a bead", interpreted as a transformation from inner to outer.




**span category**

$\vec{\mathcal{R}}(\mathrm{a}_1, \mathrm{b}_1) \circ \vec{\mathcal{R}}(\mathrm{a}_2, \mathrm{b}_2) \Rightarrow \vec{\mathcal{R}}(\mathrm{a}_1\mathrm{a}_2, \mathrm{b}_1\mathrm{b}_2)$



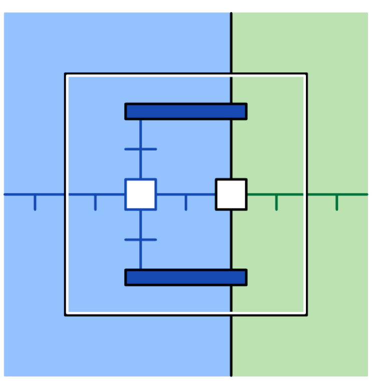

**span profunctor**

$i(\mathrm{f}, \mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a}, \mathrm{b}) \Rightarrow i(\mathrm{fa}, \mathrm{gb})$

We introduce the concept of *displayed profunctor* 1.2, and show the double category of span categories $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ to be equivalent to that of displayed categories $\mathcal{R} : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$. The matrices $\mathcal{R}(\mathrm{A}, \mathrm{B})$ are the basic data of the co/descent calculus.

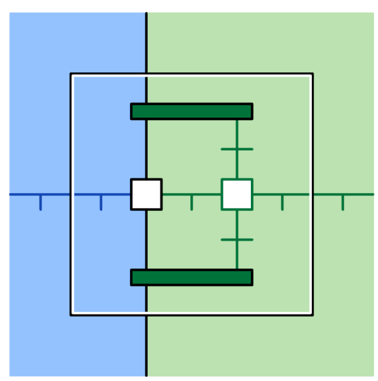
**Chapter 2: Matrix categories.**

A *matrix category* $\mathcal{R} : \mathbb{A} \parallel \mathbb{B}$ is a span of categories, with actions by both arrows and "op-arrows" in $\mathbb{A}$ and $\mathbb{B}$: the *weave double category* $\langle \mathbb{A} \rangle$ is the coproduct of the arrow double category and its opposite $\overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$, forming a logic, and $\mathcal{R}$ is a bimodule from $\langle \mathbb{A} \rangle$ to $\langle \mathbb{B} \rangle$. In the terminology of [21], a matrix category is a *two-sided bifibration*.



**matrix category**

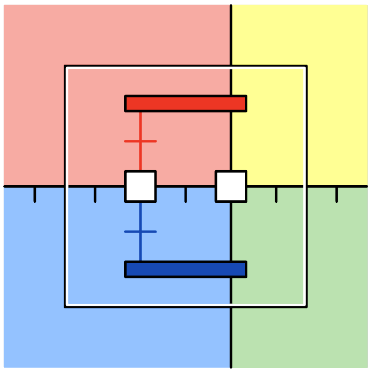
$\odot_{\mathbb{A}} : \langle \mathbb{A} \rangle(\mathrm{A}_0, \mathrm{A}_1) \times \mathcal{R}(\mathrm{A}_1, \mathrm{B}) \rightarrow \mathcal{R}(\mathrm{A}_0, \mathrm{B})$



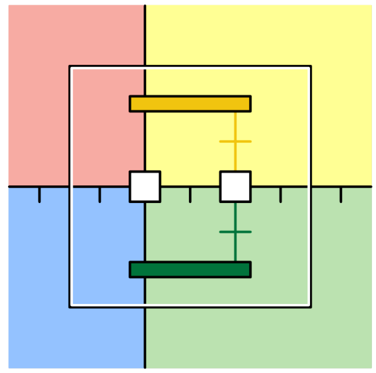
**matrix category**

$\odot_{\mathbb{B}} : \mathcal{R}(\mathrm{A}, \mathrm{B}_0) \times \langle \mathbb{B} \rangle(\mathrm{B}_0, \mathrm{B}_1) \rightarrow \mathcal{R}(\mathrm{A}, \mathrm{B}_1)$

This generalizes from categories to profunctors: the *arrow profunctor* $\vec{f} : \vec{\mathbb{X}} \mid \vec{\mathbb{A}}$ consists of commutative squares $\mathrm{f}_0 \cdot \mathrm{a} = \mathrm{x} \cdot \mathrm{f}_1$. The *weave vertical profunctor* $\langle f \rangle : \langle \mathbb{A} \rangle \mid \langle \mathbb{B} \rangle$ is the union of $\vec{f}$ and its opposite. A *matrix profunctor* $i(f, g) : \mathcal{Q}(\mathbb{X}, \mathbb{Y}) \mid \mathcal{R}(\mathbb{A}, \mathbb{B})$ is a span of profunctors $f \leftarrow i \rightarrow g$, which is a bimodule from $\langle f \rangle$ to $\langle g \rangle$.

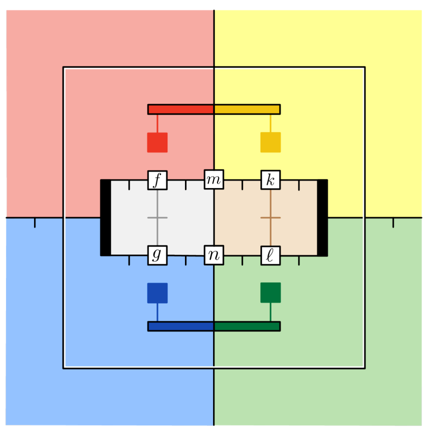


**matrix profunctor**
$\odot_f : \langle f \rangle(\mathrm{f}_0, \mathrm{f}_1) \times i(\mathrm{f}_1, \mathrm{g}) \Rightarrow i(\mathrm{f}_0, \mathrm{g})$



**matrix profunctor**
$\odot_g : i(\mathrm{f}, \mathrm{g}_0) \times \langle g \rangle(\mathrm{g}_0, \mathrm{g}_1) \Rightarrow i(\mathrm{f}, \mathrm{g}_1)$

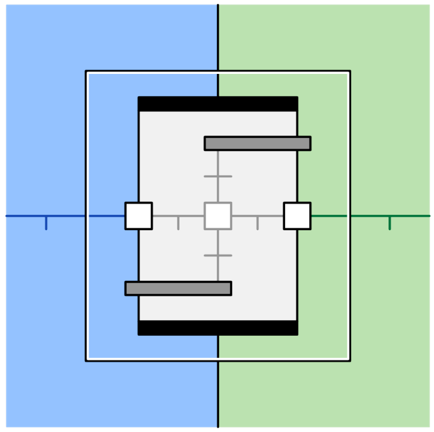
Morphisms of matrix categories and matrix profunctors are *matrix functors* and *matrix transformations*. These form a double category $\mathrm{Mat}\mathbb{C}\mathrm{at}$ over $\mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$. Sequential composition of matrix profunctors over that of profunctors is defined by a coequalizer, which nullifies the parallel action of zig-zags reassociating $[(\mathrm{f}_0, \mathrm{g}_0)] = [(\mathrm{f}_1, \mathrm{g}_1)] : \langle f \circ g \rangle$ and $[(\mathrm{k}_0, \mathrm{l}_0)] = [(\mathrm{k}_1, \mathrm{l}_1)] : \langle k \circ \ell \rangle$. (Definition 44)




**sequential composite**
$(m, n) \;\equiv\; (\mathrm{f} \odot m \odot \mathrm{k}, \mathrm{g} \odot n \odot \mathrm{k}) \;:\; m \diamond n$

Moreover, $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is a logic, and $\mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$ is a *double fibration* [3]: sequential composition of matrix profunctors preserves substitution of transformations (starting at Prop. 47). Hence we call the structure $\mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$ a *fibered logic*.

To complete the three-dimensional structure of $\mathbb{C}\mathrm{at} \leftarrow \mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathbb{C}\mathrm{at}$, we define *parallel composition* of matrix categories in Section 2.5. While profunctors compose by coequalizer, matrix categories compose by *codescent object* [21], which adjoins an associator isomorphism for the action by arrows and oparrows of the middle category.



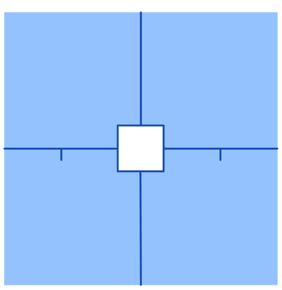
**parallel composition**
$\alpha : (R, \bar{\mathrm{b}} \odot S) \cong (R \odot \bar{\mathrm{b}}, S)$

Dually, the category of matrix functors is constructed as a *descent object* [20]. So composition and transformation of matrix categories are dual, just as in the co/end calculus (Theorem 57).

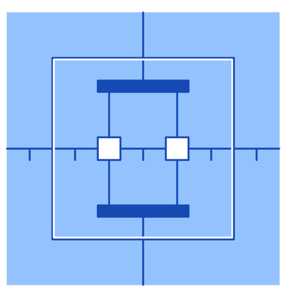
$$\begin{aligned} \mathcal{R} \otimes \mathcal{S} &= \vec{\Sigma}\mathrm{B}. \quad \mathcal{R}(-, \mathrm{B}) \times \mathcal{S}(\mathrm{B}, -) \\ [\mathcal{P}, \mathcal{Q}] &= \vec{\Pi}\mathrm{X}, \mathrm{Y}. \quad \mathcal{P}(\mathrm{X}, \mathrm{Y}) \to \mathcal{Q}(\mathrm{X}, \mathrm{Y}) \end{aligned}$$

However, parallel composition does not preserve sequential composition of matrix profunctors: because both dimensions are bimodules, both compositions involve colimits which the other cannot represent. So $\mathbb{C}\mathrm{at} \leftarrow \mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathbb{C}\mathrm{at}$ is like a triple category without interchange, a structure on span categories: we define a *metalogic* to be a fibered logic $\mathbb{M} \to \mathbb{C} \times \mathbb{C}$, which forms a 2-weak category in $\mathrm{Span}\mathbb{C}\mathrm{at}$. [Definition 55]

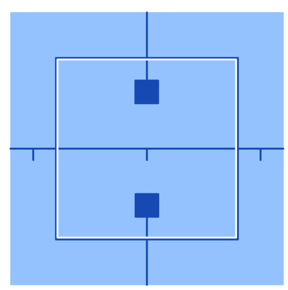
**Chapter 3: The metalogic of logics.**
A bifibrant double category, i.e. a logic, is a pseudomonad in $\mathrm{Mat}\mathbb{C}\mathrm{at}$.



**logic**

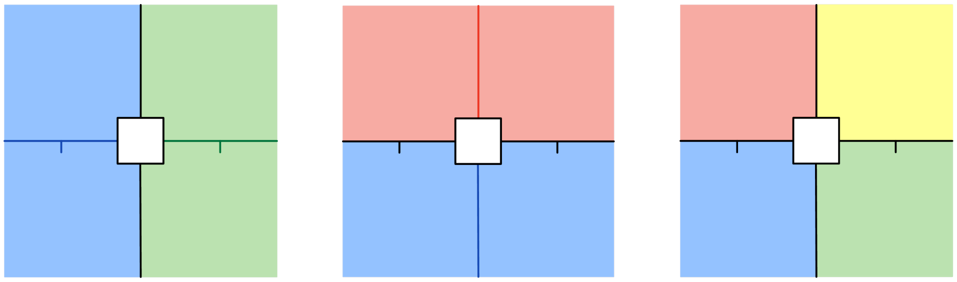


composition



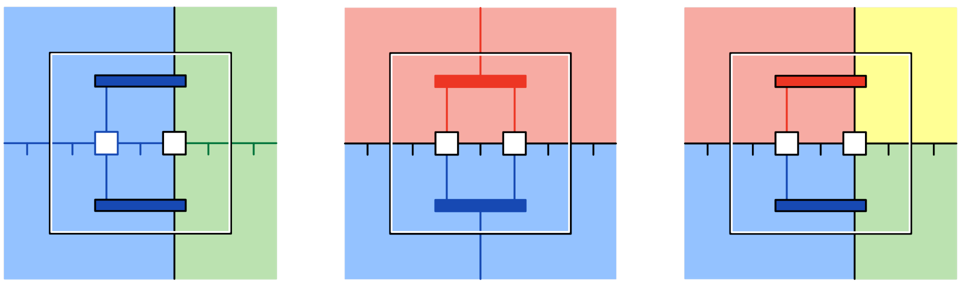
unit

Because a logic is two-dimensional, there are *two* kinds of relations between logics: a *vertical profunctor* consists of processes between logics, and a *horizontal profunctor* consists of relations between logics. Two pairs are connected by a *double profunctor*, which consists of inferences between relations, along processes.



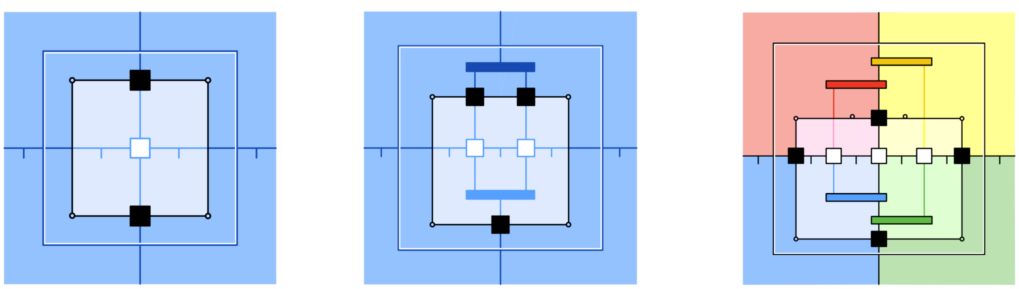
meta relation [horiz. profunctor] | meta process [vert. profunctor] | meta inference [dbl. profunctor]

For horizontal profunctors, parallel composition is a familiar *bimodule* action. Yet because vertical profunctors are orthogonal, parallel composition defines a *monad* structure, and so double profunctors are bimodules thereof.



H-prof. composition | V-prof. composition | D-prof. composition

So logics have two kinds of "relations", and one kind of "function": a *double functor* $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ maps squares of $\mathbb{A}_0$ to squares of $\mathbb{A}_1$, preserving relation composition and unit up to coherent isomorphism. This generalizes to transformations of vertical, horizontal, and double profunctors; all four are defined by mapping squares in a way that coheres with parallel composition.



double functor | preserves composition; | double transformation

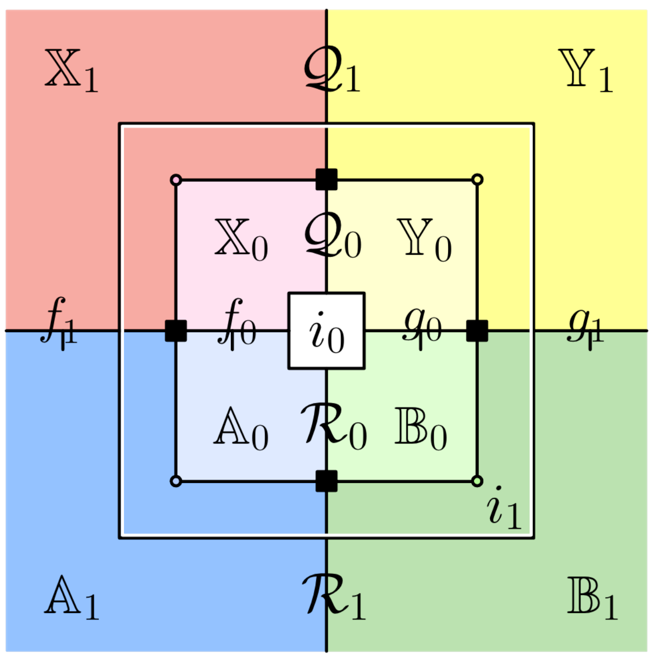
All together, logics form a metalogic: morphisms are functors, profunctors, and matrix categories; squares are vertical transformations, horizontal transformations, and double profunctors; and cubes are double transformations.

Below, the outline: we construct the metalogic of matrix categories, then apply the "horizontal pseudomonad" construction to form the metalogic of bifibrant double categories; and we give a metalogical interpretation of this structure.

| | MatℂCat | H.PsMnd(−) | $bf$.DblℂCat | Logic |
|---|---|---|---|---|
| 0 | category | (H)-pseudomonad | bifibrant double category | logic |
| V | profunctor | (H)-vertical monad | vertical profunctor | meta process |
| H | matrix category | (H)-pseudobimodule | horizontal profunctor | meta relation |
| VH | matrix profunctor | (H)-vertical bimodule | double profunctor | meta inference |
| T | functor | ps. mnd. morphism | double functor | flow type |
| TV | transformation | v. mnd. morphism | vertical transformation | flow process |
| TH | matrix functor | ps. bim. morphism | horizontal transformation | flow relation |
| TVH | matrix transformation | v. bim. morphism | double transformation | flow inference |

As a double profunctor consists of inferences between logics, a double transformation is a “flow” of *meta*-reasoning, a way to transform one system of reasoning into another.

In this sense, the language of $bf$.DblℂCat is the language of metalogic.




## 0.3 Logic in Color

This language is the basis for an education and research program. First priorities:

- We need a good *drawing app*. it can be simple: a grid of points, tools for lines and rectangles, auto-filling colors, and substitution of syntax. Plus an organized database.
- A research seminar. The project is too big for one person, so the sooner we can begin collaboration the better. If you’re interested, send me a message.

Contents

# 1 Spans of categories: strings, beads, and flows

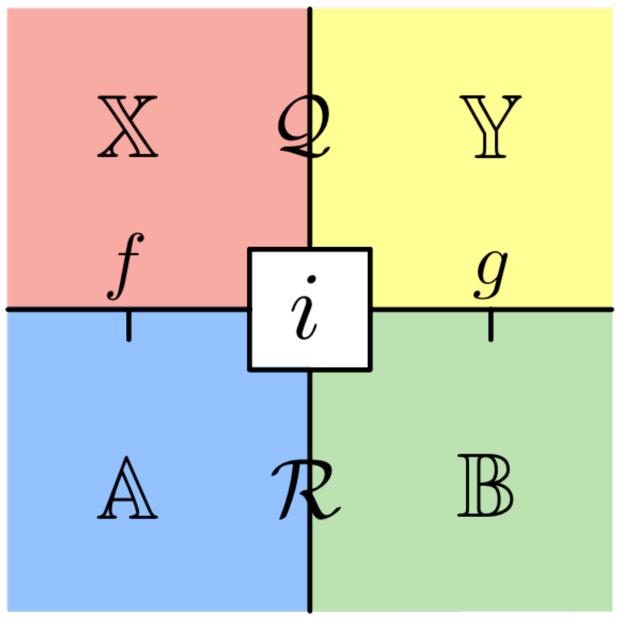
We can share a language for *all kinds of thinking*, if we determine the basic structure of a "kind of thinking", and understand the language of these structures.

So, we proceed from the introduction: what is the underlying structure of a *logic*?

We can understand this by each *dimension*: type and process, relation and inference.

Types form a category, and processes form a profunctor. So the language of categories, of objects and morphisms, forms dimensions 0 and V of logic.

Now, *relations* form a category as well; yet a relation *depends* on a pair of types, and an inference depends on a pair of processes. So, the language of *matrices of categories* forms the dimensions H and VH of logic. This is the middle column below.

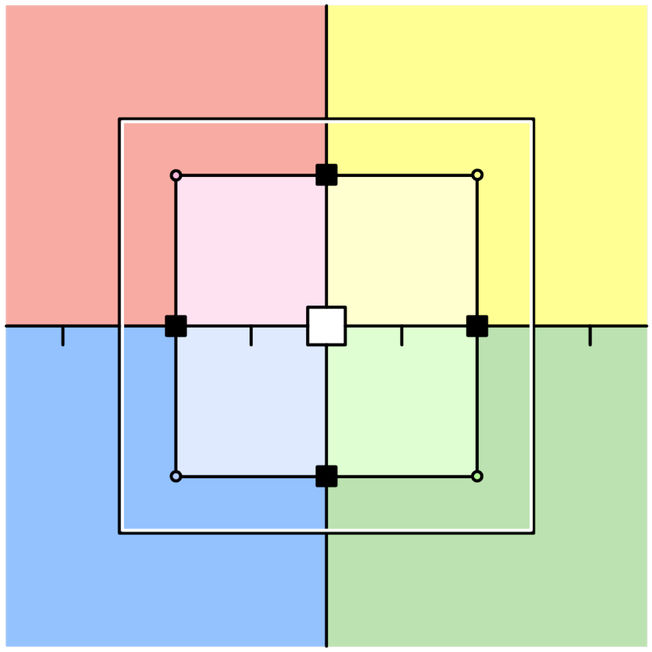



| | | |
|---|---|---|
| **type**<br>category<br>$\mathbb{X} : \mathbb{C}\mathrm{at}$ | **relation**<br>span category<br>$\mathbb{X} \leftarrow \mathcal{Q} \rightarrow \mathbb{Y}$<br>$\sim \mathcal{Q}(\mathrm{X}, \mathrm{Y}) : \mathrm{Cat}$ | **type**<br>category<br>$\mathbb{Y} : \mathbb{C}\mathrm{at}$ |
| **process**<br>profunctor<br>$f : \mathbb{X} \mid \mathbb{A}$<br>$\sim f(\mathrm{X}, \mathrm{A}) : \mathrm{Set}$ | **inference**<br>span profunctor<br>$f \leftarrow i \rightarrow g$<br>$\sim i(\mathrm{f}, \mathrm{g}) : \mathrm{Prof}$ | **process**<br>profunctor<br>$g : \mathbb{Y} \mid \mathbb{B}$<br>$\sim g(\mathrm{Y}, \mathrm{B}) : \mathrm{Set}$ |
| **type**<br>category<br>$\mathbb{A} : \mathbb{C}\mathrm{at}$ | **relation**<br>span category<br>$\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$<br>$\sim \mathcal{R}(\mathrm{A}, \mathrm{B}) : \mathrm{Cat}$ | **type**<br>category<br>$\mathbb{B} : \mathbb{C}\mathrm{at}$ |

In this way, the language of dependent categories can be understood as *metalogic*: each span profunctor is a *kind of thinking*, and each span transformation is... well, a way to *transform* one kind of thinking into another.

Because a span profunctor is a system of inference, it can be imagined as a bead. Now, the third dimension is *inner to outer*: a span transformation is a bead within a bead.

We define the three-dimensional visual language of dependent categories.



span transformation $[\![i]\!] : i_0 \Rightarrow i_1$

Imagery is *dual* to syntax: a color is dual to a point, and a string is dual to an arrow. So they complement each other; in fact, the visual and symbolic unite in one language, which we call **color syntax**: a string diagram is a general concept, and *substituting* syntax determines a specific instance of the concept.

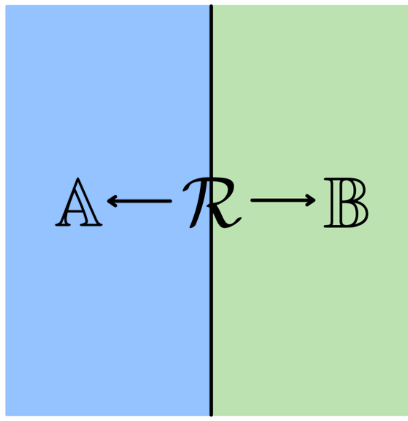
Yet this is the very basis of dependent category theory: a span of categories $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ is equivalent to a matrix of categories $\mathcal{R}(\mathrm{A}, \mathrm{B})$. Miraculously, not only is the language of dependent categories visualized in the simple form of colors, strings, beads, and flows — its key principle is somehow *immanent* in the very idea of uniting imagery and syntax.

Now we define the logic of span categories $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$, as a visual language, and then by applying syntax, we form the equivalent logic of *displayed categories* $\mathcal{R} : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$.

## 1.1 Span categories

Let $\mathbb{A}$ and $\mathbb{B}$ be categories. A **span category** from $\mathbb{A}$ to $\mathbb{B}$ is a category $\mathcal{R}$ with functors $\pi_{\mathbb{A}}^{\mathcal{R}} : \mathcal{R} \rightarrow \mathbb{A}$ and $\pi_{\mathbb{B}}^{\mathcal{R}} : \mathcal{R} \rightarrow \mathbb{B}$; we can denote the span by $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$, or $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$. Note this data is equivalent to a functor $(\pi_{\mathbb{A}}^{\mathcal{R}}, \pi_{\mathbb{B}}^{\mathcal{R}}) : \mathcal{R} \rightarrow \mathbb{A} \times \mathbb{B}$. $\mathbb{A}$ and $\mathbb{B}$ are the **base categories**, and $\mathcal{R}$ is the **total category**; we may refer to the span simply as $\mathcal{R}$.

We can draw a span category $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ simply as a *string*.




span category $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$

We can see a span category as a *matrix of categories*, by inverse image along $\mathcal{R} \to \mathbb{A} \times \mathbb{B}$. The notion of inverse image along a functor $\mathcal{S} \to \mathbb{C}$ has been given by Street in [19]; the resulting map $\mathcal{S} : \mathbb{C} \to \mathbb{C}\mathrm{at}$ is called a *normal lax functor*. The notion was later developed for use in type theory, and rebranded as "displayed category" [1].

**Definition 1.** A **displayed category** $\mathcal{R} : \mathbb{A} \times \mathbb{B} \to \mathbb{C}\mathrm{at}$ gives, for each pair of:

| | |
|---|---|
| objects | a category |
| $\mathrm{A} : \mathbb{A}, \mathrm{B} : \mathbb{B}$ | $\mathcal{R}(\mathrm{A}, \mathrm{B}) : \mathbb{C}\mathrm{at}$ |
| morphisms | a profunctor |
| $\mathrm{a} : \mathbb{A}(\mathrm{A}_0, \mathrm{A}_1), \mathrm{b} : \mathbb{B}(\mathrm{B}_0, \mathrm{B}_1)$ | $\vec{\mathcal{R}}(\mathrm{a}, \mathrm{b}) : \mathcal{R}(\mathrm{A}_0, \mathrm{B}_0) \mid \mathcal{R}(\mathrm{A}_1, \mathrm{B}_1)$ |
| composable pairs | a transformation |
| $(\mathrm{a}_1, \mathrm{b}_1), (\mathrm{a}_2, \mathrm{b}_2)$ | $r \cdot r : \vec{\mathcal{R}}(\mathrm{a}_1, \mathrm{b}_1) \circ \vec{\mathcal{R}}(\mathrm{a}_2, \mathrm{b}_2) \Rightarrow \vec{\mathcal{R}}(\mathrm{a}_1\mathrm{a}_2, \mathrm{b}_1\mathrm{b}_2)$ |
| objects | an equality |
| $\mathrm{A} : \mathbb{A}, \mathrm{B} : \mathbb{B}$ | $\mathcal{R}(\mathrm{A}, \mathrm{B})(-, -) = \vec{\mathcal{R}}(\mathrm{id}_\mathrm{A}, \mathrm{id}_\mathrm{B})$ |

so that composition is associative and unital, i.e. $(r \cdot r) \cdot r = r \cdot (r \cdot r)$ and $\mathrm{id}_\mathcal{R} \cdot r = r = r \cdot \mathrm{id}_\mathcal{R}$.

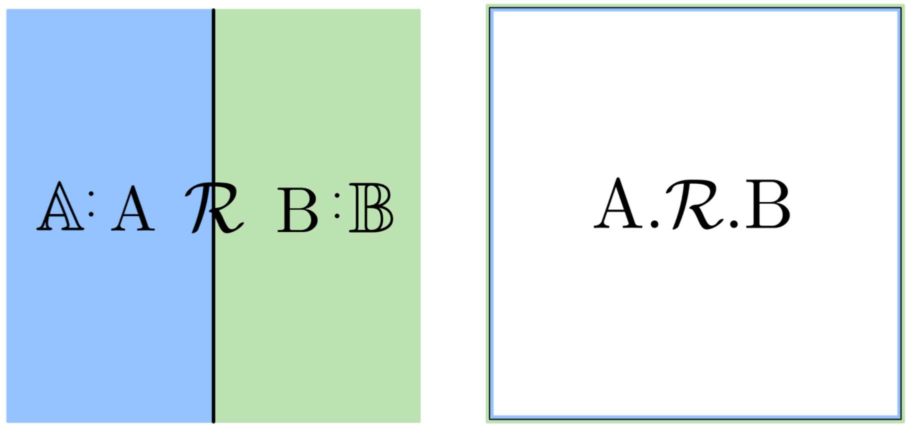
We give the proposition, and then expound in the visual language of span categories.

**Proposition 2.** Let $\mathbb{A}, \mathbb{B}$ be categories, and let $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ be a span of categories. Inverse image along $\mathcal{R} \to \mathbb{A} \times \mathbb{B}$ determines a displayed category $\mathcal{R} : \mathbb{A} \times \mathbb{B} \to \mathbb{C}\mathrm{at}$. [19]

For each pair of objects $\mathrm{A} : \mathbb{A}, \mathrm{B} : \mathbb{B}$ there is a category $\mathcal{R}(\mathrm{A}, \mathrm{B})$ of objects $R : \mathcal{R}$ which map to $(\mathrm{A}, \mathrm{B})$, also known as the "fiber over" $(\mathrm{A}, \mathrm{B})$; this may also be denoted $\mathcal{R}^\mathrm{A}_\mathrm{B}$. This is given by pullback in $\mathrm{Cat}$, of $\mathcal{R}$ along the functor which selects the pair $(\mathrm{A}, \mathrm{B})$.

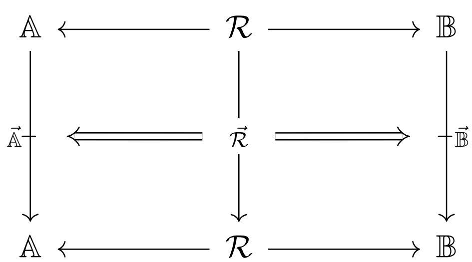
$$\begin{array}{ccc} \mathcal{R}(\mathrm{A}, \mathrm{B}) & \longrightarrow & \mathcal{R} \\ \big\downarrow & \lrcorner & \big\downarrow \\ * & \xrightarrow{(\mathrm{A},\mathrm{B})} & \mathbb{A} \times \mathbb{B} \end{array}$$

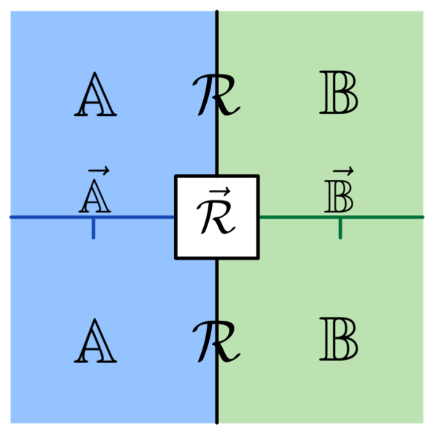
We introduce *color syntax*, a visual language that unifies string diagrams and syntax. The basic principle is *substitution*: by writing a pair of objects $\mathrm{A}, \mathrm{B}$ in the color of each category $\mathbb{A}, \mathbb{B}$, we determine the fiber category $\mathcal{R}(\mathrm{A}, \mathrm{B})$. An entry is drawn on the right as a type in $\mathbb{C}\mathrm{at}$, which we color white as the "ambient" logic, outlined in blue and green to indicate that it is a diagram indexed by categories $\mathbb{A}$ and $\mathbb{B}$.




category $\mathcal{R}(\mathrm{A}, \mathrm{B})$

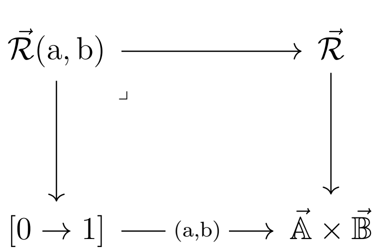
Now for the *morphisms* of a span category $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$, we consider the span of *hom-profunctors* $\vec{\mathbb{A}} \leftarrow \vec{\mathcal{R}} \rightarrow \vec{\mathbb{B}}$. Profunctors are drawn as strings pointing downward, and the hom of $\mathcal{R}$ is drawn as a *bead* from the $\mathcal{R}$ string to itself, along the homs of $\mathbb{A}$ and $\mathbb{B}$.

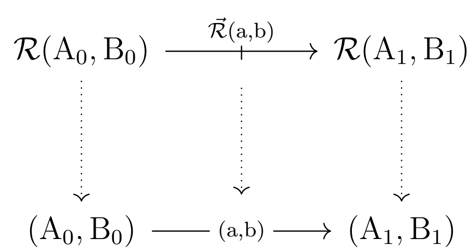




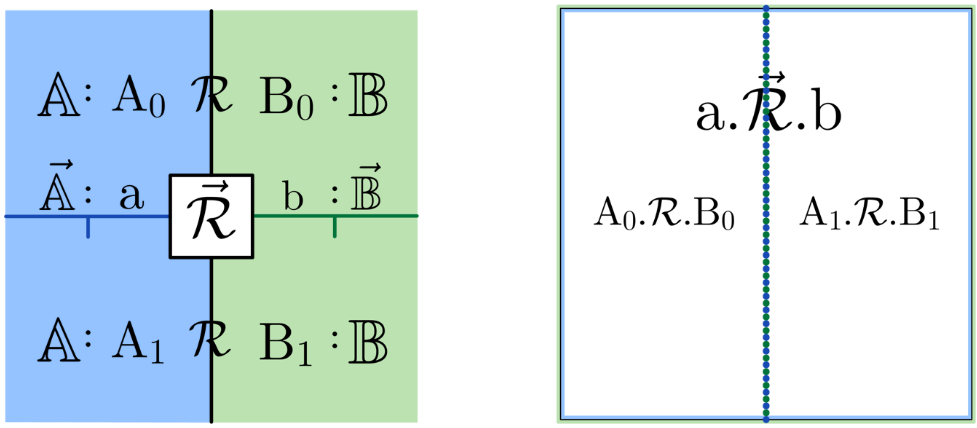

span hom-profunctor $\vec{\mathbb{A}} \leftarrow \vec{\mathcal{R}} \rightarrow \vec{\mathbb{B}}$

Just as $\mathcal{R} \rightarrow \mathbb{A} \times \mathbb{B}$ determines a matrix of categories, $\vec{\mathcal{R}} \Rightarrow \vec{\mathbb{A}} \times \vec{\mathbb{B}}$ determines a matrix of *profunctors*: for each pair $\mathrm{a} : \mathbb{A}(\mathrm{A}_0, \mathrm{A}_1), \mathrm{b} : \mathbb{B}(\mathrm{B}_0, \mathrm{B}_1)$ there is a profunctor $\vec{\mathcal{R}}(\mathrm{a}, \mathrm{b})$ from the category $\mathcal{R}(\mathrm{A}_0, \mathrm{B}_0)$ to $\mathcal{R}(\mathrm{A}_1, \mathrm{B}_1)$. This is given by pullback in $\mathrm{Prof}$ of the hom of $\mathcal{R}$ along the functor which maps the walking arrow to $(\mathrm{a}, \mathrm{b}) : \vec{\mathbb{A}} \times \vec{\mathbb{B}}$.



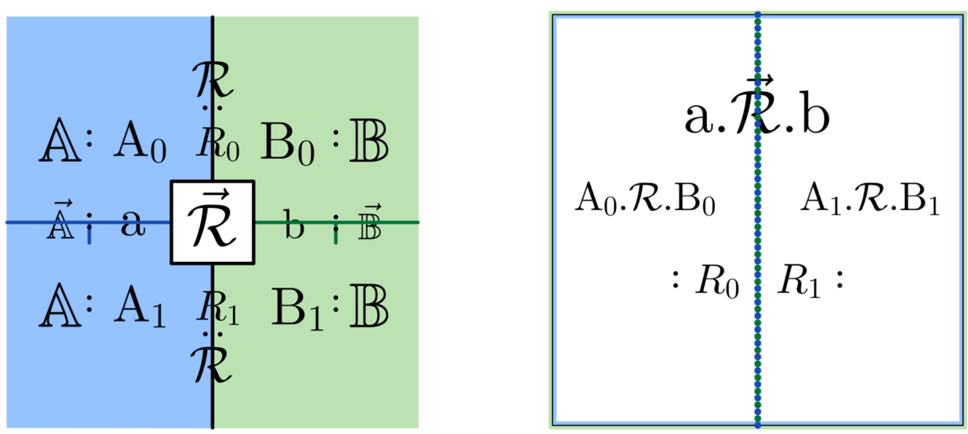


In color syntax, this pullback is given by substitution of the pair of morphisms $\mathrm{a}, \mathrm{b}$ into the hom-profunctors $\vec{\mathbb{A}}, \vec{\mathbb{B}}$. This gives a diagram of categories and profunctors $\vec{\mathcal{R}} : \vec{\mathbb{A}} \times \vec{\mathbb{B}} \to \mathrm{Prof}$, depicted on the right. Each profunctor is drawn as a blue and green "string of beads", as its elements can be understood as two-dimensional morphisms.




profunctor $\vec{\mathcal{R}}(\mathrm{a}, \mathrm{b}) : \mathcal{R}(\mathrm{A}_0, \mathrm{B}_0) \mid \mathcal{R}(\mathrm{A}_1, \mathrm{B}_1)$

Now we can go one level further, to the morphisms of the span category. Given $R_0 : \mathcal{R}(\mathrm{A}_0, \mathrm{B}_0)$ and $R_1 : \mathcal{R}(\mathrm{A}_1, \mathrm{B}_1)$, then $\vec{\mathcal{R}}(\mathrm{a}, \mathrm{b})(R_0, R_1)$ is the set of morphisms $r : \mathcal{R}(R_0, R_1)$ over $(\mathrm{a}, \mathrm{b})$.



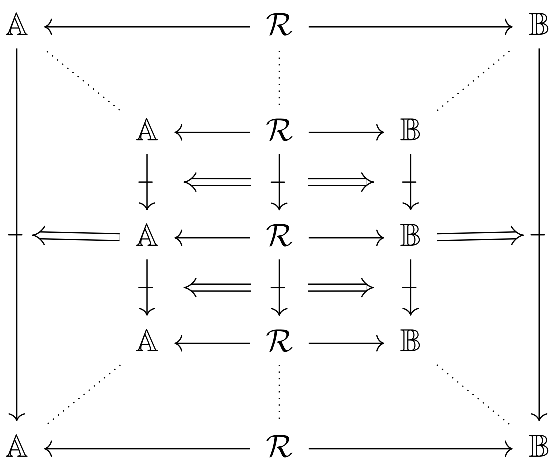

set $\vec{\mathcal{R}}(\mathrm{a}, \mathrm{b})(R_0, R_1)$

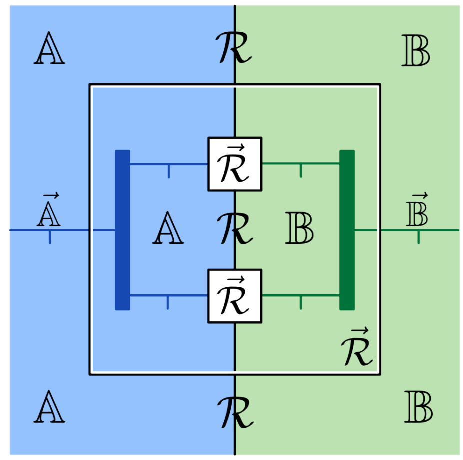
As the string diagram suggests, we can interpret objects and morphisms of $\mathcal{R}$ as *relations* and *transformations*, i.e. horizontal morphisms and squares in a double category. Once we define matrix categories, with horizontal composition, this interpretation will be literal.

This completes the data of a span category, which as we see is two-dimensional; we now consider its structure of composition and unit, which is *three-dimensional*.

A span category has a **composition** transformation $r \cdot r : \vec{\mathcal{R}} \circ \vec{\mathcal{R}} \Rightarrow \vec{\mathcal{R}}$ over composition of $\mathbb{A}$ and $\mathbb{B}$. We draw equalities as dotted lines, and the transformation as going *outward*.

We draw a three-dimensional string diagram "head on" in the same way, but now we see more: because the source and target span profunctors are drawn as "beads", the target can be depicted as a large "hollow" bead. Intuitively, we are looking at the front of a box and "poking a hole" to look inside.

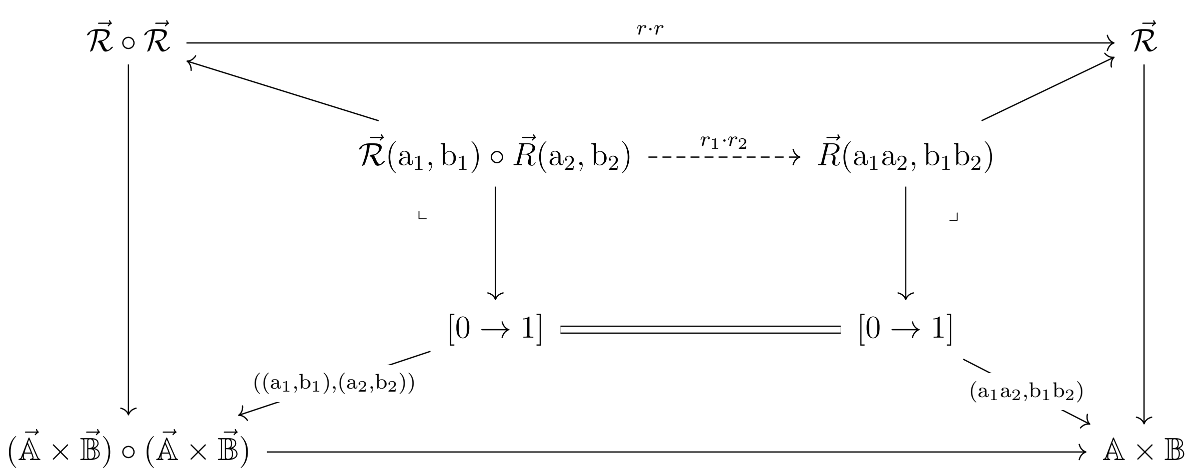




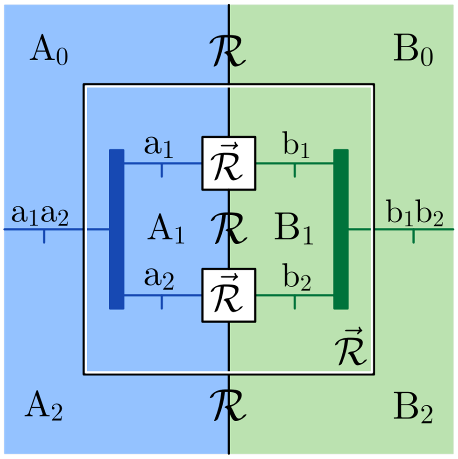

$r \cdot r : \vec{\mathcal{R}} \circ \vec{\mathcal{R}} \Rightarrow \vec{\mathcal{R}}$

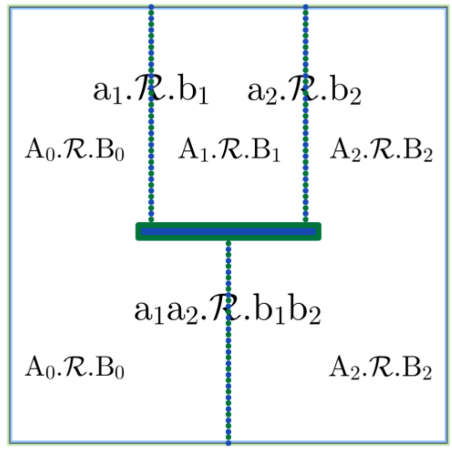
Yet to see the actual 3-morphism, we still need to "slice down the middle". As we do so, we draw the middle slice as its "displayed category" equivalent.

The span transformation $r \cdot r : \vec{\mathcal{R}} \circ \vec{\mathcal{R}} \Rightarrow \vec{\mathcal{R}}$ determines a matrix of transformations: for each composable pair of pairs $(\mathrm{a}_1, \mathrm{b}_1) : \mathbb{A}(\mathrm{A}_0, \mathrm{A}_1) \times \mathbb{B}(\mathrm{B}_0, \mathrm{B}_1)$ and $(\mathrm{a}_2, \mathrm{b}_2) : \mathbb{A}(\mathrm{A}_1, \mathrm{A}_2) \times \mathbb{B}(\mathrm{B}_1, \mathrm{B}_2)$, there is a transformation $r_1 \cdot r_2 : \vec{\mathcal{R}}(\mathrm{a}_1, \mathrm{b}_1) \circ \vec{\mathcal{R}}(\mathrm{a}_2, \mathrm{b}_2) \Rightarrow \vec{\mathcal{R}}(\mathrm{a}_1\mathrm{a}_2, \mathrm{b}_1\mathrm{b}_2)$. This is given by functoriality of pullback in $\mathrm{Prof}$.



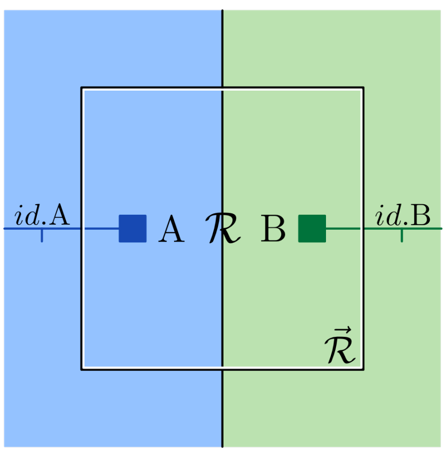

Again, this is given in color syntax by substituting morphisms $(\mathrm{a}_1, \mathrm{b}_1)$ and $(\mathrm{a}_2, \mathrm{b}_2)$ into the homs of $\mathbb{A}$ and $\mathbb{B}$. As diagrams become more complex, we may leave types implicit when they can be inferred in context. We may also use $\mathcal{R}(\mathrm{a}, \mathrm{b})$, rather than $\vec{\mathcal{R}}(\mathrm{a}, \mathrm{b})$.

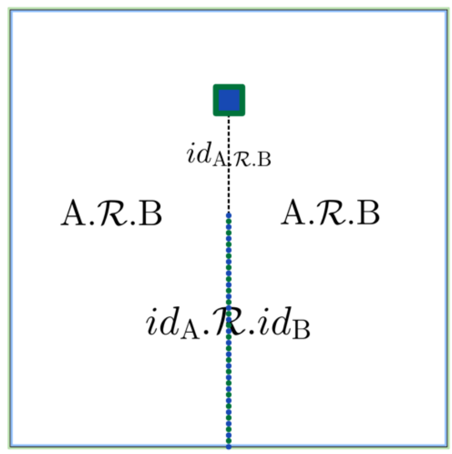



$r_1 \cdot r_2 : \mathcal{R}(a_1, b_1) \circ \mathcal{R}(a_2, b_2) \Rightarrow \mathcal{R}(a_1a_2, b_1b_2)$

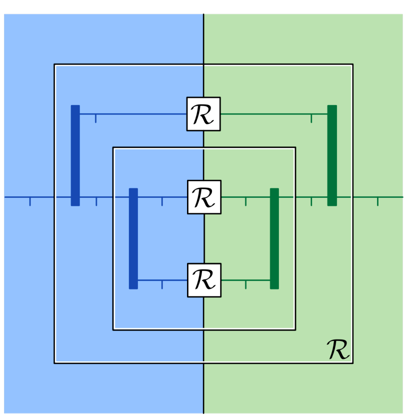
The second structure of a span category $\mathcal{R}$ is a **unit** transformation $\mathrm{id}_{\mathcal{R}} \Rightarrow \vec{\mathcal{R}}$. For each pair of objects, there is an *equality* from the hom of $\mathcal{R}(A, B)$ to the profunctor $\vec{\mathcal{R}}(\mathrm{id}_A, \mathrm{id}_B)$. So, identities of $\mathcal{R}(A, B)$ become identities of $\vec{\mathcal{R}}(\mathrm{id}_A, \mathrm{id}_B)$.

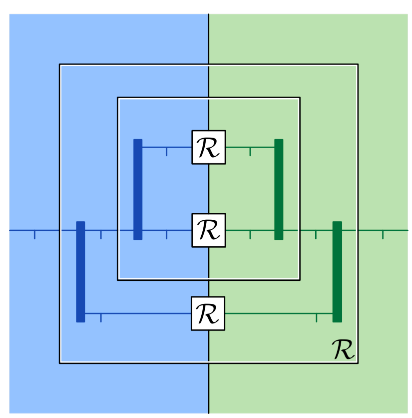



$\mathrm{id}_{\mathcal{R}} : \mathcal{R}^A_B \Rightarrow \mathcal{R}^A_B(-,-) = \mathcal{R}^{\mathrm{id}.A}_{\mathrm{id}.B}$

Finally, this structure satisfies two properties: composition is associative and unital. For any composable triple $r_1, r_2, r_3$ we have $r_1 \cdot (r_2 \cdot r_3) = (r_1 \cdot r_2) \cdot r_3$.

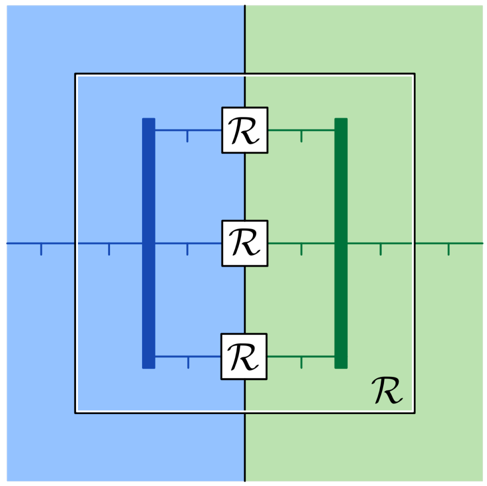



$r_1 \cdot (r_2 \cdot r_3) \quad = \quad (r_1 \cdot r_2) \cdot r_3$

We introduce the **coherence principle** for three-dimensional string diagrams:

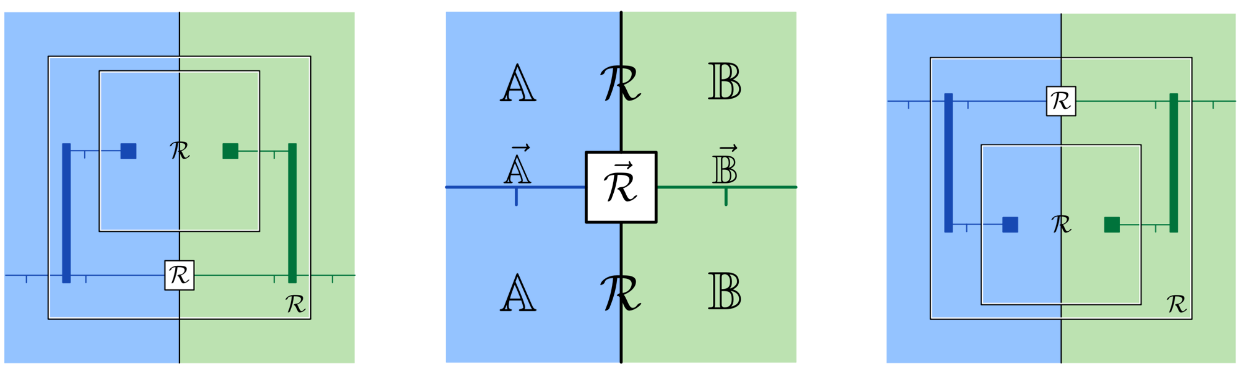
In a definition, each cube is interpreted as well-defined; so if there are multiple ways of constructing the cube from the given structure, they are equal. Hence the above equation of associativity can be drawn as a single cube.




**associativity**

Finally, composition with identities is unital.

**unitality**




$\mathrm{id}.R_0 \cdot r \quad = \quad r \quad = \quad r \cdot \mathrm{id}.R_1$

By the coherence principle, the middle diagram alone expresses the two equations.

In summary, a span of categories $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ determines a displayed category $\mathcal{R} : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$: this is a matrix of categories $\mathcal{R}(\mathrm{A}, \mathrm{B})$ and profunctors $\vec{\mathcal{R}}(\mathrm{a}, \mathrm{b})$, with composition $\vec{\mathcal{R}}(\mathrm{a}_1, \mathrm{b}_1) \circ \vec{\mathcal{R}}(\mathrm{a}_2, \mathrm{b}_2) \Rightarrow \vec{\mathcal{R}}(\mathrm{a}_1\mathrm{a}_2, \mathrm{b}_1\mathrm{b}_2)$ which is associative and unital.

Conversely, the *collage* of a displayed category $\mathcal{R} : \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$ is a span category $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$; this gives the equivalence $\mathrm{Span}\mathbb{C}\mathrm{at} \simeq \mathrm{Dis}\mathbb{C}\mathrm{at}$. [Theorem 14]

The relations of a logic form such a matrix of categories $\mathcal{R}(\mathrm{A}, \mathrm{B})$; hence span categories provide the basic infrastructure for metalogic. Once equipped with horizontal composition, a span category will be a "metarelation", i.e. horizontal profunctor, between logics. **??**

### 1.1.1 Span functors

Let $\mathcal{R}_0 : \mathbb{A}_0 \parallel \mathbb{B}_0$ and $\mathcal{R}_1 : \mathbb{A}_1 \parallel \mathbb{B}_1$ be span categories. A **span functor** from $\mathcal{R}_0$ to $\mathcal{R}_1$ is a pair of functors $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ and $[\![\mathbb{B}]\!] : \mathbb{B}_0 \to \mathbb{B}_1$, and a functor $[\![\mathcal{R}]\!] : \mathcal{R}_0 \to \mathcal{R}_1$ such that the two squares commute, i.e. for any $R : \mathcal{R}_0$ over $(\mathrm{A}, \mathrm{B})$ we have that $[\![R]\!] : \mathcal{R}_1$ lies over $([\![\mathrm{A}]\!], [\![\mathrm{B}]\!])$.

$$\begin{array}{ccccc} \mathbb{A}_0 & \longleftarrow & \mathcal{R}_0 & \longrightarrow & \mathbb{B}_0 \\ \downarrow{\scriptstyle [\![\mathbb{A}]\!]} & & \downarrow{\scriptstyle [\![\mathcal{R}]\!]} & & \downarrow{\scriptstyle [\![\mathbb{B}]\!]} \\ \mathbb{A}_1 & \longleftarrow & \mathcal{R}_1 & \longrightarrow & \mathbb{B}_1 \end{array} \qquad \begin{array}{ccc} \mathcal{R}_0 & \xrightarrow{[\![\mathcal{R}]\!]} & \mathcal{R}_1 \\ \downarrow & & \downarrow \\ \mathbb{A}_0 \times \mathbb{B}_0 & \xrightarrow[{[\![\mathbb{A}]\!] \times [\![\mathbb{B}]\!]}]{} & \mathbb{A}_1 \times \mathbb{B}_1 \end{array}$$

Just as a span category is a matrix of categories, a span functor is a matrix of functors.

**Definition 3.** Let $\mathcal{R}_0 : \mathbb{A}_0 \times \mathbb{B}_0 \to \mathbb{C}\mathrm{at}$ and $\mathcal{R}_1 : \mathbb{A}_1 \times \mathbb{B}_1 \to \mathbb{C}\mathrm{at}$ be displayed categories, and $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$, $[\![\mathbb{B}]\!] : \mathbb{B}_0 \to \mathbb{B}_1$ be functors. A **displayed functor** $[\![\mathcal{R}]\!] : \mathcal{R}_0 \Rightarrow \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])$ gives for each pair:

$$\begin{array}{ccc} \mathbb{A}_0 \times \mathbb{B}_0 & \xrightarrow{\mathcal{R}_0} & \mathbb{C}\mathrm{at} \\ \downarrow{\scriptstyle [\![\mathbb{A}]\!] \times [\![\mathbb{B}]\!]} & \Downarrow{\scriptstyle [\![\mathcal{R}]\!]} & \| \\ \mathbb{A}_1 \times \mathbb{B}_1 & \xrightarrow[\mathcal{R}_1]{} & \mathbb{C}\mathrm{at} \end{array}$$

| | |
|---|---|
| objects | a functor |
| $\mathrm{A} : \mathbb{A}, \mathrm{B} : \mathbb{B}$ | $[\![R]\!](\mathrm{A}, \mathrm{B}) : \mathcal{R}_0(\mathrm{A}, \mathrm{B}) \to \mathcal{R}_1([\![\mathrm{A}]\!], [\![\mathrm{B}]\!])$ |
| morphisms | a transformation |
| $\mathrm{a} : \vec{\mathbb{A}}, \mathrm{b} : \vec{\mathbb{B}}$ | $[\![r]\!](\mathrm{a}, \mathrm{b}) : \vec{\mathcal{R}}_0(\mathrm{a}, \mathrm{b}) \Rightarrow \vec{\mathcal{R}}_1([\![\mathrm{a}]\!], [\![\mathrm{b}]\!])$ |
| composable pairs | an equality |
| $(\mathrm{a}_1, \mathrm{b}_1), (\mathrm{a}_2, \mathrm{b}_2)$ | $[\![r]\!](\mathrm{a}_1\mathrm{a}_2, \mathrm{b}_1\mathrm{b}_2) = [\![r]\!](\mathrm{a}_1, \mathrm{b}_1) \cdot [\![r]\!](\mathrm{a}_2, \mathrm{b}_2)$ |
| objects | an equality |
| $R : \mathcal{R}_0(\mathrm{A}, \mathrm{B})$ | $[\![\mathrm{id}_R]\!] = \mathrm{id}_{[\![R]\!]}$ |

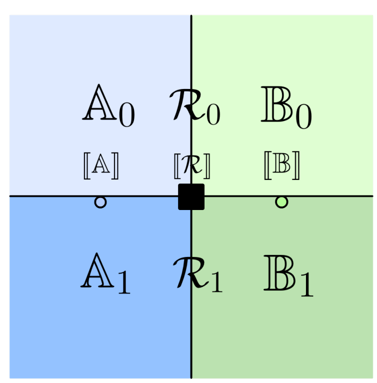
**Proposition 4.** Let $\mathbb{A}_0 \leftarrow \mathcal{R}_0 \to \mathbb{B}_0$ and $\mathbb{A}_1 \leftarrow \mathcal{R}_1 \to \mathbb{B}_1$ be span categories. Let $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ and $[\![\mathbb{B}]\!] : \mathbb{B}_0 \to \mathbb{B}_1$ be functors, and let $[\![\mathcal{R}]\!] : \mathcal{R}_0 \to \mathcal{R}_1$ be a span functor over $[\![\mathbb{A}]\!], [\![\mathbb{B}]\!]$.

Inverse image along $[\![\mathcal{R}]\!]$ determines a displayed functor $[\![\mathcal{R}]\!] : \mathcal{R}_0 \Rightarrow \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])$.

We now expound this idea, in color syntax.

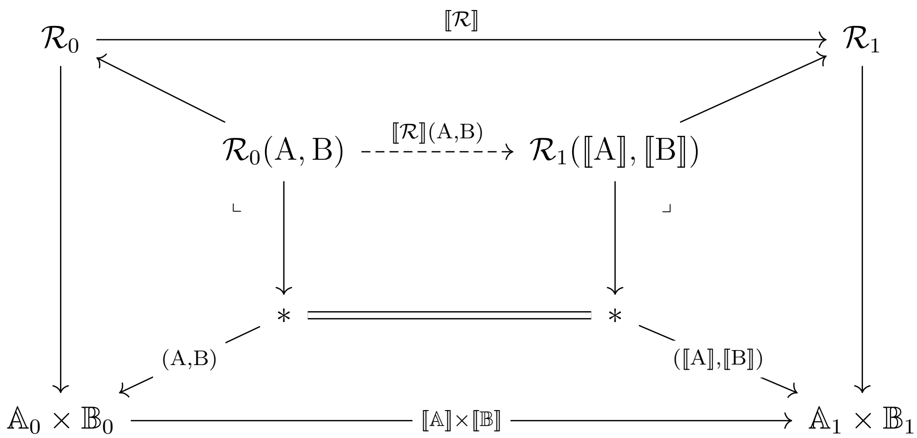
A functor is a *transversal* morphism in $\mathrm{Span}\mathbb{C}\mathrm{at}$, drawn as a string with a small "bubble" pointer, filled with the color of its source. A span functor, like a transformation, is drawn as a solid black bead, to distinguish from the "open" bead of a span profunctor.

$$\begin{array}{ccccc} \mathbb{A}_0 & \longleftarrow & \mathcal{R}_0 & \longrightarrow & \mathbb{B}_0 \\ \downarrow{\scriptstyle [\![\mathbb{A}]\!]} & & \downarrow{\scriptstyle [\![\mathcal{R}]\!]} & & \downarrow{\scriptstyle [\![\mathbb{B}]\!]} \\ \mathbb{A}_1 & \longleftarrow & \mathcal{R}_1 & \longrightarrow & \mathbb{B}_1 \end{array}$$

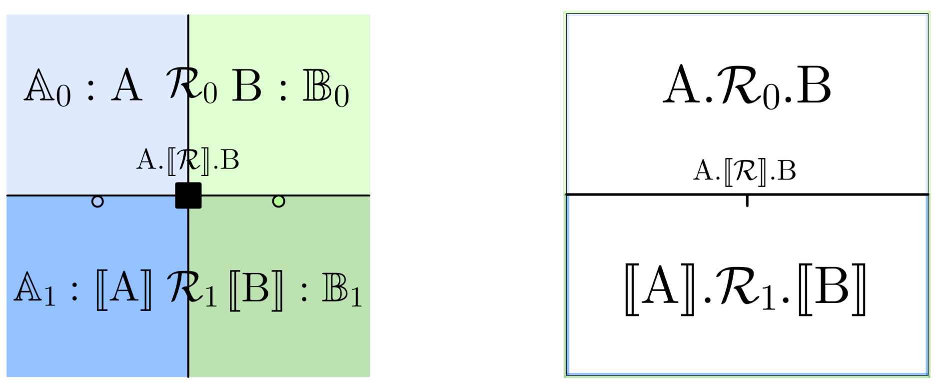



span functor $[\![\mathcal{R}]\!] : \mathcal{R}_0 \to \mathcal{R}_1$

Inverse image defines a matrix of functors $[\![\mathcal{R}]\!](\mathrm{A},\mathrm{B}) : \mathcal{R}_0(\mathrm{A},\mathrm{B}) \to \mathcal{R}_1([\![\mathrm{A}]\!],[\![\mathrm{B}]\!])$, by functoriality of pullback.

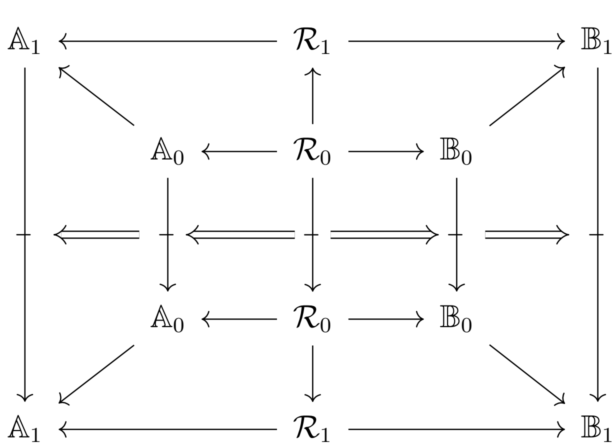


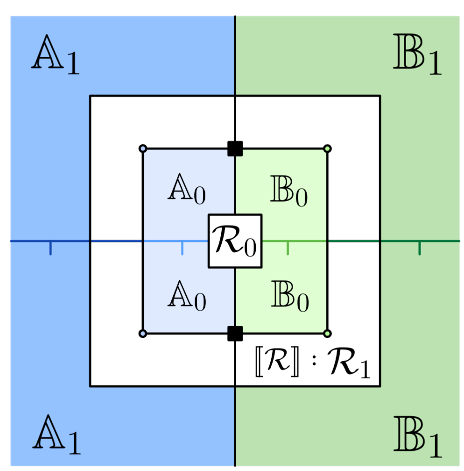

Each functor is determined in color syntax by substituting a pair of objects $\mathrm{A},\mathrm{B}$ into the base categories $\mathbb{A}_0, \mathbb{B}_0$ of the source span category $\mathcal{R}_0$.



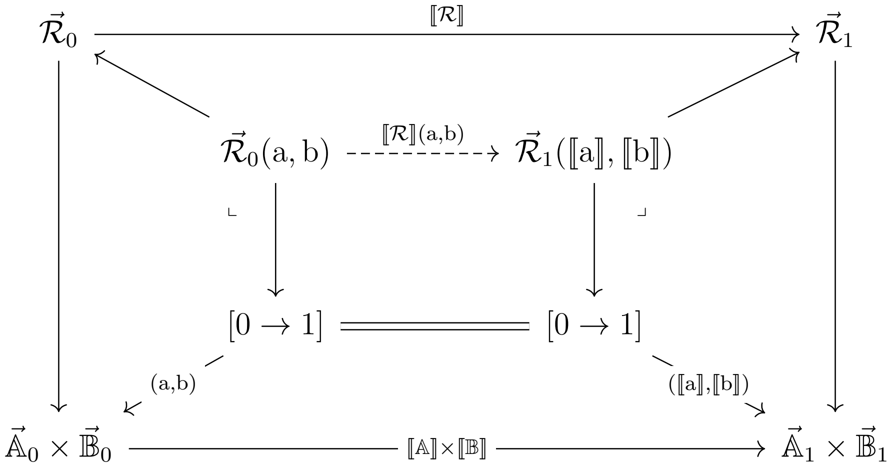

$[\![\mathcal{R}]\!](\mathrm{A},\mathrm{B}) : \mathcal{R}_0(\mathrm{A},\mathrm{B}) \to \mathcal{R}_1([\![\mathrm{A}]\!],[\![\mathrm{B}]\!])$

The span functor induces a transformation of span profunctors. As span profunctors are two-dimensional, this transformation is *three-dimensional*, depicted below on the right. To distinguish this transformation in the diagram, we may designate white space between the span functor and the hom of the target span category.

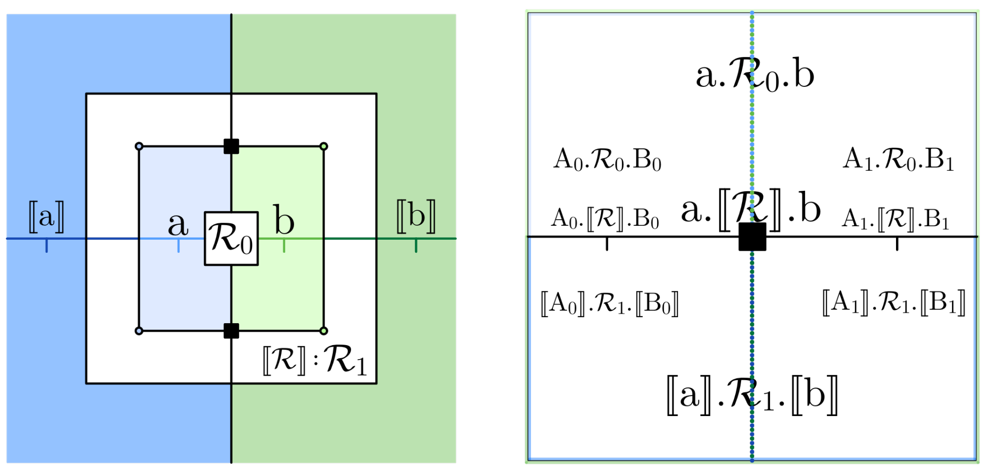


Inverse image determines a matrix of transformations $[\![\mathcal{R}]\!](\mathrm{a},\mathrm{b}) : \vec{\mathcal{R}}_0(\mathrm{a},\mathrm{b}) \Rightarrow \vec{\mathcal{R}}_1([\![\mathrm{a}]\!],[\![\mathrm{b}]\!])$, by functoriality of pullback in $\mathrm{Prof}$.

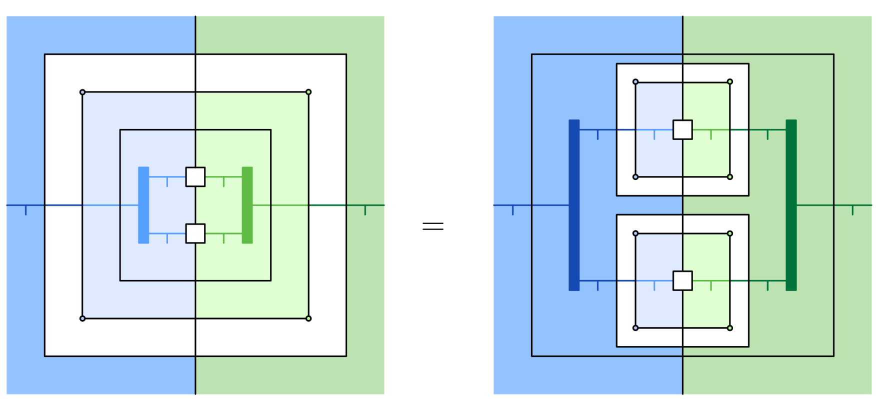



Again, this is represented in color syntax by substitution.



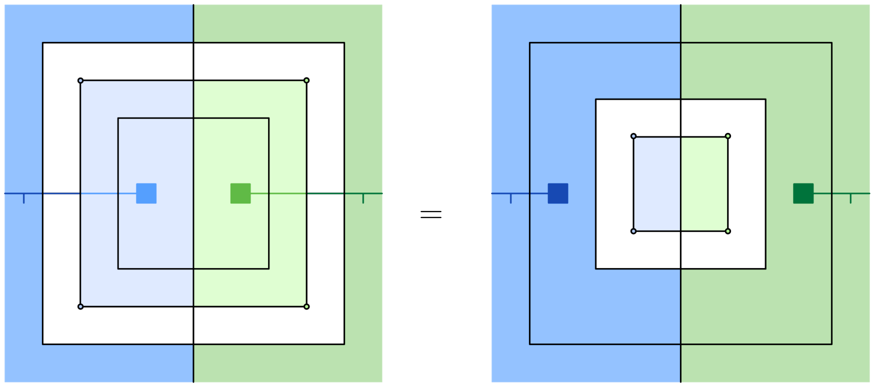

$[\![\mathcal{R}]\!](\mathrm{a},\mathrm{b}) : \mathcal{R}_0(\mathrm{a},\mathrm{b}) \Rightarrow \mathcal{R}_1([\![\mathrm{a}]\!],[\![\mathrm{b}]\!])$

This completes the structure of a displayed functor $[\![\mathcal{R}]\!] : \mathcal{R}_0 \Rightarrow \mathcal{R}_1([\![\mathbb{A}]\!],[\![\mathbb{B}]\!])$. Lastly: the property that it preserves the composition and unit of the displayed categories $\mathcal{R}_0, \mathcal{R}_1$.



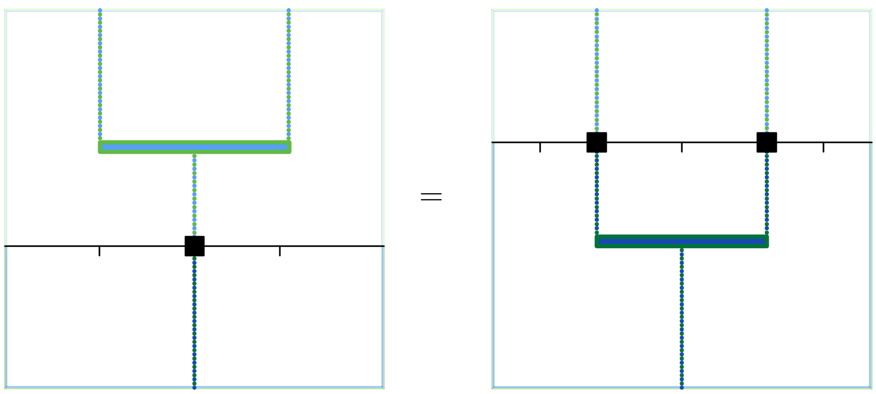

$[\![r_1 \cdot r_2]\!]$ $\qquad$ $[\![r_1]\!] \cdot [\![r_2]\!]$

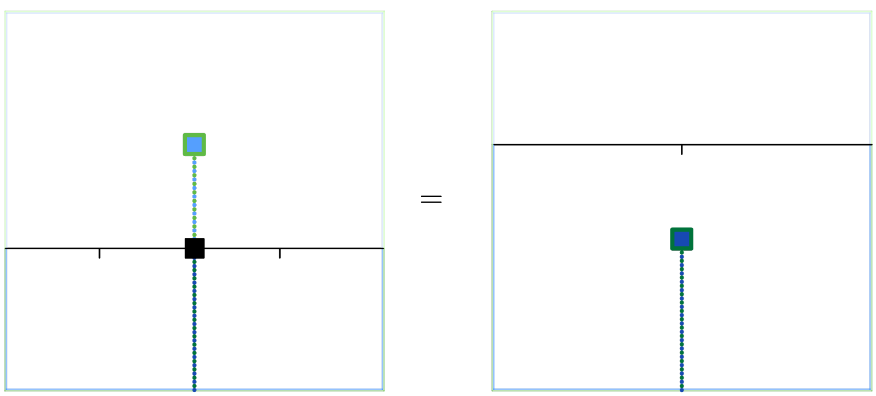
Contents



$[\![\mathrm{id}.R]\!]$ $\qquad$ $\mathrm{id}.[\![R]\!]$

This defines a transformation of lax functors, i.e. displayed functor $[\![\mathcal{R}]\!] : \mathcal{R}_0 \Rightarrow \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])$.





So, a span functor $[\![\mathcal{R}]\!] : \mathcal{R}_0 \to \mathcal{R}_1$ over functors $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ and $[\![\mathbb{B}]\!] : \mathbb{B}_0 \to \mathbb{B}_1$ is equivalent to a matrix of functors $[\![\mathcal{R}]\!](\mathrm{A}, \mathrm{B}) : \mathcal{R}_0(\mathrm{A}, \mathrm{B}) \to \mathcal{R}_1([\![\mathrm{A}]\!], [\![\mathrm{B}]\!])$ and transformations $[\![\mathcal{R}]\!](\mathrm{a}, \mathrm{b}) : \vec{\mathcal{R}}_0(\mathrm{a}, \mathrm{b}) \Rightarrow \vec{\mathcal{R}}_1([\![\mathrm{a}]\!], [\![\mathrm{b}]\!])$, which preserves the composition and unit of $\mathcal{R}_0$ and $\mathcal{R}_1$.

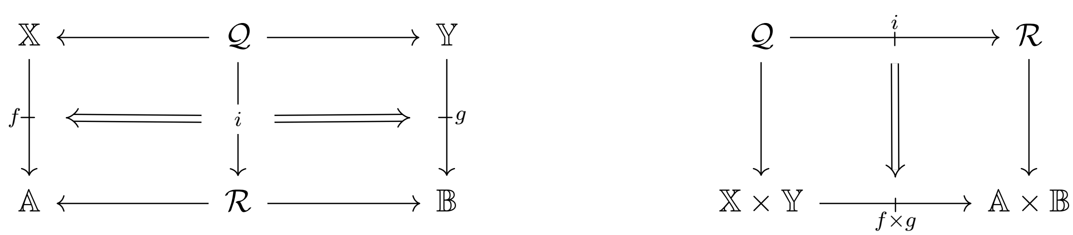
## 1.2 Span profunctors

The previous section gave the known equivalence between span categories $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ and displayed categories $\mathcal{R}\colon \mathbb{A}\times\mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$. We now extend this idea to *spans of profunctors*; surprisingly this has not been explored, and yet it works in the exact same way.

We introduce the concept of *displayed profunctor*, a bimodule of displayed categories, formed by inverse image along a transformation. These are the relations in the logic of displayed categories, which we prove equivalent to the logic of span categories.

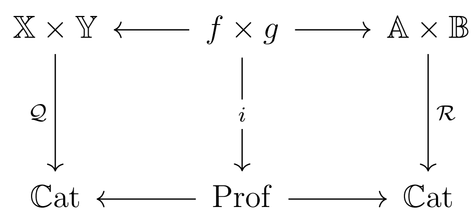
**Definition 5.** Let $\mathbb{X} \leftarrow \mathcal{Q} \rightarrow \mathbb{Y}$ and $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ be span categories. A **span profunctor** from $\mathcal{Q}$ to $\mathcal{R}$ is a pair of profunctors $f\colon \mathbb{X} \mid \mathbb{A}$ and $g\colon \mathbb{Y} \mid \mathbb{B}$, and a profunctor $i\colon \mathcal{Q} \mid \mathcal{R}$ with transformations $\pi^i_f\colon i \Rightarrow f(\pi^{\mathcal{Q}}_{\mathbb{X}}, \pi^{\mathcal{R}}_{\mathbb{A}})$ and $\pi^i_g\colon i \Rightarrow g(\pi^{\mathcal{Q}}_{\mathbb{Y}}, \pi^{\mathcal{R}}_{\mathbb{B}})$, denoted $i(f,g)\colon \mathcal{Q}(\mathbb{X},\mathbb{Y}) \mid \mathcal{R}(\mathbb{A},\mathbb{B})$.

Note this data is equivalent to a transformation $(\pi^i_f, \pi^i_g)\colon i \Rightarrow (f\times g)(\pi^{\mathcal{Q}}_{\mathbb{X}} \times \pi^{\mathcal{Q}}_{\mathbb{Y}}, \pi^{\mathcal{R}}_{\mathbb{A}} \times \pi^{\mathcal{R}}_{\mathbb{B}})$.



**Definition 6.** Let $\mathcal{Q}\colon \mathbb{X}\times\mathbb{Y} \rightarrow \mathbb{C}\mathrm{at}$ and $\mathcal{R}\colon \mathbb{A}\times\mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$ be displayed categories.

A **displayed profunctor** $i\colon f\times g \rightarrow \mathrm{Prof}$ from $\mathcal{Q}$ to $\mathcal{R}$ is a map



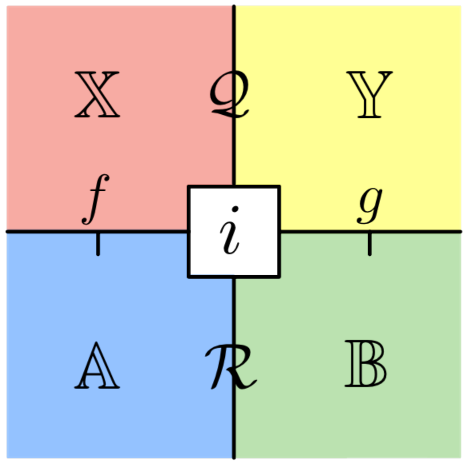
which gives for each pair:

| | |
|---|---|
| elements $\mathrm{f}\colon f(\mathrm{X},\mathrm{A}), \mathrm{g}\colon g(\mathrm{Y},\mathrm{B})$ | a profunctor $i(\mathrm{f},\mathrm{g})\colon \mathcal{Q}(\mathrm{X},\mathrm{A}) \mid \mathcal{R}(\mathrm{Y},\mathrm{B})$ |
| composable pairs $(\mathrm{x},\mathrm{y}),(\mathrm{f},\mathrm{g})$ | a transformation $q\cdot i\colon \vec{\mathcal{Q}}(\mathrm{x},\mathrm{y}) \circ i(\mathrm{f},\mathrm{g}) \Rightarrow i(\mathrm{xf},\mathrm{yg})$ |
| composable pairs $(\mathrm{f},\mathrm{g}),(\mathrm{a},\mathrm{b})$ | a transformation $i\cdot r\colon i(\mathrm{f},\mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a},\mathrm{b}) \Rightarrow i(\mathrm{fa},\mathrm{gb})$ |
| with associativity | $(q\cdot i)\cdot r = q\cdot(i\cdot r)$ |
| and unitality | $\mathrm{id}.Q\cdot i = i = i\cdot \mathrm{id}.R$ . |

**Proposition 7.** Let $\mathcal{Q}(\mathbb{X},\mathbb{Y})$ and $\mathcal{R}(\mathbb{A},\mathbb{B})$ be span categories, and $i(f,g)$ a span profunctor. Inverse image along $i \Rightarrow f\times g$ determines a displayed profunctor $i\colon f\times g \rightarrow \mathrm{Prof}$.

Just as a displayed category is a map $\mathcal{R}\colon \mathbb{A}\times\mathbb{B} \rightarrow \mathbb{C}\mathrm{at}$ with a "monad" structure for composition, i.e. a "lax functor", a displayed profunctor is a *bimodule* of such monads [17]. We now expound the concept, continuing to expand the visual language of $\mathrm{Span}\mathbb{C}\mathrm{at}$.

Generalizing the hom of a span category, a span profunctor can be drawn as a bead which connects the string of one span category to another, along the profunctors $f$ and $g$ drawn as downward "pointer strings".

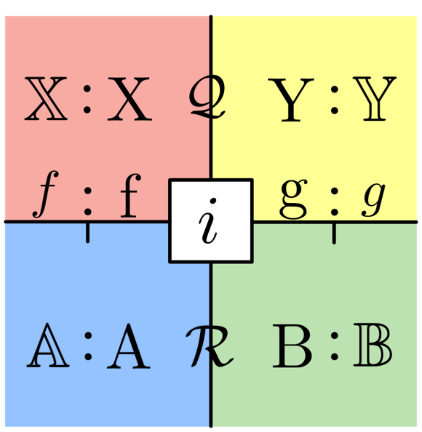


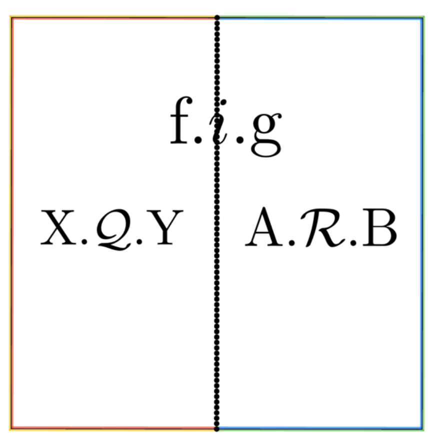

span profunctor $i(f,g) : \mathcal{Q} \,|\, \mathcal{R}$

Inverse image along the transformation $i \Rightarrow f \times g$ determines a *matrix of profunctors*: for each $\mathrm{f} : f(\mathrm{X},\mathrm{A})$ and $\mathrm{g} : g(\mathrm{Y},\mathrm{B})$ there is a profunctor $i(\mathrm{f},\mathrm{g})$ from category $\mathcal{Q}(\mathrm{X},\mathrm{Y})$ to category $\mathcal{R}(\mathrm{A},\mathrm{B})$. This is given by pullback in $\mathrm{Prof}$ of $i \to f \times g$ along the transformation which maps the walking arrow to the pair $(\mathrm{f},\mathrm{g}) : f \times g$.

$$\begin{array}{ccc} i(\mathrm{f},\mathrm{g}) & \longrightarrow & i \\ \downarrow & \lrcorner & \downarrow \\ [0 \to 1] & \xrightarrow{(\mathrm{f},\mathrm{g})} & f \times g \end{array}$$

This is represented in color syntax by substituting a pair $\mathrm{f},\mathrm{g}$ into the strings of $f$ and $g$. The resulting profunctor is a relation in the logic of $\mathbb{C}\mathrm{at}$, drawn on the right.

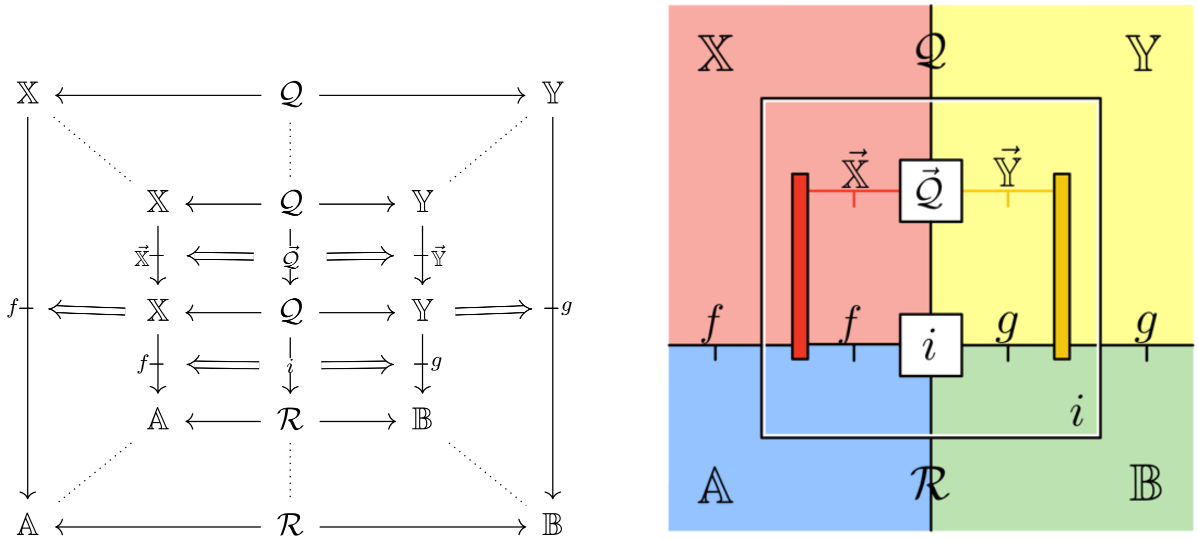


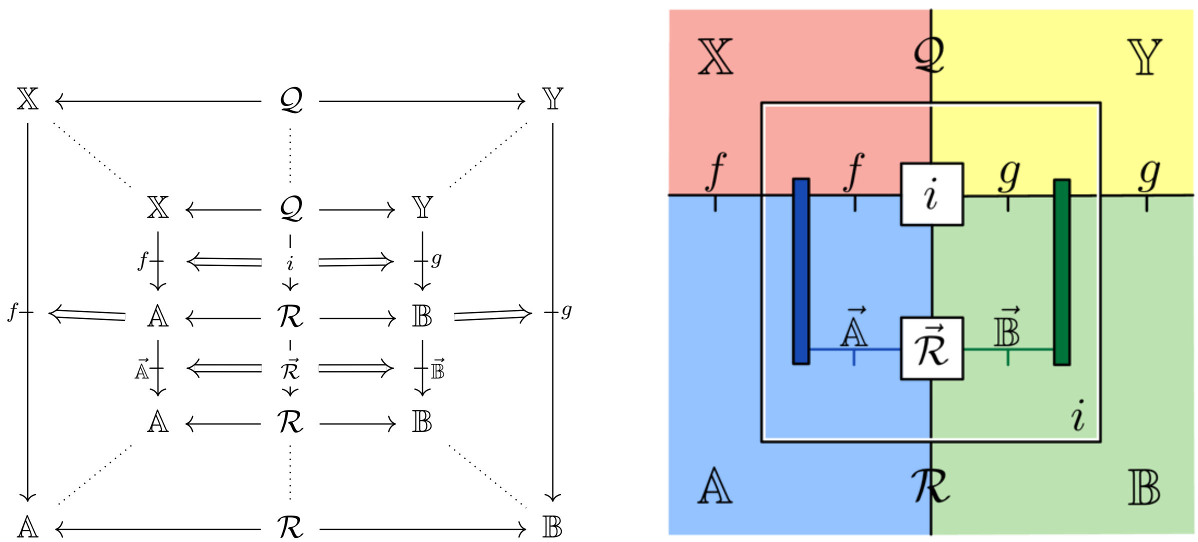

profunctor $i(\mathrm{f},\mathrm{g}) : \mathcal{Q}(\mathrm{X},\mathrm{Y}) \,|\, \mathcal{R}(\mathrm{A},\mathrm{B})$

So the above is the data of a span profunctor, which is two-dimensional. Now we explicate its *structure*, sequential composition, which is *three-dimensional*.

A span profunctor $i : \mathcal{Q} \mid \mathcal{R}$ has a *precompose* action $\vec{\mathcal{Q}} \circ i \to i$, and a *postcompose* action $i \circ \mathcal{R} \to i$. Below, these are given in conventional diagrams and string diagrams.

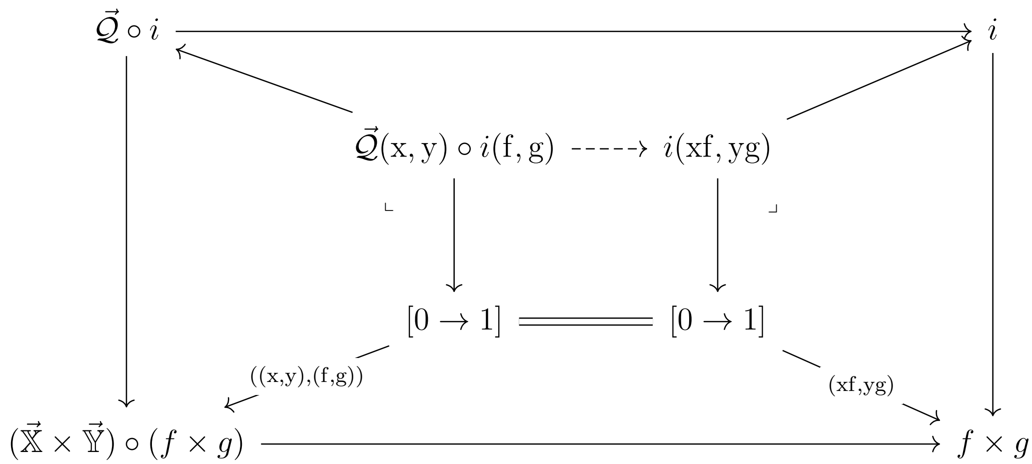


precompose action $\vec{\mathcal{Q}} \circ i \Rightarrow i$

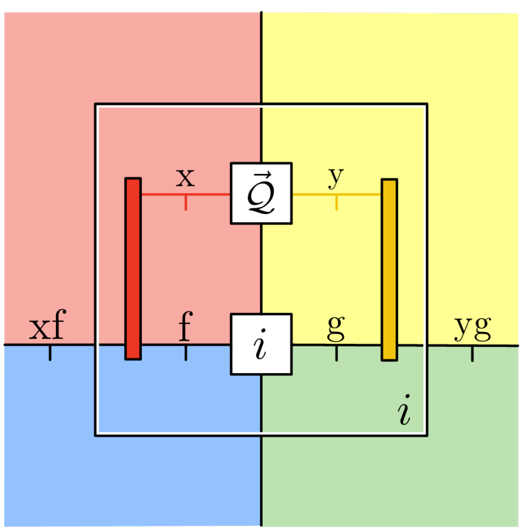


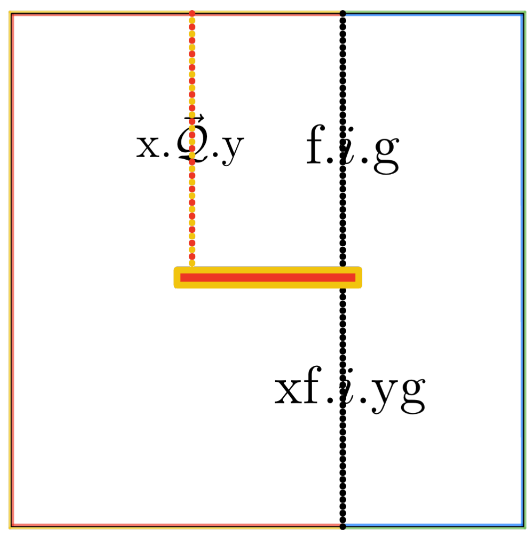
postcompose action $i \circ \vec{\mathcal{R}} \Rightarrow i$

Precomposition by $\mathcal{Q}$ is a matrix of transformations (indexed by composable pairs) $\vec{Q}(\mathrm{x},\mathrm{y}) \circ i(\mathrm{f},\mathrm{g}) \Rightarrow i(\mathrm{xf},\mathrm{yg})$. This is given by the functoriality of pullback in $\mathrm{Prof}$.



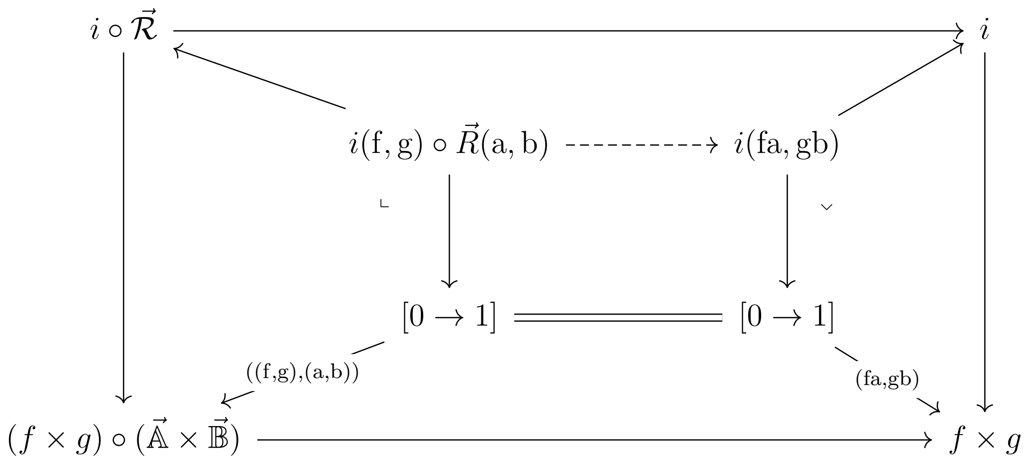

So, substitution in the string diagram for composition determines a transformation in $\mathbb{C}\mathrm{at}$.

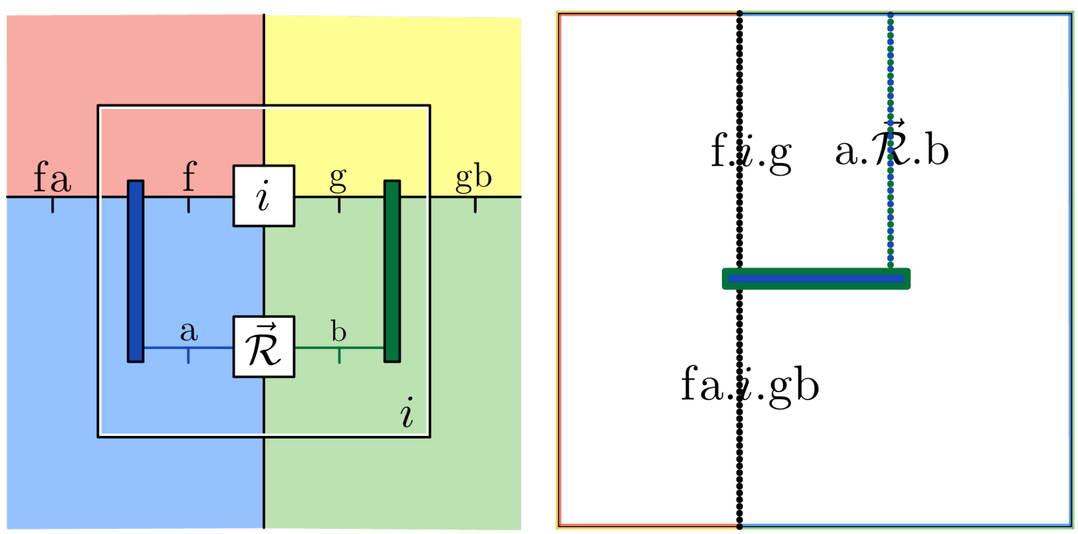



$q \cdot i : \vec{\mathcal{Q}}(\mathrm{x},\mathrm{y}) \circ i(\mathrm{f},\mathrm{g}) \Rightarrow i(\mathrm{xf},\mathrm{yg})$

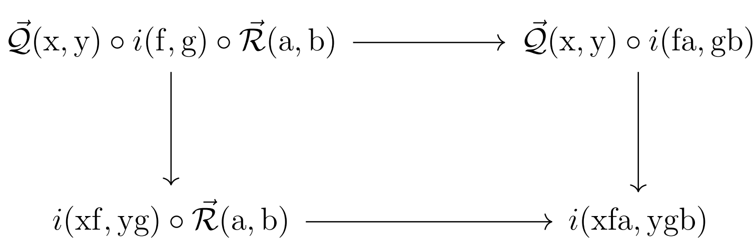
Postcomposition by $\vec{\mathcal{R}}$ is a matrix of transformations $i(\mathrm{f},\mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a},\mathrm{b}) \Rightarrow i(\mathrm{fa},\mathrm{gb})$.



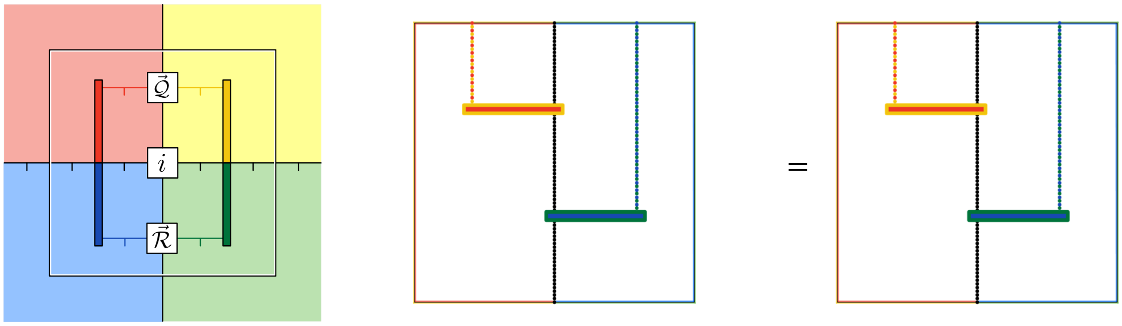




$i \cdot r : i(\mathrm{f},\mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a},\mathrm{b}) \Rightarrow i(\mathrm{fa},\mathrm{gb})$

Hence the structure of precomposition by $\vec{\mathcal{Q}}$ and postcomposition by $\vec{\mathcal{R}}$ is given by matrices of transformations $\vec{\mathcal{Q}}(\mathrm{x},\mathrm{y}) \circ i(\mathrm{f},\mathrm{g}) \Rightarrow i(\mathrm{xf},\mathrm{yg})$ and $i(\mathrm{f},\mathrm{g}) \circ \vec{\mathcal{R}}(\mathrm{a},\mathrm{b}) \Rightarrow i(\mathrm{fa},\mathrm{gb})$. To complete the exposition, this structure satisfies the property of associativity and unitality.

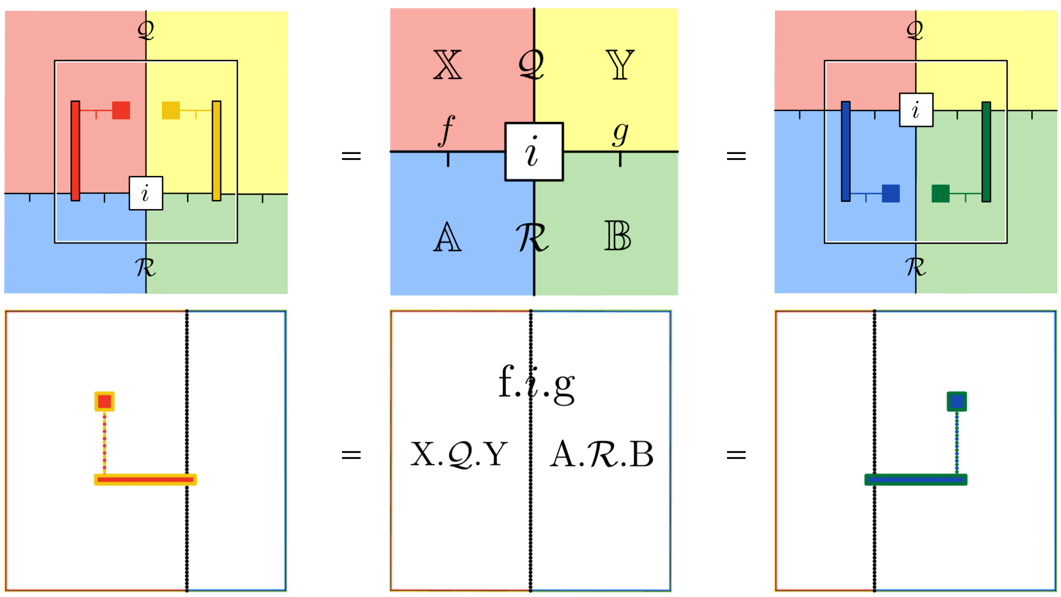



By the "coherence principle" of string diagrams, introduced for span categories, associativity can be depicted simply by drawing the cube $\mathcal{Q} \circ i \circ \mathcal{R} \to i$. This expresses that the cube is "coherent" or well-defined, i.e. the two transformations $\vec{\mathcal{Q}} \circ i \circ \vec{\mathcal{R}} \to i$ are equal.




$\mathcal{Q}(\mathrm{x},\mathrm{y})\cdot(i(\mathrm{f},\mathrm{g})\cdot\mathcal{R}(\mathrm{a},\mathrm{b}))$ $\quad$ $(\mathcal{Q}(\mathrm{x},\mathrm{y})\cdot i(\mathrm{f},\mathrm{g}))\cdot\mathcal{R}(\mathrm{a},\mathrm{b})$

Finally, composition is unital.




$\mathrm{id}_Q\cdot i$ $\quad$ $i$ $\quad$ $i\cdot\mathrm{id}_R$

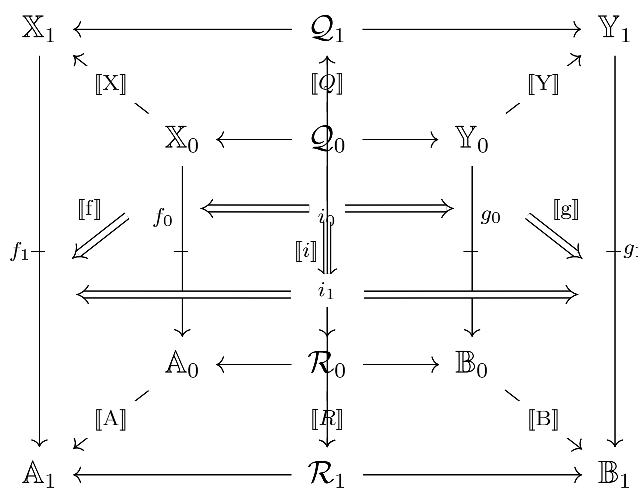
So, a span profunctor determines a matrix of profunctors $i(\mathrm{f},\mathrm{g}) : \mathcal{Q}(\mathrm{X},\mathrm{Y}) \mid \mathcal{R}(\mathrm{A},\mathrm{B})$, with actions for sequential composition $\vec{\mathcal{Q}}(\mathrm{x},\mathrm{y})\circ i(\mathrm{f},\mathrm{g}) \Rightarrow i(\mathrm{xf},\mathrm{yg})$ and $i(\mathrm{f},\mathrm{g})\circ\vec{\mathcal{R}}(\mathrm{a},\mathrm{b}) \Rightarrow i(\mathrm{fa},\mathrm{gb})$, which are associative and unital.

This concept is precisely what was needed to complete the framework for metalogic: the *inferences* or “transformations” of a logic form a matrix of profunctors. Once we add parallel composition, span profunctors form “meta inferences”, i.e. double profunctors, between logics. Metalogic is the language of metainferences and their transformations.

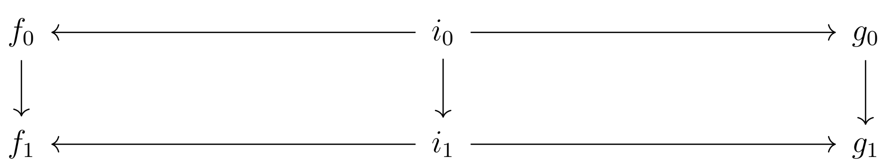
### 1.2.1 Span Transformations

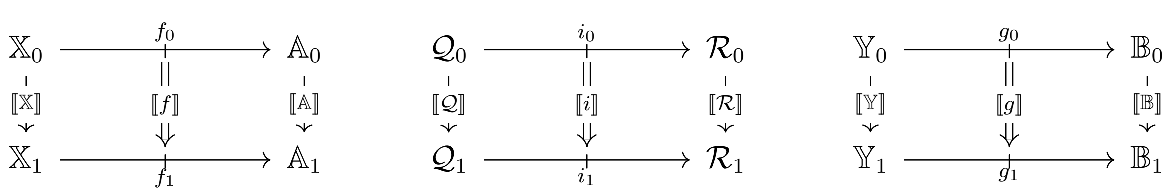
We have seen thus far that a span profunctor can be understood as a system of inference. Now, *meta*logic is the language of such systems of inference, and *transformations* thereof. Completing the logic of span categories, we define span transformations.

**Definition 8.** Let $\mathcal{Q}_0 : \mathbb{X}_0 \parallel \mathbb{Y}_0, \mathcal{R}_0 : \mathbb{A}_0 \parallel \mathbb{B}_0, \mathcal{Q}_1 : \mathbb{X}_1 \parallel \mathbb{Y}_1, \mathcal{R}_1 : \mathbb{A}_1 \parallel \mathbb{B}_1$ be span categories.
Let $[\![\mathcal{Q}]\!] : \mathcal{Q}_0 \to \mathcal{Q}_1$ and $[\![\mathcal{R}]\!] : \mathcal{R}_0 \to \mathcal{R}_1$ be span functors over $[\![\mathbb{X}]\!], [\![\mathbb{Y}]\!]$ and $[\![\mathbb{A}]\!], [\![\mathbb{B}]\!]$.
Let $i_0(f_1, g_1) : \mathcal{Q}_0 \mid \mathcal{R}_0$, $i_1(f_1, g_1) : \mathcal{Q}_1 \mid \mathcal{R}_1$ be span profunctors.
A **span transformation** $[\![i]\!] : i_0 \Rightarrow i_1$ is a pair of transformations $[\![f]\!] : f_0 \Rightarrow f_1$ over $[\![\mathbb{X}]\!], [\![\mathbb{A}]\!]$ and $[\![g]\!] : g_0 \Rightarrow g_1$ over $[\![\mathbb{Y}]\!], [\![\mathbb{B}]\!]$, and a transformation $[\![i]\!] : i_0 \Rightarrow i_1$ over $[\![\mathcal{Q}]\!], [\![\mathcal{R}]\!]$, such that the two squares commute.

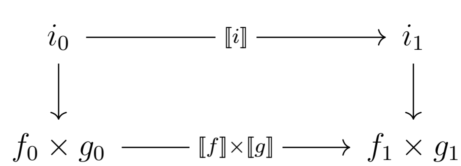



In two dimensions, the morphism of span profunctors is drawn as follows.



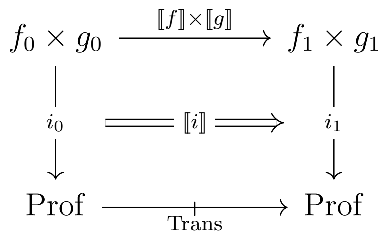




Note this is equivalent to one commutative square of transformations.



Just as a span profunctor determines a matrix of profunctors, a span transformation determines a matrix of transformations.

**Definition 9.** A **displayed transformation** $[\![i]\!] : i_0 \Rightarrow i_1([\![f]\!], [\![g]\!])$ is a map



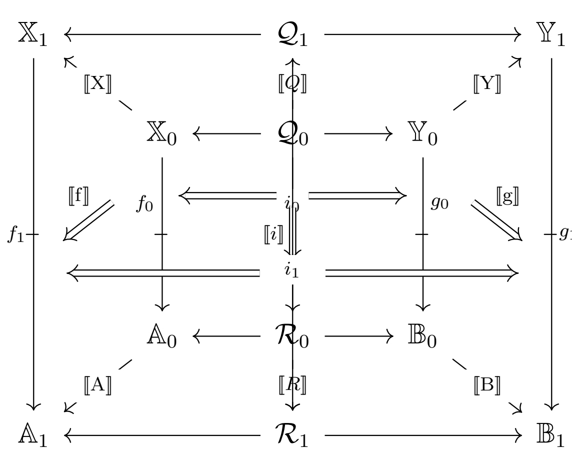

which gives for each

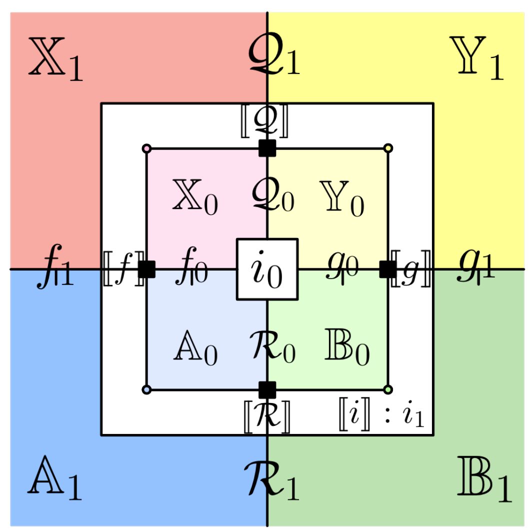
| | |
|---|---|
| pair of morphisms | a transformation |
| $\mathrm{f} : f_0(\mathrm{X}, \mathrm{A}), \mathrm{g} : g_0(\mathrm{Y}, \mathrm{B})$ | $[\![i]\!](\mathrm{f}, \mathrm{g}) : i_0(\mathrm{f}, \mathrm{g}) \Rightarrow i_1([\![\mathrm{f}]\!], [\![\mathrm{g}]\!])$ |
| composable pair | an equality |
| $q : \vec{\mathcal{Q}}_0(\mathrm{x}, \mathrm{y}), i : i_0(\mathrm{f}, \mathrm{g})$ | $[\![q \cdot i]\!] = [\![q]\!] \cdot [\![i]\!] : i_1([\![\mathrm{xf}]\!], [\![\mathrm{yg}]\!])$ |
| composable pair | an equality |
| $i : i_0(\mathrm{f}, \mathrm{g}), r : \vec{\mathcal{R}}_0(\mathrm{a}, \mathrm{b})$ | $[\![i \cdot r]\!] = [\![i]\!] \cdot [\![r]\!] : i_1([\![\mathrm{fa}]\!], [\![\mathrm{gb}]\!])$ . |

We now expound the idea, completing the visual language of $\mathrm{Span}\mathbb{C}\mathrm{at}$.

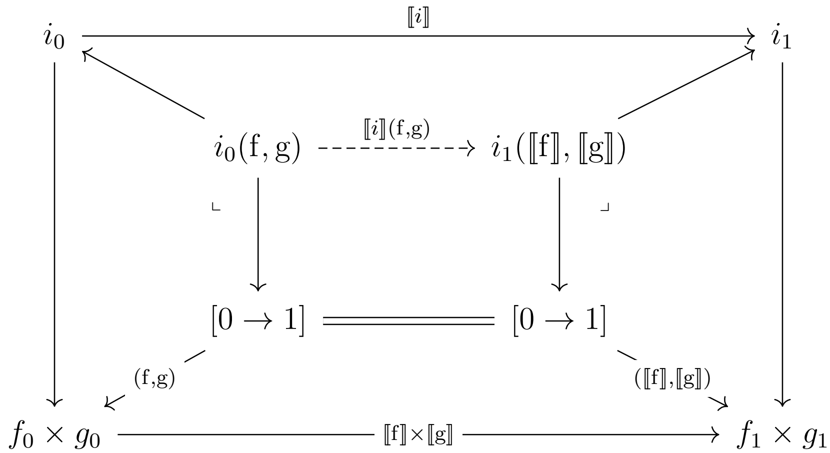
A span transformation is a cube: the inner face is the source span profunctor $i_0$, and the outer face is the target span profunctor $i_1$. The left and right faces are transformations $[\![f]\!] : f_0 \Rightarrow f_1$ and $[\![g]\!] : g_0 \to g_1$, and the top and bottom are span functors $[\![\mathcal{Q}]\!] : \mathcal{Q}_0 \to \mathcal{Q}_1$ and $[\![\mathcal{R}]\!] : \mathcal{R}_0 \to \mathcal{R}_1$. The span transformation $[\![i]\!] : i_0 \Rightarrow i_1$ fills the cube.

Note that the string diagram is exactly dual to the conventional diagram.

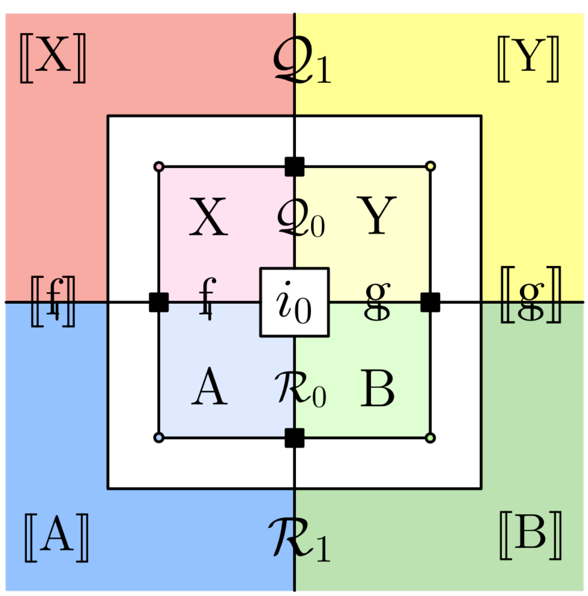


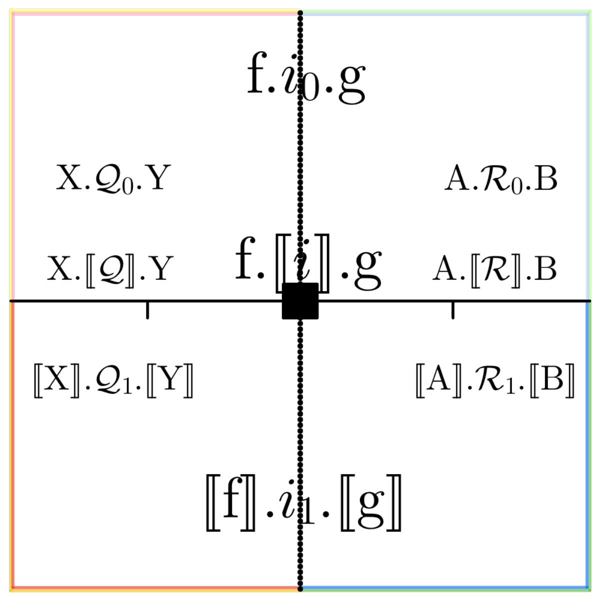


Substitution determines a matrix of transformations, by functoriality of pullback.



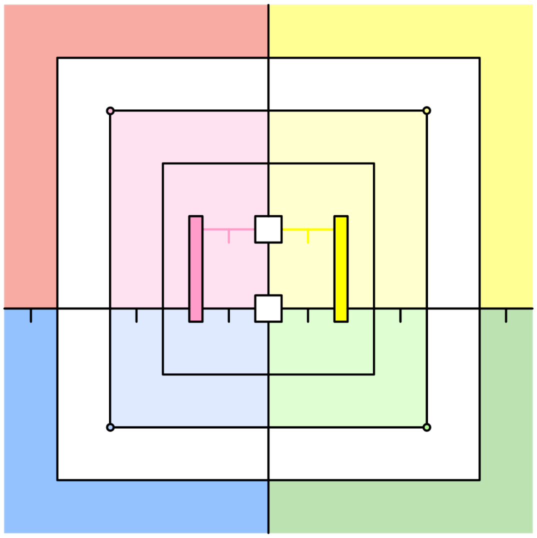

This is given in color syntax by substituting elements $\mathrm{f}: f_0, \mathrm{g}: g_0$ into the profunctors of the source span profunctor $i_0$, determining the transformation $[\![i]\!](\mathrm{f},\mathrm{g}) : i_0(\mathrm{f},\mathrm{g}) \Rightarrow i_1([\![\mathrm{f}]\!],[\![\mathrm{g}]\!])$.



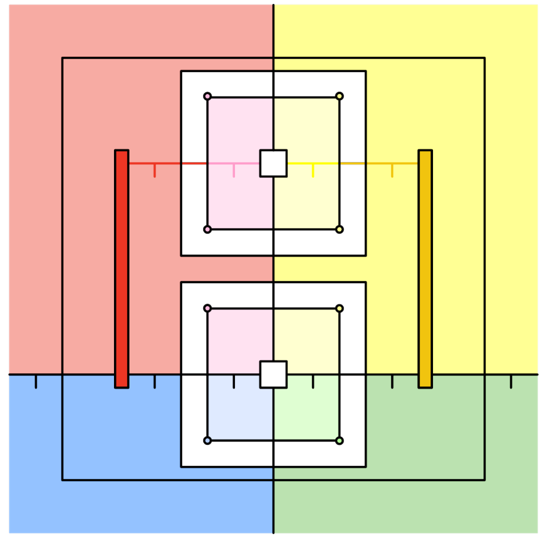

So, the structure of a span transformation is three-dimensional: maps of squares.

Finally, it just has one property: the transformation is *natural* with respect to the actions of $i_0$ and $i_1$, i.e. it preserves sequential composition of inference.

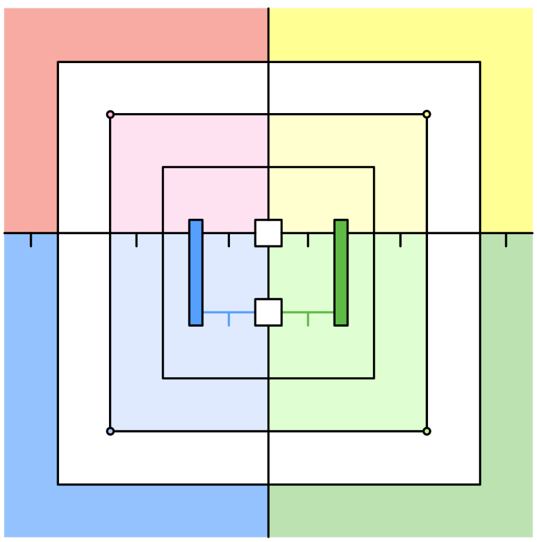


$[\![q \cdot i]\!]$

=

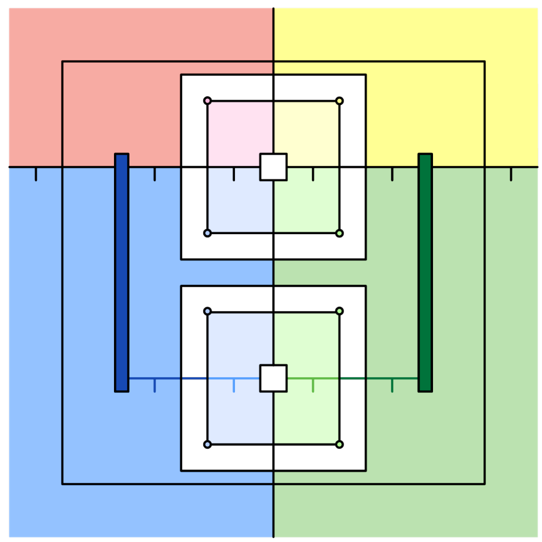


$[\![q]\!] \cdot [\![i]\!]$



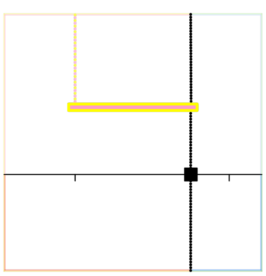
$[\![i \cdot r]\!]$

=



$[\![i]\!] \cdot [\![r]\!]$

This is a transformation of bimodules of lax functors, i.e. displayed transformation.



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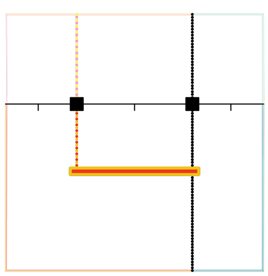

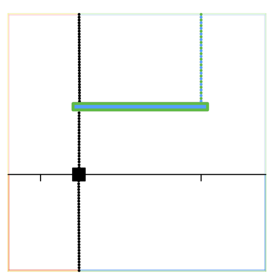

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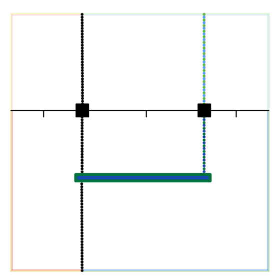
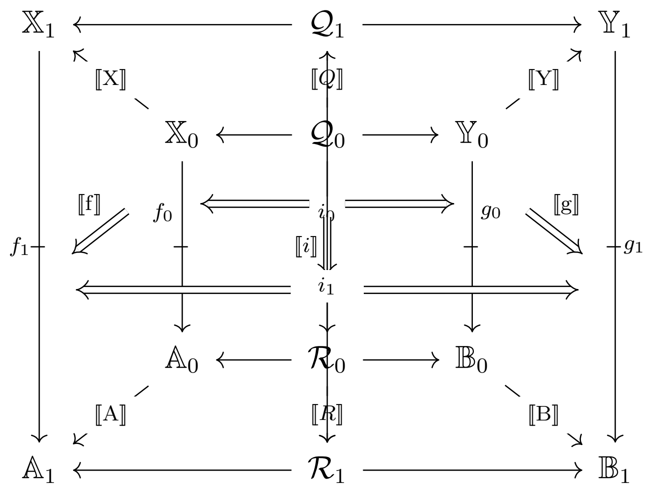


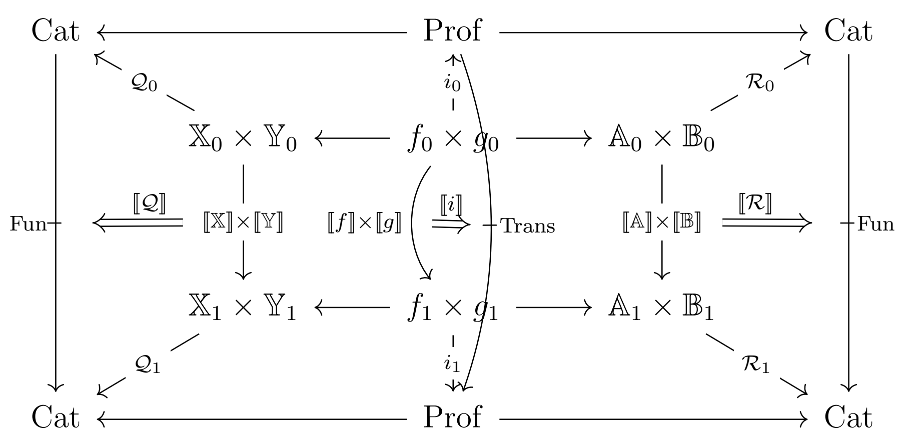
Thus, a span transformation $[\![i]\!] : i_0 \Rightarrow i_1$ determines a matrix of transformations

$$[\![i]\!](\mathrm{f}, \mathrm{g}) : i_0(\mathrm{f}, \mathrm{g}) \Rightarrow i_1([\![\mathrm{f}]\!], [\![\mathrm{g}]\!]).$$

In giving the correspondence of span categories and displayed categories, we have essentially turned $\mathrm{Span}\mathbb{C}\mathrm{at}$ "inside out": given a span transformation



span categories become lax functors, and and span functors become transformations thereof; span profunctors become bimodules of lax functors, and span transformations become transformations thereof.



Ultimately, this fact is the realization of the *collage completeness* of $\mathbb{C}\mathrm{at}$ [5], by which $\mathbb{C}\mathrm{at}$ forms a classifying object for spans of categories.

**Definition 10.** Let $i : f \times g \to \mathrm{Prof}$ be a displayed profunctor. The **collage** $\Sigma i$ is the set

$$\Sigma i \quad \equiv \quad \Sigma \mathrm{f} : f \, \Sigma \mathrm{g} : g. \;\; i(\mathrm{f}, \mathrm{g})$$

which is equipped with projections to $f$ and $g$.

The actions $\vec{\mathcal{Q}}(\mathrm{x}, \mathrm{y}) \circ i(\mathrm{f}, \mathrm{g}) \Rightarrow i(\mathrm{xf})$ define an action $\Sigma\vec{\mathcal{Q}} \circ \Sigma i \Rightarrow \Sigma i$; and similarly for $\mathcal{R}$. The associativity and unitality of the former give that of the latter.

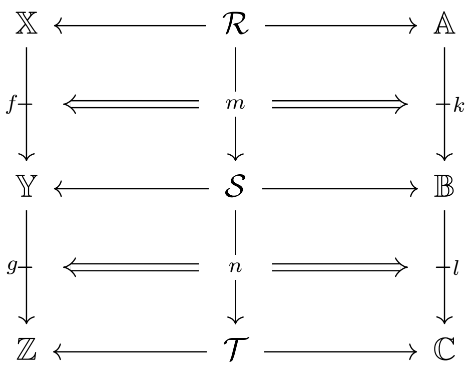
This construction is functorial: given a displayed transformation $[\![i]\!] : i_0 \Rightarrow i_1$, as above, the collage $\Sigma[\![i]\!] : \Sigma i_0 \Rightarrow \Sigma i_1$ is a span transformation.

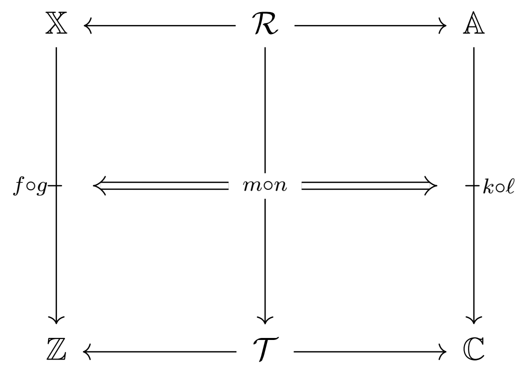
This gives a bijection of squares between the double category $\mathrm{Span}\mathbb{C}\mathrm{at}$ and $\mathrm{Dis}\mathbb{C}\mathrm{at}$.

## 1.3 Sequential composition

We now complete the equivalence of double categories $\mathrm{Span}\mathbb{C}\mathrm{at} \simeq \mathrm{Dis}\mathbb{C}\mathrm{at}$. It remains to define *sequential composition* of span profunctors and displayed profunctors, as this is the horizontal composition of each double category.

**Definition 11.** Let $m(f,k) : \mathcal{R}(\mathbb{X},\mathbb{A}) \mid \mathcal{S}(\mathbb{Y},\mathbb{B})$ and $n(g,l) : \mathcal{S}(\mathbb{Y},\mathbb{B}) \mid \mathcal{T}(\mathbb{Z},\mathbb{C})$ be a composable pair of span profunctors. The **sequential composite** $(m \circ n)(f \circ g, k \circ \ell) : \mathcal{R}(\mathbb{X},\mathbb{A}) \mid \mathcal{T}(\mathbb{Z},\mathbb{C})$ is the span of profunctor composites.





An element of $f \circ g$ is an indexed pair $\mathrm{Y}.(\mathrm{f},\mathrm{g}) : f(\mathrm{X},\mathrm{Y}) \times g(\mathrm{Y},\mathrm{Z})$, and of $k \circ \ell$ is $\mathrm{B}.(\mathrm{k},\mathrm{l}) : k(\mathrm{A},\mathrm{B}) \times \ell(\mathrm{B},\mathrm{C})$. Then an element of $m \circ n$ over $((\mathrm{f},\mathrm{g}),(\mathrm{k},\mathrm{l}))$ is a pair

$$S.(m,n) : m(\mathrm{f},\mathrm{k})(R,S) \times n(\mathrm{g},\mathrm{l})(S,T)$$

quotiented by associativity: for any $s : \mathcal{S}(S_0,S_1)$ we have $S_0.(m, s\cdot n) = S_1.(m \cdot s, n)$.

The sequential composite of span transformations $[\![m]\!] : m_0 \Rightarrow m_1$ and $[\![n]\!] : n_0 \Rightarrow n_1$ is given by horizontal composition of transformations: $[\![m]\!] \circ [\![n]\!] : m_0 \circ n_0 \Rightarrow m_1 \circ n_1$ maps

$$S.(m,n) : m_0(\mathrm{f},\mathrm{k})(R,S) \times n_0(\mathrm{g},\mathrm{l})(S,T)$$

to

$$[\![S]\!].([\![m]\!],[\![n]\!]) : m_1([\![\mathrm{f}]\!],[\![\mathrm{k}]\!])([\![R]\!],[\![S]\!]) \times n_1([\![\mathrm{g}]\!],[\![\mathrm{l}]\!])([\![S]\!],[\![T]\!])\ .$$

This defines horizontal composition of the double category of span categories.

**Proposition 12.** Span categories and span functors, span profunctors and span transformations form a double category $\mathrm{Span}\mathbb{C}\mathrm{at}$.

In the same way, displayed categories form a double category.

**Proposition 13.** Displayed categories and displayed functors, displayed profunctors and displayed transformations form a double category $\mathrm{Dis}\mathbb{C}\mathrm{at}$.

*Proof.* Sequential composition of displayed profunctors is defined: given $m : f \times k \to \mathrm{Prof}$ and $n : g \times l \to \mathrm{Prof}$, the composite $(m \circ n) : (f \circ g) \times (k \circ l) \to \mathrm{Prof}$ is

$$(m \circ n)((\mathrm{f},\mathrm{g}),(\mathrm{k},\mathrm{l})) = m(\mathrm{f},\mathrm{k}) \times n(\mathrm{g},\mathrm{l}).$$

This is functorial, defining parallel composition of the double category $\mathrm{Dis}\mathbb{C}\mathrm{at}$. □

Thus, we summarize the exposition of the section: the double category of span categories is equivalent to that of displayed categories.

**Theorem 14.** $\mathrm{Span}\mathbb{C}\mathrm{at} \simeq \mathrm{Dis}\mathbb{C}\mathrm{at}$.

## 1.4 Parallel composition

As explored in this section, we can understand span categories to consist of *relations*, and span profunctors to consist of *inferences*. The composition thus far defined has been *sequential* composition of inference.

Now to complete the underlying structure of metalogic, we just need to define *parallel* composition of these "meta"-relations and "meta"-inferences.

**Definition 15.** Let $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ and $\mathcal{S} : \mathbb{B} \,\|\, \mathbb{C}$ be span categories. The **parallel composite** $\mathcal{R} * \mathcal{S} : \mathbb{A} \,\|\, \mathbb{C}$ is a span category defined by composition of spans in $\mathbb{C}\mathrm{at}$.

This means that an object of $\mathcal{R} * \mathcal{S}$ over $\mathrm{A} : \mathbb{A}, \mathrm{C} : \mathbb{C}$ is a pair $R : \mathcal{R}(\mathrm{A}, \mathrm{B}), S : \mathcal{S}(\mathrm{B}, \mathrm{C})$ for some $\mathrm{B} : \mathbb{B}$. Hence the composite is equivalent to the matrix of categories

$$(\mathcal{R} * \mathcal{S})(\mathrm{A}, \mathrm{C}) = \Sigma \mathrm{B} : \mathbb{B}.\ \mathcal{R}(\mathrm{A}, \mathrm{B}) \times \mathcal{S}(\mathrm{B}, \mathrm{C})$$

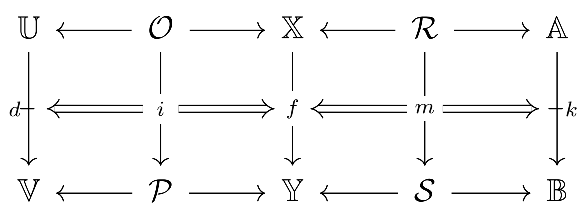
and similarly for morphisms.

$$(\vec{\mathcal{R}} * \vec{\mathcal{S}})(\mathrm{a}, \mathrm{c}) = \Sigma \mathrm{b} : \mathbb{B}.\ \vec{\mathcal{R}}(\mathrm{a}, \mathrm{b}) \times \vec{\mathcal{S}}(\mathrm{b}, \mathrm{c})$$

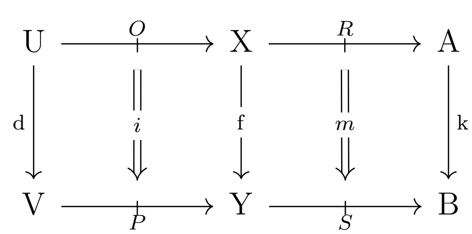
Composition and unit of $\mathcal{R} * \mathcal{S}$ are given by that of $\mathcal{R}$ and $\mathcal{S}$; this is associative and unital.

In the same way, we define parallel composition of span profunctors.

**Definition 16.** Let $i(d, f) : \mathcal{O}(\mathbb{U}, \mathbb{X}) \,|\, \mathcal{P}(\mathbb{V}, \mathbb{Y})$ and $m(f, k) : \mathcal{R}(\mathbb{X}, \mathbb{A}) \,|\, \mathcal{S}(\mathbb{Y}, \mathbb{B})$ be span profunctors. The **parallel composite** $(i * m)(d, k) : (\mathcal{Q} * \mathcal{S})(\mathbb{U}, \mathbb{A}) \,|\, (\mathcal{R} * \mathcal{T})(\mathbb{V}, \mathbb{B})$ is the span composite in $\mathrm{Prof}$.




So an element of $i * m$ over $\mathrm{d} : d(\mathrm{U}, \mathrm{V}), \mathrm{k} : k(\mathrm{A}, \mathrm{B})$ is a pair $i : i(\mathrm{d}, \mathrm{f})$ and $m : m(\mathrm{f}, \mathrm{k})$ for some $\mathrm{f} : f(\mathrm{X}, \mathrm{Y})$. This can be understood as pairs of parallel-composable squares.




Hence the composite is equivalent to the following matrix of profunctors.

$$(i * m)(\mathrm{d}, \mathrm{k}) = \Sigma \mathrm{f} : f(\mathrm{X}, \mathrm{Y}).\ i(\mathrm{d}, \mathrm{f}) \times m(\mathrm{f}, \mathrm{k})$$

Parallel composition of span functors and of span transformations is defined analogously.

Parallel composition is *lax* functorial with respect to sequential composition of span profunctors, because sequential composition is a quotient: given a diagram

$$\begin{array}{ccccccccc}
\mathbb{U} & \longleftarrow & \mathcal{O} & \longrightarrow & \mathbb{X} & \longleftarrow & \mathcal{R} & \longrightarrow & \mathbb{A} \\
{\scriptstyle d}\downarrow & \Longleftarrow & \downarrow{\scriptstyle i} & \Longrightarrow & \downarrow{\scriptstyle f} & \Longleftarrow & \downarrow{\scriptstyle m} & \Longrightarrow & \downarrow{\scriptstyle k} \\
\mathbb{V} & \longleftarrow & \mathcal{P} & \longrightarrow & \mathbb{Y} & \longleftarrow & \mathcal{S} & \longrightarrow & \mathbb{B} \\
{\scriptstyle e}\downarrow & \Longleftarrow & \downarrow{\scriptstyle j} & \Longrightarrow & \downarrow{\scriptstyle g} & \Longleftarrow & \downarrow{\scriptstyle n} & \Longrightarrow & \downarrow{\scriptstyle \ell} \\
\mathbb{W} & \longleftarrow & \mathcal{Q} & \longrightarrow & \mathbb{Z} & \longleftarrow & \mathcal{T} & \longrightarrow & \mathbb{C}
\end{array}$$

there is a canonical transformation $(i * m) \circ (j * n) \Rightarrow (i \circ j) * (m \circ n)$; yet this may be noninvertible, because the quotient formed by $f \circ g$ introduces *more* parallel composability.

Hence $\mathrm{Span}\mathbb{C}\mathrm{at}$ is an *intercategory* [6], or “lax triple category”. But this is not yet our metalanguage, as “meta”relations and “meta”inferences (span categories and profunctors) are not yet equipped with *actions* of parallel composition.

The present notion of “double category” is defined to be a pseudomonad in $\mathrm{Span}\mathbb{C}\mathrm{at}$; yet the monad-and-bimodules construction is *not well-defined*, because composition by an arbitrary span of categories does not preserve colimits.

Yet when equipped with parallel composition, a *matrix category* is exponentiable [19]; hence the metalanguage of *bifibrant double categories* is well-defined, and *higher-order*.

So next, we determine how *a category* $\mathbb{A}$ *forms a logic* $\langle\mathbb{A}\rangle$, its “logic of equations”, and define a *matrix category* or *two-sided bifibration* to be a span category $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ with bimodule actions by the equational logics $\mathbb{A} \leftarrow \langle\mathbb{A}\rangle \rightarrow \mathbb{A}$ and $\mathbb{B} \leftarrow \langle\mathbb{B}\rangle \rightarrow \mathbb{B}$.

# 2 Bifibered categories: push and pull

We establish in Chapter 1 the basic data of a logic: a category of types and processes $\underline{\mathbb{A}}$, and a span category of relations and inferences $\underline{\mathbb{A}} \leftarrow \mathbb{A} \rightarrow \underline{\mathbb{A}}$. Now, we add the basic structure: processes *act* on relations, i.e. morphisms of $\underline{\mathbb{A}}$ act on objects of $\mathbb{A}$.

So in this chapter, we define the concept of *matrix category* or *two-sided bifibration*: a span category $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ equipped with both "push and pull" actions by $\mathbb{A}$ and $\mathbb{B}$. Matrix categories form a double category $\mathrm{Mat}\mathbb{C}\mathrm{at}$, which is fibered over $\mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$; then $\mathbb{C}\mathrm{at} \leftarrow \mathrm{Mat}\mathbb{C}\mathrm{at} \rightarrow \mathbb{C}\mathrm{at}$ forms a three-dimensional structure which we call a *metalogic*.

**Section 2.1: Fibrations and bifibrations.** In order to act on span categories, we first determine how a category $\mathbb{A}$ forms a logic $\mathbb{A} \leftarrow \langle\mathbb{A}\rangle \rightarrow \mathbb{A}$. Two-sided fibrations have been defined as bimodules of arrow double categories [21]; yet these are not logics, because they lack conjoints. So, we define the logic of the *weave double category* to be the coproduct of the arrow double category with its opposite, $\langle\mathbb{A}\rangle \equiv \overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$.

**Section 2.2: Matrix categories.** We define a matrix category to be a bimodule of weave double categories. The "weave construction" extends to profunctors, giving the new concept of *matrix profunctor* [2.3]. This is a relation of dependent categories; and to the author's knowledge, such a notion has yet to be defined nor explored. We show why *bi*fibrations are essential to define relation composition.

Matrix categories and matrix functors, matrix profunctors and matrix transformations form a double category $\mathrm{Mat}\mathbb{C}\mathrm{at}$, which is fibered over $\mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$ [2.4].

**Section 2.5: Parallel composition.** Just as a profunctor consists of processes, a matrix category consists of relations, and hence forms a "meta relation" between logics. So to conclude the chapter, we define *composition* of these relations.

While profunctors compose by forming a coequalizer for associativity, matrix categories compose by forming a *codescent object*, a weak colimit which adjoins a coherent associator isomorphism. We show this defines the horizontal composition of a three-dimensional category; but it does *not* preserve sequential composition of matrix profunctors, so we define a *metalogic* to be a "bifibrant triple category without interchange" [2.5.1].

The language of $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is powerful, because composition is dual to the *descent object* which forms each category of matrix functors. We derive formulae for extensions and lifts, which allow for systematic derivation of weighted limits and colimits [2.5.2].

## 2.1 Fibrations and bifibrations

A category is seen as a 1-dimensional structure of objects and morphisms; yet *reasoning* in a category consists of 2-dimensional equalities between composites of morphisms.

Every category forms a double category, in fact three double categories, whose squares are commuting squares. Two are known: the *arrow double category* $\overrightarrow{\mathbb{A}}$ and its opposite $\overleftarrow{\mathbb{A}}$; their modules are *fibrations* and *opfibrations*.

Yet $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$ are *not* logics; so we define the *weave double category* $\langle\mathbb{A}\rangle$ to be the union $\overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$. It is a logic, and its modules are *bifibrations*.

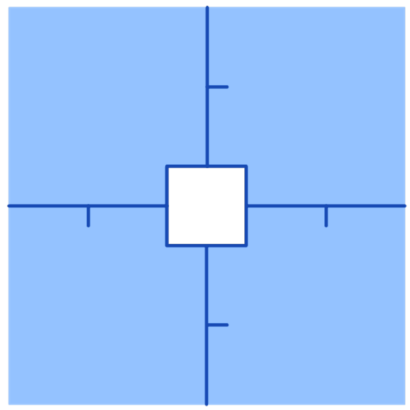
### 2.1.1 Arrow double category

**Definition 17.** Let $\mathbb{A}$ be a category. The **arrow double category** $\overrightarrow{\mathbb{A}}$ is as follows: the base category is $\mathbb{A}$; a loose morphism is a morphism of $\mathbb{A}$, and a square is a commutative square.

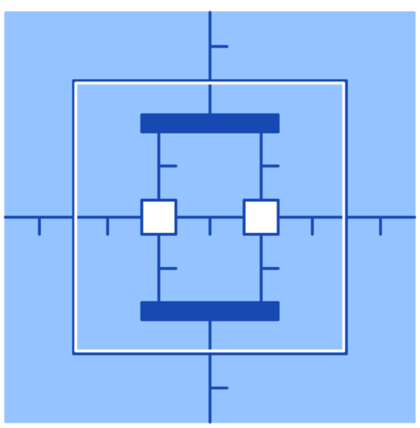
We denote (vertical) processes by $\mathrm{a}$, and (horizontal) relations by $\hat{\mathrm{a}}$.

$$\begin{array}{ccc} \mathrm{A}_0^0 & \xrightarrow{\hat{\mathrm{a}}_0^1} & \mathrm{A}_0^1 \\ {\scriptstyle \mathrm{a}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{a}_1} \\ \mathrm{A}_1^0 & \xrightarrow{\hat{\mathrm{a}}_1^1} & \mathrm{A}_1^1 \end{array}$$

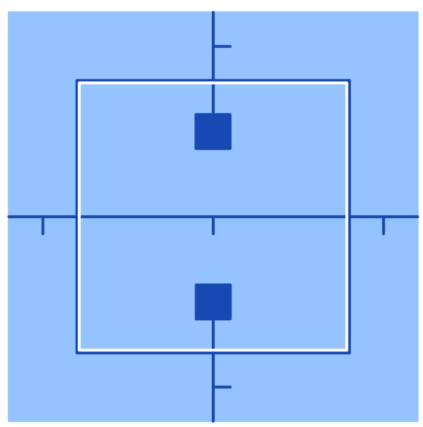
$$(\mathrm{a}_0, \mathrm{a}_1) : \overrightarrow{\mathbb{A}}(\hat{\mathrm{a}}_0^1 \to \hat{\mathrm{a}}_1^1)$$



Horizontal composition is that of squares, and horizontal units are identities.



composition



unit

By forming a double category, $\mathbb{A} \leftarrow \overrightarrow{\mathbb{A}} \rightarrow \mathbb{A}$ can *act* on span categories. If an object of $\mathcal{R} : \mathbb{A} \,||\, \mathbb{B}$ is to be a relation from an $\mathbb{A}$-type to a $\mathbb{B}$-type, then such relations should *vary* over processes of $\mathbb{A}$ and $\mathbb{B}$ — this is a *module* of arrow double categories.

**Definition 18.** Let $\mathbb{A}$ and $\mathbb{B}$ be categories.

A **fibered category** over $\mathbb{A}$ is a left module of the arrow double category $\overrightarrow{\mathbb{A}}$. This is a span category $\mathcal{R} : \mathbb{A} \| 1$, with a span functor $\odot : \overrightarrow{\mathbb{A}} * \mathcal{R} \to \mathcal{R}$, and coherent isomorphisms for associativity and unitality. The action, called **substitution**, is a matrix of functors

$$\hat{\mathrm{a}} \odot R : \overrightarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1) \times \mathcal{R}(\mathrm{A}_1) \to \mathcal{R}(\mathrm{A}_0)$$

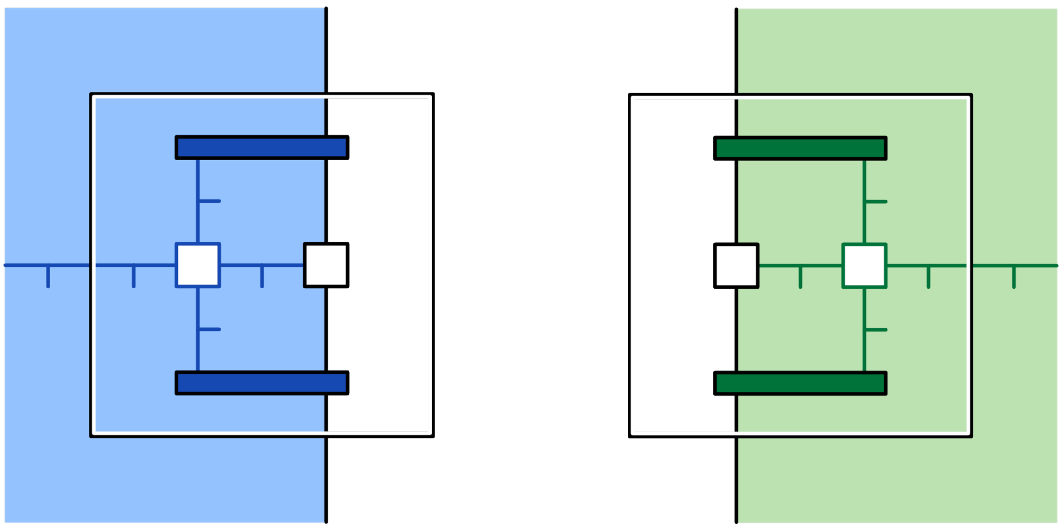
which is contravariant in A. It is also known as "pullback", and often denoted by $\mathrm{a}^*(R_1)$.

An **opfibered category** over $\mathbb{B}$ is a right module of the arrow double category $\overrightarrow{\mathbb{B}}$. This is a span category $\mathcal{R} : 1 \| \mathbb{B}$, with a span functor $\odot : \mathcal{R} * \overrightarrow{\mathbb{B}} \to \mathcal{R}$, and coherent isomorphisms for associativity and unitality. The action, called **image**, is a matrix of functors

$$R \odot \hat{\mathrm{b}} : \mathcal{R}(\mathrm{B}_0) \times \overrightarrow{\mathbb{B}}(\mathrm{B}_0, \mathrm{B}_1) \to \mathcal{R}(\mathrm{B}_1)$$

which is covariant in B. It is also known as "pushforward", and often denoted by $\mathrm{b}_!(R_0)$.

In string diagrams, with category 1 as white space, the actions are drawn as follows.



**substitution**
$\overrightarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1) \times \mathcal{R}(\mathrm{A}_1) \to \mathcal{R}(\mathrm{A}_0)$

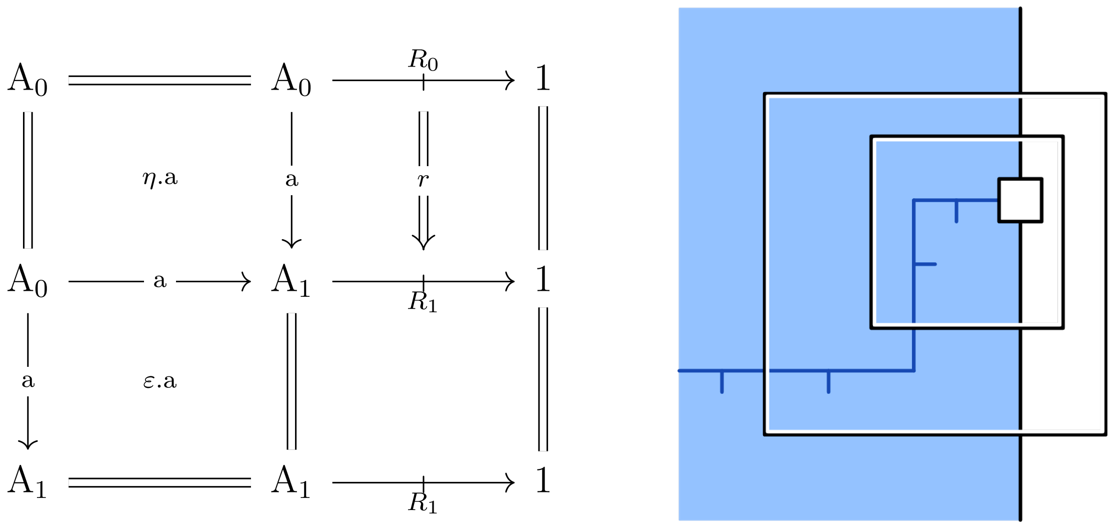
**image**
$\mathcal{R}(\mathrm{B}_0) \times \overrightarrow{\mathbb{B}}(\mathrm{B}_0, \mathrm{B}_1) \to \mathcal{R}(\mathrm{B}_1)$

Arrow double categories are special, because every process has a *companion*: there are two squares which "bend" the process up or down into a relation.

**Definition 19.** Let $\mathbb{A}$ be a category, with $\overrightarrow{\mathbb{A}}$ the arrow double category. Each morphism $\mathrm{a} : \mathbb{A}(\mathrm{A}_0, \mathrm{A}_1)$ induces two squares: the **cartesian** square $\varepsilon.\mathrm{a}$ and the **opcartesian** square $\eta.\mathrm{a}$, drawn below.

Fibered and opfibered categories are usually defined via the notions of cartesian and opcartesian morphism [9, Ch. 1,9]. These morphisms are given by the actions of squares in the arrow double category, as follows.

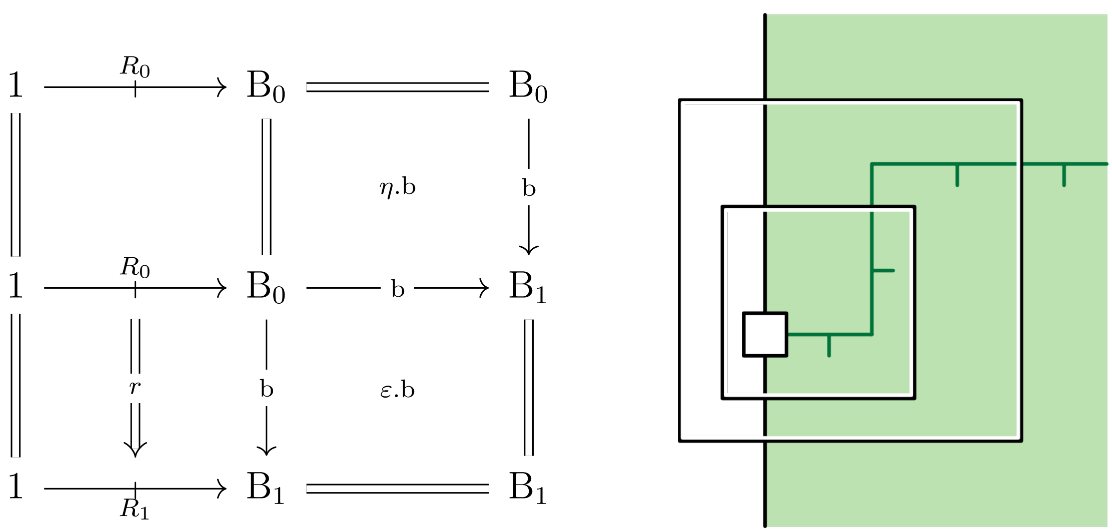
**Proposition 20.** In a fibered category $\mathcal{R}$ over $\mathbb{A}$, a morphism $r : R_0 \to R_1$ over $\mathrm{a} : \mathbb{A}(\mathrm{A}_0, \mathrm{A}_1)$ is equivalent to $\eta.\mathrm{a} \circ r : R_0 \to \mathrm{a} \odot R_1$ over $\mathrm{id}.\mathrm{A}_0$, by factoring through $\varepsilon.\mathrm{a} \circ \mathrm{id}.R_1 : \mathrm{a} \odot R_1 \to R_1$, the **cartesian** morphism of $R_1$ over $\mathrm{a}$.




This gives a contravariant representation of morphisms over $\mathrm{a}$.

$$\vec{\mathcal{R}}(\mathrm{a})(R_0, R_1) \cong \mathcal{R}(R_0, \mathrm{a} \odot R_1)$$

In an opfibered category $\mathcal{R}$ over $\mathbb{B}$, a morphism $r : R_0 \to R_1$ over $\mathrm{b} : \mathbb{B}(\mathrm{B}_0, \mathrm{B}_1)$ is equivalent to a morphism $r \circ \varepsilon.\mathrm{b} : R_0 \odot \mathrm{b} \to R_1$ over $\mathrm{id}.\mathrm{B}_1$, by factoring through $\mathrm{id}.R_0 \circ \eta.\mathrm{b} : R_0 \to R_0 \odot \mathrm{b}$, the **opcartesian** morphism of $R_0$ over $\mathrm{b}$.

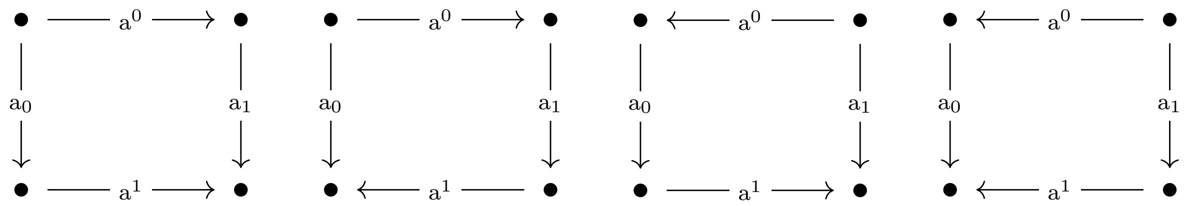



This gives a covariant representation of morphisms over $\mathrm{b}$.

$$\vec{\mathcal{R}}(\mathrm{b})(R_0, R_1) \cong \mathcal{R}(R_0 \odot \mathrm{b}, \mathcal{R}_1)$$

However, there is a limitation to the arrow double category: it is not a *logic*, because there are no backwards-pointing arrows to be conjoints. This may seem like a technicality — surely all equational reasoning of $\mathbb{A}$ can be expressed in $\overrightarrow{\mathbb{A}}$, right? Actually, no.

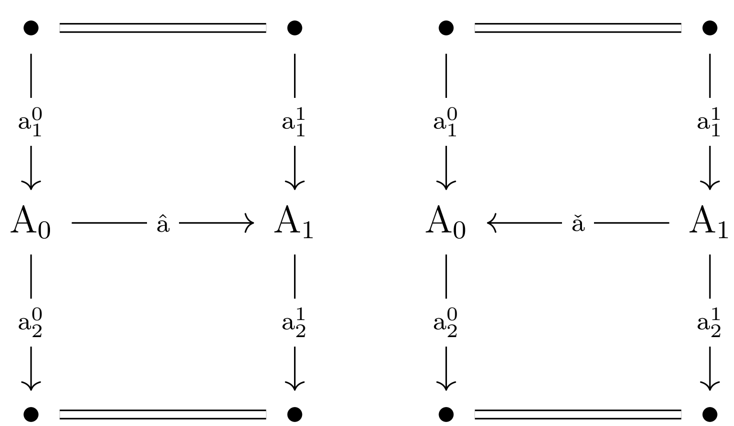
In two dimensions, we distinguish between morphisms as *processes* and as *relations*. Based on how processes act on relations, there are four basic kinds of equations.




**natural** $a^0 \cdot a_1 = a_0 \cdot a^1$ **factorization** $a_0 = a^0 \cdot a_1 \cdot a^1$ **composition** $a^0 \cdot a_0 \cdot a^1 = a_1$ **conatural** $a^0 \cdot a_0 = a_1 \cdot a^1$

Of course, each of the above equations can be expressed as a “natural” commutative square in an arrow double category. However, there is an obstruction to reasoning about sequential composition. The notion of a *composable pair* is determined only up to associativity.

Associativity has two forms, “forward” and “backward”: suppose that two pairs $(a_1^0, a_2^0)$ and $(a_1^1, a_2^1)$ are equal in the composite profunctor $\mathbb{A}\circ\mathbb{A}$; then there is a “zig-zag” connecting the pair: a sequence of morphisms $\hat{a} : \mathbb{A}(A_i, A_{i+1})$ *or* $\check{a} : \mathbb{A}(A_{i+1}, A_i)$, so that the squares commute. The two unary cases are below.




**forward associativity** $a_1^0 \cdot \hat{a} = a_1^1$, $a_2^0 = \hat{a} \cdot a_2^1$ **backward associativity** $a_1^0 = a_1^1 \cdot \check{a}$, $\check{a} \cdot a_2^0 = a_2^1$

Forward associativity, on the left, is the composite of two “natural” squares, which can be expressed in the arrow double category. Backwards associativity, on the right, is the composite of a “factorization” and a “composition” — this *cannot be expressed* in the arrow double category.

Hence we identify the following limitation.

**Proposition 21.** Let $\mathbb{A}$ be a category. In the arrow double category $\overrightarrow{\mathbb{A}}$, factorization and composition squares do not compose in sequence; so backward associativity cannot be expressed.

This causes an obstruction to defining sequential composition of profunctors between "two-sided fibrations", i.e. bimodules of arrow double categories — this lesson the author learned the hard way.

We could accept this limitation and use this structure to construct logics, but it would be more complex than necessary. Instead, we see the problem to be that *arrow double categories are not logics*, and we determine the logic which a category *does* form.

### 2.1.2 Weave double category

Every category $\mathbb{A}$ defines a logic, called the *weave double category* $\langle\mathbb{A}\rangle$. It is the union, i.e. coproduct, of the arrow double category and its opposite: $\langle\mathbb{A}\rangle \equiv \overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$. As we will see, $\langle\mathbb{A}\rangle$ can be understood simply as the *equational logic* of $\mathbb{A}$.

A relation is a *zig-zag* in $\mathbb{A}$, and an inference is a *weave*: a composite of squares in $\overrightarrow{\mathbb{A}}$, opsquares in $\overleftarrow{\mathbb{A}}$, and *unit isomorphisms* — the units of $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$ are "united" by adjoining isomorphisms between each identity arrow and oparrow.

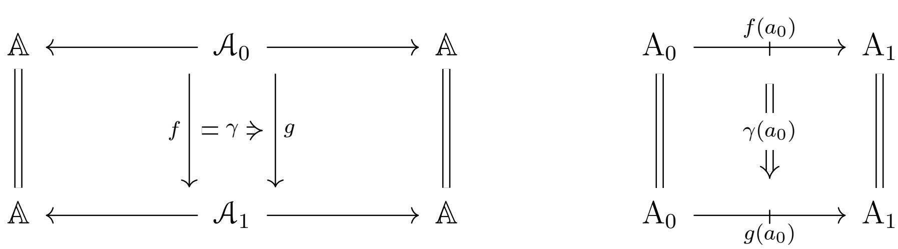
**Definition 22.** Let $\mathbb{A}$ be a category, with arrow double category $\overrightarrow{\mathbb{A}}$.

The **op-arrow double category** $\overleftarrow{\mathbb{A}}$ is the horizontal opposite: $\overleftarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1) \equiv \overrightarrow{\mathbb{A}}(\mathrm{A}_1, \mathrm{A}_0)$.

We denote an **arrow** by $\hat{\mathrm{a}} : \overrightarrow{\mathbb{A}}(\mathrm{A}_0, \mathrm{A}_1)$, and an **op-arrow** by $\check{\mathrm{a}} : \overleftarrow{\mathbb{A}}(\mathrm{A}_1, \mathrm{A}_0)$; and we use $\bar{\mathrm{a}} : \overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$. A square of $\overrightarrow{\mathbb{A}}$ is a **square**, and a square of $\overleftarrow{\mathbb{A}}$ is an **opsquare**.

**Definition 23.** Define $\mathrm{Dbl}_{\mathbb{A}}$ to be the 2-category of double categories on $\mathbb{A}$, double functors over $\mathrm{id}.\mathbb{A}$, and identity-component transformations, a.k.a. icons [12].

Given double categories $\mathcal{A}_0$ and $\mathcal{A}_1$ on $\mathbb{A}$, and double functors $f, g : \mathcal{A}_0 \to \mathcal{A}_1$ over $\mathrm{id}.\mathbb{A}$, an icon $\gamma : f \Rightarrow g$ is a natural family of 2-morphisms $\gamma : \Pi a_0 : \mathcal{A}_0.\ f(a_0) \Rightarrow g(a_0)$.




**Definition 24.** Let $\mathbb{A}$ be a category. Define the **weave double category** $\langle\mathbb{A}\rangle$ to be the 2-coproduct of the arrow and oparrow double categories in $\mathrm{Dbl}_{\mathbb{A}}$.

$$\langle\mathbb{A}\rangle \equiv \overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$$

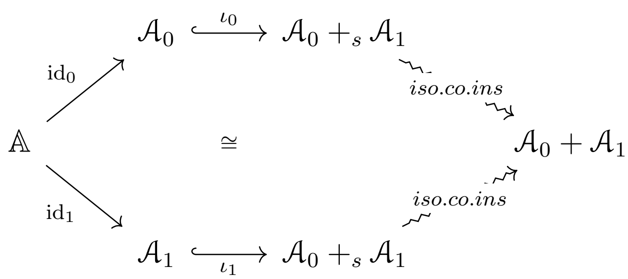
So for every double category $\mathbb{A} \leftarrow \mathcal{A} \to \mathbb{A}$ there is the following natural equivalence.

$$\mathrm{Dbl}_{\mathbb{A}}(\langle\mathbb{A}\rangle, \mathcal{A}) \simeq \mathrm{Dbl}_{\mathbb{A}}(\overrightarrow{\mathbb{A}}, \mathcal{A}) \times \mathrm{Dbl}_{\mathbb{A}}(\overleftarrow{\mathbb{A}}, \mathcal{A})$$

For this abstract definition, we now provide an explicit construction: the coproduct $\overrightarrow{\mathbb{A}} + \overleftarrow{\mathbb{A}}$ is generated by squares of $\overrightarrow{\mathbb{A}}$, opsquares of $\overleftarrow{\mathbb{A}}$, and natural isomorphisms of their identities $\hat{\mathrm{id}}.\mathrm{A} \cong \check{\mathrm{id}}.\mathrm{A}$. We can assume composition to be strictly associative.

Let $\mathbb{A}$ and $\mathrm{Dbl}_{\mathbb{A}}$ be as above. Let $\mathrm{sDbl}_{\mathbb{A}}$ be the 2-category of *semi*-double categories, i.e. span categories $\mathbb{A} \leftarrow \mathcal{A} \rightarrow \mathbb{A}$ with an associative composition.

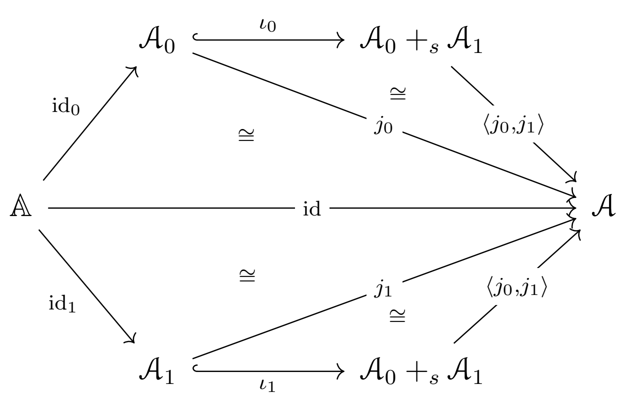
**Theorem 25.** The coproduct $\mathcal{A}_0 + \mathcal{A}_1$ in $\mathrm{Dbl}_{\mathbb{A}}$ can be constructed as the coproduct in $\mathrm{sDbl}_{\mathbb{A}}$, followed by an *iso-coinserter* of their units, the 2-initial object of the following form.



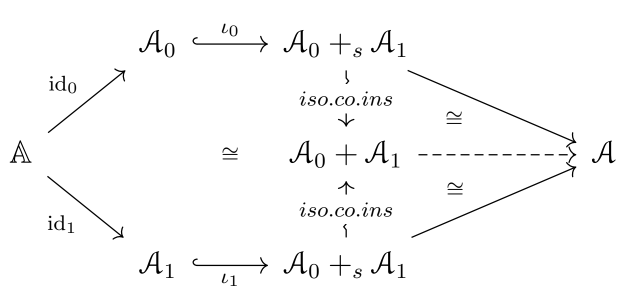

A square of $\mathcal{A}_0 +_s \mathcal{A}_1$ is a composite of squares in $\mathcal{A}_0$ and squares in $\mathcal{A}_1$; then the iso-coinserter adjoins a natural family of isomorphisms $\mathrm{id}_0.\mathrm{A} \cong \mathrm{id}_1.\mathrm{A}$, so that $\mathcal{A}_0 + \mathcal{A}_1$ is a double category (with unit either $\mathrm{id}_0$ or $\mathrm{id}_1$), and in fact the coproduct in $\mathrm{Dbl}_{\mathbb{A}}$.

*Proof.* Let $\mathcal{A}$ be a double category on $\mathbb{A}$ equipped with double functors $j_0 : \mathcal{A}_0 \rightarrow \mathcal{A}$ and $j_1 : \mathcal{A}_1 \rightarrow \mathcal{A}$. Then for the underlying semi-double categories, there is a semi-double functor $\langle j_0, j_1 \rangle : \mathcal{A}_0 +_s \mathcal{A}_1 \rightarrow \mathcal{A}$, equipped with 2-isomorphisms $\langle j_0, j_1 \rangle(\iota_0) \cong j_0$ and $\langle j_0, j_1 \rangle(\iota_1) \cong j_1$.

Yet because $j_0$ and $j_1$ are double functors, there is also $j_0(\mathrm{id}_0) \cong \mathrm{id} \cong j_1(\mathrm{id}_1)$.




The composite isomorphism is exactly the kind which factors through the iso-coinserter, giving both the 1- and 2-dimensional universal property of coproduct.




□

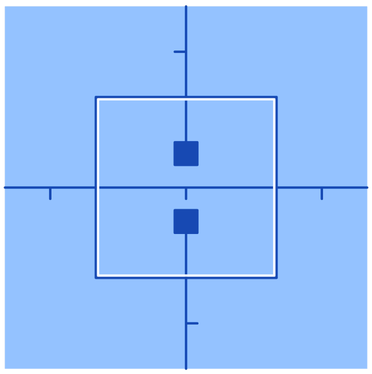
A relation in $\langle \mathbb{A} \rangle$ is a **zig-zag** in $\mathbb{A}$: a nonempty sequence of morphisms $(\mathrm{A}_0, \bar{a}_1, \ldots, \bar{a}_k, \mathrm{A}_k)$ alternating with each $\bar{a}_i$ either an arrow $\hat{a}_i : \overrightarrow{\mathbb{A}}(\mathrm{A}_{i-1}, \mathrm{A}_i)$ or an op-arrow $\check{a}_i : \overleftarrow{\mathbb{A}}(\mathrm{A}_{i-1}, \mathrm{A}_i)$.

$$\mathrm{A}_0 \xrightarrow{\hat{a}_1} \mathrm{A}_1 \xleftarrow{\check{a}_2} \mathrm{A}_2 \longrightarrow \cdots \longleftarrow \mathrm{A}_{k-2} \xrightarrow{\hat{a}_{k-1}} \mathrm{A}_{k-1} \xleftarrow{\check{a}_k} \mathrm{A}_k$$

We may abbreviate a zig-zag by $\langle \bar{a}_1, \ldots, \bar{a}_k \rangle$ or simply by $\langle \bar{a}_k \rangle$.

An inference is a **weave**: a composite of squares of $\overrightarrow{\mathbb{A}}$, opsquares of $\overleftarrow{\mathbb{A}}$, and isomorphisms of identities. These can be fairly complex, but it should be possible to give a normal form.

The natural isomorphism of identities can be drawn as follows.



$$\begin{array}{ccc} \mathrm{A}_0 & \leftarrow & \mathrm{A}_0 \\ \downarrow a & & \downarrow a \\ \mathrm{A}_1 & \leftarrow & \mathrm{A}_1 \\ \| & \cong & \| \\ \mathrm{A}_1 & \rightarrow & \mathrm{A}_1 \end{array} \quad = \quad \begin{array}{ccc} \mathrm{A}_0 & \leftarrow & \mathrm{A}_0 \\ \| & \cong & \| \\ \mathrm{A}_0 & \rightarrow & \mathrm{A}_0 \\ \downarrow a & & \downarrow a \\ \mathrm{A}_1 & \rightarrow & \mathrm{A}_1 \end{array}$$

The weave double category contains *all equational reasoning* of $\mathbb{A}$, in that it contains the four kinds of squares and their composites: the sequential composite of a factorization and a composition square is below.

$$\begin{array}{ccc} \mathrm{A}_0^0 & \xrightarrow{\hat{a}_0} & \mathrm{A}_1^0 \\ a_1^0 \downarrow & & \downarrow a_1^1 \\ \mathrm{A}_0^1 & \xleftarrow{\check{a}_1} & \mathrm{A}_1^1 \\ a_2^0 \downarrow & & \downarrow a_2^1 \\ \mathrm{A}_0^2 & \xrightarrow{\hat{a}_2} & \mathrm{A}_1^2 \end{array} \quad = \quad \begin{array}{ccccccc} \mathrm{A}_0^0 & \xrightarrow{\hat{a}_0} & \mathrm{A}_1^0 & \leftarrow & \mathrm{A}_1^0 & \rightarrow & \mathrm{A}_1^0 \\ & & a_1^1 \downarrow & & \downarrow a_1^1 & & \\ a_1^0 \downarrow & & \mathrm{A}_1^1 & \leftarrow & \mathrm{A}_1^1 & & \downarrow a_1^1 \\ & & a_1 \downarrow & & \| & & \\ \mathrm{A}_0^1 & \rightarrow & \mathrm{A}_0^1 & \xleftarrow{\check{a}_1} & \mathrm{A}_1^1 & \rightarrow & \mathrm{A}_1^1 \\ & & \| & & \downarrow a_1 & & \\ a_2^0 \downarrow & & \mathrm{A}_0^1 & \leftarrow & \mathrm{A}_0^1 & & \downarrow a_2^1 \\ & & a_2^0 \downarrow & & \downarrow a_2^0 & & \\ \mathrm{A}_0^2 & \rightarrow & \mathrm{A}_0^2 & \leftarrow & \mathrm{A}_0^2 & \xrightarrow{a_2} & \mathrm{A}_1^2 \end{array}$$

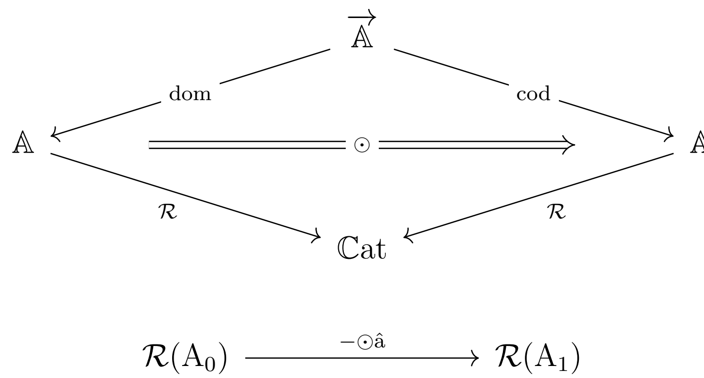
This is because the arrows and oparrows of $\langle \mathbb{A} \rangle$ give both companions and conjoints.

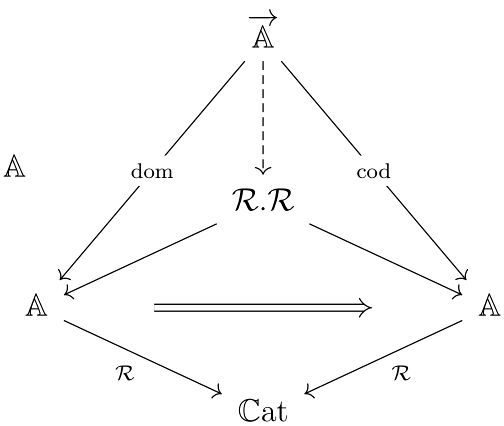
**Proposition 26.** $\langle \mathbb{A} \rangle$ is a bifibrant double category, i.e. a logic.

By coproduct, actions by the weave double category $\langle\mathbb{A}\rangle$ are equivalent to pairs of actions by the arrow and oparrow double categories $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$: so left modules are right modules are *bifibrations*.

To show this by universality, we determine how $\mathcal{R}$ forms a double category on $\mathbb{A}$ that represents actions on $\mathcal{R}$. The key is to see that an action $\odot : \overrightarrow{\mathbb{A}}(A_0, A_1) \times \mathcal{R}(A_0) \to \mathcal{R}(A_1)$ is equivalent to a displayed functor of the following form.

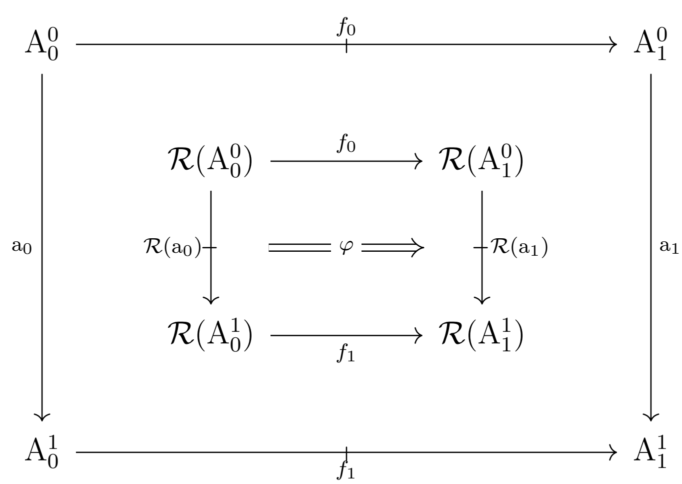
So, we define the double category as the universal comma square.



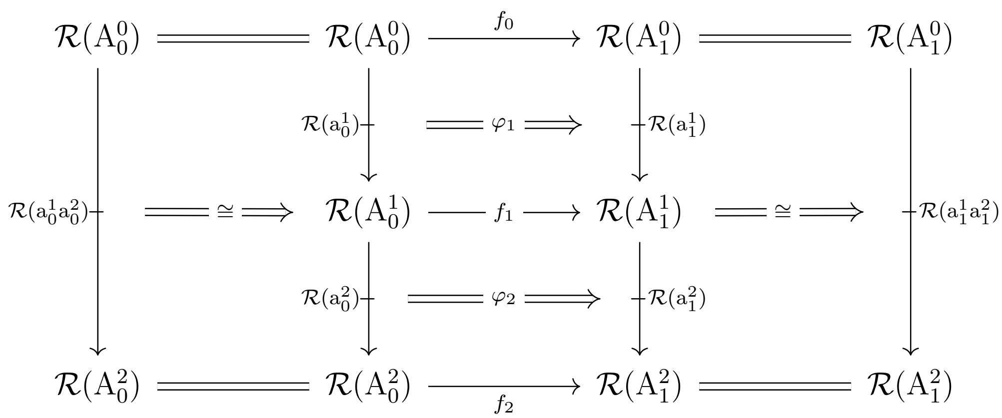

To define sequential composition of $\mathcal{R}.\mathcal{R}$, the displayed category $\mathcal{R} : \mathbb{A} \to \mathbb{C}\mathrm{at}$ must be a *pseudofunctor*, i.e. the composition transformation $\mathcal{R}(a_1) \circ \mathcal{R}(a_2) \Rightarrow \mathcal{R}(a_1 a_2)$ must be *invertible*. This is known as an **exponentiable category** [19], a shared generalization of fibered and opfibered category.

**Definition 27.** Let $\mathcal{R} \to \mathbb{A}$ be exponentiable. Define the **fiber-hom double category** $\mathbb{A} \leftarrow \mathcal{R}.\mathcal{R} \to \mathbb{A}$ to be the collage of the comma object of $\mathcal{R} : \mathbb{A} \to \mathbb{C}\mathrm{at}$ along itself.




The base category is $\mathbb{A}$; a loose morphism over $(A_0, A_1)$ is a functor $f : \mathcal{R}(A_0) \to \mathcal{R}(A_1)$, and a square over $(a_0, a_1)$ is a transformation $\varphi(f_0, f_1) : \mathcal{R}(a_0) \Rightarrow \mathcal{R}(a_1)$. Parallel composition is sequential composition in $\mathbb{C}\mathrm{at}$.

Sequential composition is parallel composition of $\mathbb{C}\mathrm{at}$, conjugated by composition isomorphisms of $\mathcal{R}$. Composing in sequence and parallel, the middle isomorphisms cancel, giving interchange.




**Proposition 28.** Let $\mathcal{R} \to \mathbb{A}$ be an exponentiable category over $\mathbb{A}$. A right action on $\mathcal{R}$ by a double category $\mathcal{A}$ over $\mathbb{A}$ is equivalent to a double functor $\mathcal{A} \to \mathcal{R}.\mathcal{R}$.

A left action $\mathcal{A} * \mathcal{R} \to \mathcal{R}$ is equivalent to a double functor $\mathcal{A}^{\mathsf{op}} \to \mathcal{R}.\mathcal{R}$.

*Proof.* Let $\mathcal{A} : \mathrm{Dbl}_{\mathbb{A}}$, and $\odot : \mathcal{R} * \mathcal{A} \to \mathcal{R}$ be a module action. Then mapping

$$\begin{array}{lll} A : \mathcal{A} & \text{to} & - \odot A : \mathcal{R}(\mathrm{A}_0) \to \mathcal{R}(\mathrm{A}_1) \quad \text{and} \\ \alpha : \mathcal{A}(A, A') & \text{to} & - \odot \alpha : \mathcal{R}(\mathrm{a}_0) \Rightarrow \mathcal{R}(\mathrm{a}_1) \end{array}$$

defines a double functor $\mathcal{A} \to \mathcal{R}.\mathcal{R}$: the associator $(R \odot A_1) \odot A_2 \cong R \odot (A_1 \circ A_2)$ defines the composition isomorphism, and the unitor $R \cong R \odot U_{\mathrm{A}}$ defines the unit isomorphism; the coherence equations correspond. □

**Theorem 29.** $\langle \mathbb{A} \rangle$-modules are equivalent to bifibrations.

*Proof.* By coproduct, we have the following equivalence.

$$\mathrm{Dbl}_{\mathbb{A}}(\overleftarrow{\mathbb{A}} + \overrightarrow{\mathbb{A}}, \mathcal{R}.\mathcal{R}) \simeq \mathrm{Dbl}_{\mathbb{A}}(\overrightarrow{\mathbb{A}}, \mathcal{R}.\mathcal{R}) \times \mathrm{Dbl}_{\mathbb{A}}(\overleftarrow{\mathbb{A}}, \mathcal{R}.\mathcal{R})$$

This means that a right action by $\langle \mathbb{A} \rangle$ is equivalent to a pair of right actions by $\overleftarrow{\mathbb{A}}$ and $\overrightarrow{\mathbb{A}}$; these give $\mathcal{R}$ the structures of a fibration and opfibration. □

We will soon define *matrix categories* to be bimodules of weave double categories. These form a double category over that of categories; so we have to determine how the "weave construction" applies to categories and functors, profunctors and transformations.

How does the notion of "arrow category" generalize to profunctors?

**Definition 30.** Let $f : \mathbb{X} \,|\, \mathbb{A}$ be a profunctor. The **arrow profunctor** $\overrightarrow{f} : \overrightarrow{\mathbb{X}} \,|\, \overrightarrow{\mathbb{A}}$ consists of commutative squares; it forms a span profunctor $f \leftarrow \overrightarrow{f} \rightarrow f$.

$$\overrightarrow{f}(\hat{x}, \hat{a}) = \{(f_0 : f(X_0, A_0), f_1 : f(X_1, A_1)) \mid a \cdot f_0 = f_1 \cdot x\}$$

Dually, the **oparrow profunctor** of $f$ is the profunctor of oparrow categories $\overleftarrow{f} : \overleftarrow{\mathbb{X}} \,|\, \overleftarrow{\mathbb{A}}$.

$$\overleftarrow{f}(\check{x}, \check{a}) = \{f_0 : f(X_0, A_0), f_1 : f(X_1, A_1) \mid x \cdot f_0 = f_1 \cdot a\}$$

$$\begin{array}{ccc} X_0 & \xrightarrow{\hat{x}} & X_1 \\ {\scriptstyle f_0}\downarrow & \overrightarrow{f} & \downarrow{\scriptstyle f_1} \\ A_0 & \xrightarrow[\hat{a}]{} & A_1 \end{array} \qquad \begin{array}{ccc} X_0 & \xleftarrow{\check{x}} & X_1 \\ {\scriptstyle f_0}\downarrow & \overleftarrow{f} & \downarrow{\scriptstyle f_1} \\ A_0 & \xleftarrow[\check{a}]{} & A_1 \end{array}$$

Note the only difference between the arrow and oparrow profunctors is which morphism acts on which element of $f$, i.e. "natural" squares versus "conatural" opsquares.

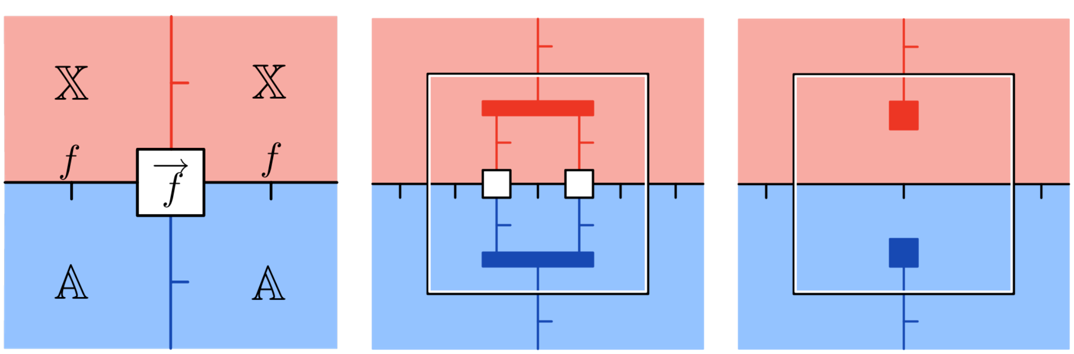
Just as commutative squares of a category compose in parallel, commutative squares of a profunctor compose in parallel.

**Proposition 31.** Let $f : \mathbb{X} \,|\, \mathbb{A}$ be a profunctor. The arrow profunctor $f \leftarrow \overrightarrow{f} \rightarrow f$ is a monad in $\mathrm{Span}(\mathrm{Prof})$. Composition $\overrightarrow{f} * \overrightarrow{f} \Rightarrow \overrightarrow{f}$ is that of commutative squares, and the unit is given by that of $\mathbb{X}$ and $\mathbb{A}$.

$$\begin{array}{ccccc} X_0 & \xrightarrow{\hat{x}_1} & X_1 & \xrightarrow{\hat{x}_2} & X_2 \\ {\scriptstyle f_0}\downarrow & f_1 & \downarrow{\scriptstyle f_1} & f_2 & \downarrow{\scriptstyle f_2} \\ A_0 & \xrightarrow{\hat{a}_1} & A_1 & \xrightarrow{\hat{a}_2} & A_2 \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\hat{id}.X} & X \\ {\scriptstyle f}\downarrow & id.f & \downarrow{\scriptstyle f} \\ A & \xrightarrow{\hat{id}.A} & A \end{array}$$

Dually, the oparrow profunctor is a monad in $\mathrm{Span}(\mathrm{Prof})$.

The arrow profunctor is drawn as follows, and the oparrow profunctor is dual.

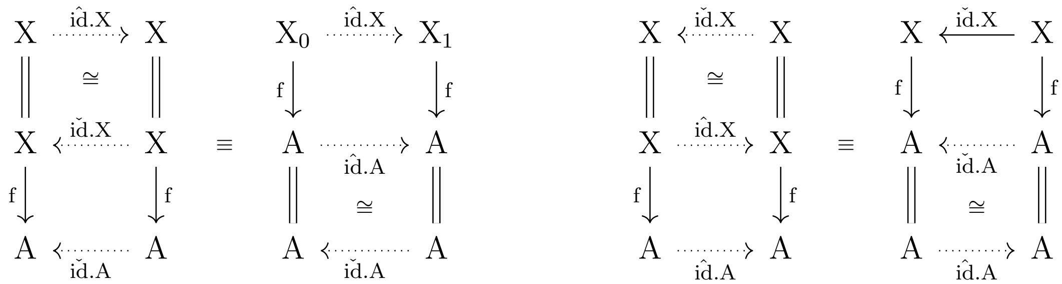



**arrow profunctor** **composition** **unit**

Now, a profunctor of categories forms a "weave profunctor" of weave double categories.

**Definition 32.** Let $f : \mathbb{X} \mid \mathbb{A}$ be a profunctor. Define the **weave vertical profunctor** between weave double categories $\langle f \rangle : \langle \mathbb{X} \rangle \mid \langle \mathbb{A} \rangle$ to be the coproduct of $\overrightarrow{f}$ and $\overleftarrow{f}$ in the 2-category of vertical profunctors on $f$, vertical transformations, and modifications.

Hence $\langle f \rangle$ is generated by squares of $\overrightarrow{f}$ and opsquares of $\overleftarrow{f}$, and the actions by squares, opsquares, and unit isomorphisms of $\langle \mathbb{X} \rangle$ and $\langle \mathbb{A} \rangle$; these are subject to associativity and unitality, plus *naturality* of unit isomorphisms with respect to elements of $f$.

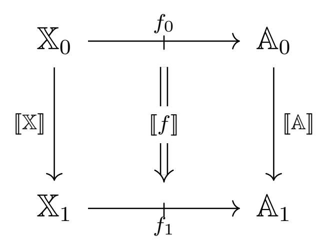



Finally, we extend the "weave construction" to functors and transformations.

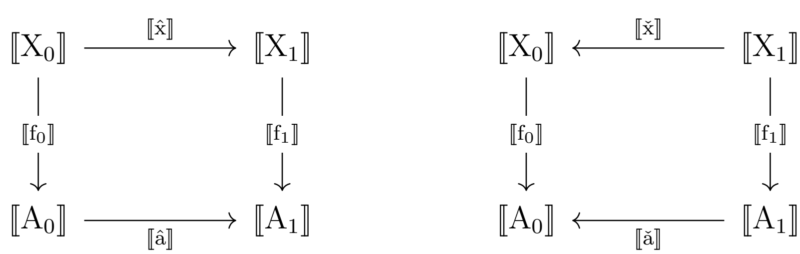
**Definition 33.** Let $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ be a functor; this induces an **arrow double functor** $\overrightarrow{[\![\mathbb{A}]\!]} : \overrightarrow{\mathbb{A}_0} \to \overrightarrow{\mathbb{A}_1}$ and an **oparrow double functor** $\overleftarrow{[\![\mathbb{A}]\!]} : \overleftarrow{\mathbb{A}_0} \to \overleftarrow{\mathbb{A}_1}$.

Define the **weave double functor** $\langle [\![\mathbb{A}]\!] \rangle : \langle \mathbb{A}_0 \rangle \to \langle \mathbb{A}_1 \rangle$ to be their coproduct. So $\langle [\![\mathbb{A}]\!] \rangle$ maps squares to squares, opsquares to opsquares, and unit isomorphisms to unit isos.

**Definition 34.** Let $[\![f]\!] : f_0 \Rightarrow f_1$ be a transformation over $[\![\mathbb{X}]\!], [\![\mathbb{A}]\!]$.




Then $[\![f]\!]$ gives a transformation of squares $\overrightarrow{[\![f]\!]} : \overrightarrow{f_0} \Rightarrow \overrightarrow{f_1}$ and opsquares $\overleftarrow{[\![f]\!]} : \overleftarrow{f_0} \Rightarrow \overleftarrow{f_1}$.




Each commutes by naturality: if $\mathrm{x} \cdot \mathrm{f}_1 = \mathrm{f}_0 \cdot \mathrm{a}$, then $[\![\mathrm{x}]\!] \cdot [\![\mathrm{f}_1]\!] = [\![\mathrm{x} \cdot \mathrm{f}_1]\!] = [\![\mathrm{f}_0 \cdot \mathrm{a}]\!] = [\![\mathrm{f}_0]\!] \cdot [\![\mathrm{a}]\!]$.

The **weave vertical transformation** $\langle [\![f]\!] \rangle : \langle f_0 \rangle(\langle \mathbb{X}_0 \rangle, \langle \mathbb{A}_0 \rangle) \Rightarrow \langle f_1 \rangle(\langle \mathbb{X}_1 \rangle, \langle \mathbb{A}_1 \rangle)$ is the coproduct of these transformations, defined by mapping squares and opsquares of $f_0$.

So, the "weave construction" is a mapping of squares from $\mathbb{C}\mathrm{at}$ to $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$ (3.1): bifibrant double categories and double functors, vertical profunctors and transformations.

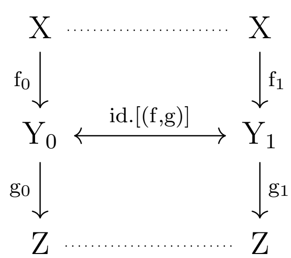
But does $\langle - \rangle$ form a double functor? i.e. how does the weave construction interact with profunctor composition? Here we find that the *associativity quotient* of $f \circ g$ introduces significant complexity.

### The complexity of weaves and composition

Let $f : \mathbb{X} \,|\, \mathbb{Y}$ and $g : \mathbb{Y} \,|\, \mathbb{Z}$ be profunctors. The composite $f \circ g : \mathbb{X} \,|\, \mathbb{Z}$ consists of pairs $(\mathrm{f}, \mathrm{g})$ quotiented by associativity: $(\mathrm{f}, \mathrm{y} \cdot \mathrm{g}) = (\mathrm{f} \cdot \mathrm{y}, \mathrm{g})$, forming equivalence classes $[(\mathrm{f}, \mathrm{g})]$.

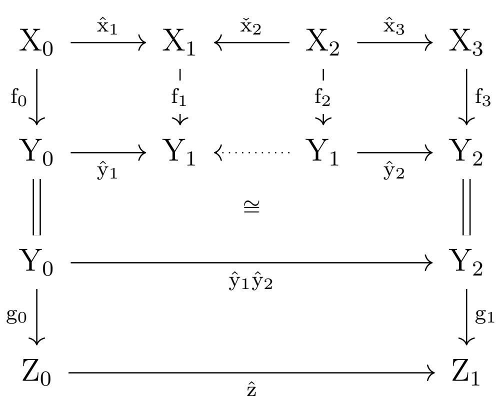
Yet two pairs $(\mathrm{f}_0, \mathrm{g}_0)$ and $(\mathrm{f}_1, \mathrm{g}_1)$ may be equivalent via many distinct zig-zags, while in the composite there is only an equality $[(\mathrm{f}_0, \mathrm{g}_0)] = [(\mathrm{f}_1, \mathrm{g}_1)]$, with no specific zig-zag. This means that all structures defined on $f \circ g$, i.e. *actions* of a matrix profunctor, must be independent of any choice of pair *and* any choice of zig-zag.

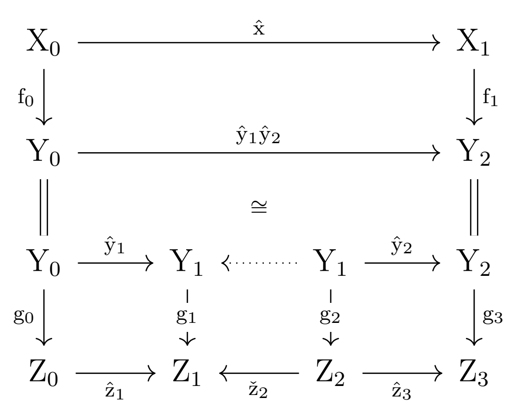
Fortunately, the associativity quotient can be clearly characterized in the weave of the composite, $\langle f \circ g \rangle$: the inner actions by zig-zags in $\mathbb{Y}$ are precisely the *identity squares*.



Hence to define sequential composition of matrix profunctors, we must *quotient* by the action of these zig-zags, to make these identity squares act as the identity; see Def. 44.

So, is $\langle - \rangle$ a double functor? The answer is *no*. Above, there are many distinct representations of each identity square, so there is no transformation $\langle f \circ g \rangle \Rightarrow \langle f \rangle \circ \langle g \rangle$. Yet the other direction is also obstructed, as the following composites of weaves cannot be expressed as squares in $\langle f \circ g \rangle$.





**Proposition 35.** Mapping a category $\mathbb{A}$ to the weave double category $\langle \mathbb{A} \rangle$ defines a span functor from $\mathbb{C}\mathrm{at}$ to $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$, which is neither a lax nor colax double functor.

## 2.2 Matrix categories

We are now ready to define the primary concepts which underlie a logic.

We simplify the presentation of structures and coherences in two ways.
(1) We denote a transformation by its components, e.g. the associator of a matrix category

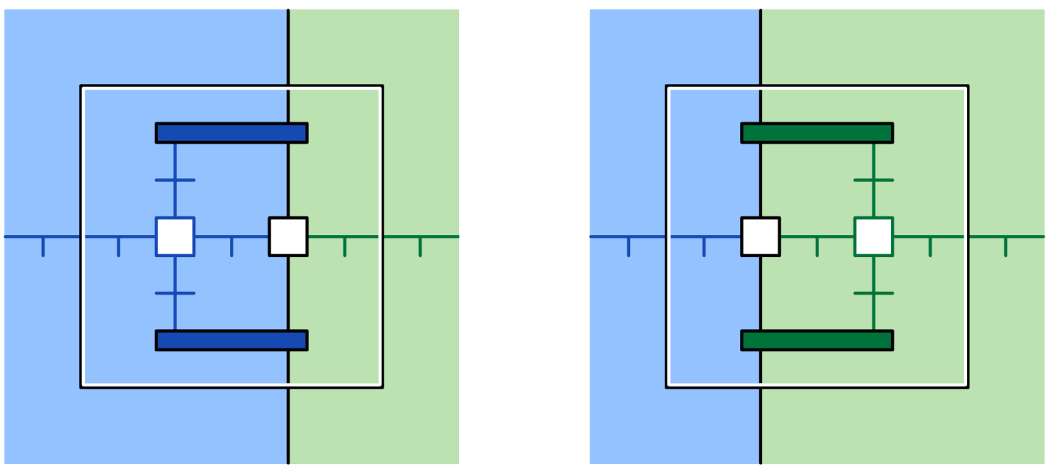
$$(\mathrm{a} \odot R) \odot \mathrm{b} \cong \mathrm{a} \odot (R \odot \mathrm{b}).$$

(2) We use the symbol $x \rightrightarrows y$ to denote that the two transformations from $x$ to $y$, inferrable from context, are equal; e.g. the two ways to reassociate four elements are equal.

$$((\mathrm{a}_1 \odot \mathrm{a}_2) \odot \mathrm{a}_3) \odot R \rightrightarrows \mathrm{a}_1 \odot (\mathrm{a}_2 \odot (\mathrm{a}_3 \odot R))$$

Additionally, we elide the associators and unitors of $\mathrm{Span}\mathbb{C}\mathrm{at}$; they can be inferred.

A *matrix category* is a span category $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ with parallel composition actions by the logics $\langle\mathbb{A}\rangle$ and $\langle\mathbb{B}\rangle$, i.e. objects of $\mathcal{R}$ are genuine *relations* because they can be pushed and pulled along processes of $\mathbb{A}$ and $\mathbb{B}$.

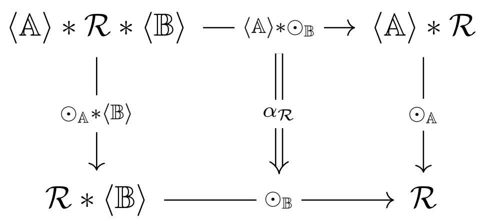


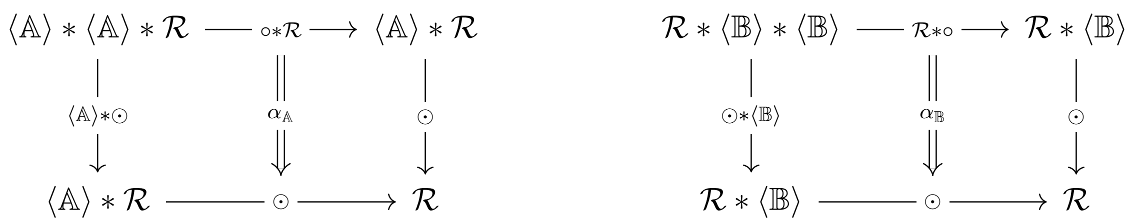
**Definition 36.** Let $\mathbb{A}$ and $\mathbb{B}$ be categories, with weave double categories $\langle\mathbb{A}\rangle$ and $\langle\mathbb{B}\rangle$.
A **matrix category** or **two-sided bifibration** $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ is a span category $\mathbb{A} \leftarrow \mathcal{R} \rightarrow \mathbb{B}$ which forms a bimodule from $\langle\mathbb{A}\rangle$ to $\langle\mathbb{B}\rangle$.

So a matrix category $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ is a span category, with a pair of actions, span functors

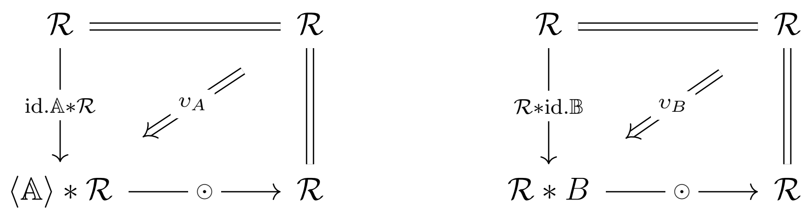
$$\begin{array}{ccccc} \mathbb{A} & \longleftarrow & \langle\mathbb{A}\rangle * \mathcal{R} & \longrightarrow & \mathbb{B} \\ \| & & \downarrow{\scriptstyle \odot_{\mathbb{A}}} & & \| \\ \mathbb{A} & \longleftarrow & \mathcal{R} & \longrightarrow & \mathbb{B} \end{array} \qquad \begin{array}{ccccc} \mathbb{A} & \longleftarrow & \mathcal{R} * \langle\mathbb{B}\rangle & \longrightarrow & \mathbb{B} \\ \| & & \downarrow{\scriptstyle \odot_{\mathbb{B}}} & & \| \\ \mathbb{A} & \longleftarrow & \mathcal{R} & \longrightarrow & \mathbb{B} \end{array}$$

and three invertible span transformations for associativity

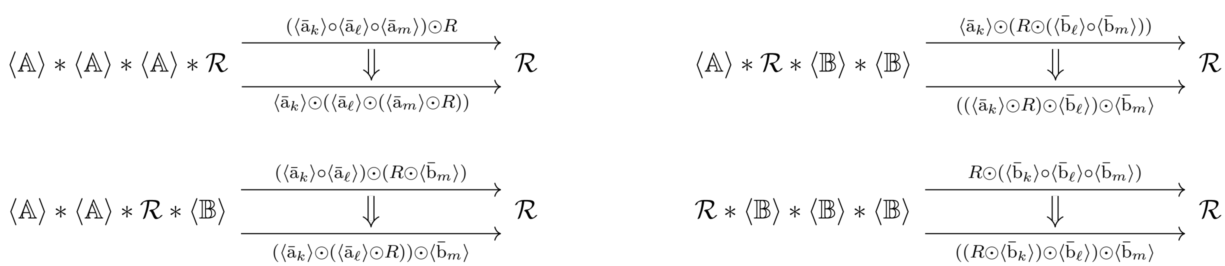



and two invertible span transformations for unitality

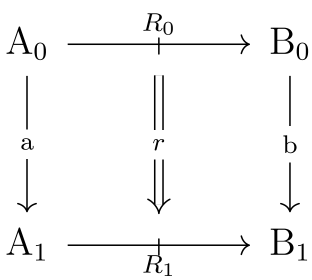


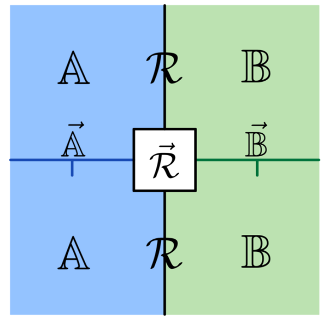

so that the following transformations are well-defined, for associativity




and for unitality.

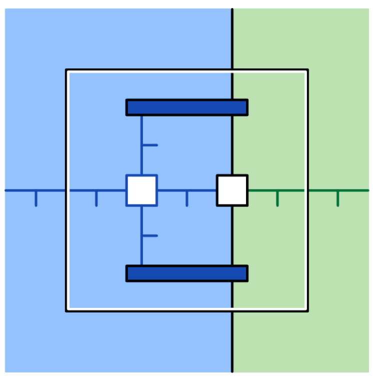
The objects and morphisms of a matrix category are the loose morphisms and squares of a bifibrant double category, i.e. relations and inferences of a logic, via the *collage* (37).



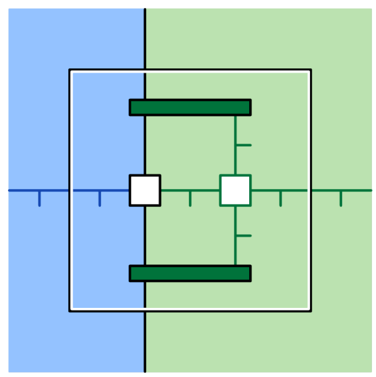




The actions by $\langle\mathbb{A}\rangle$ and $\langle\mathbb{B}\rangle$ define parallel composition of this double category.

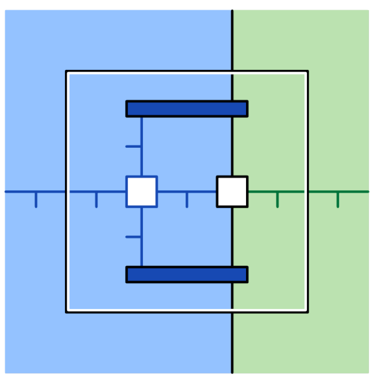
Because a weave double category is a coproduct, an action by $\langle\mathbb{A}\rangle$ defines a pair of actions by $\overrightarrow{\mathbb{A}}$ and $\overleftarrow{\mathbb{A}}$, and a bimodule structure defines *four* actions.



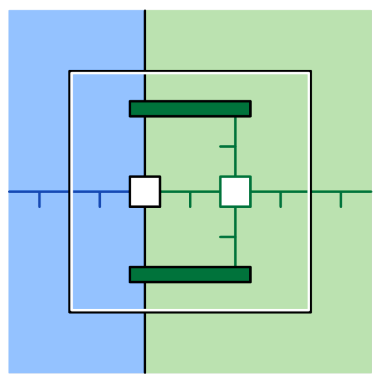
$\overrightarrow{\mathbb{A}}$-substitution



$\overrightarrow{\mathbb{B}}$-image



$\overleftarrow{\mathbb{A}}$-image

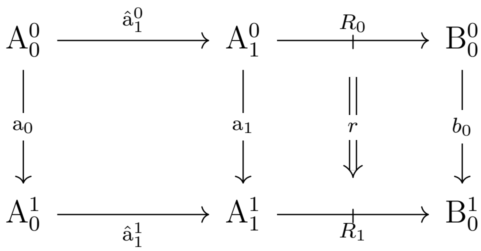


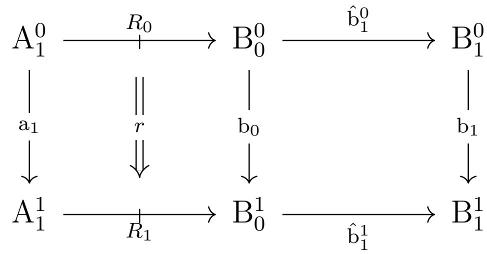
$\overleftarrow{\mathbb{B}}$-substitution

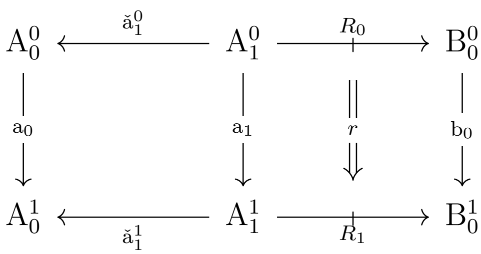
Combining these pairwise, there are four distinct bimodule structures.

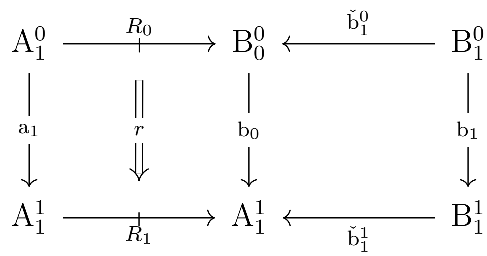
| $\overrightarrow{\mathbb{A}}, \overrightarrow{\mathbb{B}}$-bimodule | $\overrightarrow{\mathbb{A}}, \overleftarrow{\mathbb{B}}$-bimodule | $\overleftarrow{\mathbb{A}}, \overrightarrow{\mathbb{B}}$-bimodule | $\overleftarrow{\mathbb{A}}, \overleftarrow{\mathbb{B}}$-bimodule |
|---|---|---|---|
| **companion** | **fibration** | **opfibration** | **conjoint** |

The actions define parallel composition by squares in $\overrightarrow{\mathbb{A}}, \overleftarrow{\mathbb{A}}$ and $\overrightarrow{\mathbb{B}}, \overleftarrow{\mathbb{B}}$.

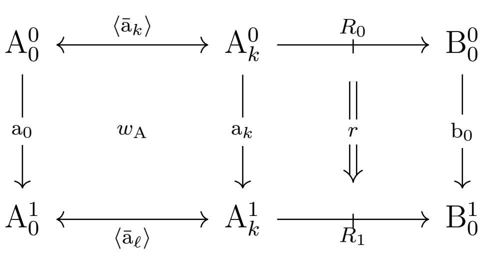


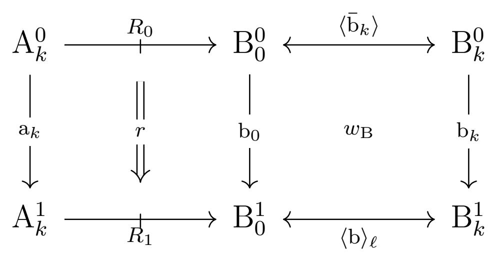




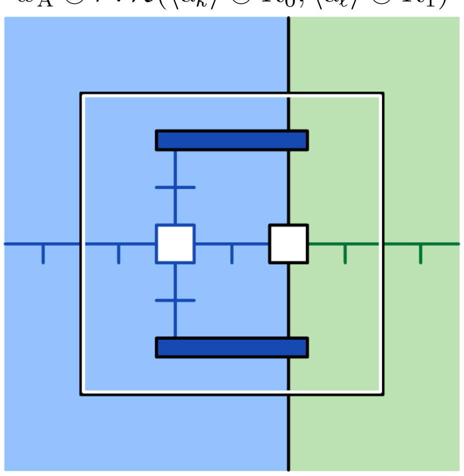


We draw a zig-zag as an arrow pointing both ways, and denote the action as follows.



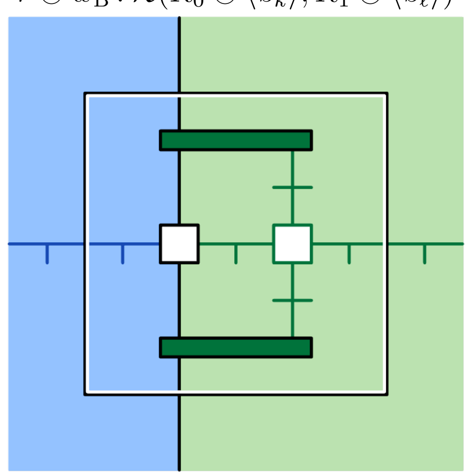


$w_{\mathrm{A}} \odot r : \mathcal{R}(\langle \bar{\mathrm{a}}_k \rangle \odot R_0, \langle \bar{\mathrm{a}}_\ell \rangle \odot R_1)$



**left action by** $\langle \mathbb{A} \rangle$

$r \odot w_{\mathrm{B}} : \mathcal{R}(R_0 \odot \langle \bar{\mathrm{b}}_k \rangle, R_1 \odot \langle \bar{\mathrm{b}}_\ell \rangle)$



**right action by** $\langle \mathbb{B} \rangle$

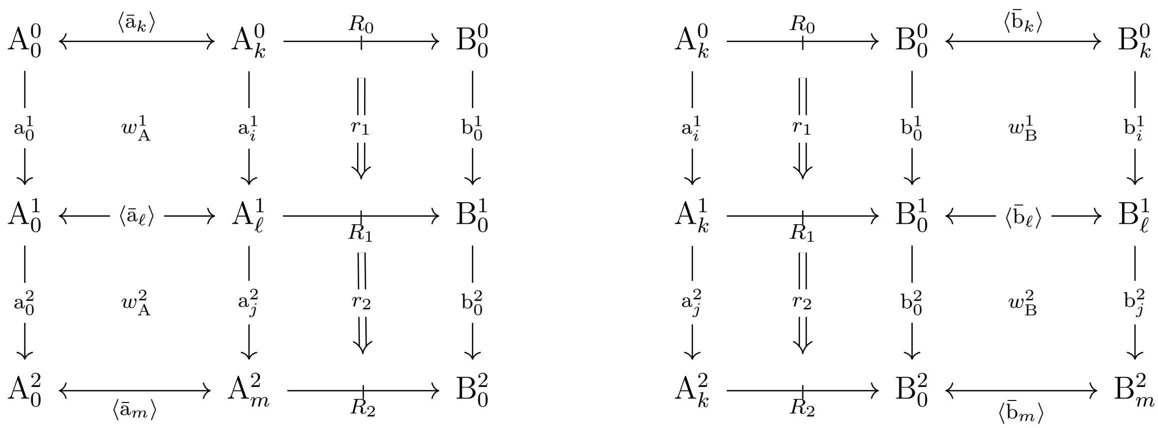
Yet apart from functoriality, which involves weaves in $\mathbb{A}$ and $\mathbb{B}$, the action is a structure on objects; and an action by zig-zags is equivalent to a pair of actions by arrows and oparrows. Hence for many definitions, particularly the coherence isomorphisms, we may simplify action notation to $\bar{\mathrm{a}} \odot R$ and $R \odot \bar{\mathrm{b}}$.

We now proceed to draw the coherences of these actions in string diagrams, and show that they define the parallel composition of a bifibrant double category.

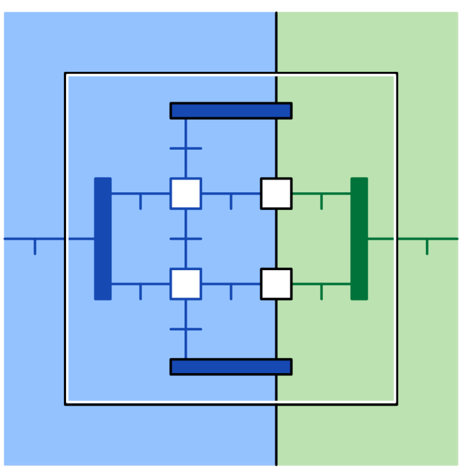
The actions of a matrix category satisfy the following coherence. First, each action is a span functor, i.e. it preserves the sequential composition of the span categories $\langle\mathbb{A}\rangle, \mathcal{R}, \langle\mathbb{B}\rangle$.

Composing in $\langle\mathbb{A}\rangle$ and $\mathcal{R}$ then acting by $\langle\mathbb{A}\rangle$, equals acting by $\langle\mathbb{A}\rangle$ then composing in $\mathcal{R}$. Composing in $\mathcal{R}$ and $\langle\mathbb{B}\rangle$ then acting by $\langle\mathbb{B}\rangle$ equals acting by $\langle\mathbb{B}\rangle$ then composing in $\mathcal{R}$.

Hence the following two composite squares are well-defined.

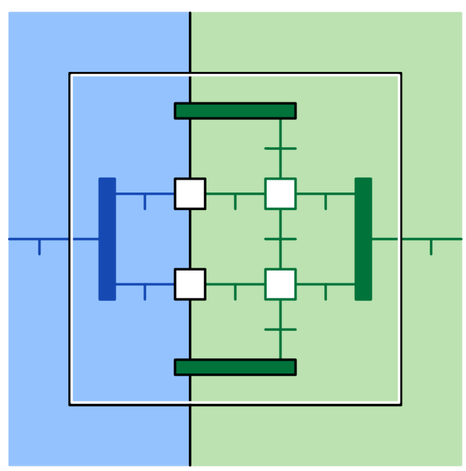



By the coherence principle, these equations can be expressed by drawing simultaneous sequential and parallel composition. This is the "interchange law" for double categories.



**left interchange**

$(w^1_A \cdot w^2_A) \odot (r_1 \cdot r_2) = (w^1_A \odot r_1) \cdot (w^2_A \odot r_2)$



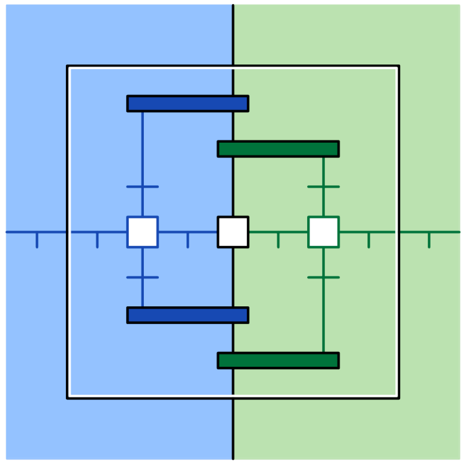
**right interchange**

$(r_1 \cdot r_2) \odot (w^1_B \cdot w^2_B) = (r_1 \odot w^1_B) \cdot (r_2 \odot w^2_B)$

Next to unpack is the three-dimensional structure. The actions are associative and unital up to coherent isomorphism: there are three "associators" for $\mathbb{A}\mathbb{A}\mathcal{R}$, $\mathbb{A}\mathcal{R}\mathbb{B}$, and $\mathcal{R}\mathbb{B}\mathbb{B}$, and two "unitors" for $\mathrm{id}_{\mathbb{A}}\mathcal{R}$ and $\mathcal{R}\mathrm{id}_{\mathbb{B}}$.

Three-dimensional string diagrams effectively depict the coherence of these isomorphisms. First, each is *natural* with respect to the morphisms of $\langle\mathbb{A}\rangle$, $\mathcal{R}$, and $\langle\mathbb{B}\rangle$.

The center associator is a span transformation $((\langle\mathbb{A}\rangle \odot \mathcal{R}) \odot \langle\mathbb{B}\rangle \cong \langle\mathbb{A}\rangle \odot (\mathcal{R} \odot \langle\mathbb{B}\rangle)$. This can be drawn as a cube, with source on top and target on bottom, connected by the homs of $\langle\mathbb{A}\rangle$, $\mathcal{R}$, and $\langle\mathbb{B}\rangle$.



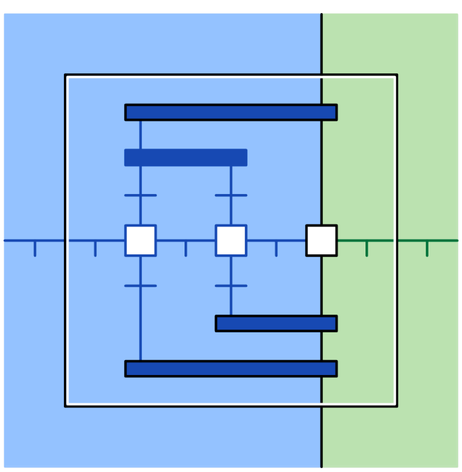
**center associator**

$\alpha_{\mathcal{R}}:\ \bar{\mathrm{a}} \odot (R \odot \bar{\mathrm{b}}) \cong (\bar{\mathrm{a}} \odot R) \odot \bar{\mathrm{b}}$

By the coherence principle, this cube expresses the naturality of the associator with respect to morphisms of $\langle\mathbb{A}\rangle, \mathcal{R}, \langle\mathbb{B}\rangle$: for every pair of weaves $w_{\mathrm{A}} : \langle\bar{\mathrm{a}}^0_k\rangle \to \langle\bar{\mathrm{a}}^1_m\rangle$ and $w_{\mathrm{B}} : \langle\bar{\mathrm{b}}^0_\ell\rangle \to \langle\bar{\mathrm{b}}^1_n\rangle$ the following commutes.

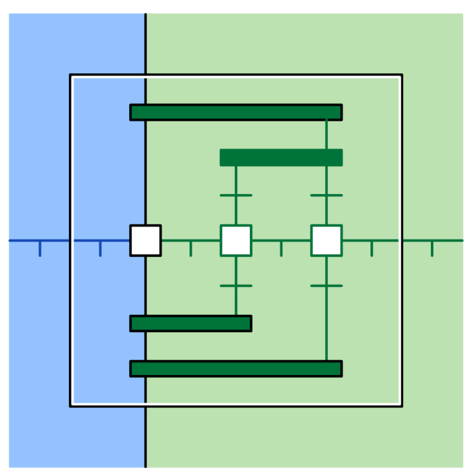
$$\begin{array}{ccc} ((\langle\bar{\mathrm{a}}^0_k\rangle \odot R) \odot \langle\bar{\mathrm{b}}^0_\ell\rangle & \xrightarrow{\alpha_{\mathcal{R}}} & \langle\bar{\mathrm{a}}^0_k\rangle \odot (R \odot \langle\bar{\mathrm{b}}^0_\ell\rangle) \\ \downarrow{\scriptstyle (w_{\mathrm{A}}\odot R)\odot w_{\mathrm{B}}} & & \downarrow{\scriptstyle w_{\mathrm{A}}\odot(R\odot w_B)} \\ ((\langle\bar{\mathrm{a}}^1_m\rangle \odot R) \odot \langle\bar{\mathrm{b}}^1_n\rangle & \xrightarrow[\alpha_{\mathcal{R}}]{} & \langle\bar{\mathrm{a}}^1_m\rangle \odot (R \odot \langle\bar{\mathrm{b}}^1_n\rangle) \end{array}$$

Continuing with the isomorphisms, there are associators for each composite action



**left associator**

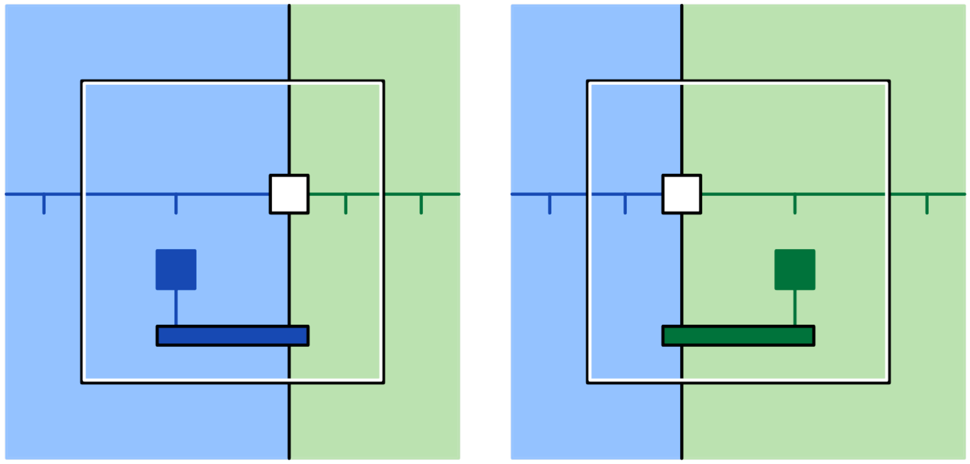
$\alpha_{\mathbb{A}}:\ (\bar{\mathrm{a}}_1 \circ \bar{\mathrm{a}}_2) \odot R \cong \bar{\mathrm{a}}_1 \odot (\bar{\mathrm{a}}_2 \odot R)$



**right associator**

$\alpha_{\mathbb{B}}:\ R \odot (\bar{\mathrm{b}}_1 \circ \bar{\mathrm{b}}_2) \cong (R \odot \bar{\mathrm{b}}_1) \odot \bar{\mathrm{b}}_2$

and the left and right unitors, which are invertible span transformations.



**left unitor**
$\upsilon_{\mathbb{A}} : R \cong \overline{\mathrm{id}}.\mathrm{A} \odot R$

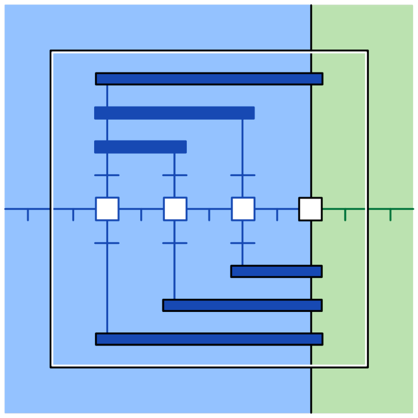
**right unitor**
$\upsilon_{\mathbb{B}} :\ R \cong R \odot \overline{\mathrm{id}}.\mathrm{B}$

Finally, we have the equations that these isomorphisms satisfy.

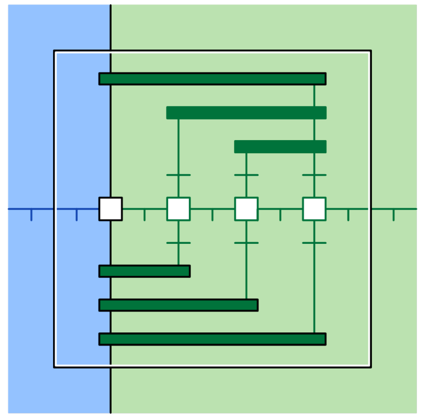
For each quadruple in $\langle\mathbb{A}\rangle * \langle\mathbb{A}\rangle * \langle\mathbb{A}\rangle * \mathcal{R}$, $\langle\mathbb{A}\rangle * \langle\mathbb{A}\rangle * \mathcal{R} * \langle\mathbb{B}\rangle$, $\langle\mathbb{A}\rangle * \mathcal{R} * \langle\mathbb{B}\rangle * \langle\mathbb{B}\rangle$, and $\mathcal{R} * \langle\mathbb{B}\rangle * \langle\mathbb{B}\rangle * \langle\mathbb{B}\rangle$, the two ways to reassociate are equal.
These are the “pentagon equations” of a double category.

Last, the left unitor coheres with the left associator, and the right unitor coheres with the right associator. These are the “triangle equations” of a double category.

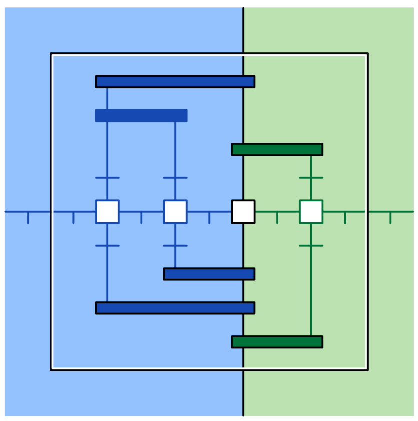
**associator coherence**



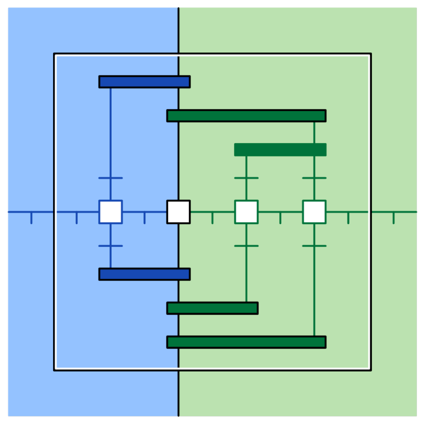
$(\langle \bar{\mathrm{a}}_k \rangle \circ \langle \bar{\mathrm{a}}_\ell \rangle \circ \langle \bar{\mathrm{a}}_m \rangle) \odot R$
$\rightrightarrows \langle \bar{\mathrm{a}}_k \rangle \odot (\langle \bar{\mathrm{a}}_\ell \rangle \odot (\langle \bar{\mathrm{a}}_m \rangle \odot R))$



$R \odot (\langle \bar{\mathrm{b}}_k \rangle \circ \langle \bar{\mathrm{b}}_\ell \rangle \circ \langle \bar{\mathrm{b}}_m \rangle)$
$\rightrightarrows ((R \odot \langle \bar{\mathrm{b}}_k \rangle) \odot \langle \bar{\mathrm{b}}_\ell \rangle) \odot \langle \bar{\mathrm{b}}_m \rangle$

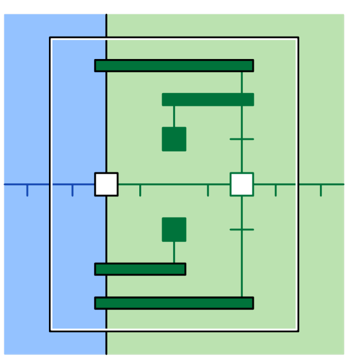


$(\langle \bar{\mathrm{a}}_k \rangle \circ \langle \bar{\mathrm{a}}_\ell \rangle) \odot (R \odot \langle \bar{\mathrm{b}}_m \rangle)$
$\rightrightarrows (\langle \bar{\mathrm{a}}_k \rangle \odot (\langle \bar{\mathrm{a}}_\ell \rangle \odot R)) \odot \langle \bar{\mathrm{b}}_m \rangle$

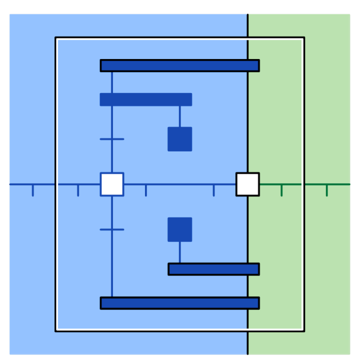


$\langle \bar{\mathrm{a}}_k \rangle \odot (R \odot (\langle \bar{\mathrm{b}}_\ell \rangle \circ \langle \bar{\mathrm{b}}_m \rangle))$
$\rightrightarrows ((\langle \bar{\mathrm{a}}_k \rangle \odot R) \odot \langle \bar{\mathrm{b}}_\ell \rangle) \odot \langle \bar{\mathrm{b}}_m \rangle$

**unitor coherence**



$(\langle \bar{\mathrm{a}}_k \rangle \circ \mathrm{id.A}) \odot R \rightrightarrows \langle \bar{\mathrm{a}}_k \rangle \odot (\mathrm{id.A} \odot R)$



$R \odot (\mathrm{id.B} \circ \langle \bar{\mathrm{b}}_k \rangle) \rightrightarrows (R \odot \mathrm{id.B}) \odot \langle \bar{\mathrm{b}}_k \rangle$

We summarize the definition, by dimension.

| | | | |
|---|---|---|---|
| 1. | **matrix category** | a span category | $\mathcal{R}:\mathbb{A}\,\|\,\mathbb{B}$ |
| 2. | precompose action | a span functor | $\langle\mathbb{A}\rangle\odot\mathcal{R}:\langle\mathbb{A}\rangle*\mathcal{R}\to\mathcal{R}$ |
| | postcompose action | a span functor | $\mathcal{R}\odot\langle\mathbb{B}\rangle:\mathcal{R}*\langle\mathbb{B}\rangle\to\mathcal{R}$ |
| 3. | associators | inv. span trans. | $\alpha_{\mathbb{A}}:(\bar{\mathrm{a}}_1\odot\bar{\mathrm{a}}_2)\odot R\;\cong\;\bar{\mathrm{a}}_1\odot(\bar{\mathrm{a}}_2\odot\mathcal{R})$ |
| | | | $\alpha_{\mathcal{R}}:(\bar{\mathrm{a}}\odot R)\odot\bar{\mathrm{a}}\;\cong\;\bar{\mathrm{a}}\odot(R\odot\bar{\mathrm{b}})$ |
| | | | $\alpha_{\mathbb{B}}:(R\odot\bar{\mathrm{b}}_1)\odot\bar{\mathrm{b}}_2\;\cong\;R\odot(\bar{\mathrm{b}}_1\odot\bar{\mathrm{b}}_2)$ |
| | unitors | inv. span trans. | $\upsilon_{\mathbb{A}}:R\;\cong\;\overline{\mathrm{id}}.\mathrm{A}\odot R$ |
| | | | $\upsilon_{\mathbb{B}}:R\;\cong\;R\odot\overline{\mathrm{id}}.\mathrm{B}$ |
| 4. | assoc. coherence | equations | $(\bar{\mathrm{a}}_1\circ\bar{\mathrm{a}}\circ\bar{\mathrm{a}}_3)\odot R\rightrightarrows\bar{\mathrm{a}}_1\odot(\bar{\mathrm{a}}_2\odot(\bar{\mathrm{a}}_3\odot R))$ |
| | | | $\bar{\mathrm{a}}_1\odot(R\odot(\bar{\mathrm{b}}_2\circ\bar{\mathrm{b}}_3))\rightrightarrows((\bar{\mathrm{a}}_1\odot R)\odot\bar{\mathrm{b}}_2)\odot\bar{\mathrm{b}}_3$ |
| | | | $(\bar{\mathrm{a}}_1\circ\bar{\mathrm{a}}_2)\odot(R\odot\bar{\mathrm{b}}_3)\rightrightarrows(\bar{\mathrm{a}}_1\odot(\bar{\mathrm{a}}_2\odot R))\odot\bar{\mathrm{b}}_3$ |
| | | | $R\odot(\bar{\mathrm{b}}_1\circ\bar{\mathrm{b}}_2\circ\bar{\mathrm{b}}_3)\rightrightarrows((R\odot\bar{\mathrm{b}}_1)\odot\bar{\mathrm{b}}_2)\odot\bar{\mathrm{b}}_3$ |
| | unit coherence | equations | $(\bar{\mathrm{a}}\circ\overline{\mathrm{id}}.\mathrm{A})\odot R\rightrightarrows\bar{\mathrm{a}}\odot(\overline{\mathrm{id}}.\mathrm{A}\odot R)$ |
| | | | $R\odot(\overline{\mathrm{id}}.\mathrm{B}\circ\bar{\mathrm{b}})\rightrightarrows(R\odot\overline{\mathrm{id}}.\mathrm{B})\odot\bar{\mathrm{b}}$ |

To complete the section, we show how matrix category forms a logic.

**Proposition 37.** Let $\mathcal{R}:\mathbb{A}\,\|\,\mathbb{B}$ be a matrix category, i.e. two-sided bifibration. The **collage** of $\mathcal{R}$, defined as follows, is a bifibrant double category. The base category is $\mathbb{A}+\mathbb{B}$, and the total category is $\langle\mathbb{A}\rangle+\mathcal{R}+\langle\mathbb{B}\rangle$, equipped with the only possible projections.

$$\mathbb{A}+\mathbb{B}\longleftarrow\langle\mathbb{A}\rangle+\mathcal{R}+\langle\mathbb{B}\rangle\longrightarrow\mathbb{A}+\mathbb{B}$$

Parallel composition is given by the actions of $\langle\mathbb{A}\rangle$ and $\langle\mathbb{B}\rangle$ on $\mathcal{R}$, and parallel composition in $\langle\mathbb{A}\rangle$ and $\langle\mathbb{B}\rangle$. The associators and unitors are given by the coherence isomorphisms of $\mathcal{R}$, and those of $\langle\mathbb{A}\rangle$ and $\langle\mathbb{B}\rangle$; all of their equations hold by fiat.

The collage is a bifibrant double category, because morphisms of $\mathbb{A}$ and $\mathbb{B}$ induce arrows and oparrows, which are companions and conjoints.

### 2.2.1 Matrix functors

So, a matrix category consists of relations and inferences in a logic. Now, a matrix functor is a *mapping* of these relations and inferences, visualized from inner to outer.

Because a matrix category is a *pseudo*bimodule, a matrix functor preserves composition actions only up to coherent isomorphism.

**Definition 38.** Let $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ and $[\![\mathbb{B}]\!] : \mathbb{B}_0 \to \mathbb{B}_1$ be functors.

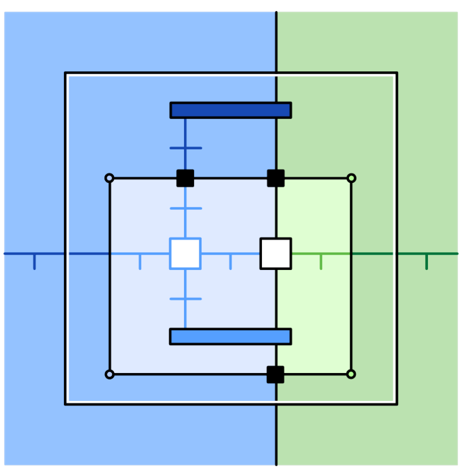
Let $\mathcal{R}_0 : \mathbb{A}_0 \,\|\, \mathbb{B}_0$ and $\mathcal{R}_1 : \mathbb{A}_1 \,\|\, \mathbb{B}_1$ be matrix categories. A **matrix functor** $[\![\mathcal{R}]\!] : \mathcal{R}_0 \to \mathcal{R}_1$ is a morphism of pseudobimodules in $\mathrm{Span}\mathbb{C}\mathrm{at}$. This is a span functor

$$\begin{array}{ccccc} \mathbb{A}_0 & \longleftarrow & \mathcal{R}_0 & \longrightarrow & \mathbb{B}_0 \\ \downarrow {\scriptstyle [\![\mathbb{A}]\!]} & & \downarrow {\scriptstyle [\![\mathcal{R}]\!]} & & \downarrow {\scriptstyle [\![\mathbb{B}]\!]} \\ \mathbb{A}_1 & \longleftarrow & \mathcal{R}_1 & \longrightarrow & \mathbb{B}_1 \end{array}$$

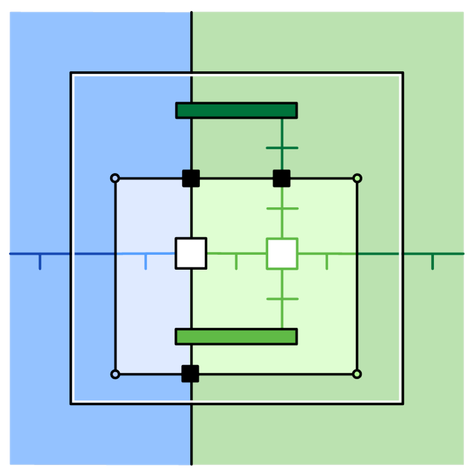
with invertible span transformations called the **left** and **right join**

$$\begin{array}{ccc} \langle \mathbb{A}_0 \rangle * \mathcal{R}_0 & \xrightarrow{\langle [\![\mathbb{A}]\!] \rangle * [\![\mathcal{R}]\!]} & \langle \mathbb{A}_1 \rangle * \mathcal{R}_1 \\ \downarrow {\scriptstyle \odot^0_{\mathbb{A}}} & \Downarrow {\scriptstyle [\![\odot_{\mathbb{A}}]\!]} & \downarrow {\scriptstyle \odot^1_{\mathbb{A}}} \\ \mathcal{R}_0 & \xrightarrow[{[\![\mathcal{R}]\!]}]{} & \mathcal{R}_1 \end{array} \qquad \begin{array}{ccc} \mathcal{R}_0 * \langle \mathbb{B}_0 \rangle & \xrightarrow{[\![\mathcal{R}]\!] * \langle [\![\mathbb{B}]\!] \rangle} & \mathcal{R}_1 * \langle \mathbb{B}_1 \rangle \\ \downarrow {\scriptstyle \odot^0_{\mathbb{B}}} & \Downarrow {\scriptstyle [\![\odot_{\mathbb{B}}]\!]} & \downarrow {\scriptstyle \odot^1_{\mathbb{B}}} \\ \mathcal{R}_0 & \xrightarrow[{[\![\mathcal{R}]\!]}]{} & \mathcal{R}_1 \end{array}$$

$$[\![\odot_{\mathbb{A}}]\!] : [\![\bar{\mathrm{a}}_0]\!] \odot [\![R_0]\!] \cong [\![\bar{\mathrm{a}}_0 \odot R_0]\!] \qquad [\![\odot_{\mathbb{B}}]\!] : [\![R_0]\!] \odot [\![\bar{\mathrm{b}}_0]\!] \cong [\![R_0 \odot \bar{\mathrm{b}}_0]\!]$$



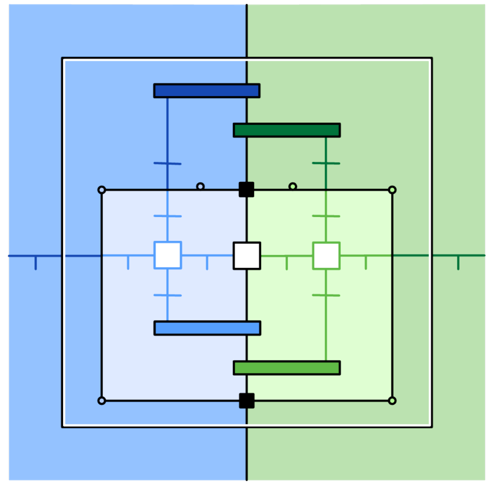
**left join**



**right join**

which together are natural with respect to the center associator:

$$
\begin{array}{ccc}
\llbracket \bar{a} \rrbracket \odot (\llbracket R \rrbracket \odot \llbracket \bar{b} \rrbracket) & \xrightarrow{\alpha_{\mathcal{R}}} & (\llbracket \bar{a} \rrbracket \odot \llbracket R \rrbracket) \odot \llbracket \bar{b} \rrbracket \\
\downarrow {\scriptstyle \llbracket \mathbb{A} \rrbracket \odot_{\mathbb{A}} \llbracket \odot_{\mathbb{B}} \rrbracket} & & \downarrow {\scriptstyle \llbracket \odot_{\mathbb{A}} \rrbracket \odot_{\mathbb{B}} \llbracket \mathbb{B} \rrbracket} \\
\llbracket \bar{a} \rrbracket \odot \llbracket R \odot \bar{b} \rrbracket & & \llbracket \bar{a} \odot R \rrbracket \odot \llbracket \bar{b} \rrbracket \\
\downarrow {\scriptstyle \llbracket \odot_{\mathbb{A}} \rrbracket} & & \downarrow {\scriptstyle \llbracket \odot_{\mathbb{B}} \rrbracket} \\
\llbracket \bar{a} \odot (R \odot \bar{b}) \rrbracket & \xrightarrow[\alpha_{\mathcal{R}}]{} & \llbracket (\bar{a} \odot R) \odot \bar{b} \rrbracket
\end{array}
$$

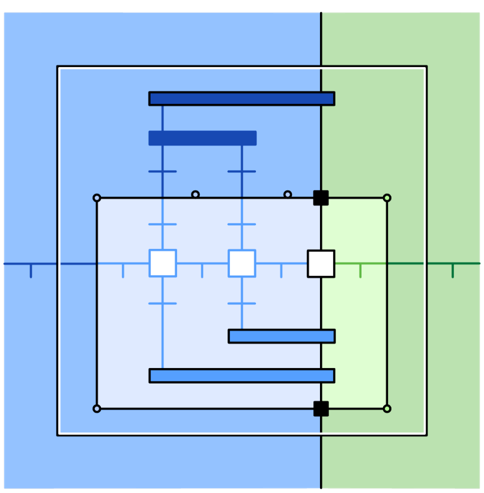


**center associator coherence**

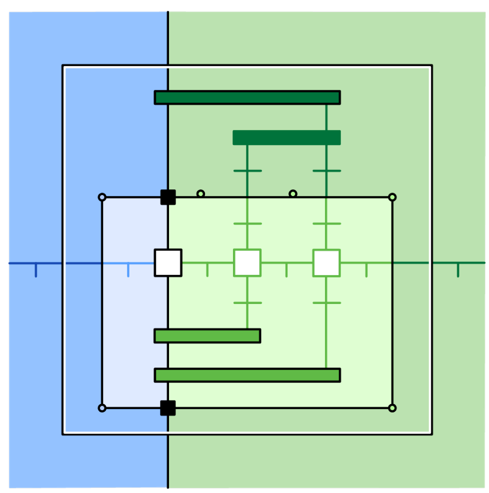
and each is natural with respect to its own associator:

$$
\begin{array}{ccc}
(\llbracket \bar{a}_1 \rrbracket \circ \llbracket \bar{a}_2 \rrbracket) \odot \llbracket R \rrbracket & \xrightarrow{\alpha_{\mathbb{A}}} & \llbracket \bar{a}_1 \rrbracket \odot (\llbracket \bar{a}_2 \rrbracket \odot \llbracket R \rrbracket) \\
\downarrow {\scriptstyle \llbracket \odot_{\mathbb{A}} \rrbracket \odot_{\mathbb{A}} \llbracket \mathcal{R} \rrbracket} & & \downarrow {\scriptstyle \llbracket \mathbb{A} \rrbracket \odot_{\mathbb{A}} \llbracket \odot_{\mathbb{A}} \rrbracket} \\
\llbracket \bar{a}_1 \circ \bar{a}_2 \rrbracket \odot \llbracket R \rrbracket & & \llbracket \bar{a}_1 \rrbracket \odot \llbracket \bar{a}_2 \odot R \rrbracket \\
\downarrow {\scriptstyle \llbracket \odot_{\mathbb{A}} \rrbracket} & & \downarrow {\scriptstyle \llbracket \odot_{\mathbb{A}} \rrbracket} \\
\llbracket (\bar{a}_1 \circ \bar{a}_2) \odot R \rrbracket & \xrightarrow[\alpha_{\mathbb{A}}]{} & \llbracket \bar{a}_1 \odot (\bar{a}_2 \odot R) \rrbracket
\end{array}
$$

$$
\begin{array}{ccc}
\llbracket R \rrbracket \odot (\llbracket \bar{b}_1 \rrbracket \circ \llbracket \bar{b}_2 \rrbracket) & \xrightarrow{\alpha_{\mathbb{B}}} & (\llbracket R \rrbracket \odot \llbracket \bar{b}_1 \rrbracket) \odot \llbracket \bar{b}_2 \rrbracket \\
\downarrow {\scriptstyle \llbracket \mathcal{R} \rrbracket \odot_{\mathbb{B}} \llbracket \odot_{\mathbb{B}} \rrbracket} & & \downarrow {\scriptstyle \llbracket \odot_{\mathbb{B}} \rrbracket \odot_{\mathbb{B}} \llbracket \mathbb{B} \rrbracket} \\
\llbracket R \rrbracket \odot \llbracket \bar{b}_1 \circ \bar{b}_2 \rrbracket & & \llbracket R \odot \bar{b}_1 \rrbracket \odot \llbracket \bar{b}_2 \rrbracket \\
\downarrow {\scriptstyle \llbracket \odot_{\mathbb{B}} \rrbracket} & & \downarrow {\scriptstyle \llbracket \odot_{\mathbb{B}} \rrbracket} \\
\llbracket R \odot (\bar{b}_1 \circ \bar{b}_2) \rrbracket & \xrightarrow[\alpha_{\mathbb{B}}]{} & \llbracket (R \odot \bar{b}_1) \odot \bar{b}_2 \rrbracket
\end{array}
$$



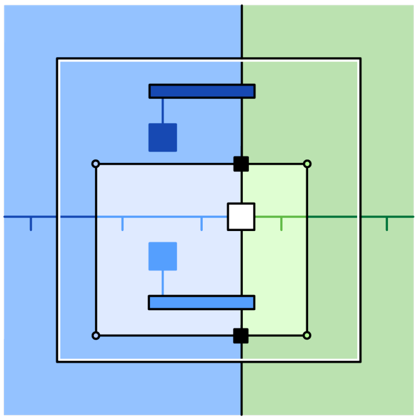
**left associator coherence**



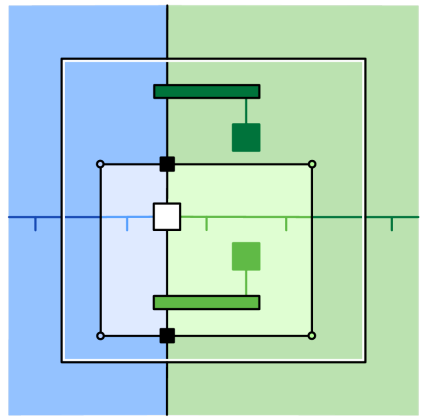
**right associator coherence**

and each is natural with respect to its own unitor.

$$\begin{array}{ccc} [\![\bar{\mathrm{id}}.\mathrm{A}]\!] \odot [\![R]\!] & \xrightarrow{\upsilon_{\mathbb{A}}} & [\![R]\!] \\ \downarrow{\scriptstyle [\![\odot_{\mathbb{A}}]\!]} & & \| \\ [\![\bar{\mathrm{id}}.\mathrm{A} \odot R]\!] & \xrightarrow[\upsilon_{\mathbb{A}}]{} & [\![R]\!] \end{array} \qquad \begin{array}{ccc} [\![R]\!] \odot [\![\bar{\mathrm{id}}.\mathrm{B}]\!] & \xrightarrow{\upsilon_{\mathbb{B}}} & [\![R]\!] \\ \downarrow{\scriptstyle [\![\odot_{\mathbb{B}}]\!]} & & \| \\ [\![R \odot \bar{\mathrm{id}}.\mathrm{B}]\!] & \xrightarrow[\upsilon_{\mathbb{B}}]{} & [\![R]\!] \end{array}$$



**left unitor coherence**



**right unitor coherence**

We summarize the concept of matrix functor.

| | | | |
|---|---|---|---|
| 2. | **matrix functor** | span functor | $[\![\mathcal{R}]\!]([\![\mathbb{A}]\!],[\![\mathbb{B}]\!]) : \mathcal{R}_0(\mathbb{A}_0,\mathbb{B}_0) \to \mathcal{R}_1(\mathbb{A}_1,\mathbb{B}_1)$ |
| 3. | left join | inv. span trans. | $[\![\odot_{\mathbb{A}}]\!] : [\![\bar{\mathrm{a}}_0]\!] \odot [\![R_0]\!] \cong [\![\bar{\mathrm{a}}_0 \odot R_0]\!]$ |
| | right join | inv. span trans. | $[\![\odot_{\mathbb{B}}]\!] : [\![R_0]\!] \odot [\![\bar{\mathrm{b}}_0]\!] \cong [\![R_0 \odot \bar{\mathrm{b}}_0]\!]$ |
| 4. | left assoc. coherence | equation | $([\![\bar{\mathrm{a}}_1]\!] \circ [\![\bar{\mathrm{a}}_2]\!]) \odot [\![R]\!] \rightrightarrows [\![\bar{\mathrm{a}}_1 \odot (\bar{\mathrm{a}}_2 \odot R)]\!]$ |
| | center assoc. coherence | equation | $[\![\bar{\mathrm{a}}]\!] \odot ([\![R]\!] \odot [\![\bar{\mathrm{b}}]\!]) \rightrightarrows [\![(\bar{\mathrm{a}} \odot R) \odot \bar{\mathrm{b}}]\!]$ |
| | right assoc. coherence | equation | $[\![R]\!] \odot ([\![\bar{\mathrm{b}}_1]\!] \circ [\![\bar{\mathrm{b}}_2]\!]) \rightrightarrows [\![(R \odot \bar{\mathrm{b}}_1) \odot \bar{\mathrm{b}}_2]\!]$ |
| | left unit coherence | equation | $[\![\bar{\mathrm{id}}.\mathrm{A}]\!] \odot [\![R]\!] \rightrightarrows [\![R]\!]$ |
| | right unit coherence | equation | $[\![R]\!] \odot [\![\bar{\mathrm{id}}.\mathrm{B}]\!] \rightrightarrows [\![R]\!]$ |

## 2.3 Matrix profunctors

Just as a matrix category is a bimodule of weave double categories, a matrix profunctor is a bimodule of weave vertical profunctors, which is coherent with the bimodule structures of the source and target matrix categories.

**Definition 39.** Let $\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}$ be categories, and $\mathcal{Q} : \mathbb{X} \,\|\, \mathbb{Y}$ $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ be matrix categories.

Let $f : \mathbb{X} \mid \mathbb{A}$ and $g : \mathbb{Y} \mid \mathbb{B}$ be profunctors, determining weave profunctors $f \leftarrow \langle f \rangle \rightarrow f$ and $g \leftarrow \langle g \rangle \rightarrow g$.

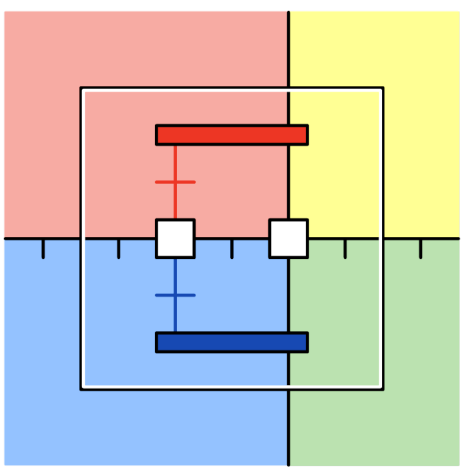
A **matrix profunctor** $i(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \mid \mathcal{R}(\mathbb{A},\mathbb{B})$ is a span profunctor which is a bimodule from $\langle f \rangle$ to $\langle g \rangle$, which coheres with the associators and unitors of $\mathcal{Q}$ and $\mathcal{R}$.

Hence a matrix profunctor is a span profunctor

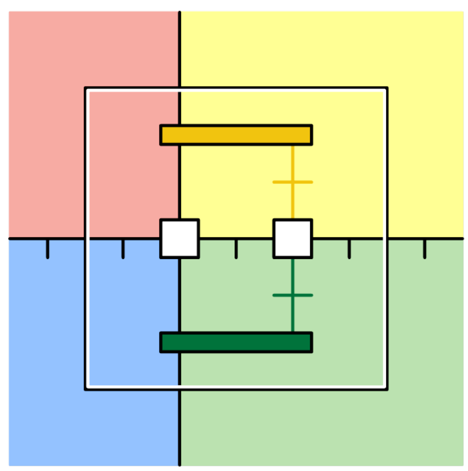
$$\begin{array}{ccccc} \mathbb{X} & \longleftarrow & \mathcal{Q} & \longrightarrow & \mathbb{Y} \\ {\scriptstyle f}\downarrow & \Leftarrow & \downarrow{\scriptstyle i} & \Rightarrow & \downarrow{\scriptstyle g} \\ \mathbb{A} & \longleftarrow & \mathcal{R} & \longrightarrow & \mathbb{B} \end{array}$$

with two span transformations, precompose action by $\langle f \rangle$ and postcompose action by $\langle g \rangle$

$$\begin{array}{ccc} \langle \mathbb{X} \rangle * \mathcal{Q} & \xrightarrow{\langle f \rangle * i} & \langle \mathbb{A} \rangle * \mathcal{R} \\ \downarrow{\scriptstyle \odot_{\mathbb{X}}} & \Downarrow{\scriptstyle \odot_f} & \downarrow{\scriptstyle \odot_{\mathbb{A}}} \\ \mathcal{Q} & \xrightarrow[i]{} & \mathcal{R} \end{array} \qquad \begin{array}{ccc} \mathcal{Q} * \langle \mathbb{Y} \rangle & \xrightarrow{i * \langle g \rangle} & \mathcal{R} * \langle \mathbb{B} \rangle \\ \downarrow{\scriptstyle \odot_{\mathbb{Y}}} & \Downarrow{\scriptstyle \odot_g} & \downarrow{\scriptstyle \odot_{\mathbb{B}}} \\ \mathcal{Q} & \xrightarrow[i]{} & \mathcal{R} \end{array}$$



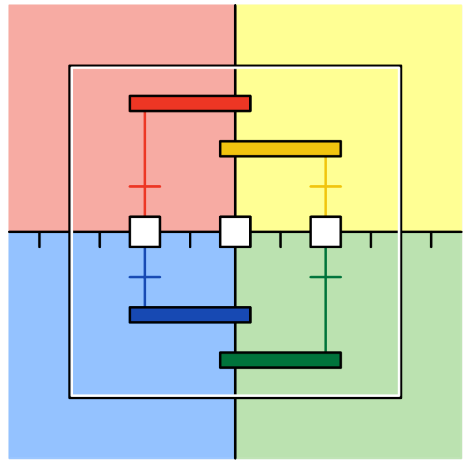
**precompose action**



**postcompose action**

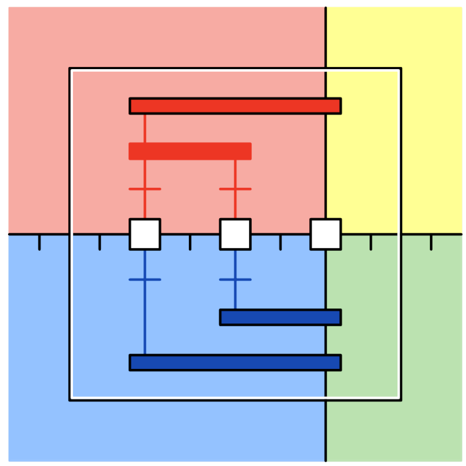
which cohere with the associators and unitors of $\mathcal{Q}$ and $\mathcal{R}$, as follows.

$$\begin{array}{ccc} \bar{x} \odot (Q \odot \bar{y}) & \xrightarrow{\alpha_{\mathcal{Q}}} & (\bar{x} \odot Q) \odot \bar{y} \\ \downarrow {\scriptstyle [f_0,f_1]\odot(i\odot[g_0,g_1])} & & \downarrow {\scriptstyle ([f_0,f_1]\odot i)\odot[g_0,g_1]} \\ \bar{a} \odot (R \odot \bar{b}) & \xrightarrow[\alpha_{\mathcal{R}}]{} & (\bar{a} \odot R) \odot \bar{b} \end{array}$$



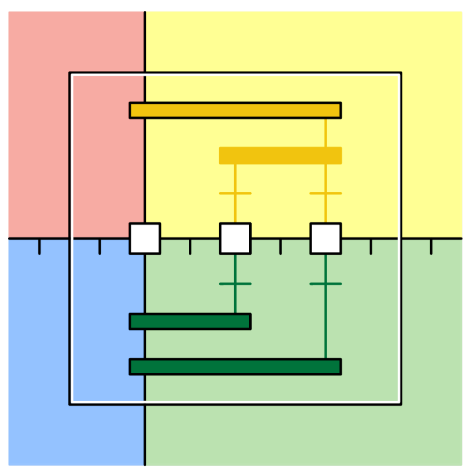
**center associator coherence**

$$\begin{array}{ccc} (\bar{x}_1 \circ \bar{x}_2) \odot Q & \xrightarrow{\alpha_{\mathbb{X}}} & \bar{x}_1 \odot (\bar{x}_2 \odot Q) \\ \downarrow {\scriptstyle [f_0,f_1,f_2]\odot i} & & \downarrow {\scriptstyle [f_0,f_1]\odot([f_1,f_2]\odot i)} \\ (\bar{a}_1 \circ \bar{a}_2) \odot R & \xrightarrow[\alpha_{\mathbb{A}}]{} & \bar{a}_1 \odot (\bar{a}_2 \odot R) \end{array}$$



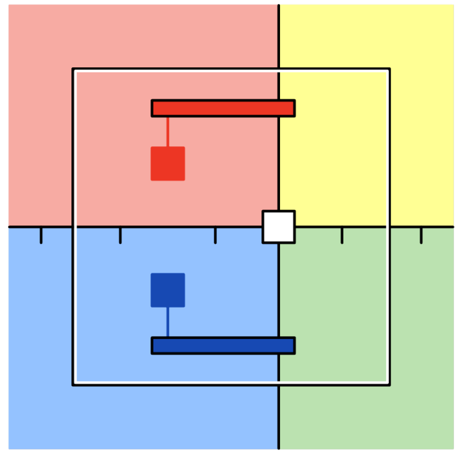
**left associator coherence**

$$\begin{array}{ccc} Q \odot (\bar{y}_1 \circ \bar{y}_2) & \xrightarrow{\alpha_{\mathbb{Y}}} & (Q \odot \bar{y}_1) \odot \bar{y}_2 \\ \downarrow {\scriptstyle i\odot[g_0,g_1,g_2]} & & \downarrow {\scriptstyle (i\odot[g_0,g_1])\odot[g_1,g_2]} \\ R \odot (\bar{b}_1 \circ \bar{b}_2) & \xrightarrow[\alpha_{\mathbb{B}}]{} & (R \odot \bar{b}_1) \odot \bar{b}_2 \end{array}$$



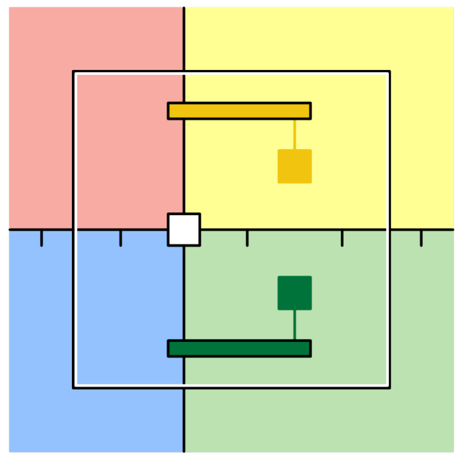
**right associator coherence**

$$\begin{array}{ccc} \overline{\mathrm{id}}.\mathrm{X} \odot Q & \xrightarrow{\upsilon_{\mathbb{X}}} & Q \\ \Big\downarrow{\scriptstyle \mathrm{id.f} \odot i} & & \Big\downarrow{\scriptstyle i} \\ \overline{\mathrm{id}}.\mathrm{A} \odot R & \xrightarrow[\upsilon_{\mathbb{A}}]{} & R \end{array}$$



**left unit coherence**

$$\begin{array}{ccc} Q \odot \overline{\mathrm{id}}.\mathrm{Y} & \xrightarrow{\upsilon_{\mathbb{Y}}} & Q \\ \Big\downarrow{\scriptstyle i \odot \mathrm{id.g}} & & \Big\downarrow{\scriptstyle i} \\ R \odot \overline{\mathrm{id}}.\mathrm{B} & \xrightarrow[\upsilon_{\mathbb{B}}]{} & R \end{array}$$



**right unit coherence**

We summarize the concept of matrix profunctor, ordered by dimension.

| | | | |
|---|---|---|---|
| 2. | **matrix profunctor** | a span profunctor | $i(f,g): \mathcal{Q}(\mathbb{X},\mathbb{Y}) \mid \mathcal{R}(\mathbb{A},\mathbb{B})$ |
| 3. | precompose action | a span transformation | $\langle f \rangle \odot i : \langle f \rangle * i \Rightarrow i$ |
| | postcompose action | a span transformation | $i \odot \langle g \rangle : i * \langle g \rangle \Rightarrow i$ |
| 4. | assoc. coherence | equations | $(\bar{\mathrm{x}}_1 \odot \bar{\mathrm{x}}_2) \odot Q \rightrightarrows \bar{\mathrm{a}}_1 \odot (\bar{\mathrm{a}}_2 \odot R)$ |
| | | | $\bar{\mathrm{x}} \odot (Q \odot \bar{\mathrm{y}}) \rightrightarrows (\bar{\mathrm{a}} \odot R) \odot \bar{\mathrm{b}}$ |
| | | | $Q \odot (\bar{\mathrm{y}}_1 \odot \bar{\mathrm{y}}_2) \rightrightarrows (R \odot \bar{\mathrm{b}}_1) \odot \bar{\mathrm{b}}_2$ |
| | unit coherence | equations | $\overline{\mathrm{id}}.\mathrm{X} \odot Q \rightrightarrows \overline{\mathrm{id}}.\mathrm{A} \odot R$ |
| | | | $Q \odot \overline{\mathrm{id}}.\mathrm{Y} \rightrightarrows R \odot \overline{\mathrm{id}}.\mathrm{B}$ |

**Note.** A matrix profunctor $i(f,g): \mathcal{Q}(\mathbb{X},\mathbb{Y}) \mid \mathcal{R}(\mathbb{A},\mathbb{B})$ does not include any action of the elements of $f$ or $g$ on $\mathcal{Q}$ or $\mathcal{R}$. Visually, this means that in general the "pointer strings" of $f$ and $g$ connecting $\mathbb{X}$ to $\mathbb{A}$ and $\mathbb{Y}$ to $\mathbb{B}$ *do not bend*; i.e. the collage is not a bifibrant double category. It is a special property when such actions do exist.

### 2.3.1 Matrix transformations

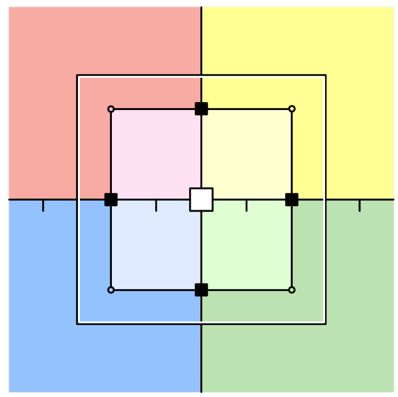
Just as a matrix profunctor is a bimodule of weave profunctors, a matrix transformation is a homomorphism of these bimodules, which coheres with the joins of the source and target matrix functors.

**Definition 40.** Let $\llbracket\mathbb{X}\rrbracket, \llbracket\mathbb{Y}\rrbracket, \llbracket\mathbb{A}\rrbracket, \llbracket\mathbb{B}\rrbracket$ be functors, $f_0 : \mathbb{X}_0 \mid \mathbb{A}_0$, $f_1 : \mathbb{X}_1 \mid \mathbb{A}_1$, $g_0 : \mathbb{Y}_0 \mid \mathbb{B}_0$, $g_1 : \mathbb{Y}_1 \mid \mathbb{B}_1$ profunctors, and $\llbracket f\rrbracket : f_0 \Rightarrow f_1$, $\llbracket g\rrbracket : g_0 \Rightarrow g_1$ transformations.

Let $\mathcal{Q}_0 : \mathbb{X}_0 \parallel \mathbb{Y}_0$, $\mathcal{Q}_1 : \mathbb{X}_1 \parallel \mathbb{Y}_1$, $\mathcal{R}_0 : \mathbb{A}_0 \parallel \mathbb{B}_0$, $\mathcal{R}_1 : \mathbb{A}_1 \parallel \mathbb{B}_1$ be matrix categories, and $\llbracket\mathcal{Q}\rrbracket, \llbracket\mathcal{R}\rrbracket$ be matrix functors. Let $i_0(f_0, g_0) : \mathcal{Q}_0 \mid \mathcal{R}_0$ and $i_1(f_1, g_1) : \mathcal{Q}_1 \mid \mathcal{R}_1$ be matrix profunctors.

A **matrix transformation** $\llbracket i\rrbracket : i_0 \to i_1$ is a span transformation

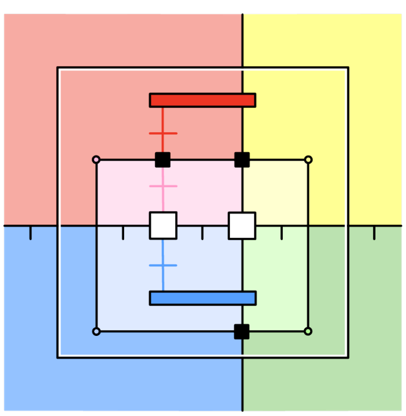
$$\begin{array}{ccccc} f_0 & \longleftarrow & i_0 & \longrightarrow & g_0 \\ \downarrow{\scriptstyle \llbracket f\rrbracket} & & \downarrow{\scriptstyle \llbracket i\rrbracket} & & \downarrow{\scriptstyle \llbracket g\rrbracket} \\ f_1 & \longleftarrow & i_1 & \longrightarrow & g_1 \end{array}$$



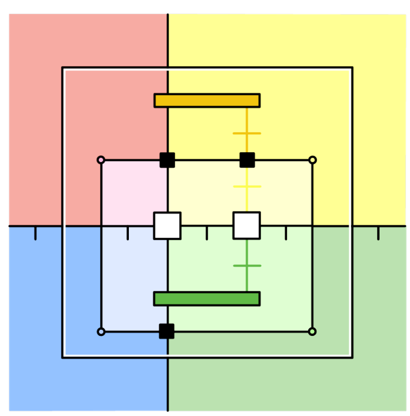
**matrix transformation**

which coheres with the left and right joins of $\llbracket\mathcal{Q}\rrbracket$ and $\llbracket\mathcal{R}\rrbracket$.

$$\begin{array}{ccc} \llbracket \mathrm{x}\rrbracket \odot \llbracket Q\rrbracket & \xrightarrow{[\odot_{\mathbb{X}}]} & \llbracket \mathrm{x} \odot Q\rrbracket \\ \downarrow{\scriptstyle \llbracket \mathrm{f}\rrbracket\odot\llbracket i\rrbracket} & & \downarrow{\scriptstyle \llbracket \mathrm{f}\odot i\rrbracket} \\ \llbracket \mathrm{a}\rrbracket \odot \llbracket R\rrbracket & \xrightarrow[{[\odot_{\mathbb{A}}]}]{} & \llbracket \mathrm{a} \odot R\rrbracket \end{array} \qquad \begin{array}{ccc} \llbracket Q\rrbracket \odot \llbracket \mathrm{y}\rrbracket & \xrightarrow{[\odot_{\mathbb{Y}}]} & \llbracket Q \odot \mathrm{y}\rrbracket \\ \downarrow{\scriptstyle \llbracket \mathrm{g}\rrbracket\odot\llbracket i\rrbracket} & & \downarrow{\scriptstyle \llbracket i\odot \mathrm{g}\rrbracket} \\ \llbracket R\rrbracket \odot \llbracket \mathrm{b}\rrbracket & \xrightarrow[{[\odot_{\mathbb{B}}]}]{} & \llbracket R \odot \mathrm{b}\rrbracket \end{array}$$



**left join coherence**



**right join coherence**

We summarize the concept of matrix transformation.

| | | | |
|---|---|---|---|
| 3. | **matrix transformation** | a span transformation | $[\![i]\!]([\![f]\!],[\![g]\!]) : i_0(f_0, g_0) \Rightarrow i_1(f_1, g_1)$ |
| 4. | left join coherence | equation | $[\![\mathrm{x}]\!] \odot [\![Q]\!] \rightrightarrows [\![\mathrm{a} \odot R]\!]$ |
| | right join coherence | equation | $[\![Q]\!] \odot [\![\mathrm{y}]\!] \rightrightarrows [\![R \odot \mathrm{b}]\!]$ |

Matrix categories and matrix functors, matrix profunctors and matrix transformations form $\mathrm{Mat}\mathbb{C}\mathrm{at}$, a bifibrant double category which is fibered over $\mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$.

## 2.4 Sequential composition

To complete the logic of matrix categories, we define its relation composition: *sequential* matrix profunctor composition, in the direction of profunctors.

First, we see the category of matrix profunctors is fibered over the category of pairs of matrix categories; so the double category will indeed be a logic.

**Definition 41.** Define $\mathrm{MatCat}$ to be the category of matrix categories and matrix functors. Composition of matrix functors is defined by that of span functors, and that of joins; one can verify this satisfies the necessary coherence, and that matrix functor composition is associative and unital.

**Definition 42.** Define $\mathrm{MatProf}$ to be the category of matrix profunctors and matrix transformations. Composition is defined by that of span transformations, and the coherence of the composite follows from that of its factors. $\mathrm{MatProf}$ is equipped with projections to $\mathrm{MatCat}$, giving a span of categories.

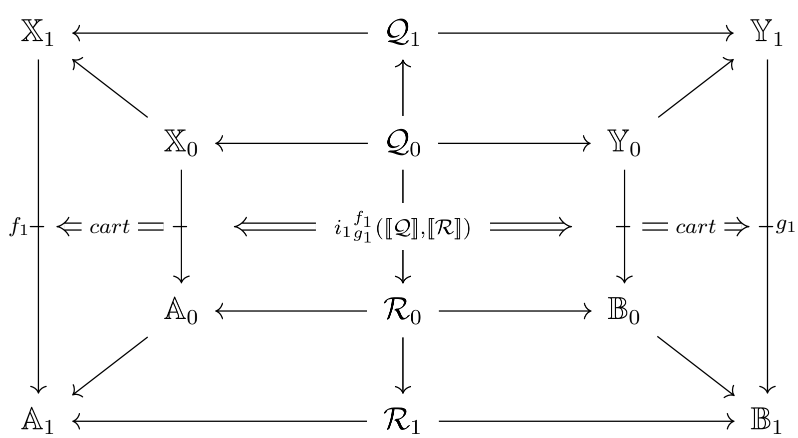
$$\mathrm{MatCat} \longleftarrow \mathrm{MatProf} \longrightarrow \mathrm{MatCat}$$

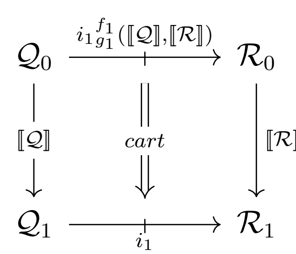
**Theorem 43.** $\mathrm{MatProf}$ is fibered over $\mathrm{MatCat} \times \mathrm{MatCat}$.

*Proof.* Let $\mathcal{Q}_0(\mathbb{X}_0, \mathbb{Y}_0)$, $\mathcal{R}_0(\mathbb{A}_0, \mathbb{B}_0)$, $\mathcal{Q}_1(\mathbb{X}_1, \mathbb{Y}_1)$ and $\mathcal{R}_1(\mathbb{A}_1, \mathbb{B}_1)$ be matrix categories.

Let $[\![\mathcal{Q}]\!]([\![\mathbb{X}]\!],[\![\mathbb{Y}]\!]) : \mathcal{Q}_0 \to \mathcal{Q}_1$ and $[\![\mathcal{R}]\!]([\![\mathbb{A}]\!],[\![\mathbb{B}]\!]) : \mathcal{R}_0 \to \mathcal{R}_1$ be matrix functors.

The **substitution** matrix profunctor $i_1(f_1, g_1)([\![\mathcal{Q}]\!],[\![\mathcal{R}]\!]) : \mathcal{Q}_0(\mathbb{X}_0, \mathbb{Y}_0) \mid \mathcal{R}_0(\mathbb{A}_0, \mathbb{B}_0)$ is given by substituting functors into profunctors: $f_1([\![\mathbb{X}]\!],[\![\mathbb{A}]\!]), i_1([\![\mathcal{Q}]\!],[\![\mathcal{Q}]\!]), g_1([\![\mathbb{Y}]\!],[\![\mathbb{B}]\!])$.

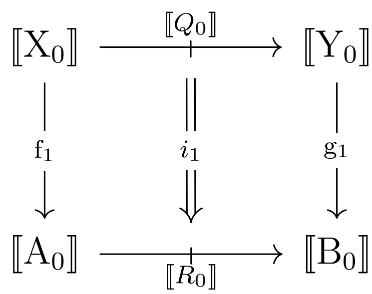




Hence it consists of elements

$$i_{1}{}^{f_1}_{g_1}([\![\mathcal{Q}]\!],[\![\mathcal{R}]\!])(\mathrm{f}_1,\mathrm{g}_1)(Q_0,R_0)=i_1(\mathrm{f}_1,\mathrm{g}_1)([\![Q_0]\!],[\![R_0]\!])$$

which can be understood as squares of the following form.

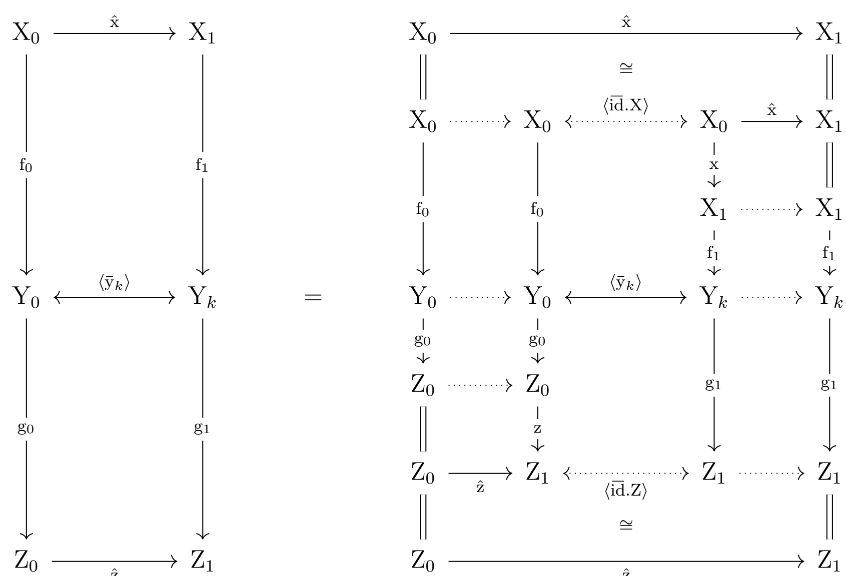



The substitution $i_{1}{}^{f_1}_{g_1}([\![\mathcal{Q}]\!],[\![\mathcal{R}]\!])$ is a matrix profunctor, because it is a restriction of the matrix profunctor $i_1$; its actions by the arrow profunctors of $f_1$ and $g_1$ are inherited, as well as their coherence. It is equipped with a cartesian morphism to $i_1$, by pullback.

Hence $\mathrm{MatProf}$ is fibered over $\mathrm{MatCat}\times\mathrm{MatCat}$. □

To compose matrix profunctors $m$ over $f$ and $n$ over $g$, we define an action by $\langle f\circ g\rangle$. We can use the actions of $\langle f\rangle$ on $m$ and $\langle g\rangle$ on $n$, because squares of $\langle f\circ g\rangle$ are composites in $\langle f\rangle\circ\langle g\rangle$, as follows.

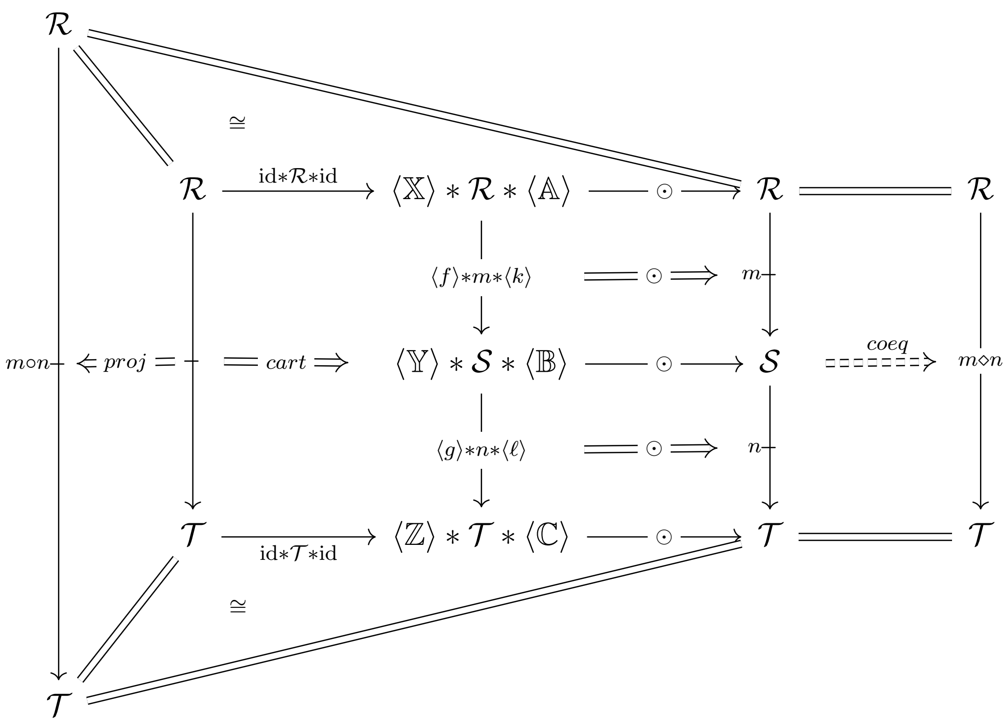
A square of $\langle f\circ g\rangle$ from $\hat{\mathrm{x}}:\langle\mathbb{X}\rangle(\mathrm{X}_0,\mathrm{X}_1)$ to $\hat{\mathrm{z}}:\langle\mathbb{Z}\rangle(\mathrm{Z}_0,\mathrm{Z}_1)$ is a pair $(\mathrm{f}_0,\mathrm{g}_0)$ and $(\mathrm{f}_1,\mathrm{g}_1)$ so that $(\mathrm{f}_0,\mathrm{g}_0\cdot\mathrm{z})=(\mathrm{x}\cdot\mathrm{f}_1,\mathrm{g}_1)$. This means there is a zig-zag of arrows and oparrows in $\langle\mathbb{Y}\rangle$ reassociating one to the other. Because $\langle\mathbb{X}\rangle$ and $\langle\mathbb{Z}\rangle$ are bifibered, the square factors as a weave in $\langle f\rangle$ and a weave in $\langle g\rangle$, by "bending" $\mathrm{x}$ and $\mathrm{z}$ to the left or right.




Hence we can define an action by a square or opsquare in $\langle f\circ g\rangle$ by the action of each factor in $\langle f\rangle$ and $\langle g\rangle$. This ensures the totality of the actions; so in fact, the crux of sequential composition is to ensure that the actions are *well-defined* over the *identities*.

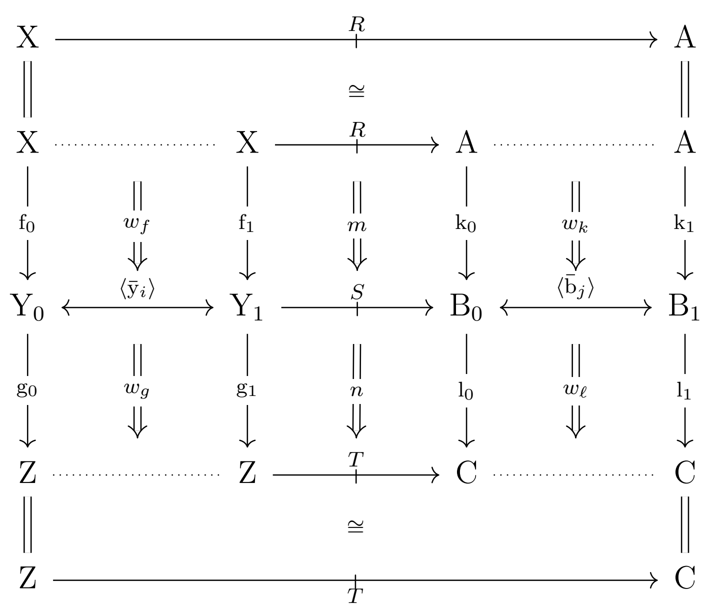
**Definition 44.** Let $m(f,k)\colon \mathcal{R}(\mathbb{X},\mathbb{A}) \mid \mathcal{S}(\mathbb{Y},\mathbb{B})$ and $n(g,\ell)\colon \mathcal{S}(\mathbb{Y},\mathbb{B}) \mid \mathcal{T}(\mathbb{Z},\mathbb{C})$ be a pair of sequential-composable matrix profunctors.

The **sequential composite** matrix profunctor $(m \diamond n)(f \circ g, k \circ \ell)\colon \mathcal{R}(\mathbb{X},\mathbb{A}) \mid \mathcal{T}(\mathbb{Z},\mathbb{C})$ is defined to be the following coequalizer.




Hence elements are equivalence classes $[S.(m,n)] : m \circ n$, such that for each pair of zig-zags, and each pair of pairs of weaves, the following are equated.

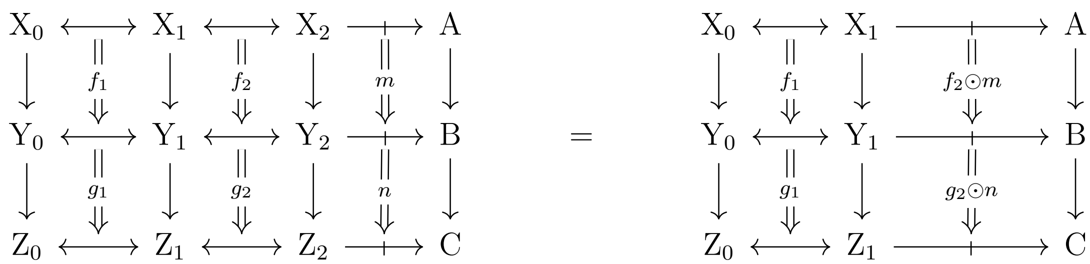
$$[S.(m,n)] \;\equiv\; [\upsilon_{\mathcal{R}} \cdot (\langle \bar{\mathrm{y}}_i \rangle \odot S \odot \langle \bar{\mathrm{b}}_j \rangle).(w_f \odot m \odot w_k,\; w_g \odot n \odot w_\ell) \cdot \upsilon_{\mathcal{T}}^{-1}]$$



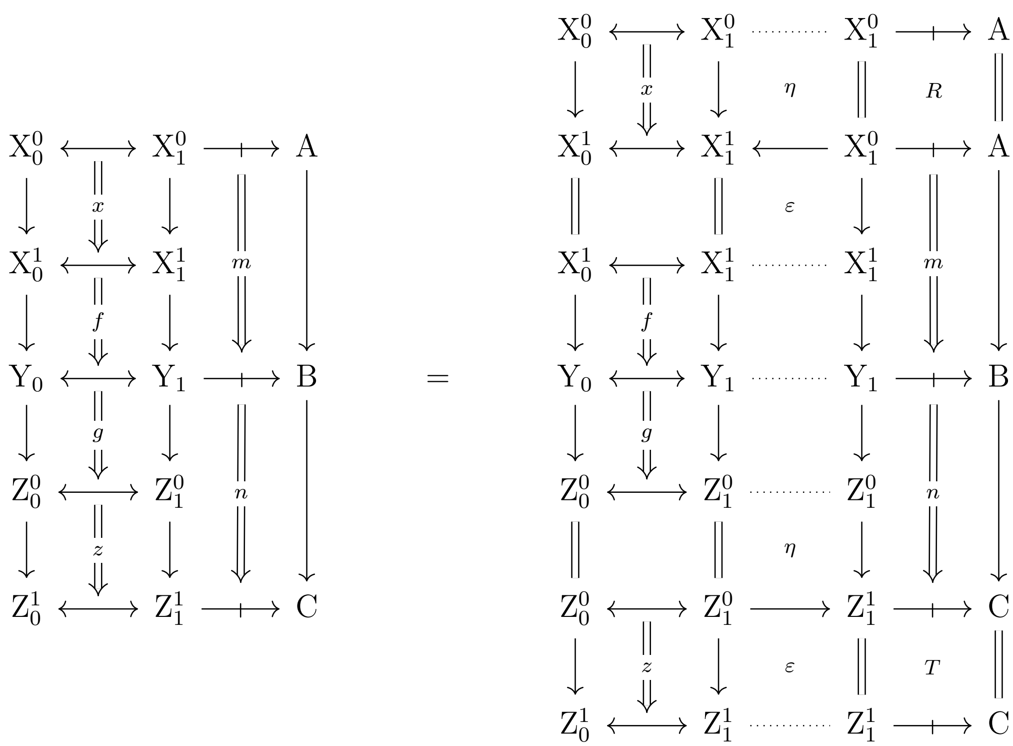
This is a span profunctor $f \circ g \leftarrow m \diamond n \rightarrow k \circ \ell$ mapping each $[S.(m,n)]$ to $[\mathrm{Y}_1.(\mathrm{f}_1, \mathrm{g}_1)]$ and $[\mathrm{B}_0.(\mathrm{k}_0, \mathrm{l}_0)]$; this is well-defined because any other representative lies over equivalent pairs $\mathrm{Y}_0.(\mathrm{f}_0, \mathrm{g}_0)$ and $\mathrm{B}_1.(\mathrm{k}_1, \mathrm{l}_1)$.

Moreover, $m \diamond n$ is a matrix profunctor from $f \circ g$ to $k \circ \ell$: we now define the action inductively over the structure of a composite weave. Then for the base generators, the quotient ensures that the action is well-defined.

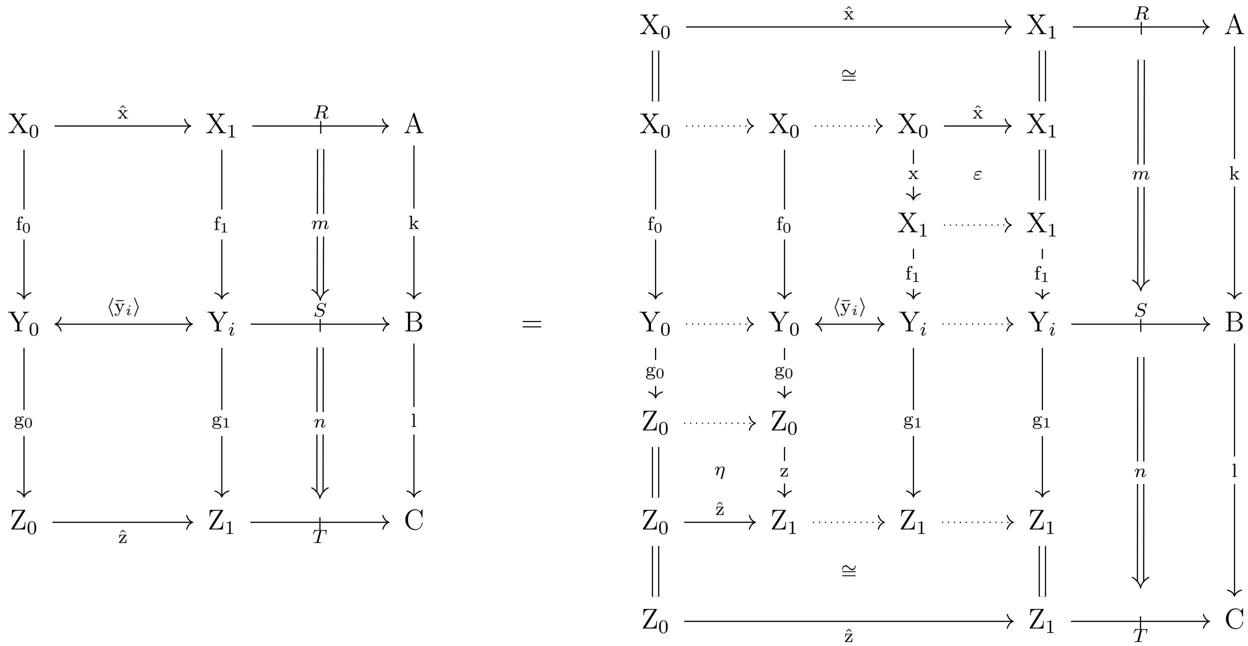
- The action of a horizontal composite is the composite of the actions of each factor.



- The action of a vertical composite of weaves in $\mathbb{X}$ and $\mathbb{Z}$ with a weave in $f \circ g$ is the vertical composite of the actions of the following factorization by op/cartesian squares.

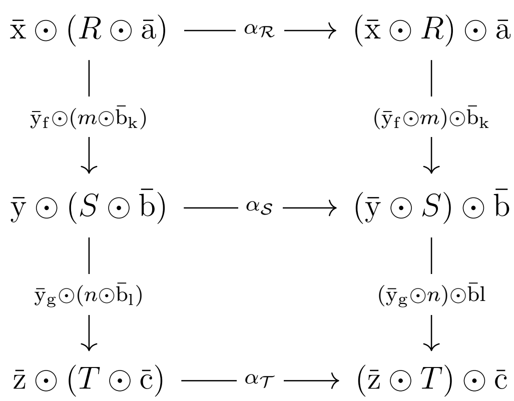


- The action by a square or opsquare is the action of its factorization into a weave in $f$ and a weave in $g$, on $m$ and $n$ respectively. The case of a square is given as follows, and an opsquare dually.




This action is well-defined by the quotient. Because squares and opsquares are the base generators of weaves, this completes the induction. Hence the actions by $\langle f \circ g \rangle$ and $\langle k \circ \ell \rangle$ are well-defined.

Finally, the coherence of $m \diamond n$ with the associators and unitors of $\mathcal{R}$ and $\mathcal{T}$ follows from that of $m$ with $\mathcal{R}$ and $\mathcal{S}$ and that of $n$ with $\mathcal{S}$ and $\mathcal{T}$. For example, the center associator.



Hence the sequential composite $(m \diamond n)(f \circ g, k \circ \ell) : \mathcal{R}(\mathbb{X}, \mathbb{A}) \mid \mathcal{T}(\mathbb{Z}, \mathbb{C})$ is a matrix profunctor.

**Theorem 45.** Matrix categories and matrix functors, matrix profunctors and matrix transformations form a bifibrant double category, i.e. logic, which we call $\mathrm{Mat}\mathbb{C}\mathrm{at}$.

*Proof.* Because matrix profunctor composition is defined by coequalizer, it is canonically functorial. Let $[\![m]\!]([\![f]\!],[\![k]\!]) : m_0(f_0,k_0) \Rightarrow m_1(f_1,k_1)$ and $[\![n]\!]([\![g]\!],[\![\ell]\!]) : n_0(g_0,\ell_0) \Rightarrow n_1(g_1,\ell_1)$ be a sequential-composable pair of matrix transformations. The composite is defined:

$$\begin{array}{lccc} ([\![m]\!] \diamond [\![n]\!]) : & (m_0 \diamond n_0)(f_0 \circ g_0, k_0 \circ \ell_0) & \Rightarrow & (m_1 \diamond n_1)(f_1 \circ g_1, k_1 \circ \ell_1) \\ & [S_0.(m_0,n_0)] & \mapsto & [[\![S_0]\!].([\![m_0]\!],[\![n_0]\!])] \end{array}$$

To be a matrix transformation, this composite must cohere with the left and right joins of the matrix functors $[\![\mathcal{R}]\!]([\![\mathbb{X}]\!],[\![\mathbb{A}]\!])$ and $\mathcal{T}([\![\mathbb{Z}]\!],[\![\mathbb{C}]\!])$; yet this follows from the coherence of $[\![m]\!]$ with respect to $[\![\mathcal{R}]\!]$ and $[\![\mathcal{S}]\!]$ and that of $[\![n]\!]$ with respect to $[\![\mathcal{S}]\!]$ and $[\![\mathcal{T}]\!]$.

$$\begin{array}{ccc} [\![\bar{\mathrm{x}}]\!] \odot [\![R]\!] & \xrightarrow{[\![\odot_{\mathbb{X}}]\!]} & [\![\bar{\mathrm{x}} \odot R]\!] \\ \downarrow {\scriptstyle [\![\mathrm{f}]\!] \odot [\![m]\!]} & & \downarrow {\scriptstyle [\![\mathrm{f} \odot m]\!]} \\ [\![\bar{\mathrm{y}}]\!] \odot [\![S]\!] & \xrightarrow{[\![\odot_{\mathbb{Y}}]\!]} & [\![\bar{\mathrm{y}} \odot S]\!] \\ \downarrow {\scriptstyle [\![\mathrm{g}]\!] \odot [\![n]\!]} & & \downarrow {\scriptstyle [\![\mathrm{g} \odot n]\!]} \\ [\![\bar{\mathrm{z}}]\!] \odot [\![T]\!] & \xrightarrow{[\![\odot_{\mathbb{Z}}]\!]} & [\![\bar{\mathrm{z}} \odot T]\!] \end{array} \qquad \begin{array}{ccc} [\![R]\!] \odot [\![\bar{\mathrm{a}}]\!] & \xrightarrow{[\![\odot_{\mathbb{A}}]\!]} & [\![R \odot \bar{\mathrm{a}}]\!] \\ \downarrow {\scriptstyle [\![m]\!] \odot [\![\mathrm{k}]\!]} & & \downarrow {\scriptstyle [\![m \odot \mathrm{k}]\!]} \\ [\![S]\!] \odot [\![\bar{\mathrm{b}}]\!] & \xrightarrow{[\![\odot_{\mathbb{B}}]\!]} & [\![S \odot \bar{\mathrm{b}}]\!] \\ \downarrow {\scriptstyle [\![n]\!] \odot [\![\mathrm{l}]\!]} & & \downarrow {\scriptstyle [\![n \odot \mathrm{l}]\!]} \\ [\![T]\!] \odot [\![\bar{\mathrm{c}}]\!] & \xrightarrow{[\![\odot_{\mathbb{C}}]\!]} & [\![T \odot \bar{\mathrm{c}}]\!] \end{array}$$

This preserves composition of matrix transformations, by functoriality of coequalizer.

The associator and unitors of $\mathrm{Mat}\mathbb{C}\mathrm{at}$ are inherited from $\mathrm{Span}\mathbb{C}\mathrm{at}$: the following

$$\begin{array}{ccc} (m \diamond n) \diamond p & \cong & m \diamond (n \diamond p) \\ [((m,n),p)] & \mapsto & [(m,(n,p))] \end{array} \qquad \begin{array}{ccccccc} m & \cong & \mathcal{R} \diamond m & & \mathcal{R} \diamond m & \cong & m \\ m & \mapsto & [(\mathrm{id}.R, m)] & & [(r,m)] & \mapsto & r \cdot m \\ m & \cong & m \diamond \mathcal{S} & & m \diamond \mathcal{S} & \cong & m \\ m & \mapsto & [(m, \mathrm{id}.S)] & & [(m,s)] & \mapsto & m \cdot s \end{array}$$

are matrix transformations, and they are well-defined on the sequential composite, because the quotient only reindexes along the base pair of morphisms.

Hence $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is a double category, and as already shown, a logic. □

We now define substitution of functors in matrix categories, and transformations in matrix profunctors: $\mathrm{MatCat}$ is fibered over $\mathrm{Cat} \times \mathrm{Cat}$, and $\mathrm{MatProf}$ over $\mathrm{Prof} \times \mathrm{Prof}$.

$$\begin{array}{ccccc} \mathrm{MatCat} & \longleftarrow & \mathrm{MatProf} & \longrightarrow & \mathrm{MatCat} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Cat} \times \mathrm{Cat} & \longleftarrow & \mathrm{Prof} \times \mathrm{Prof} & \longrightarrow & \mathrm{Cat} \times \mathrm{Cat} \end{array}$$

**Definition 46.** A **double fibration** is a category in the 2-category of fibrations. See [3].

**Proposition 47.** Let $\mathrm{Cat}$ be the category of categories and functors, and $\mathrm{MatCat}$ the category of matrix categories and matrix functors. Then $\mathrm{MatCat} \to \mathrm{Cat} \times \mathrm{Cat}$ is a fibration.

*Proof.* Let $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1, [\![\mathbb{B}]\!] : \mathbb{B}_0 \to \mathbb{B}_1$ be functors, and let $\mathcal{R}_1 : \mathbb{A}_1 \,\|\, \mathbb{B}_1$ be a matrix category. We define the **substitution** matrix category $\mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) : \mathbb{A}_0 \,\|\, \mathbb{B}_0$ as follows.

1. The span category $\mathbb{A}_0 \leftarrow \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) \to \mathbb{B}_0$ is the pullback of $\mathcal{R}_1$ along the functors $[\![\mathbb{A}]\!], [\![\mathbb{B}]\!]$. So the category over $\mathrm{A}_0 : \mathbb{A}_0, \mathrm{B}_0 : \mathbb{B}_0$ is $\mathcal{R}_1([\![\mathrm{A}_0]\!], [\![\mathrm{B}_0]\!])$, and similarly for morphisms.

$$\begin{array}{ccccc} \mathbb{A}_0 & \longleftarrow & \mathcal{R}_1([\![\mathbb{A}_0]\!], [\![\mathbb{B}_0]\!]) & \longrightarrow & \mathbb{B}_0 \\ \big\downarrow{\scriptstyle [\![\mathbb{A}]\!]} & \llcorner & \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle [\![\mathbb{B}]\!]} \\ \mathbb{A}_1 & \longleftarrow & \mathcal{R}_1 & \longrightarrow & \mathbb{B}_1 \end{array}$$

Hence $\mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])(\mathrm{a}_0, \mathrm{b}_0)(R_1^0, R_1^1)$ consists of squares $r_1 : \mathcal{R}_1$ over $([\![\mathrm{a}_0]\!], [\![\mathrm{b}_0]\!])$.

$$\begin{array}{ccc} [\![\mathrm{A}_0^0]\!] & \xrightarrow{\;R_1^0\;} & [\![\mathrm{B}_0^0]\!] \\ \big\downarrow{\scriptstyle [\![\mathrm{a}_0]\!]} & \Downarrow{\scriptstyle r_1} & \big\downarrow{\scriptstyle [\![\mathrm{b}_0]\!]} \\ [\![\mathrm{A}_0^1]\!] & \xrightarrow[\;R_1^1\;]{} & [\![\mathrm{B}_0^1]\!] \end{array}$$

2. The actions of $\mathbb{A}_0$ and $\mathbb{B}_0$ on $\mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])$, span functors

$$\begin{array}{lcccc} \langle \mathbb{A}_0 \rangle \odot - & : & \langle \mathbb{A}_0 \rangle * \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) & \to & \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) \\ - \odot \langle \mathbb{B}_0 \rangle & : & \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) * \langle \mathbb{B}_0 \rangle & \to & \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) \end{array}$$

are those induced by pullback: map the arrow or oparrow by the functor, then act on $\mathcal{R}_1$.

$$\begin{array}{cccc} \bar{\mathrm{a}}_0 : \langle \mathbb{A}_0 \rangle(\mathrm{A}_0^0, \mathrm{A}_0^1) & R_1 : \mathcal{R}_1([\![\mathrm{A}_0^1]\!], [\![\mathrm{B}_0^0]\!]) & \mapsto & [\![\bar{\mathrm{a}}_0]\!] \odot R_1 \;:\; R_1([\![\mathrm{A}_0^0]\!], [\![\mathrm{B}_0^0]\!]) \\ R_1 : \mathcal{R}_1([\![\mathrm{A}_0^1]\!], [\![\mathrm{B}_0^0]\!]) & \bar{\mathrm{b}}_0 : \langle \mathbb{B}_0 \rangle(\mathrm{B}_0^0, \mathrm{B}_0^1) & \mapsto & R_1 \odot [\![\bar{\mathrm{b}}_0]\!] \;:\; \mathcal{R}_1([\![\mathrm{A}_0^1]\!], [\![\mathrm{B}_0^1]\!]) \end{array}$$

$$[\![\mathrm{A}_0^0]\!] \xleftrightarrow{\;[\![\bar{\mathrm{a}}_0]\!]\;} [\![\mathrm{A}_0^1]\!] \xrightarrow{\;R_1\;} [\![\mathrm{B}_0^0]\!] \xleftrightarrow{\;[\![\bar{\mathrm{b}}_0]\!]\;} [\![\mathrm{B}_0^1]\!]$$

3,4. The associators and unitors are inherited from $\mathcal{R}_1$, satisfying the coherence.

The substitution matrix category $\mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])$ is equipped with a projection matrix functor to $\mathcal{R}_1$, and this is a cartesian morphism over functors $[\![\mathbb{A}]\!], [\![\mathbb{B}]\!]$, by pullback. $\square$

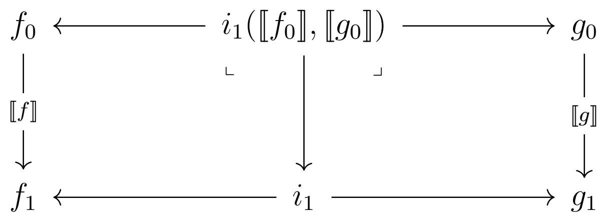
In the same way, we define substitution of transformations in a matrix profunctor.

**Theorem 48.** $\mathrm{MatProf} \to \mathrm{Prof} \times \mathrm{Prof}$ is a fibration.

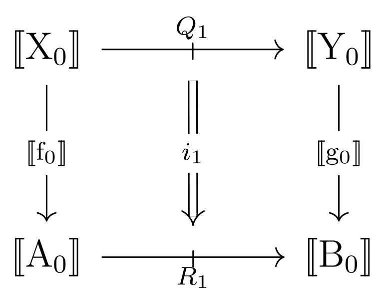
*Proof.* Let $[\![\mathbb{X}]\!], [\![\mathbb{Y}]\!], [\![\mathbb{A}]\!], [\![\mathbb{B}]\!]$ be functors, and let $\mathcal{Q}_1 : \mathbb{X}_1 \,\|\, \mathbb{Y}_1$ and $\mathcal{R}_1 : \mathbb{A}_1 \,\|\, \mathbb{B}_1$ be matrix categories, with $\mathcal{Q}_1([\![\mathbb{X}]\!], [\![\mathbb{Y}]\!]) : \mathbb{X}_0 \,\|\, \mathbb{Y}_0$ and $\mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) : \mathbb{A}_0 \,\|\, \mathbb{B}_0$.

Let $f_0 : \mathbb{X}_0 \mid \mathbb{A}_0$, $f_1 : \mathbb{X}_1 \mid \mathbb{A}_1$, $g_0 : \mathbb{Y}_0 \mid \mathbb{B}_0$, $g_1 : \mathbb{Y}_1 \mid \mathbb{B}_1$ be profunctors, and $[\![f]\!] : f_0 \Rightarrow f_1$ and $[\![g]\!] : g_0 \Rightarrow g_1$ be transformations. For $i_1(f_1, g_1) : \mathcal{Q}_1 \mid \mathcal{R}_1$, define the **substitution** matrix profunctor $i_1([\![f]\!], [\![g]\!]) : \mathcal{Q}_1([\![\mathbb{X}]\!], [\![\mathbb{Y}]\!]) \mid \mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])$ from $f_0$ to $g_0$ as follows.

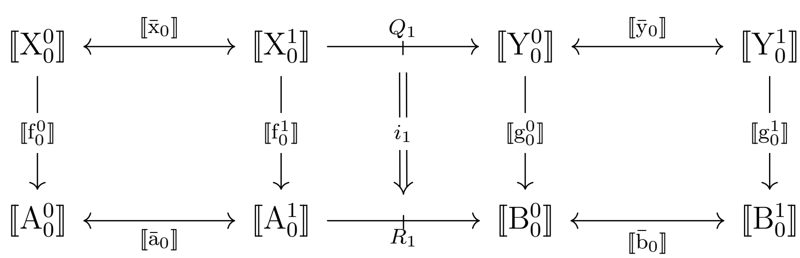
2. The span profunctor $f_0 \leftarrow i_1([\![f]\!], [\![g]\!]) \to g_0$ is the pullback of $i_1$ along $[\![f]\!] \times [\![g]\!]$.




So the profunctor over $\mathrm{f}_0 : f_0(\mathrm{X}_0, \mathrm{A}_0), \mathrm{g}_0 : g_0(\mathrm{Y}_0, \mathrm{B}_0)$ is $i_1([\![\mathrm{f}_0]\!], [\![\mathrm{g}_0]\!]) : \mathcal{Q}_1([\![\mathrm{X}_0]\!], [\![\mathrm{Y}_0]\!]) \mid \mathcal{R}_1([\![\mathrm{A}_0]\!], [\![\mathrm{B}_0]\!])$, consisting of squares of the following form.




3. The actions by the weave profunctors $\langle f_0 \rangle$ and $\langle g_0 \rangle$ are those induced by pullback.



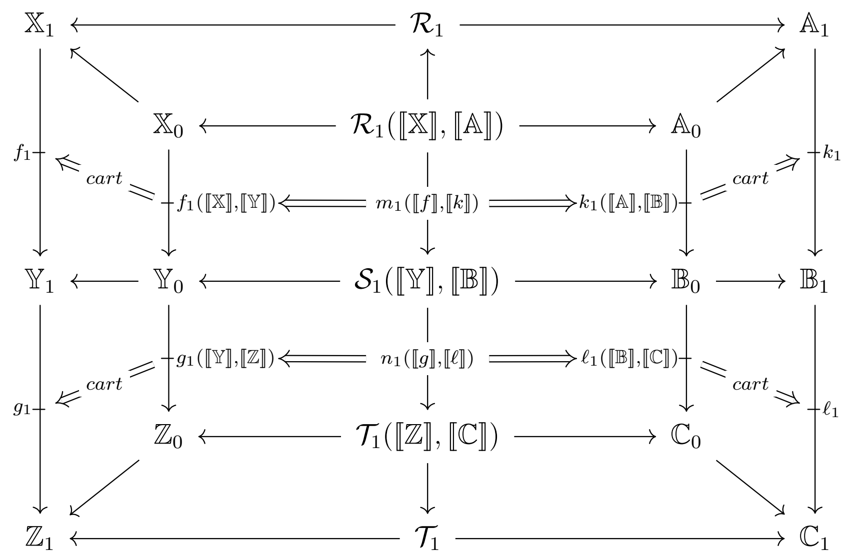

4. Because the associators and unitors of $\mathcal{Q}_1([\![\mathbb{X}]\!], [\![\mathbb{Y}]\!])$ and $\mathcal{R}_1([\![\mathbb{A}]\!], [\![\mathbb{B}]\!])$ are inherited from $\mathcal{Q}_1$ and $\mathcal{R}_1$, their coherence with $i_1([\![f]\!], [\![g]\!])$ is inherited.

$\square$

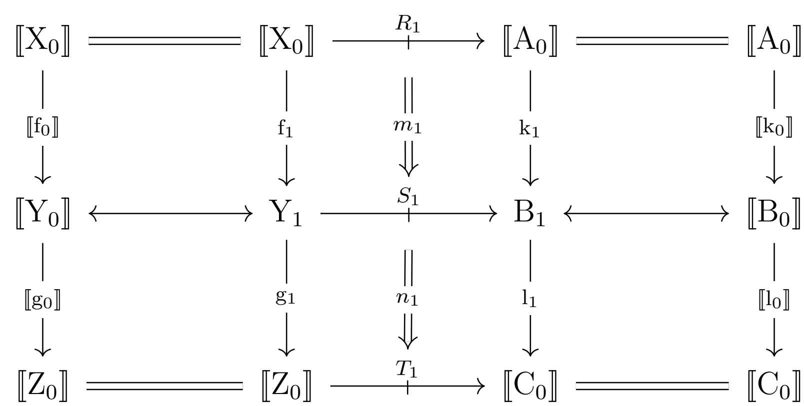
**Theorem 49.** $\mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$ is a double fibration.

*Proof.* We show that matrix profunctor composition preserves substitution.

Let $m_i(f,k) : \mathcal{R}(\mathbb{X},\mathbb{A}) \mid \mathcal{S}(\mathbb{Y},\mathbb{B})$ and $n_i(g,\ell) : \mathcal{S}(\mathbb{Y},\mathbb{B}) \mid \mathcal{T}(\mathbb{Z},\mathbb{C})$, for $i : \{0,1\}$, be matrix profunctors. Let $[\![m]\!] : m_0 \Rightarrow m_1$ and $[\![n]\!] : n_0 \Rightarrow n_1$ be matrix transformations, and form the substitution.



The composite $m_1([\![f]\!],[\![k]\!]) \diamond n_1([\![g]\!],[\![\ell]\!])$ consists of equivalence classes $[S_1.(m_1,n_1)]$ over $[([\![\mathrm{f}_0]\!],[\![\mathrm{g}_0]\!])]$ and $[([\![\mathrm{k}_0]\!],[\![\mathrm{l}_0]\!])]$. By comparison, the substitution $(m_1 \diamond n_1)([\![f]\!] \circ [\![g]\!], [\![k]\!] \circ [\![\ell]\!])$ consists of equivalence classes $[S_1.(m_1,n_1)]$ over pairs $[(\mathrm{f}_1,\mathrm{g}_1)]$ and $[(\mathrm{k}_1,\mathrm{l}_1)]$ which are equal to pairs $[([\![\mathrm{f}_0]\!],[\![\mathrm{g}_0]\!])]$ and $[([\![\mathrm{k}_0]\!],[\![\mathrm{l}_0]\!])]$ by associativity.



Hence the two are isomorphic.

$$m_1([\![f]\!],[\![g]\!]) \diamond n_1([\![k]\!],[\![\ell]\!]) \cong (m_1 \diamond n_1)([\![f]\!] \circ [\![g]\!], [\![k]\!] \circ [\![\ell]\!])$$

Thus, sequential composition preserves substitution, so $\mathrm{Mat}\mathbb{C}\mathrm{at}$ a *double fibration* or "fibered double category" [3].

$$
\begin{array}{ccccc}
\mathrm{MatCat} & \longleftarrow & \mathrm{MatProf} & \longrightarrow & \mathrm{MatCat} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Cat}\times\mathrm{Cat} & \longleftarrow & \mathrm{Prof}\times\mathrm{Prof} & \longrightarrow & \mathrm{Cat}\times\mathrm{Cat}
\end{array}
$$

□

As $\mathbb{C}\mathrm{at}$ and $\mathrm{Mat}\mathbb{C}\mathrm{at}$ are bifibrant double categories, we call this a **fibered logic**.

## 2.5 Parallel composition

We now define composition of matrix categories, and prove that $\mathbb{C}\mathrm{at} \leftarrow \mathrm{Mat}\mathbb{C}\mathrm{at} \rightarrow \mathbb{C}\mathrm{at}$ is a *metalogic* [Def. 55], i.e. a "bifibrant triple category without interchange".

Matrix categories compose in essentially the same way as profunctors; but rather than a coequalizer, the composite is formed by *codescent* [11] which adjoins to $\mathbb{A} \leftarrow \mathcal{R} * \mathcal{S} \rightarrow \mathbb{C}$ a coherent associator of the inner actions of $\langle\mathbb{B}\rangle$.

**Definition 50.** Let $\mathcal{R} : \mathbb{A} \,||\, \mathbb{B}$ and $\mathcal{S} : \mathbb{B} \,||\, \mathbb{C}$ be matrix categories. The **composite** matrix category $\mathcal{R}\otimes\mathcal{S} : \mathbb{A} \,||\, \mathbb{C}$ is defined as follows. To the composite span category $\mathbb{A} \leftarrow \mathcal{R}*\mathcal{S} \rightarrow \mathbb{C}$, an associator isomorphism is adjoined, by the following *iso-coinserter*.

$$
\begin{array}{ccccc}
(\mathcal{R} * \langle\mathbb{B}\rangle) * \mathcal{S} & \xrightarrow{\odot * \mathcal{S}} & \mathcal{R} * \mathcal{S} & \overset{\iota}{\rightsquigarrow} & \\
\cong \downarrow & \Downarrow \alpha_{\mathcal{R}\mathcal{S}} & & & (\mathcal{R} * \mathcal{S})_\alpha \\
\mathcal{R} * (\langle\mathbb{B}\rangle * \mathcal{S}) & \xrightarrow[\mathcal{R} * \odot]{} & \mathcal{R} * \mathcal{S} & \overset{\iota}{\rightsquigarrow} &
\end{array}
$$

This associator is natural by its universal construction, so for every weave $w_{\mathrm{B}} : \langle\mathbb{B}\rangle(\langle\bar{\mathrm{b}}_k\rangle, \langle\bar{\mathrm{b}}_\ell\rangle)$ and $r : \mathcal{R}(R_0, R_1)$, $s : \mathcal{S}(S_0, S_1)$ the following commutes.

$$
\begin{array}{ccc}
(R_0, \langle\bar{\mathrm{b}}_k\rangle \odot S_0) & \xrightarrow{\alpha_{\mathcal{R}\mathcal{S}}} & (R_0 \odot \langle\bar{\mathrm{b}}_k\rangle, S_0) \\
\downarrow (r, w_{\mathrm{B}} \odot s) & & \downarrow (r \odot w_{\mathrm{B}}, s) \\
(R_1, \langle\bar{\mathrm{b}}_\ell\rangle \odot S_1) & \xrightarrow{\alpha_{\mathcal{R}\mathcal{S}}} & (R_1 \odot \langle\bar{\mathrm{b}}_\ell\rangle, S_1)
\end{array}
$$

Then, two equations are imposed by *coequifier*, for reassociating a composite and a unit.

$$\mathcal{R} * \langle\mathbb{B}\rangle * \langle\mathbb{B}\rangle * \mathcal{S} \;\underset{B_2.((R\odot\bar{\mathrm{b}}_1)\odot\bar{\mathrm{b}}_2,S)}{\overset{B_0.(R,\bar{\mathrm{b}}_1\odot(\bar{\mathrm{b}}_2\odot S))}{\rightrightarrows}}\; (\mathcal{R}*\mathcal{S})_\alpha \xrightarrow{co.equif} (\mathcal{R}*\mathcal{S})_\beta$$

$$\mathcal{R} * \mathcal{S} \;\underset{\mathrm{B}.(R\odot\overline{\mathrm{id}}.\mathrm{B},S)}{\overset{\mathrm{B}.(R,\overline{\mathrm{id}}.\mathrm{B}\odot S)}{\rightrightarrows}}\; (\mathcal{R}*\mathcal{S})_\beta \xrightarrow{co.equif} \mathcal{R}\otimes\mathcal{S}$$

All together, the composite matrix category $\mathcal{R}\otimes\mathcal{S} : \mathbb{A} \parallel \mathbb{C}$ is the following codescent object.

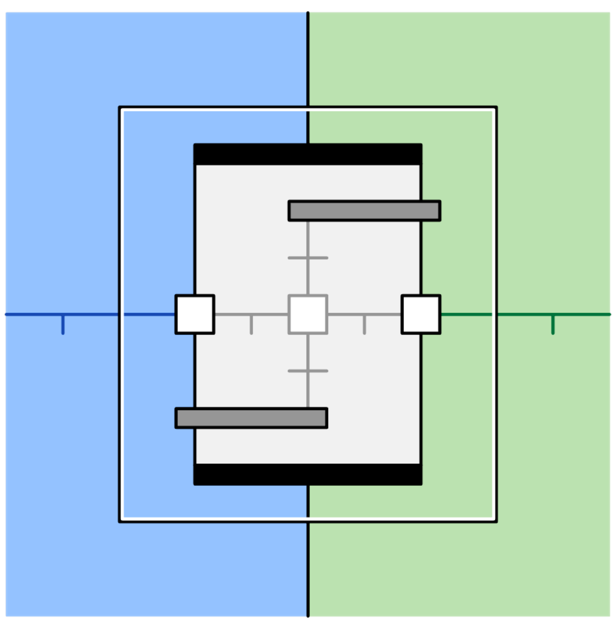
$$\mathcal{R} * \langle\mathbb{B}\rangle * \langle\mathbb{B}\rangle * \mathcal{S} \;\substack{\xrightarrow{\odot*\langle\mathbb{B}\rangle*\mathcal{S}} \\ \xrightarrow{\mathcal{R}*\circ*\mathcal{S}} \\ \xrightarrow{\mathcal{R}*\langle\mathbb{B}\rangle*\odot}}\; \mathcal{R} * \langle\mathbb{B}\rangle * \mathcal{S} \;\substack{\xrightarrow{\odot*\mathcal{S}} \\ \xleftarrow{\mathcal{R}*\mathrm{id}*\mathcal{S}} \\ \xrightarrow{\mathcal{R}*\odot}}\; \mathcal{R}*\mathcal{S} \xrightarrow{co.desc} \mathcal{R}\otimes\mathcal{S}$$

We denote the codescent object by the following "arrow sum" notation.

$$(\mathcal{R}\otimes\mathcal{S})(\mathrm{A},\mathrm{C}) \quad\equiv\quad \vec{\Sigma}\mathrm{B}:\mathbb{B}.\; \mathcal{R}(\mathrm{A},\mathrm{B}) \times \mathcal{S}(\mathrm{B},\mathrm{C})$$

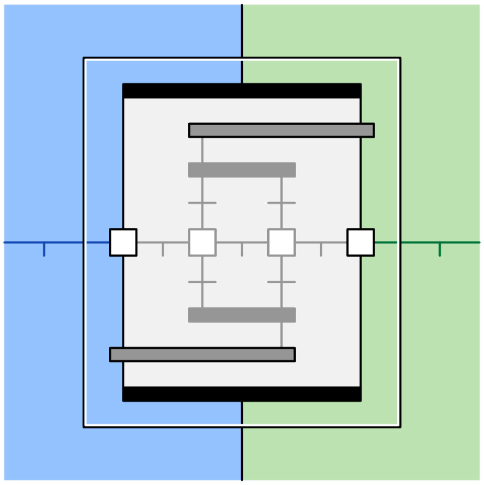
So, the parallel composite $\mathcal{R}\otimes\mathcal{S} : \mathbb{A} \parallel \mathbb{C}$ consists of pairs $\mathrm{b}.(r,s) : \mathrm{B}_0.(R_0,S_0) \to \mathrm{B}_1.(R_1,S_1)$, plus a coherent associator $\alpha_{\mathcal{R}\mathcal{S}} : \mathrm{B}_0.(R,\bar{\mathrm{b}}\odot S) \cong \mathrm{B}_1.(R\odot\bar{\mathrm{b}},S)$.

The iso-coinserter which constructs the associator is drawn in string diagrams as follows: the black bead is the colimiting span functor from $(\mathcal{R}*\mathcal{S})$ to $(\mathcal{R}*\mathcal{S})_\alpha$, and the inner face is the associator isomorphism.

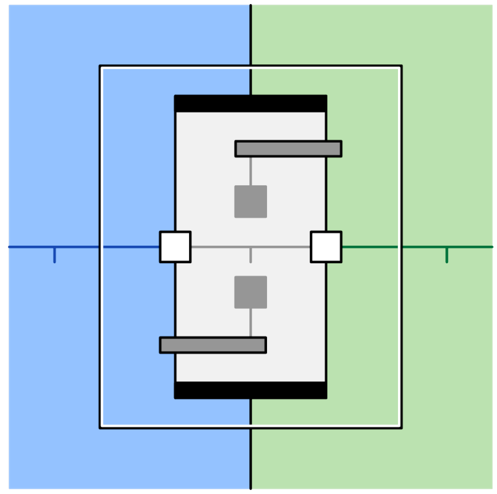


$\alpha_{\mathcal{R}\mathcal{S}} : \mathrm{B}_0.(R,\bar{\mathrm{b}}\odot S) \;\cong\; \mathrm{B}_1.(R\odot\bar{\mathrm{b}},S)$

Each coequifier on the associator can be drawn as the cube which it makes well-defined.



**associator coherence**
$(R, \bar{\mathrm{b}}_1 \odot (\bar{\mathrm{b}}_2 \odot S)) \rightrightarrows ((R \odot \bar{\mathrm{b}}_1) \odot \bar{\mathrm{b}}_2), S)$



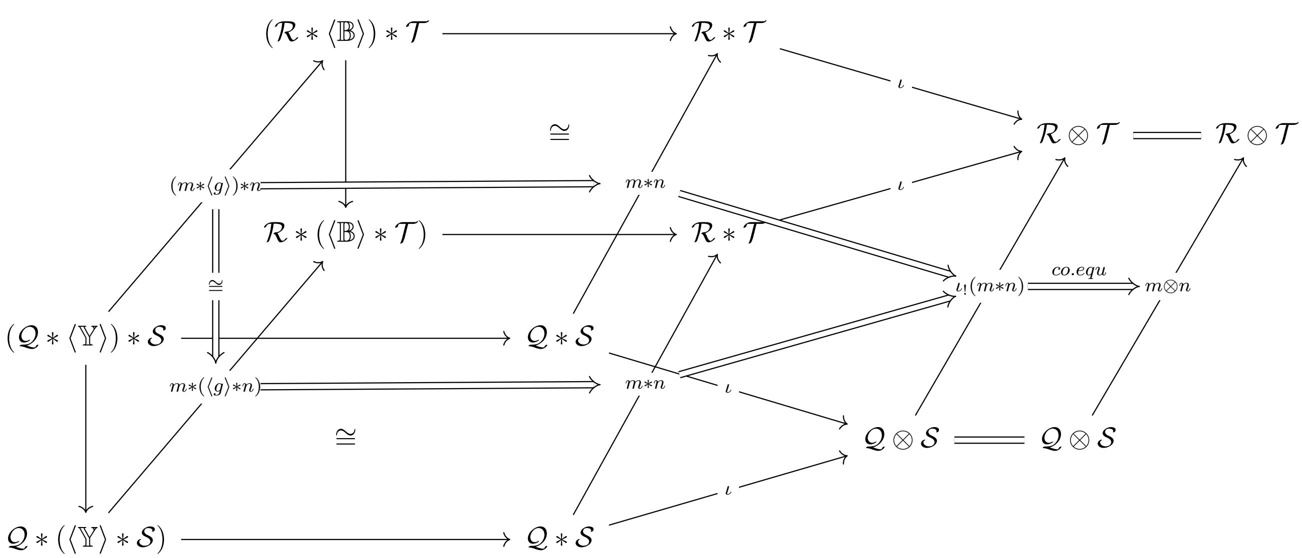
**unitor coherence**
$(R, \overline{\mathrm{id}}.\mathrm{B} \odot S) \rightrightarrows (R \odot \overline{\mathrm{id}}.\mathrm{B}, S)$

We compose matrix profunctors analogously; yet we need only impose one equation, for *naturality* of the adjoined associators.

**Definition 51.** Let $m(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \,|\, \mathcal{R}(\mathbb{A},\mathbb{B})$ and $n(g,h) : \mathcal{S}(\mathbb{Y},\mathbb{Z}) \,|\, \mathcal{T}(\mathbb{B},\mathbb{C})$ be a pair of parallel-composable matrix profunctors.

$$\begin{array}{ccccccccc}
\mathbb{X} & \longleftarrow & \mathcal{Q} & \longrightarrow & \mathbb{Y} & \longleftarrow & \mathcal{S} & \longrightarrow & \mathbb{Z} \\
{\scriptstyle f}\downarrow & \Longleftarrow & \downarrow{\scriptstyle m} & \Longrightarrow & \downarrow{\scriptstyle g} & \Longleftarrow & \downarrow{\scriptstyle n} & \Longrightarrow & \downarrow{\scriptstyle h} \\
\mathbb{A} & \longleftarrow & \mathcal{R} & \longrightarrow & \mathbb{B} & \longleftarrow & \mathcal{T} & \longrightarrow & \mathbb{C}
\end{array}$$

The **composite** matrix profunctor $m \otimes n : Q \otimes \mathcal{S} \,|\, \mathcal{R} \otimes \mathcal{T}$ is the following coequalizer.



The profunctor $\iota_!(m * n)$ forms all composites of elements $\mathrm{g}.(m,n)$ and the morphisms of $\mathcal{Q} \otimes \mathcal{S}$ and $\mathcal{R} \otimes \mathcal{T}$. Then, the coequalizer imposes that the associators are natural with respect to the elements. So the elements of $(m \otimes n)(f,h) : (\mathcal{Q} \otimes \mathcal{S})(\mathbb{X},\mathbb{Z}) \,|\, (\mathcal{R} \otimes \mathcal{T})(\mathbb{A},\mathbb{C})$ are composites of:

| | | |
|---|---|---|
| morphisms | $\mathrm{y}.(q,s):$ | $(\mathcal{Q} \otimes \mathcal{S})(\mathrm{Y}_0.(Q_0,S_0), \mathrm{Y}_1.(Q_1,S_1))$ |
| associators | $\alpha_{\mathcal{Q}\mathcal{S}}:$ | $(\mathcal{Q} \otimes \mathcal{S})(\mathrm{Y}_0.(Q,\bar{\mathrm{y}} \odot S), \mathrm{Y}_1.(Q \odot \bar{\mathrm{y}}, S))$ |
| elements | $\mathrm{g}.(m,n):$ | $(m * n)(\mathrm{Y}.(Q,S), \mathrm{B}.(R,T))$ |
| associators | $\alpha_{\mathcal{R}\mathcal{T}}:$ | $(\mathcal{R} \otimes \mathcal{T})(\mathrm{B}_0.(R,\bar{\mathrm{b}} \odot T), \mathrm{B}_1.(R \odot \bar{\mathrm{b}}, T))$ |
| morphisms | $\mathrm{b}.(r,t):$ | $(\mathcal{R} \otimes \mathcal{T})(\mathrm{B}_0.(R_0,T_0), \mathrm{B}_1.(R_1,T_1))$ |

such that for any $[\mathrm{g}_0,\mathrm{g}_1] : \langle g \rangle(\bar{\mathrm{y}},\bar{\mathrm{b}})$ and $m : m(\mathrm{f},\mathrm{g}_0)$, $n : n(\mathrm{g}_1,\mathrm{h})$ the following commutes.

$$\begin{array}{ccc}
\mathrm{Y}_0.(Q,\bar{\mathrm{y}} \odot S) & \xrightarrow{\alpha_{\mathcal{Q}\mathcal{S}}} & \mathrm{Y}_1.(Q \odot \bar{\mathrm{y}}, S) \\
\Big\downarrow {\scriptstyle \mathrm{g}_0.(m,[\mathrm{g}_0,\mathrm{g}_1]\odot n)} & & \Big\downarrow {\scriptstyle \mathrm{g}_1.(m\odot[\mathrm{g}_0,\mathrm{g}_1],n)} \\
\mathrm{B}_0.(R,\bar{\mathrm{b}} \odot T) & \xrightarrow[\alpha_{\mathcal{R}\mathcal{T}}]{} & \mathrm{B}_1.(R \odot \bar{\mathrm{b}}, T)
\end{array}$$

We denote the composite by the same "arrow sum" notation as for matrix categories.

$$(m \otimes n)(\mathrm{f},\mathrm{h}) \quad \equiv \quad \vec{\Sigma}\mathrm{g} : g.\ m(\mathrm{f},\mathrm{g}) \times n(\mathrm{g},\mathrm{h})$$

We now see that composition defines a span functor $\otimes : \mathrm{Mat}\mathbb{C}\mathrm{at} * \mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathrm{Mat}\mathbb{C}\mathrm{at}$, but *not* a double functor.

**Proposition 52.** Parallel composition of matrix categories defines a span functor

$$\otimes : \mathrm{MatCat} * \mathrm{MatCat} \to \mathrm{MatCat}.$$

*Proof.* As composition is defined by colimit, it is canonically functorial. Let $[\![\mathcal{R}]\!] : \mathcal{R}_0(\mathbb{A}_0,\mathbb{B}_0) \to \mathcal{R}_1(\mathbb{A}_1,\mathbb{B}_1)$ and $[\![\mathcal{S}]\!]([\![\mathbb{B}]\!],[\![\mathbb{C}]\!]) : \mathcal{S}_0(\mathbb{B}_0,\mathbb{C}_0) \to \mathcal{S}_1(\mathbb{B}_1,\mathbb{C}_1)$ be matrix functors. The composite

$$([\![\mathcal{R}]\!] \otimes [\![\mathcal{S}]\!]) : (\mathcal{R}_0 \otimes \mathcal{S}_0)(\mathbb{A}_0,\mathbb{C}_0) \to (\mathcal{R}_1 \otimes \mathcal{S}_1)(\mathbb{A}_1,\mathbb{C}_1)$$

is defined by applying the functors $[\![\mathcal{R}]\!]$ and $[\![\mathcal{S}]\!]$ in parallel

$$([\![\mathcal{R}]\!] \otimes [\![\mathcal{S}]\!])(\mathrm{B}_0.(R_0,S_0)) = [\![\mathrm{B}_0]\!].([\![R_0]\!],[\![S_0]\!])$$

and mapping the "inner associator" of $\mathcal{R}_0 \otimes \mathcal{S}_0$ to that of $\mathcal{R}_1 \otimes \mathcal{S}_1$.

$$([\![\mathcal{R}]\!] \otimes [\![\mathcal{S}]\!])(\alpha(\mathrm{b}_0.(R_0,S_0))) = \alpha([\![\mathrm{b}_0]\!].([\![R_0]\!],[\![S_0]\!]))$$

The joins of this matrix functor are inherited from those of $[\![\mathcal{R}]\!]$ and $[\![\mathcal{S}]\!]$.

$$[\![\mathrm{a}_0]\!] \odot ([\![\mathrm{B}_0]\!].([\![R_0]\!],[\![S_0]\!])) \odot [\![\mathrm{c}_0]\!] \;=\; [\![\mathrm{B}_0]\!].([\![\mathrm{a}_0]\!] \odot [\![R_0]\!], [\![S_0]\!] \odot [\![\mathrm{c}_0]\!]) \;\cong\; [\![\mathrm{B}_0]\!].([\![\mathrm{a}_0 \odot R_0]\!], [\![S_0 \odot \mathrm{c}_0]\!])$$

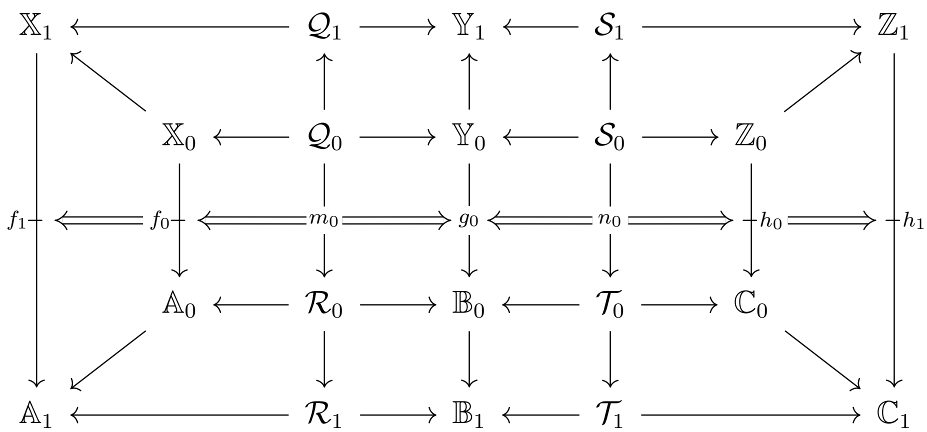
Finally, $- \otimes -$ clearly preserves matrix functor composition and identity. Hence it defines a span functor $\mathrm{MatCat} * \mathrm{MatCat} \to \mathrm{MatCat}$. □

**Proposition 53.** Parallel composition of matrix profunctors defines a span functor

$$\otimes : \mathrm{MatProf} * \mathrm{MatProf} \to \mathrm{MatProf}.$$

*Proof.* Let $m(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \,|\, \mathcal{R}(\mathbb{A},\mathbb{B})$ and $n(g,h) : \mathcal{S}(\mathbb{Y},\mathbb{Z}) \,|\, \mathcal{T}(\mathbb{B},\mathbb{C})$ be matrix profunctors with subscripts $0,1$.

Let $[\![m]\!]([\![f]\!],[\![g]\!]) : m_0(f_0,g_0) \Rightarrow m_1(f_1,g_1)$ and $[\![n]\!]([\![g]\!],[\![h]\!]) : n_0(g_0,h_0) \Rightarrow n_1(g_1,h_1)$ be matrix transformations.



Then the composite matrix transformation

$$([\![m]\!] \otimes [\![n]\!]) : (m_0 \otimes n_0)(f_0,h_0) \Rightarrow (m_1 \otimes n_1)(f_1,h_1)$$

is defined by applying the transformations $[\![m]\!]$ and $[\![n]\!]$ in parallel.

$$([\![m]\!] \otimes [\![n]\!])(\mathrm{g}_0.(\mathrm{f}_0,\mathrm{h}_0)) = [\![\mathrm{g}_0]\!].([\![\mathrm{f}_0]\!],[\![\mathrm{h}_0]\!])$$

The coherence of $[\![m]\!] \otimes [\![n]\!]$ with the joins of $[\![\mathcal{Q}]\!] \otimes [\![\mathcal{S}]\!]$ and $[\![\mathcal{R}]\!] \otimes [\![\mathcal{T}]\!]$ follows from that of $[\![m]\!]$ with $[\![\mathcal{Q}]\!]$ and $[\![\mathcal{R}]\!]$, and $[\![n]\!]$ with $[\![\mathcal{S}]\!]$ and $[\![\mathcal{T}]\!]$.

Finally, $- \otimes -$ clearly preserves matrix transformation composition and identity. Hence it defines a span functor $\mathrm{MatProf} * \mathrm{MatProf} \to \mathrm{MatProf}$. □

We have defined parallel composition of matrix categories, and matrix profunctors.

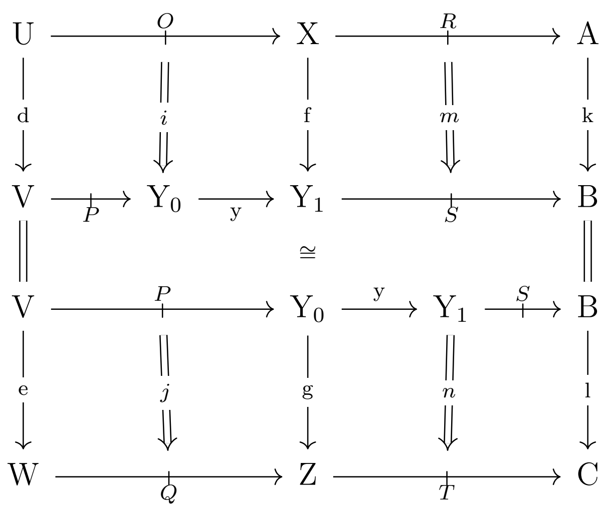
Now: is parallel composition a *double functor*? In fact, *no* — parallel composition does not preserve sequential composition of matrix profunctors; it is neither lax nor colax.

$$(i \otimes m) \diamond (j \otimes n) \qquad \not\leftrightarrow \qquad (i \diamond j) \otimes (m \diamond n)$$

The is due to the combination of *strict* and *weak* colimits: weak-to-strict (lax, left-to-right above) is not total, while strict-to-weak (colax, right-to-left above) is not well-defined.

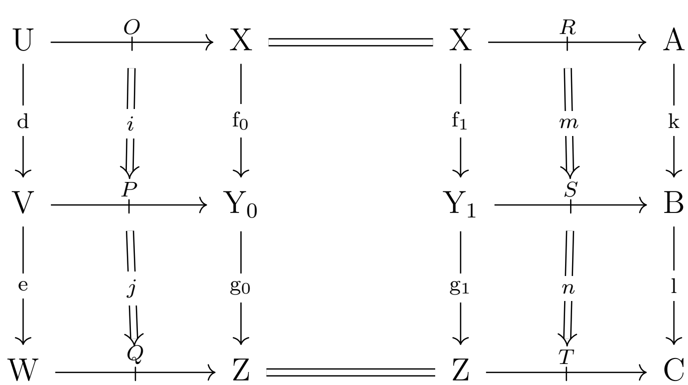
Sequential composition is given by coequalizer, while parallel composition is given by codescent object. The former *equates* elements, while the latter *creates* an isomorphism.

So, sequence-of-parallel $(i \otimes m) \diamond (j \otimes n)$ contains composites with associators which cannot be expressed as a parallel-of-sequence composite $(i \diamond j) \otimes (m \diamond n)$.



Hence there is no transformation $(i \otimes m) \diamond (j \otimes n) \Rightarrow (i \diamond j) \otimes (m \diamond n)$.

Yet in the other direction, there is a dual obstruction. To define sequential composition, each associativity zig-zag $(\bar{\mathrm{y}}_i) : (\mathrm{f}_0, \mathrm{g}_0) = (\mathrm{f}_1, \mathrm{g}_1)$ in $\langle f \circ g \rangle$ is given as a composite in $\langle f \rangle \circ \langle g \rangle$; yet elements of $(i \diamond j) \otimes (m \diamond n)$ are just "parallel-composable pairs" along $(\mathrm{f}_0, \mathrm{g}_0) = (\mathrm{f}_1, \mathrm{g}_1)$, without any specific choice of zig-zag.



So a transformation $(i \diamond j) \otimes (m \diamond n) \Rightarrow (i \otimes m) \diamond (j \diamond n)$ would have to be independent of the choice of zig-zag. Yet there is no canonical choice; there are many distinct zig-zags which reassociate from $(\mathrm{f}_0, \mathrm{g}_0)$ to $(\mathrm{f}_1, \mathrm{g}_1)$, and they each give distinct actions on the parallel pairs.

Thus, parallel composition is *neither* lax nor colax for sequential composition; there is no interchange between the two operations. Recall also that that the weave construction $\langle - \rangle$ is not lax nor colax (2.1.2). So while $\mathbb{C}\mathrm{at}$ and $\mathrm{Mat}\mathbb{C}\mathrm{at}$ are double categories, parallel composition of $\mathbb{C}\mathrm{at} \leftarrow \mathrm{Mat}\mathbb{C}\mathrm{at} \rightarrow \mathbb{C}\mathrm{at}$ is a structure on *span categories*.

### 2.5.1 Metalogic

We define a *metalogic* to be a "bifibrant triple category without interchange", with parallel composition weakly associative and unital like that of a tricategory [8].

Lastly, what ensures that parallel composition has coherent associator and unitors? Matrix categories and matrix profunctors are *exponentiable*, meaning composition has a right adjoint, and hence preserves the colimits which define parallel composition.

It is known that two-sided fibrations are exponentiable [21], and so matrix categories are as well. We show in Theorem 57 that matrix profunctors are exponentiable, by the duality of composition-by-codescent and transformation-by-*descent*.

**Definition 54.** A **metalogic** is a logic $\mathbb{C}$ and a fibered logic $\mathbb{M} \to \mathbb{C} \times \mathbb{C}$, with the structure of a 2-weak category internal to $\mathrm{Span}\mathbb{C}\mathrm{at}$.

This structure is expounded in the following theorem.

**Theorem 55.** $\mathrm{Mat}\mathbb{C}\mathrm{at} \to \mathbb{C}\mathrm{at} \times \mathbb{C}\mathrm{at}$ forms a metalogic.

*Proof.* As we showed, $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is a fibered span of logics equipped with span functors, for composition and identity

$$\begin{array}{ccccc} \mathbb{C} & \longleftarrow & \mathbb{MC} & \longrightarrow & \mathbb{C} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{P} & \longleftarrow & \mathbb{MP} & \longrightarrow & \mathbb{P} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C} & \longleftarrow & \mathbb{MC} & \longrightarrow & \mathbb{C} \end{array} \qquad \begin{array}{ccccc} \mathbb{MC} *_h \mathbb{MC} & \xrightarrow{\ \otimes\ } & \mathbb{MC} & \xleftarrow{\ \langle - \rangle\ } & \mathbb{C} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{MP} *_h \mathbb{MP} & \xrightarrow{\ \otimes\ } & \mathbb{MP} & \xleftarrow{\ \langle - \rangle\ } & \mathbb{P} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{MC} *_h \mathbb{MC} & \xrightarrow{\ \otimes\ } & \mathbb{MC} & \xleftarrow{\ \langle - \rangle\ } & \mathbb{C} \end{array}$$

with invertible span transformations for associativity,

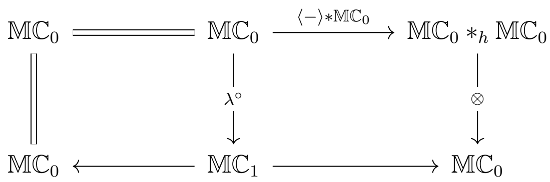
$$\begin{array}{ccccc} \mathbb{MC}_0 *_h \mathbb{MC}_0 & \xleftarrow{\ \mathbb{MC}_0 * \otimes\ } & \mathbb{MC}_0 *_h \mathbb{MC}_0 *_h \mathbb{MC}_0 & \xrightarrow{\ \otimes * \mathbb{MC}_0\ } & \mathbb{MC}_0 *_h \mathbb{MC}_0 \\ \downarrow \otimes & & \downarrow \alpha & & \downarrow \otimes \\ \mathbb{MC}_0 & \longleftarrow & \mathbb{MC}_1 & \longrightarrow & \mathbb{MC}_0 \end{array}$$

$$\alpha : \mathcal{R} \otimes (\mathcal{S} \otimes \mathcal{T}) \cong (\mathcal{R} \otimes \mathcal{S}) \otimes \mathcal{T}$$

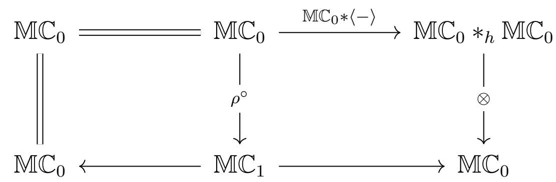
for both matrix categories and matrix profunctors

$$\alpha : m \otimes (n \otimes p) \cong (m \otimes n) \otimes p$$

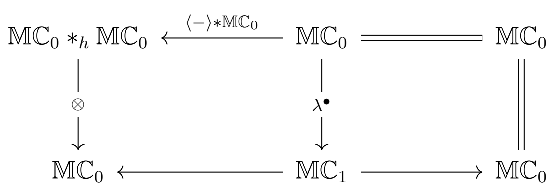
and span transformations for left and right unitality



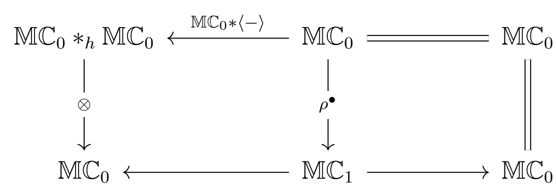
$\lambda^\circ = R.(\mathrm{id.A}, R) : \mathcal{R} \to \langle \mathbb{A} \rangle \otimes \mathcal{R}$



$\rho^\circ = R.(R, \mathrm{id.B}) : \mathcal{R} \to \mathcal{R} \otimes \langle \mathbb{B} \rangle$

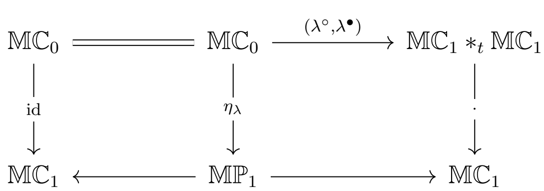


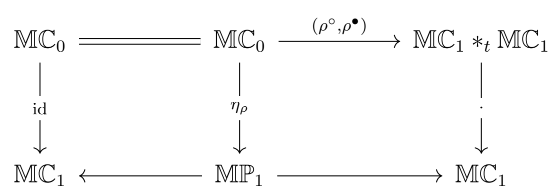
$\lambda^\bullet = \odot_{\mathbb{A}} : \langle \mathbb{A} \rangle \otimes \mathcal{R} \to \mathcal{R}$

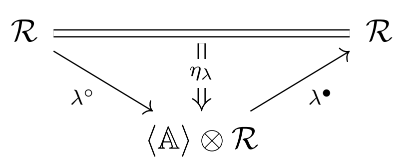


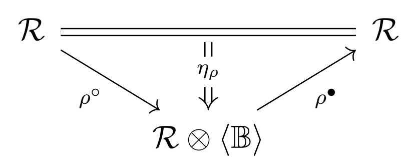
$\rho^\bullet = \odot_{\mathbb{B}} : \mathcal{R} \otimes \langle \mathbb{B} \rangle \to \mathcal{R}$

so that $(\lambda^\circ, \lambda^\bullet)$ and $(\rho^\circ, \rho^\bullet)$ form adjoint equivalences;



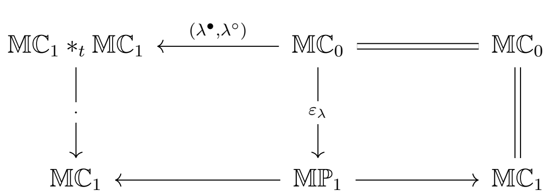


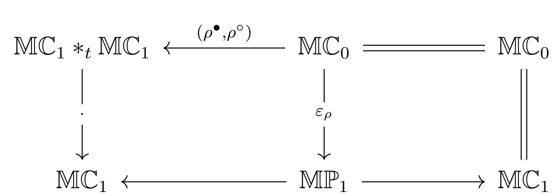


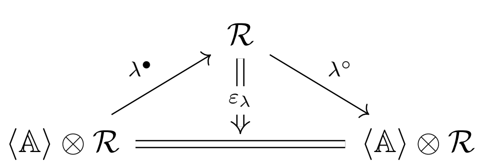


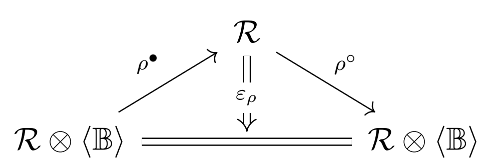
$\eta_\lambda = \upsilon_{\mathbb{A}} : R \cong \overline{\mathrm{id}}.\mathrm{A} \odot R$

$\eta_\rho = \upsilon_{\mathbb{B}} : R \cong R \odot \overline{\mathrm{id}}.\mathrm{B}$









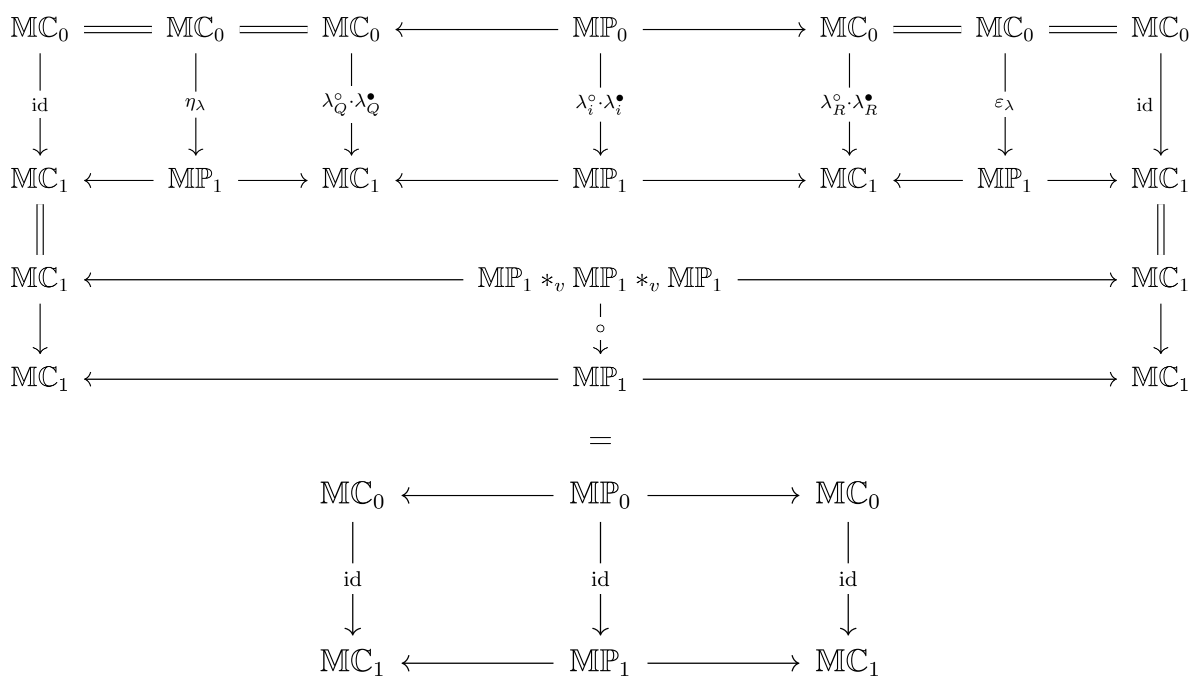
$\varepsilon_\lambda = \alpha_{\mathbb{A}} : (\mathrm{id.A}_0, \bar{\mathrm{a}} \odot R) \cong (\bar{\mathrm{a}}, R)$

$\varepsilon_\rho = \alpha_{\mathbb{B}} : (R \odot \bar{\mathrm{b}}, \mathrm{id.B}_1) \cong (R, \bar{\mathrm{b}})$

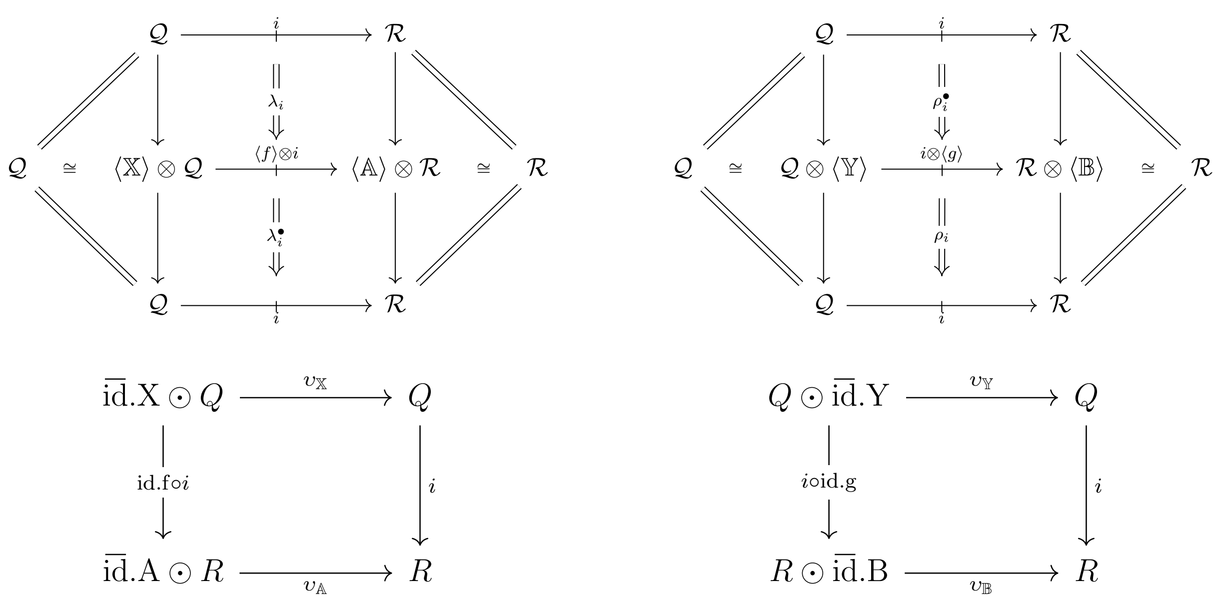
similarly for each matrix profunctor there are span transformations

$$\lambda^\circ = \quad i.(\mathrm{id.f}, i) : i \Rightarrow \langle f \rangle \otimes i \qquad \rho^\circ = \quad i.(i, \mathrm{id.g}) : i \Rightarrow i \otimes \langle g \rangle$$
$$\lambda^\bullet = \quad \odot_f : i \otimes \langle f \rangle \Rightarrow i \qquad \rho^\bullet = \quad \odot_g : i \otimes \langle g \rangle \Rightarrow i$$

so that the unitor isomorphisms cohere with these transformations, as in a modification:

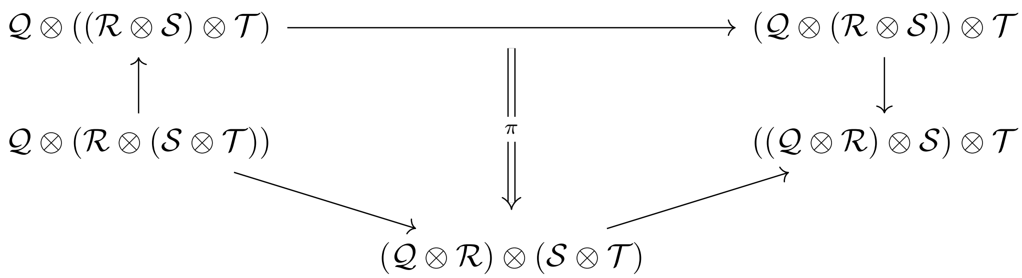


and this is given by the naturality of the unitors with respect to matrix profunctor elements;



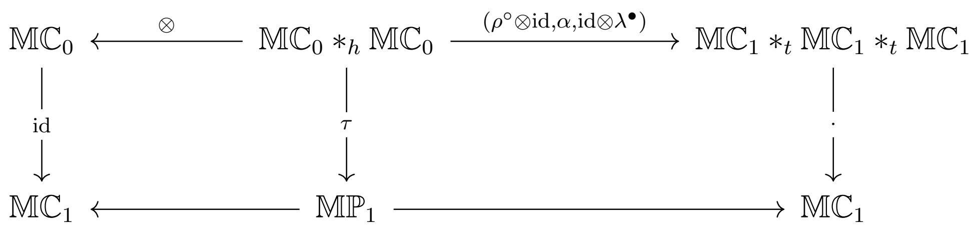
the analogous coherence holds for the right unitor $\rho$.

The “pentagon identity” for reassociating a composite is replaced by a “pentagonator”.

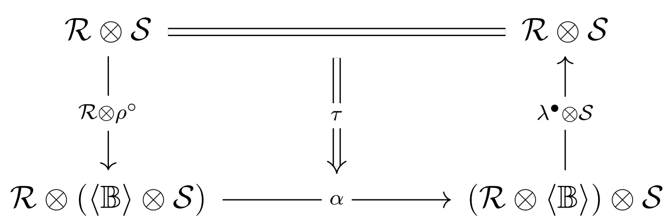



In our case, this isomorphism is an *equality*, as the associator simply moves parentheses. Hence it satisfies the coherence equation, which is given for a tricategory in [8].

Last, the unitors respect parallel composition by the “triangulator” isomorphism:

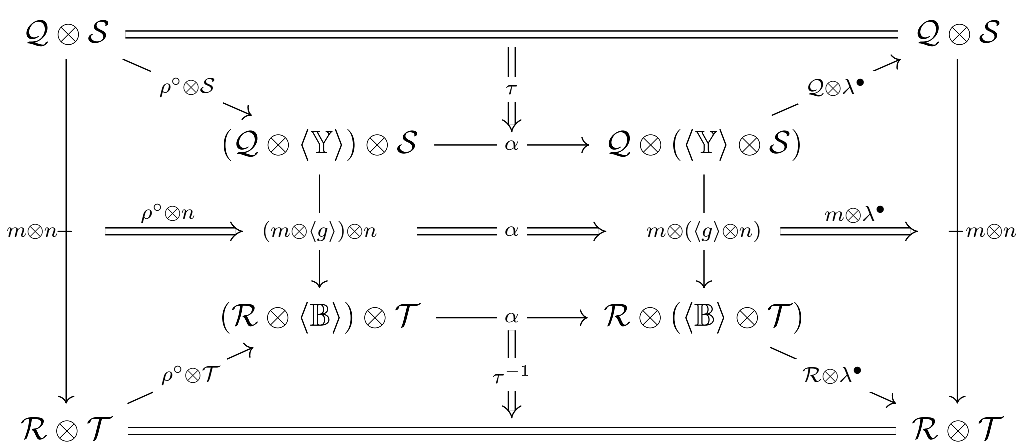



which is given by the unitor

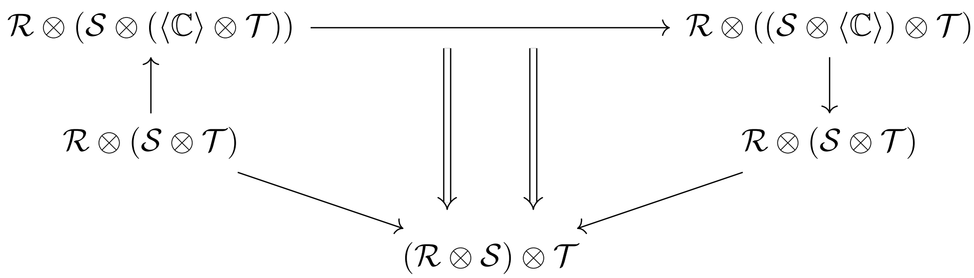



$$\tau = \upsilon_{\mathbb{B}} : (R, S) \cong (R \odot \mathrm{id}.\mathrm{B}, S)$$

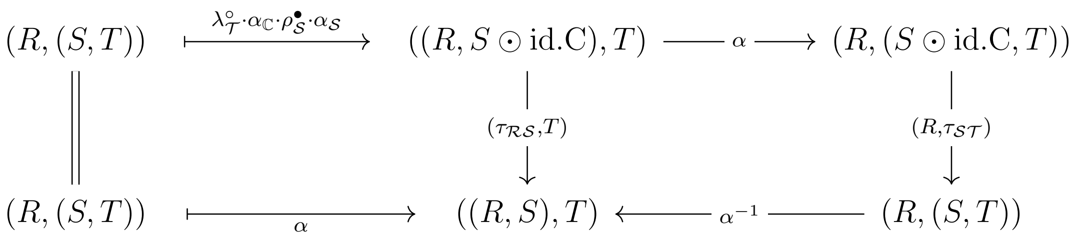
and which coheres with matrix profunctors, as in a modification.



For its coherence, the two ways to transform the top composite to the associator are equal:




meaning that applying the triangulator commutes with reassociating. This holds by the naturality of the unitor with respect to the associator.




The same coherence holds for applying the triangulator on the other side of the associator.

Hence $\mathrm{Mat}\mathbb{C}\mathrm{at}$ is a metalogic, i.e. a "bifibrant triple category without interchange".

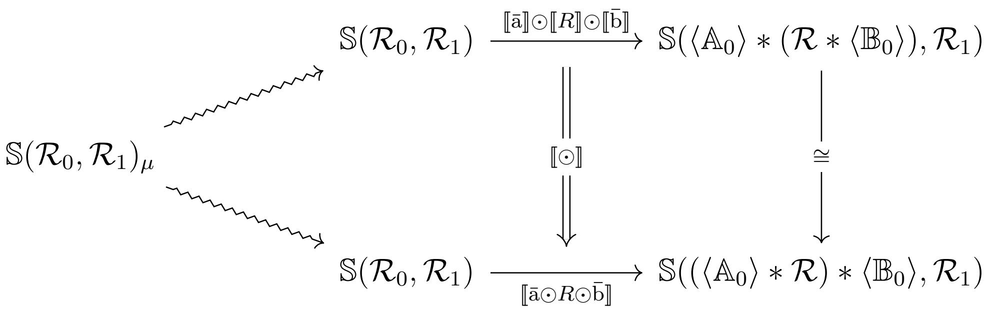
□

We now show that the metalogic of matrix categories is *higher-order*: composition is dual to *transformation*, giving $\mathrm{Mat}\mathbb{C}\mathrm{at}$ closure.

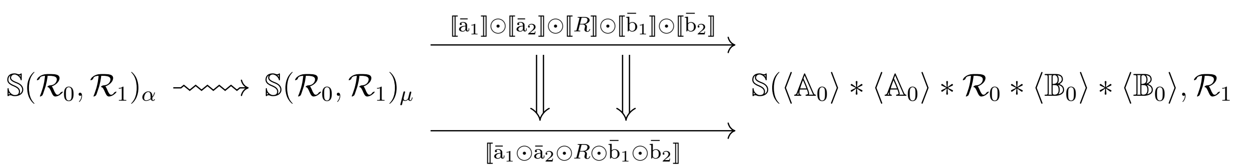
### 2.5.2 Descent

In the same way that the set of transformations between profunctors forms an *end* [15], the category of matrix functors between matrix categories forms a *descent object* [20].

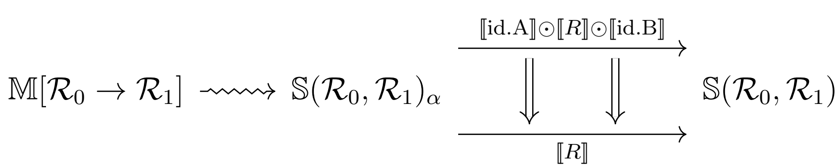
The set of transformations is an equalizer, for the equation of naturality. Yet a matrix functor is only "natural" up to isomorphism: the following *iso-inserter* forms the category of span functors equipped with a pair of joins $\odot_{\mathbb{A}}$ and $\odot_{\mathbb{B}}$. (We shorten $\mathrm{Span}\mathbb{C}\mathrm{at}$ to $\mathbb{S}$.)



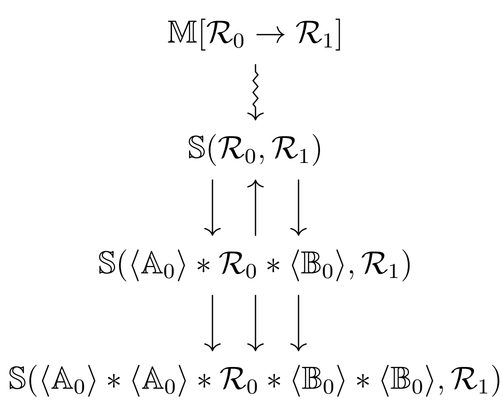

Each coherence equation is then imposed by an *equifier*. For joining composites:




and for joining units.




All together, this forms the *descent object* in $\mathbb{C}\mathrm{at}$ of the above functors and transformations.

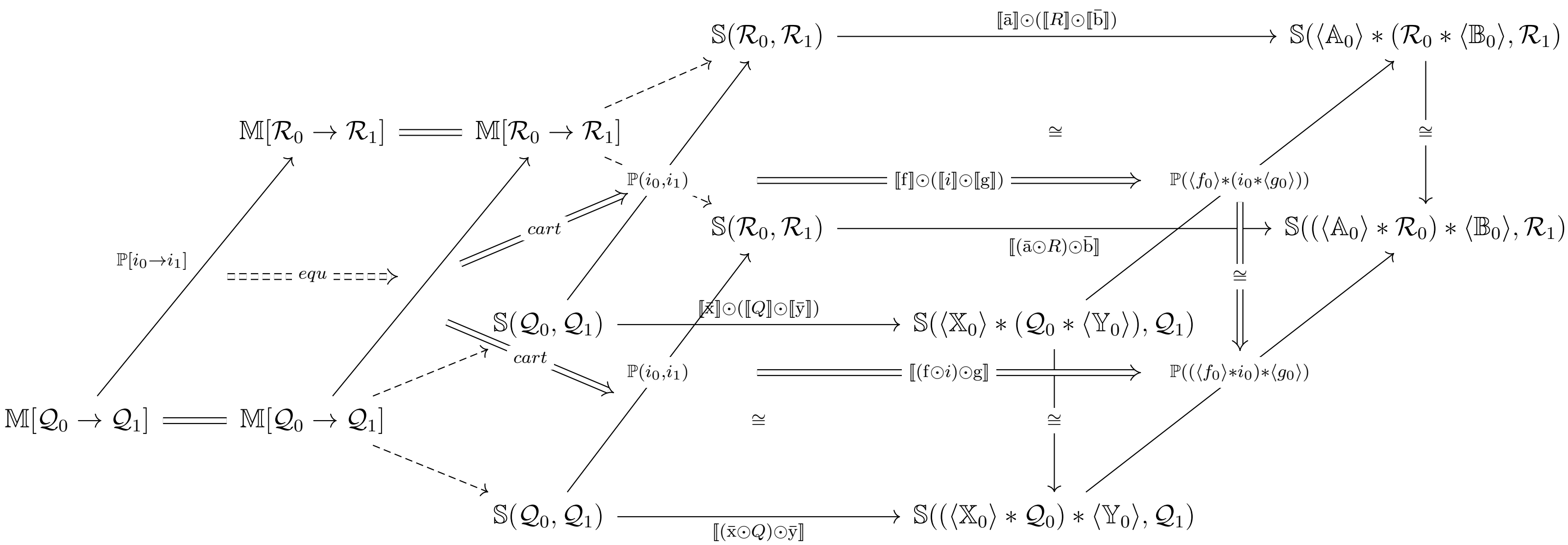



We denote the descent object by an "arrow product" notation.

$$\mathrm{Mat}\mathbb{C}\mathrm{at}[\mathcal{R}_0 \to \mathcal{R}_1] \quad \equiv \quad \vec{\Pi} \mathrm{A} : \mathbb{A}_0, \mathrm{B} : \mathbb{B}_0 \;\; \mathbb{C}\mathrm{at}[\mathcal{R}_0(\mathrm{A}, \mathrm{B}) \to \mathcal{R}_1([\![\mathrm{A}]\!], [\![\mathrm{B}]\!])]$$

Based on the hom of matrix categories, we define the hom of matrix profunctors.

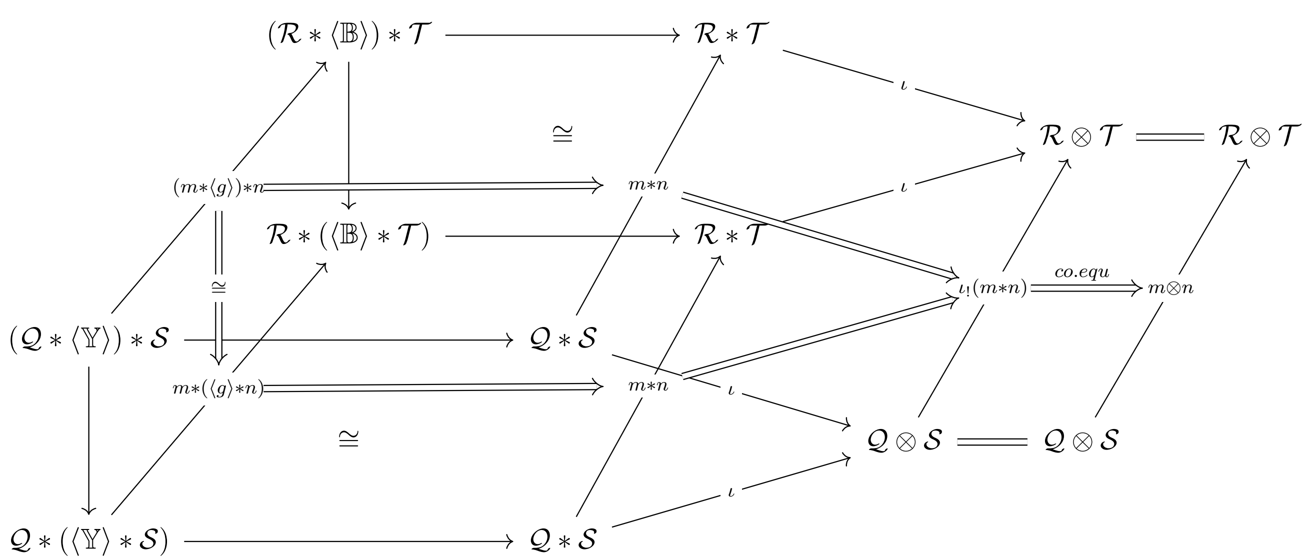
**Definition 56.** Let $i_0(f_0, g_0) : \mathcal{Q}_0(\mathbb{X}_0, \mathbb{Y}_0) \mid \mathcal{R}_0(\mathbb{A}_0, \mathbb{B}_0)$ and $i_1(f_1, g_1) : \mathcal{Q}_1(\mathbb{X}_1, \mathbb{Y}_1) \mid \mathcal{R}_1(\mathbb{A}_1, \mathbb{B}_1)$ be matrix profunctors. Matrix transformations $i_0 \Rightarrow i_1$ form a profunctor from $\mathbb{M}[\mathcal{Q}_0 \to \mathcal{Q}_1]$ to $\mathbb{M}[\mathcal{R}_0 \to \mathcal{R}_1]$, which is constructed by the following equalizer.




We denote this equalizer as follows.

$$\mathbb{M}[i_0 \to i_1] = \vec{\Pi}\mathrm{f} : f_0,\ \mathrm{g} : g_0\ \ i_0(\mathrm{f}, \mathrm{g}) \to i_1([\![\mathrm{f}]\!], [\![\mathrm{g}]\!])$$

Yet this construction is exactly *dual* to composition of matrix profunctors (2.5).




A composite matrix category forms a coequifier of an iso-coinserter of the actions of the inner category. Matrix functors form an equifier of an iso-inserter of the actions of the outer categories. A composite matrix profunctor forms a coequalizer along those iso-coinserters, and matrix transformations form an equalizer along those iso-inserters.

Hence, we show the duality of composition-by-codescent and transformation-by-descent.

**Theorem 57.** $\mathbb{M}\mathrm{at}\mathbb{C}\mathrm{at}$ is closed: for each pair of matrix categories $\mathcal{R} : \mathbb{A} \parallel \mathbb{B}$ and $\mathcal{S} : \mathbb{B} \parallel \mathbb{C}$ and matrix category $\mathcal{T} : \mathbb{A} \parallel \mathbb{C}$, there is an *extension*

$$\mathrm{Mat}\mathbb{C}\mathrm{at}(\mathbb{A},\mathbb{C})(\mathcal{R} \otimes \mathcal{S}, \mathcal{T}) \simeq \mathrm{Mat}\mathbb{C}\mathrm{at}(\mathbb{B},\mathbb{C})(\mathcal{S}, [\mathcal{R} \to \mathcal{T}])$$

with the following explicit formula:

$$[\mathcal{R} \to \mathcal{T}] \;\; \equiv \;\; \vec{\Pi}\mathrm{A} \;\; \mathcal{R}(\mathrm{A}, -) \to \mathcal{T}(\mathrm{A}, -)$$

and dually, there is a *lift*

$$\mathrm{Mat}\mathbb{C}\mathrm{at}(\mathbb{A},\mathbb{C})(\mathcal{R} \otimes \mathcal{S}, \mathcal{T}) \simeq \mathrm{Mat}\mathbb{C}\mathrm{at}(\mathbb{A},\mathbb{B})(\mathcal{R}, [\mathcal{T} \leftarrow \mathcal{S}])$$

with the following explicit formula:

$$[\mathcal{T} \leftarrow \mathcal{S}] \;\; \equiv \;\; \vec{\Pi}\mathrm{C} \;\; \mathcal{S}(-, \mathrm{C}) \to \mathcal{T}(-, \mathrm{C})$$

For each pair of matrix profunctors $m(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \mid \mathcal{R}(\mathbb{A},\mathbb{B})$ and $n(g,h) : \mathcal{S}(\mathbb{Y},\mathbb{Z}) \mid \mathcal{T}(\mathbb{B},\mathbb{C})$, and matrix profunctor $p(f,h) : (\mathcal{Q} \otimes \mathcal{S})(\mathbb{X},\mathbb{Z}) \mid (\mathcal{R} \otimes \mathcal{T})(\mathbb{A},\mathbb{C})$, there is an extension and lift.

$$[m \to p] \equiv \vec{\Pi}\mathrm{g} \;\; m(\mathrm{f}, -) \to p(\mathrm{f}, -) \qquad\qquad [p \leftarrow n] \equiv \vec{\Pi}\mathrm{h} \;\; n(-, \mathrm{h}) \to p(-, \mathrm{h})$$

*Proof.* The composite $\mathcal{R} \otimes \mathcal{S}$ is a coequifier of an iso-coinserter, while the hom $[(\mathcal{R} \otimes \mathcal{S}), \mathcal{T}]$ is an equifier of an iso-inserter. These are constructed pointwise in $\mathbb{C}\mathrm{at}$; the first coordinate of $\mathbb{C}\mathrm{at}(-,-)$ converts 2-colimits into 2-limits, while the second preserves 2-limits [10]. The Fubini equivalence is given in [2]. Hence we have the following equivalence.

$$\begin{array}{rcll}
\mathrm{Mat}\mathbb{C}\mathrm{at}(\mathcal{R} \otimes \mathcal{S}, \mathcal{T}) & = & \vec{\Pi}\mathrm{A},\mathrm{C} & \mathbb{C}\mathrm{at}((\mathcal{R} \otimes \mathcal{S})(\mathrm{A},\mathrm{C}), \mathcal{T}(\mathrm{A},\mathrm{C})) \\
& = & \vec{\Pi}\mathrm{A},\mathrm{C} & \mathbb{C}\mathrm{at}(\vec{\Sigma}\mathrm{B} \;\; \mathcal{R}(\mathrm{A},\mathrm{B}) \times \mathcal{S}(\mathrm{B},\mathrm{C}), \mathcal{T}(\mathrm{A},\mathrm{C})) \\
& \simeq & \vec{\Pi}\mathrm{A},\mathrm{C} \;\; \vec{\Pi}\mathrm{B} & \mathbb{C}\mathrm{at}(\mathcal{R}(\mathrm{A},\mathrm{B}) \times \mathcal{S}(\mathrm{B},\mathrm{C}), \mathcal{T}(\mathrm{A},\mathrm{C})) \\
& \simeq & \vec{\Pi}\mathrm{A},\mathrm{B},\mathrm{C} & \mathbb{C}\mathrm{at}(\mathcal{S}(\mathrm{B},\mathrm{C}), [\mathcal{R}(\mathrm{A},\mathrm{B}) \to \mathcal{T}(\mathrm{A},\mathrm{C})]) \\
& \simeq & \vec{\Pi}\mathrm{B},\mathrm{C} & \mathbb{C}\mathrm{at}(\mathcal{S}(\mathrm{B},\mathrm{C}), \vec{\Pi}\mathrm{C} \;\; [\mathcal{R}(\mathrm{A},\mathrm{B}) \to \mathcal{T}(\mathrm{A},\mathrm{C})]) \\
\\
& = & & \mathrm{Mat}\mathbb{C}\mathrm{at}(\mathcal{S}, [\mathcal{R} \to \mathcal{T}])
\end{array}$$

In the same way, the equalizer formed by matrix transformations is dual to the coequalizer formed by composition of matrix profunctors; hence we have the following isomorphism.

$$\begin{array}{rcll}
\mathrm{MatProf}(m \otimes n, p) & = & \vec{\Pi}\mathrm{f},\mathrm{h} & \mathrm{Prof}((m \otimes n)(\mathrm{f},\mathrm{h}), p(\mathrm{f},\mathrm{h})) \\
& = & \vec{\Pi}\mathrm{f},\mathrm{h} & \mathrm{Prof}(\vec{\Sigma}\mathrm{g} \;\; m(\mathrm{f},\mathrm{g}) \times n(\mathrm{g},\mathrm{h}), p(\mathrm{f},\mathrm{h})) \\
& \cong & \vec{\Pi}\mathrm{f},\mathrm{h} \;\; \vec{\Pi}\mathrm{g} & \mathrm{Prof}(m(\mathrm{f},\mathrm{g}) \times n(\mathrm{g},\mathrm{h}), p(\mathrm{f},\mathrm{h})) \\
& \cong & \vec{\Pi}\mathrm{f},\mathrm{g},\mathrm{h} & \mathrm{Prof}(n(\mathrm{g},\mathrm{h}), [m(\mathrm{f},\mathrm{g}) \to p(\mathrm{f},\mathrm{h})]) \\
& \cong & \vec{\Pi}\mathrm{g},\mathrm{h} & \mathrm{Prof}(n(\mathrm{g},\mathrm{h}), \vec{\Pi}\mathrm{f} \;\; [m(\mathrm{f},\mathrm{g}) \to p(\mathrm{f},\mathrm{h})]) \\
\\
& = & & \mathrm{MatProf}(n, [m \to p])
\end{array}$$

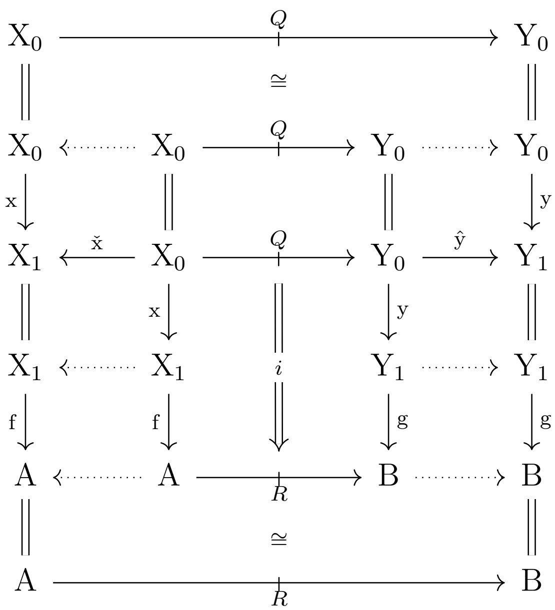
These define a weak right adjoint [7] to composition $\mathcal{R} \otimes -$ and $m \otimes -$; dually for lifts. The detailed definition of *closed metalogic* will be given in a forthcoming draft. □

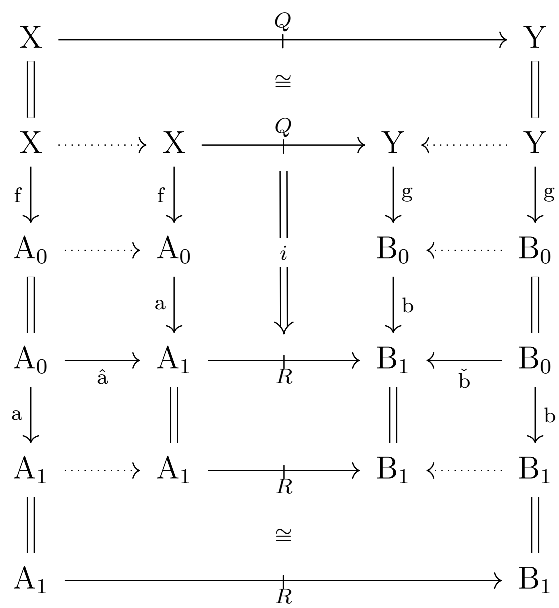
Hence matrix profunctors are *exponentiable*, ensuring the coherence of $\mathrm{Mat}\mathbb{C}\mathrm{at}$.

The above theorem gives the formula for the right adjoint; but first we follow the reasoning of Street in [19]: let $i(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \,|\, \mathcal{R}(\mathbb{A},\mathbb{B})$ be a matrix profunctor, determining a displayed profunctor $i : f \times g \to \mathrm{Prof}$ with actions

$$\mathcal{Q}(\mathrm{x},\mathrm{y}) \circ i(\mathrm{f},\mathrm{g}) \Rightarrow i(\mathrm{xf},\mathrm{yg}) \quad \text{and} \quad i(\mathrm{f},\mathrm{g}) \circ \mathcal{R}(\mathrm{a},\mathrm{b}) \Rightarrow i(\mathrm{fa},\mathrm{gb}).$$

These actions are invertible, as $\mathcal{Q}$ and $\mathcal{R}$ are bifibered: each $i : i(\mathrm{xf},\mathrm{yg})$ and each $i : i(\mathrm{fa},\mathrm{gb})$ factor as the following elements of $\mathcal{Q}(\mathrm{x},\mathrm{y}) \circ i(\mathrm{f},\mathrm{g})$ and $i(\mathrm{f},\mathrm{g}) \circ \mathcal{R}(\mathrm{a},\mathrm{b})$, respectively.

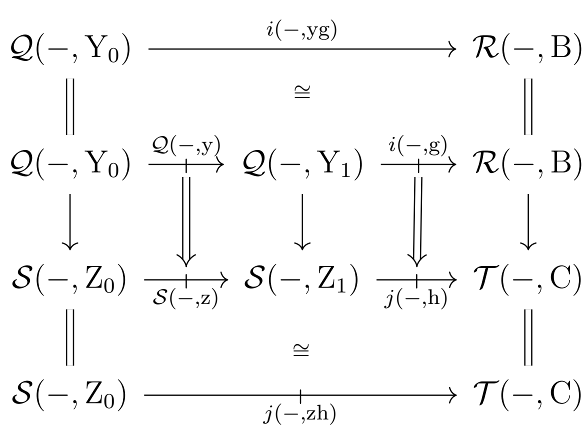


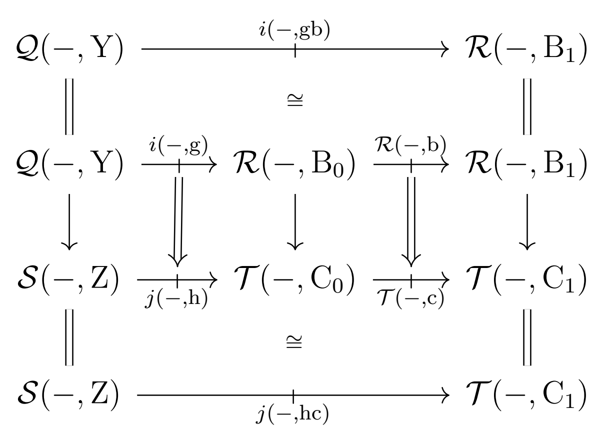




Then for $j(f,h) : \mathcal{S}(\mathbb{X},\mathbb{Z}) \,|\, \mathcal{T}(\mathbb{A},\mathbb{C})$, the extension

$$[i \to j](g,h) : [\mathcal{Q} \to \mathcal{S}](\mathbb{Y},\mathbb{Z}) \;\; | \;\; [\mathcal{R} \to \mathcal{T}](\mathbb{B},\mathbb{C})$$

consists of transformations $i(-,\mathrm{g}) \Rightarrow j(-,\mathrm{h})$ and actions as follows.



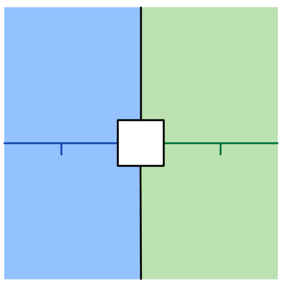



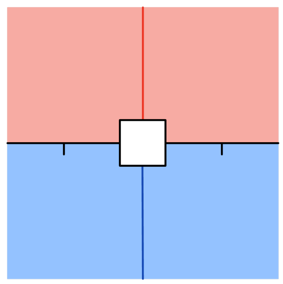

Hence by reasoning exactly analogous to [19], matrix profunctors are exponentiable.

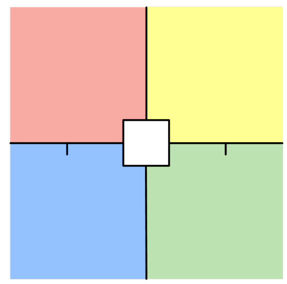
# 3 Metalogic: the language of bifibrant double categories

We can now define a *logic*, or *bifibrant double category*: a matrix category $\mathbb{A} : \underline{\mathbb{A}} \,\|\, \underline{\mathbb{A}}$ with composition $\circ : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ and unit $\mathrm{id} : \underline{\mathbb{A}} \to \mathbb{A}$, with coherent associator and unitors — a *pseudomonad* in $\mathrm{Mat}\mathbb{C}\mathrm{at}$. So, this chapter is simply the *completion* of the previous chapter.

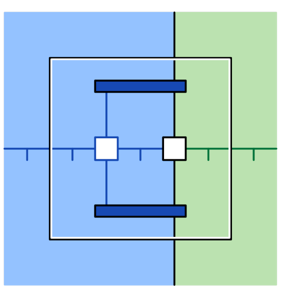
Because we developed all underlying structure, we can define the whole "multiverse" of logics. As a logic is two-dimensional, there are *two* kinds of relations between logics: a *vertical profunctor* consists of processes between logics, and a *horizontal profunctor* consists of relations between logics. Two pairs are connected by a *double profunctor*, which consists of inferences between relations, along processes.

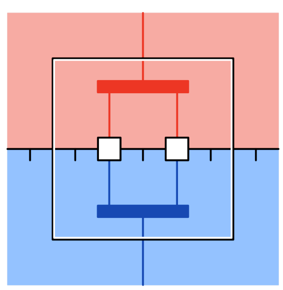

meta relation
[horiz. profunctor]

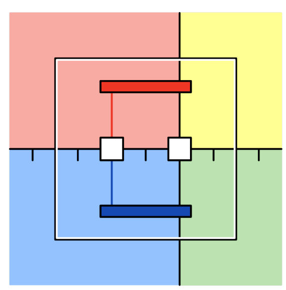

meta process
[vert. profunctor]


meta inference
[dbl. profunctor]

Because $\mathrm{Mat}\mathbb{C}\mathrm{at}$ consists of categories and profunctors, the above profunctors already have sequential composition; so we only need to add the structure of *parallel* composition. For horizontal profunctors, this is a familiar *bimodule* action. But as vertical profunctors are orthogonal, parallel composition defines a *monad* structure, and double profunctors are bimodules thereof.

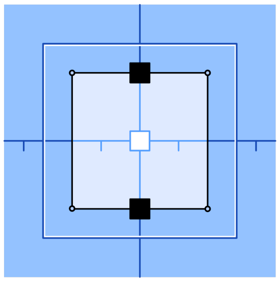

H-prof. composition


V-prof. composition

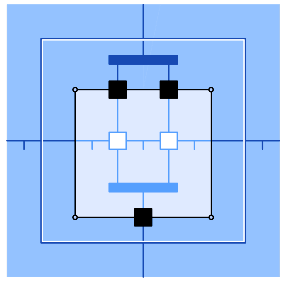

D-prof. composition

So logics have two kinds of "relations", and one kind of "function": a *double functor* $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ maps squares of $\mathbb{A}_0$ to squares of $\mathbb{A}_1$, preserving relation composition and unit up to coherent isomorphism. This generalizes to transformations of vertical, horizontal, and double profunctors; all four are defined by mapping squares in a way that coheres with parallel composition.

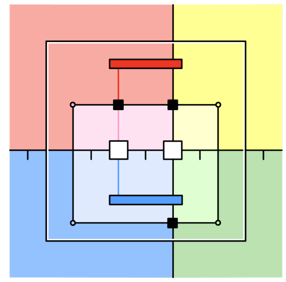
Contents



double functor



preserves composition;



double transformation

Logics form a metalogic: the three kinds of 1-morphism are v-profunctor, h-profunctor, and double functor; the three kinds of 2-morphism are double profunctor, v-transformation, and h-transformation; and the 3-morphism is a double transformation.

| | MatℂCat | H.PsMnd(−) | $bf$.DblℂCat | Logic |
|---|---|---|---|---|
| 0 | category | (H)-pseudomonad | bifibrant double category | logic |
| V | profunctor | (H)-vertical monad | vertical profunctor | meta process |
| H | matrix category | (H)-pseudobimodule | horizontal profunctor | meta relation |
| VH | matrix profunctor | (H)-vertical bimodule | double profunctor | meta inference |
| T | functor | ps. mnd. morphism | double functor | flow type |
| TV | transformation | v. mnd. morphism | vertical transformation | flow process |
| TH | matrix functor | ps. bim. morphism | horizontal transformation | flow relation |
| TVH | matrix transformation | v. bim. morphism | double transformation | flow inference |

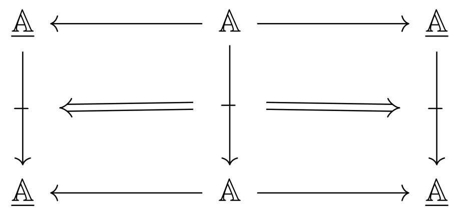
We construct the double category $bf$.DblℂCat of bifibrant double categories and double functors, vertical profunctors and vertical transformations.

We construct the double category $bf$.DblProf of horizontal profunctors and horizontal transformations, double profunctors and double transformations.

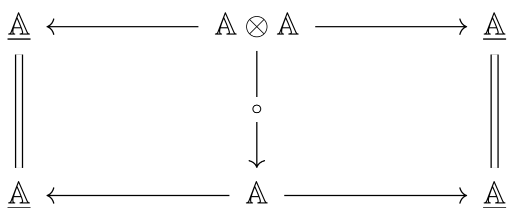
Then we define parallel composition of horizontal profunctors. As for matrix categories in 2.5, the composite forms a codescent object, which adjoins a coherent associator for the middle action. We show that this defines the structure of a metalogic.

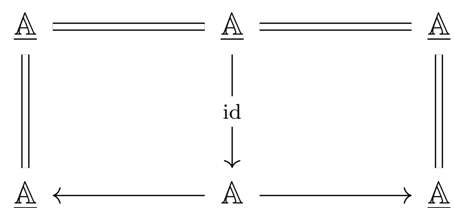
## 3.1 Logic [Bifibrant double category]

**Definition 58.** A **logic** $\mathbb{A}$, a.k.a. *bifibrant double category*, is a pseudomonad in $\mathrm{Mat}\mathbb{C}\mathrm{at}$.
Hence a logic is a category $\underline{\mathbb{A}}$ with a matrix category $\mathbb{A} : \underline{\mathbb{A}} \| \underline{\mathbb{A}}$

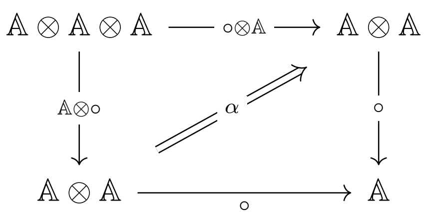


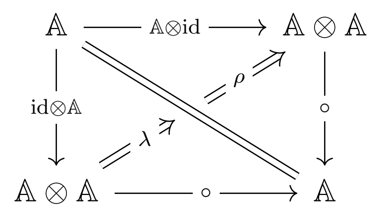
with matrix functors $\circ : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ for composition and $\mathrm{id} : \underline{\mathbb{A}} \to \mathbb{A}$ for unit



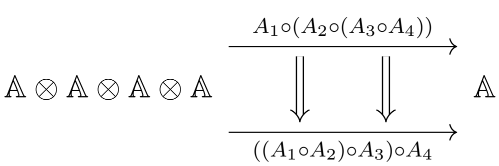



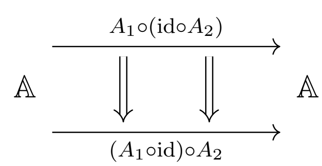

and invertible matrix transformations for associativity and unit

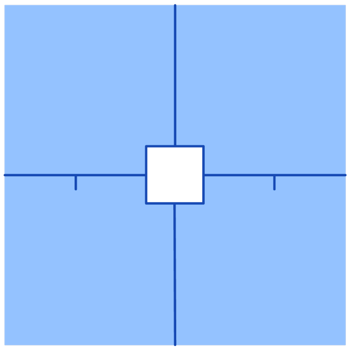


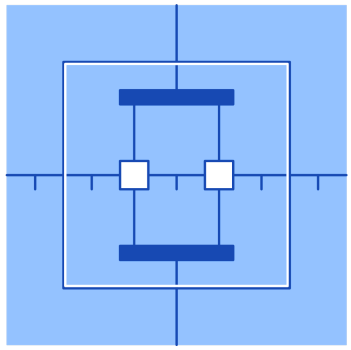

which satisfy the associator and unitor coherence.



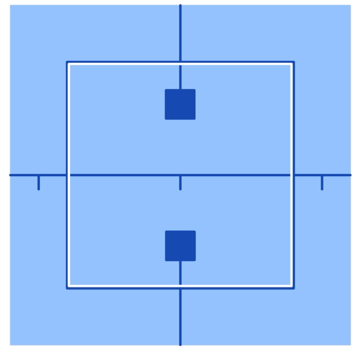





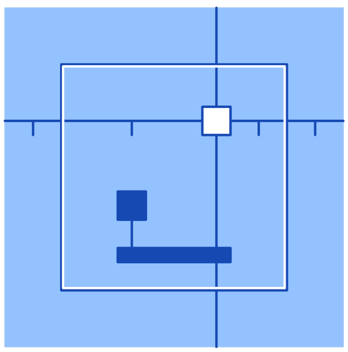
**bifibrant double category**
matrix category
$\mathbb{A} : \underline{\mathbb{A}} \,\|\, \underline{\mathbb{A}}$



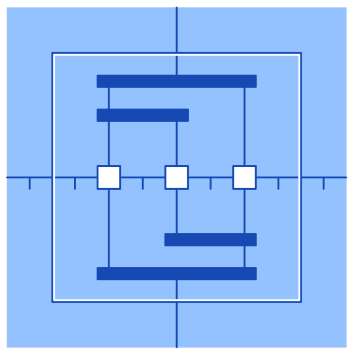
**composition**
matrix functor
$\circ : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$



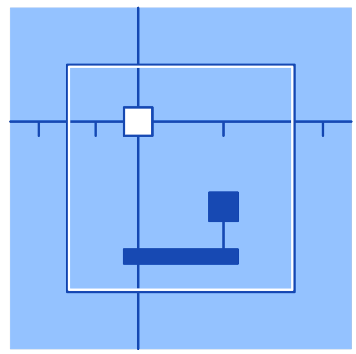
**unit**
matrix functor
$\mathrm{id} : \underline{\mathbb{A}} \to \mathbb{A}$



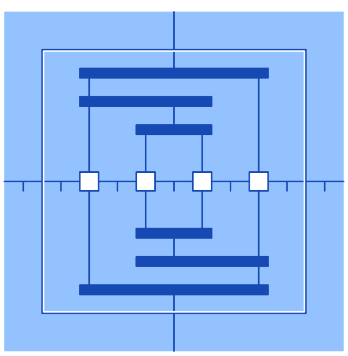
**left unitor**
matrix transformation
$\lambda : \mathbb{A} \cong \mathrm{id} \circ \mathbb{A}$



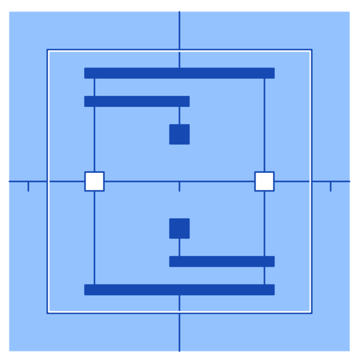
**associator**
matrix transformation
$\alpha : (\mathbb{A} \circ \mathbb{A}) \circ \mathbb{A} \cong \mathbb{A} \circ (\mathbb{A} \circ \mathbb{A})$



**right unitor**
matrix transformation
$\rho : \mathbb{A} \cong \mathbb{A} \circ \mathrm{id}$



associator coherence
$((\mathrm{A}_0 \circ \mathrm{A}_1) \circ \mathrm{A}_2) \circ \mathrm{A}_3 \rightrightarrows \mathrm{A}_0 \circ (\mathrm{A}_1 \circ (\mathrm{A}_2 \circ \mathrm{A}_3))$

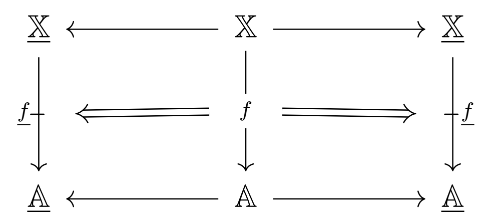


unitor coherence
$(\mathrm{A}_1 \circ \mathrm{id}) \circ \mathrm{A}_2 \rightrightarrows \mathrm{A}_1 \circ (\mathrm{id} \circ \mathrm{A}_2)$

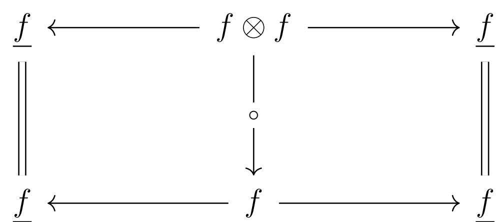
## 3.2 Relations [Double profunctor]

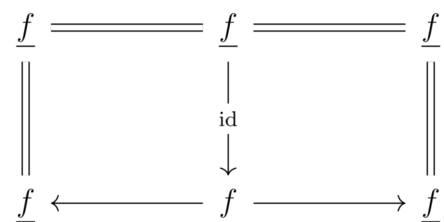
**Definition 59.** Let $\mathbb{X}, \mathbb{A}$ be bifibrant double categories. A **vertical profunctor** $f : \mathbb{X} \mid \mathbb{A}$, i.e. **meta process**, is a *vertical monad* between pseudomonads in $\mathrm{Mat}\mathbb{C}\mathrm{at}$.

Hence it is a profunctor $\underline{f} : \underline{\mathbb{X}} \mid \underline{\mathbb{A}}$ and a matrix profunctor $f(\underline{f}, \underline{f}) : \mathbb{X}(\underline{\mathbb{X}}, \underline{\mathbb{X}}) \mid \mathbb{A}(\underline{\mathbb{A}}, \underline{\mathbb{A}})$

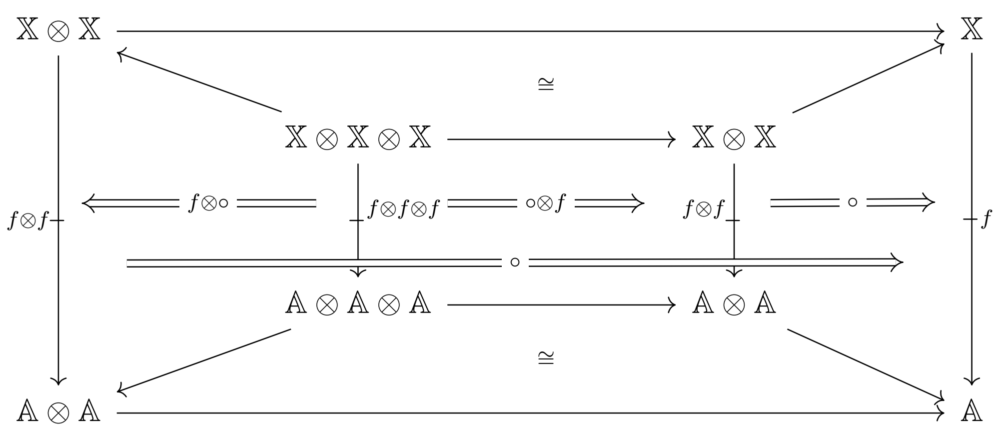


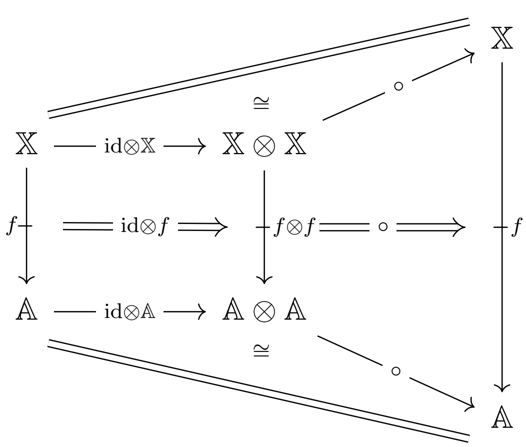

with matrix transformations $\circ : f * f \Rightarrow f$ for composition and $\mathrm{id} : \underline{f} \Rightarrow f$ for unit

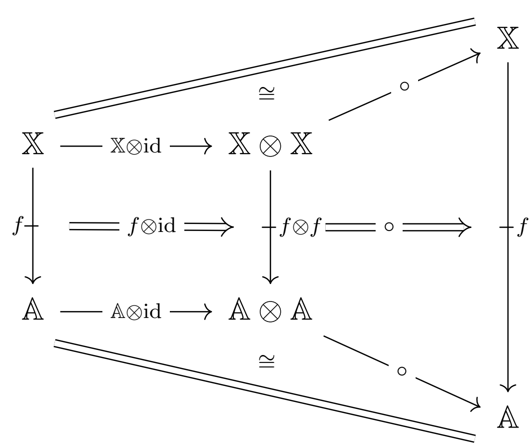


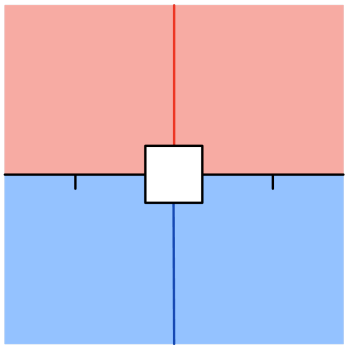




which cohere with the associators and unitors of $\mathbb{X}$ and $\mathbb{A}$.

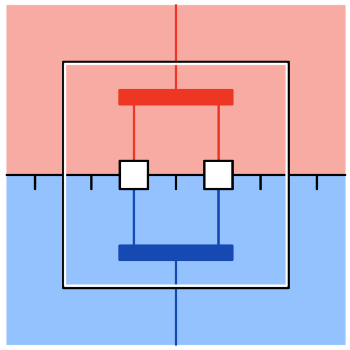


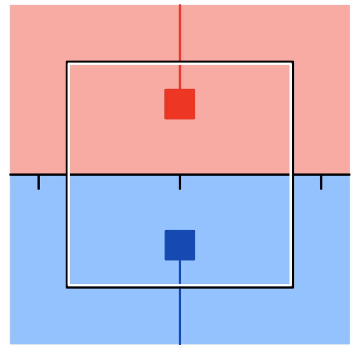





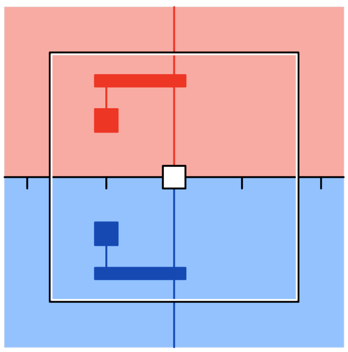
**vertical profunctor**
matrix profunctor
$f(\underline{f}, \underline{f}) : \mathbb{X}(\underline{\mathbb{X}}, \underline{\mathbb{X}}) \mid \mathbb{A}(\underline{\mathbb{A}}, \underline{\mathbb{A}})$



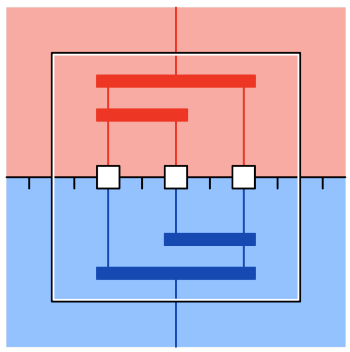
**composition**
matrix transformation
$\circ : f * f \to f$



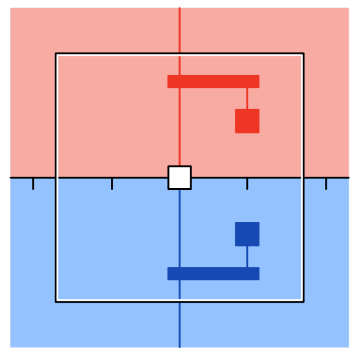
**unit**
matrix transformation
$\mathrm{id} : \underline{f} \to f$



**left unit coherence**
$\mathrm{id}.\mathbb{X} \circ \mathbb{X} \rightrightarrows \mathrm{id}.\mathbb{A} \circ \mathbb{A}$



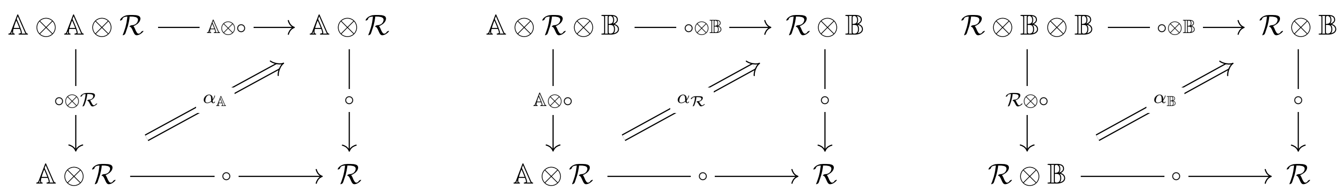
**assoc coherence**
$(\mathbb{X} \circ \mathbb{X}) \circ \mathbb{X} \rightrightarrows \mathbb{A} \circ (\mathbb{A} \circ \mathbb{A})$



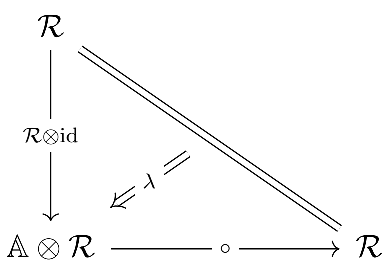
**right unit coherence**
$\mathbb{X} \circ \mathrm{id}.\mathbb{X} \rightrightarrows \mathbb{A} \circ \mathrm{id}.\mathbb{A}$

**Definition 60.** Let $\mathbb{A}$ and $\mathbb{B}$ be bifibrant double categories. A **horizontal profunctor** $\mathcal{R} : \mathbb{A} \parallel \mathbb{B}$, i.e. **meta relation**, is a matrix category forming a bimodule of pseudomonads.

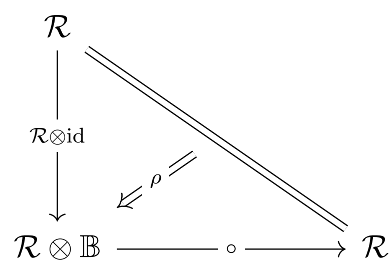
Hence it is a matrix category $\mathcal{R} : \underline{\mathbb{A}} \parallel \underline{\mathbb{B}}$, with action matrix functors $\mathbb{A} \otimes \mathcal{R} \to \mathcal{R}$ and $\mathcal{R} \otimes \mathbb{B} \to \mathcal{R}$, and invertible matrix transformations for associators and unitors




$(A_1 \circ A_2) \circ R \cong A_1 \circ (A_2 \circ R) \quad A \circ (R \circ B) \cong (A \circ R) \circ B \quad R \circ (B_1 \circ B_2) \cong (R \circ B_1) \circ B_2$

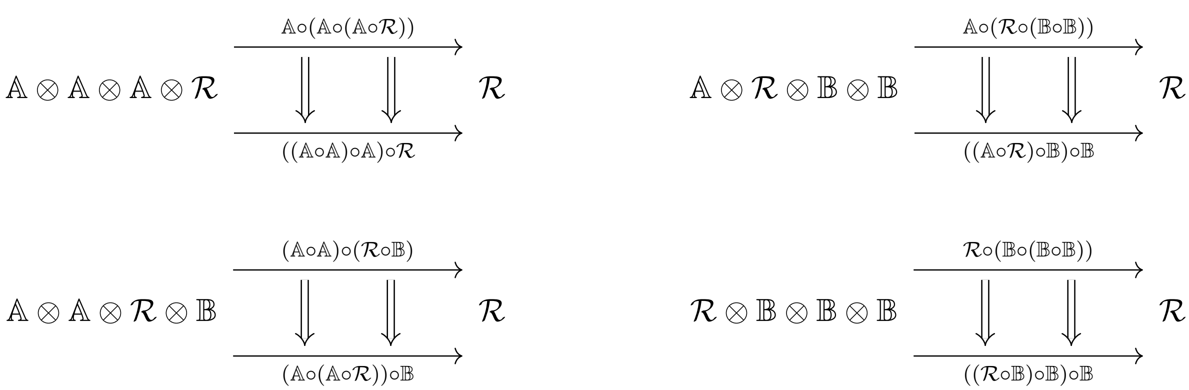



$\upsilon_{\mathbb{A}} : R \cong \mathrm{id.A} \circ R$

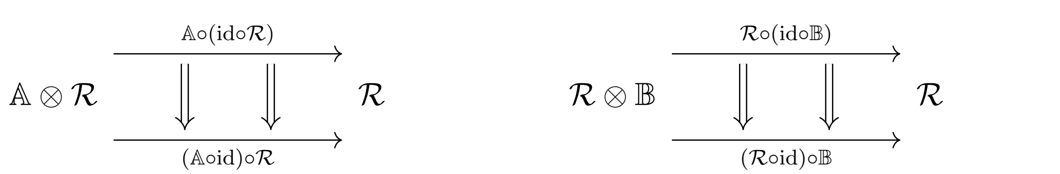


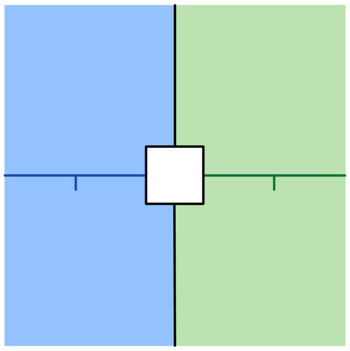

$\upsilon_{\mathbb{B}} : R \cong R \circ \mathrm{id.B}$

satisfying the associator coherence

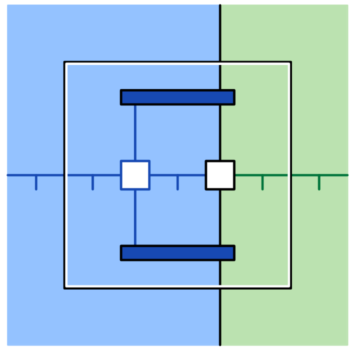



and unitor coherence.

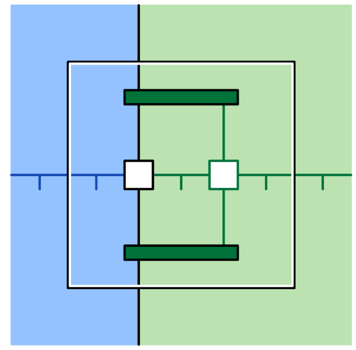




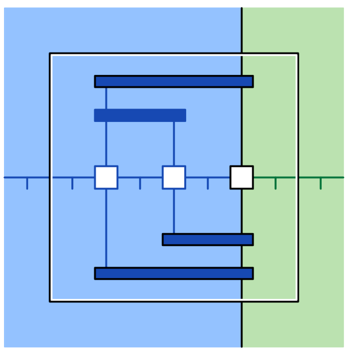
**horizontal profunctor**
matrix category
$\mathcal{R} : \underline{\mathbb{A}} \,\|\, \underline{\mathbb{B}}$



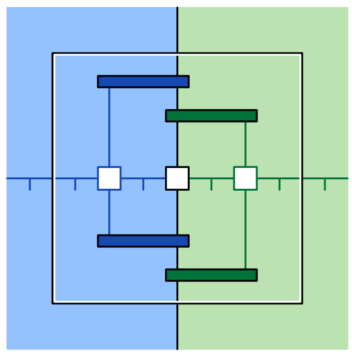
**left composition**
matrix functor
$\circ : \mathbb{A} * \mathcal{R} \to \mathcal{R}$



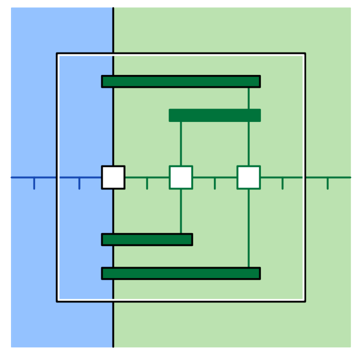
**right composition**
matrix functor
$\circ : \mathcal{R} * \mathbb{B} \to \mathcal{R}$



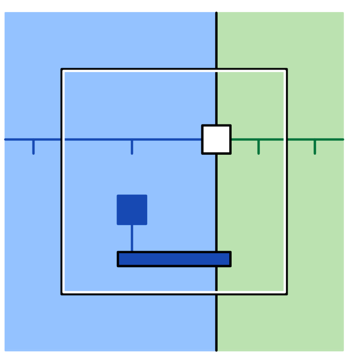
**left associator**
matrix transformation
$\alpha_{\mathbb{A}} : (\mathbb{A} \circ \mathbb{A}) \circ \mathcal{R} \cong \mathbb{A} \circ (\mathbb{A} \circ \mathcal{R})$



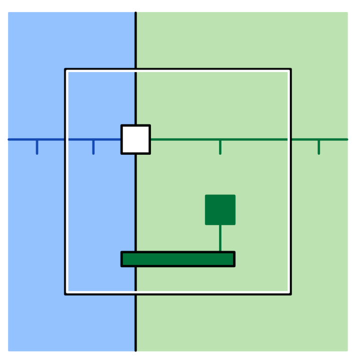
**center associator**
matrix transformation
$\alpha_{\mathcal{R}} : \mathbb{A} \circ (\mathcal{R} \circ \mathbb{B}) \cong (\mathbb{A} \circ \mathcal{R}) \circ \mathbb{B}$



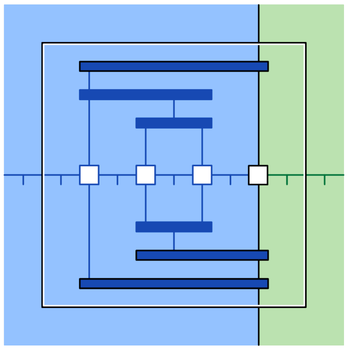
**right associator**
matrix transformation
$\alpha_{\mathbb{B}} : \mathcal{R} \circ (\mathbb{B} \circ \mathbb{B}) \cong (\mathcal{R} \circ \mathbb{B}) \circ \mathbb{B}$



**left unitor**
$\lambda : \mathcal{R} \cong \mathbb{A} \circ \mathcal{R}$

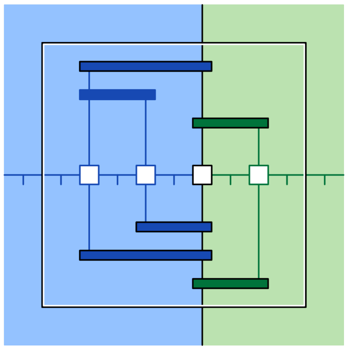


**right unitor**
$\rho : \mathcal{R} \cong \mathcal{R} \circ \mathbb{B}$



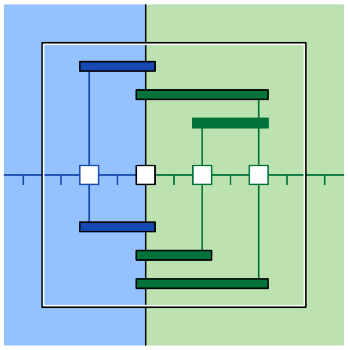
**$\mathbb{A}$-assoc coherence**

$((\mathbb{A} \circ \mathbb{A}) \circ \mathbb{A}) \circ \mathcal{R} \rightrightarrows \mathbb{A} \circ (\mathbb{A} \circ (\mathbb{A} \circ \mathcal{R}))$



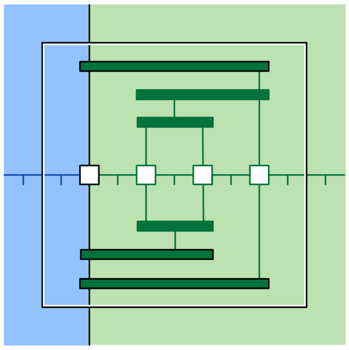
**$\mathbb{AAB}$-assoc coherence**

$(\mathbb{A} \circ \mathbb{A}) \circ (\mathcal{R} \circ \mathbb{B}) \rightrightarrows (\mathbb{A} \circ (\mathbb{A} \circ \mathcal{R})) \circ \mathbb{B}$



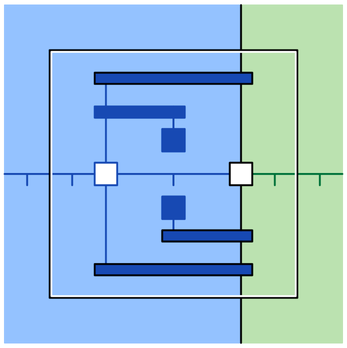
**$\mathbb{ABB}$-assoc coherence**

$\mathbb{A} \circ (\mathcal{R} \circ (\mathbb{B} \circ \mathbb{B})) \rightrightarrows ((\mathbb{A} \circ \mathcal{R}) \circ \mathbb{B}) \circ \mathbb{B}$



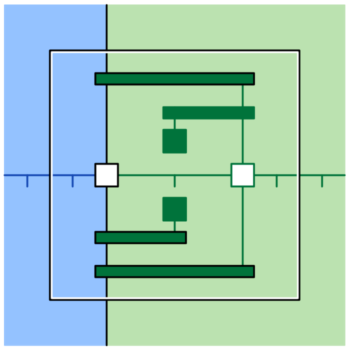
**$\mathbb{B}$-assoc coherence**

$\mathcal{R} \circ (\mathbb{B} \circ (\mathbb{B} \circ \mathbb{B})) \rightrightarrows ((\mathcal{R} \circ \mathbb{B}) \circ \mathbb{B}) \circ \mathbb{B}$



**$\mathbb{A}$-unit coherence**

$(\mathbb{A} \circ \mathrm{id}) \circ \mathcal{R} \rightrightarrows \mathbb{A} \circ (\mathrm{id} \circ \mathcal{R})$

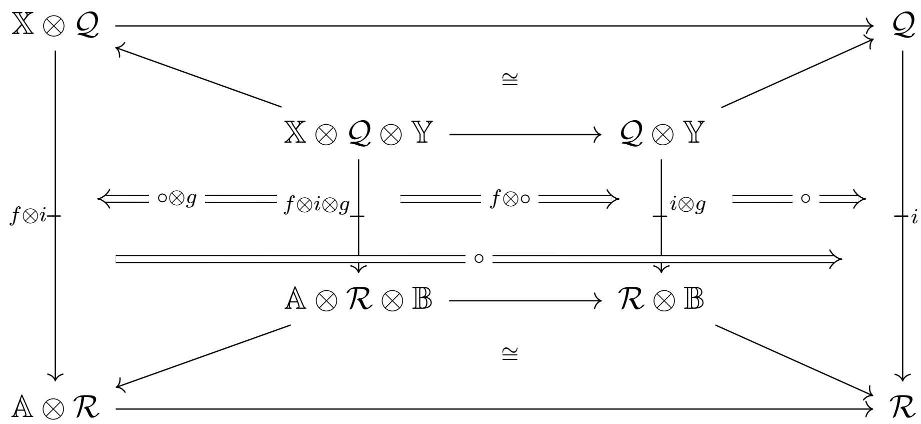


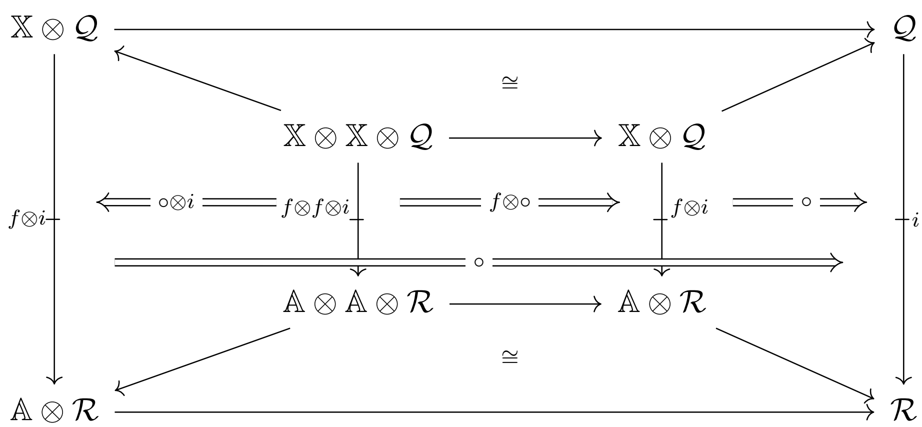
**$\mathbb{B}$-unit coherence**

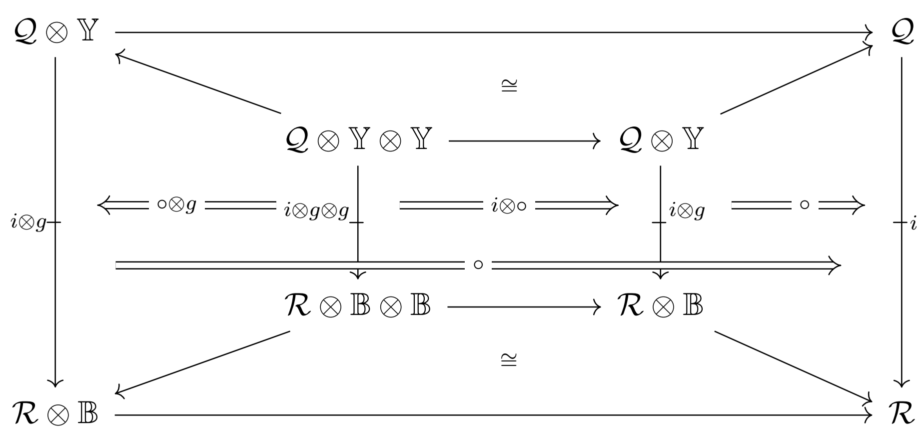
$\mathcal{R} \circ (\mathrm{id} \circ \mathbb{B}) \rightrightarrows (\mathcal{R} \circ \mathrm{id}) \circ \mathbb{B}$

**Definition 61.** Let $\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}$ be bifibrant double categories, let $\mathcal{Q} : \mathbb{X} \,\|\, \mathbb{Y}$ and $\mathcal{R} : \mathbb{A} \,\|\, \mathbb{B}$ be horizontal profunctors, and let $f : \mathbb{X} \,|\, \mathbb{A}$ and $g : \mathbb{Y} \,|\, \mathbb{B}$ be vertical profunctors.

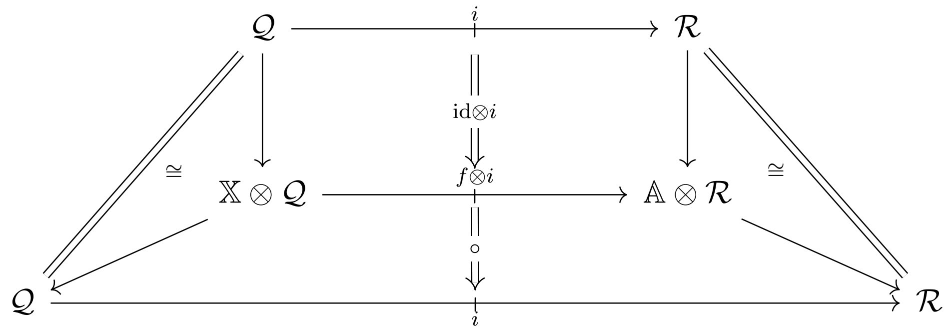
A **double profunctor** or **meta inference** $i(f,g) : \mathcal{Q}(\mathbb{X},\mathbb{Y}) \,|\, \mathcal{R}(\mathbb{A},\mathbb{B})$ is a matrix profunctor which forms a "vertical bimodule" of weak bimodules. So it is equipped with action matrix transformations $\circ : f \otimes i \Rightarrow i$ and $\circ : i \otimes g \Rightarrow i$ which cohere with the associators:

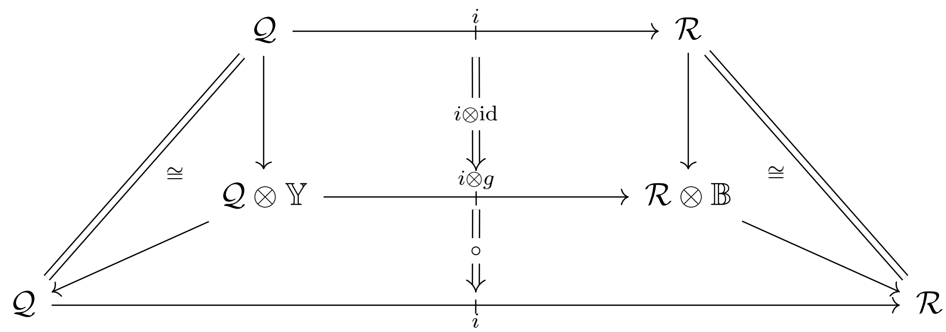


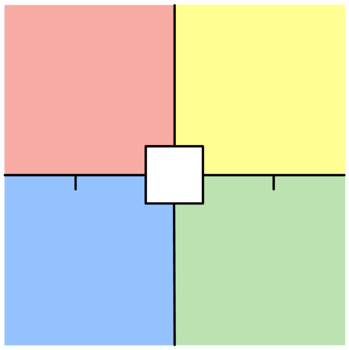



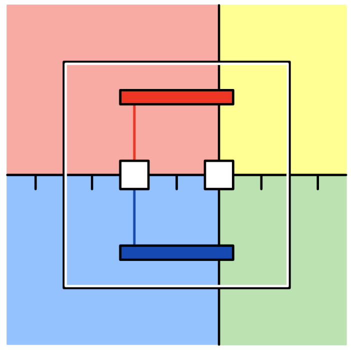
and cohere with the unitors of $\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}$.



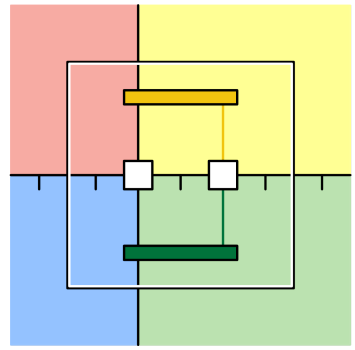





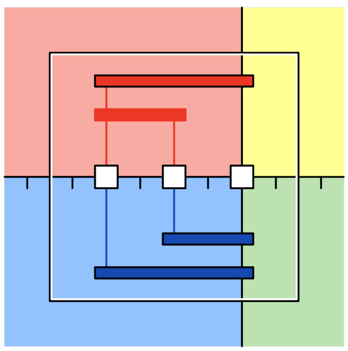
**double profunctor**
matrix profunctor
$i(f,g): \mathcal{Q}(\mathbb{X},\mathbb{Y}) \,|\, \mathcal{R}(\mathbb{A},\mathbb{B})$



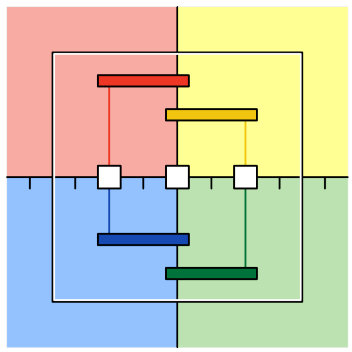
**left composition**
matrix transformation
$\circ : f \otimes i \to i$



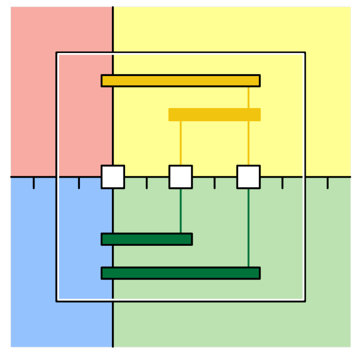
**right composition**
matrix transformation
$\circ : i \otimes g \to i$



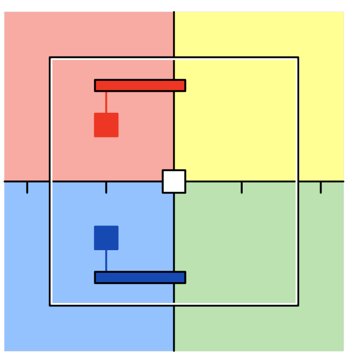
**l-assoc coherence**
$(\mathbb{X} \circ \mathbb{X}) \circ \mathcal{Q} \rightrightarrows \mathbb{A} \circ (\mathbb{A} \circ \mathcal{R})$



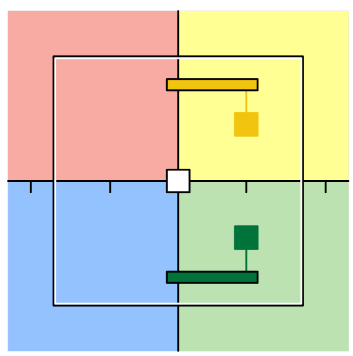
**c-assoc coherence**
$(\mathbb{X} \circ \mathcal{Q}) \circ \mathbb{Y} \rightrightarrows \mathbb{A} \circ (\mathcal{R} \circ \mathbb{B})$



**r-assoc coherence**
$(\mathcal{Q} \circ \mathbb{Y}) \circ \mathbb{Y} \rightrightarrows \mathcal{R} \circ (\mathbb{B} \circ \mathbb{B})$



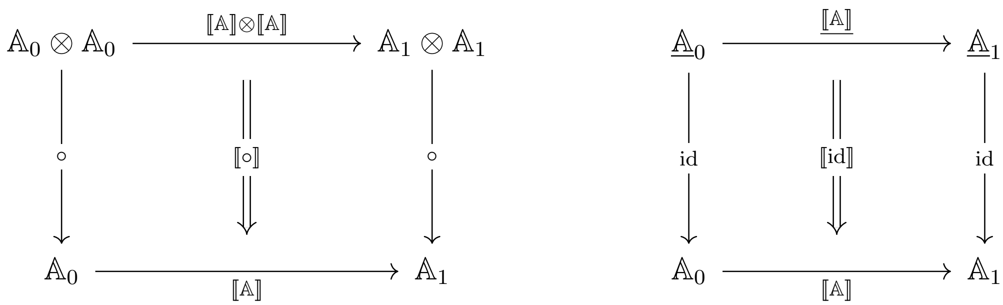
**l-unit coherence**
$\mathrm{id}.\mathbb{X} \circ \mathcal{Q} \rightrightarrows \mathrm{id}.\mathbb{A} \circ \mathcal{R}$



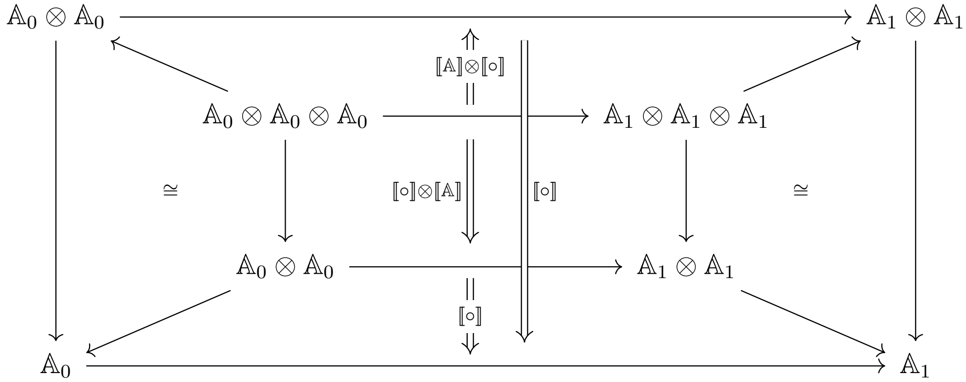
**r-unit coherence**
$\mathcal{Q} \circ \mathrm{id}.\mathbb{Y} \rightrightarrows \mathcal{R} \circ \mathrm{id}.\mathbb{B}$

## 3.3 Morphisms [Double transformation]

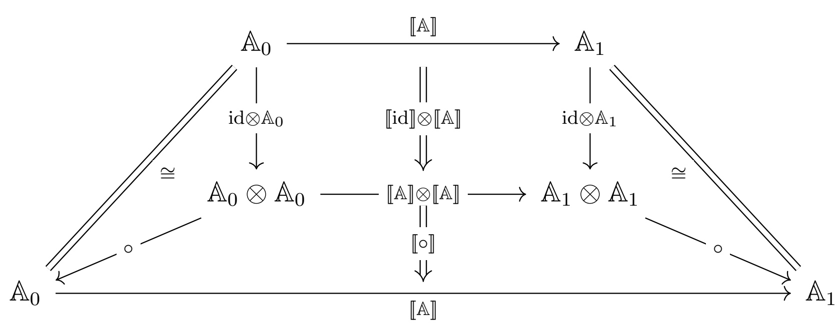
**Definition 62.** Let $\mathbb{A}_0, \mathbb{A}_1$ be bifibrant double categories. A **double functor**, i.e. **flow type**, is a morphism of pseudomonads. Hence it is a matrix functor $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ with invertible matrix transformations called the **join** and **unit**

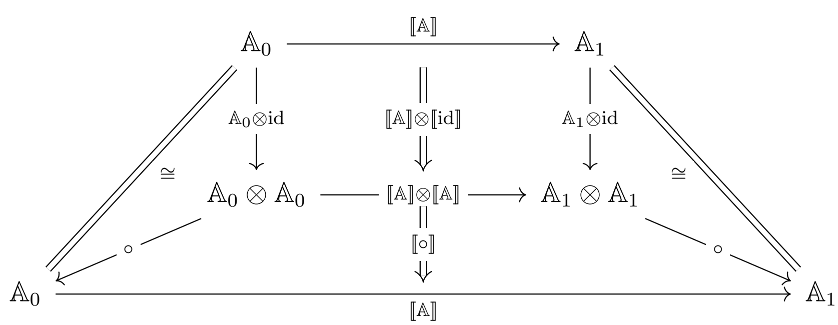


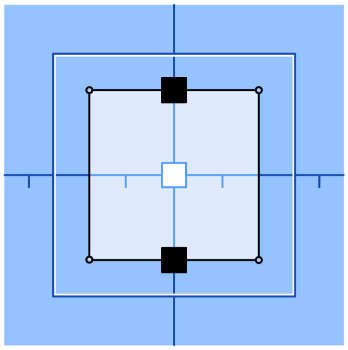

which cohere with the associators of $\mathbb{A}_0, \mathbb{A}_1$




and the unitors of $\mathbb{A}_0, \mathbb{A}_1$.

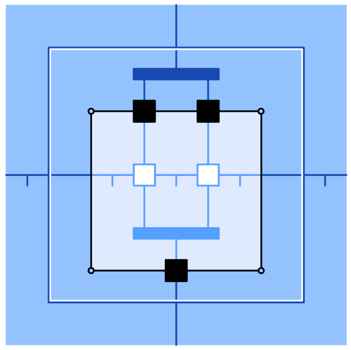




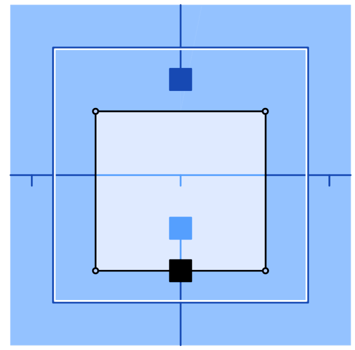
**double functor**

$[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$



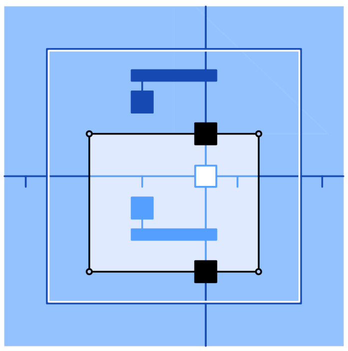
**join**

$[\![\circ]\!] : [\![\mathbb{A}]\!] \circ [\![\mathbb{A}]\!] \cong [\![\mathbb{A} \circ \mathbb{A}]\!]$



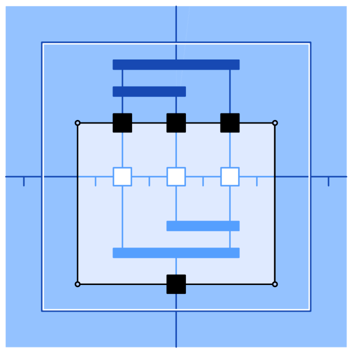
**unit**

$[\![\mathrm{id}]\!] : \mathrm{id}.[\![\mathbb{A}]\!] \cong [\![\mathrm{id}.\mathbb{A}]\!]$



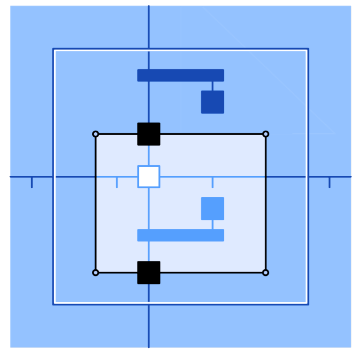
**left unit coherence**

$\mathrm{id} \circ [\![\mathbb{A}]\!] \rightrightarrows [\![\mathrm{id} \circ \mathbb{A}]\!]$



**associator coherence**

$([\![\mathbb{A}]\!] \circ [\![\mathbb{A}]\!]) \circ [\![\mathbb{A}]\!] \rightrightarrows [\![\mathbb{A} \circ (\mathbb{A} \circ \mathbb{A})]\!]$



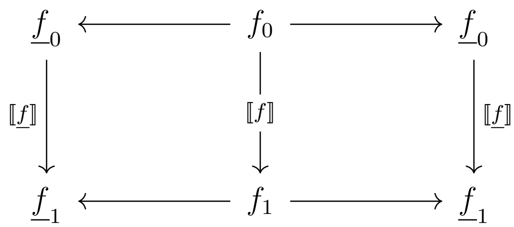
**right unit coherence**

$[\![\mathbb{A}]\!] \circ \mathrm{id} \rightrightarrows [\![\mathbb{A} \circ \mathrm{id}]\!]$

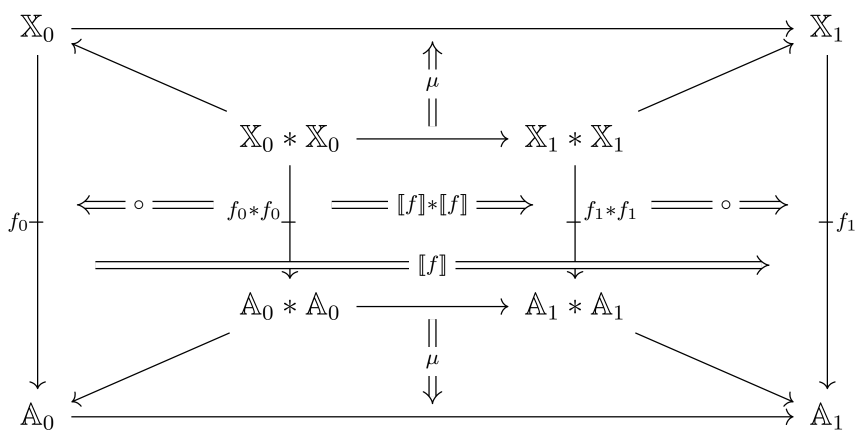
**Definition 63.** Let $\mathbb{X}_0, \mathbb{X}_1, \mathbb{A}_0, \mathbb{A}_1$ be bifibrant double categories, let $[\![\mathbb{X}]\!] : \mathbb{X}_0 \to \mathbb{X}_1$ and $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ be double functors, and let $f_0 : \mathbb{X}_0 \mid \mathbb{A}_0$ and $f_1 : \mathbb{X}_1 \mid \mathbb{A}_1$ be vertical profunctors.

A **vertical transformation**, i.e. **flow process**, $[\![f]\!]([\![\mathbb{X}]\!], [\![\mathbb{A}]\!]) : f_0(\mathbb{X}_0, \mathbb{A}_0) \Rightarrow f_1(\mathbb{X}_1, \mathbb{A}_1)$ is a transformation of vertical modules.

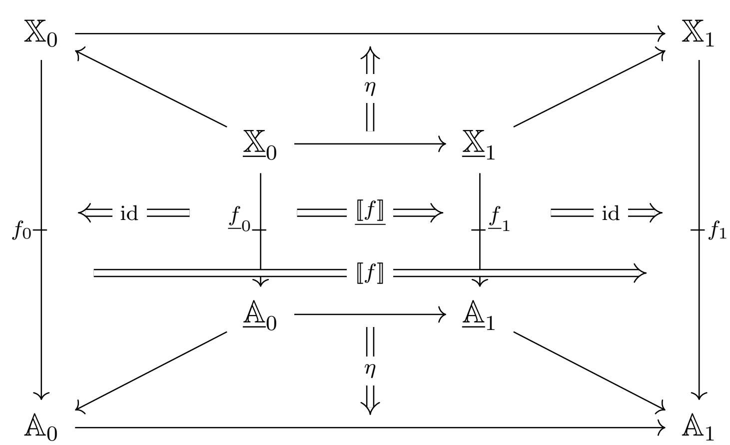
Hence it is a transformation $[\![\underline{f}]\!] : \underline{f}_0 \Rightarrow \underline{f}_1$ and a matrix transformation $[\![f]\!] : f_0 \Rightarrow f_1$

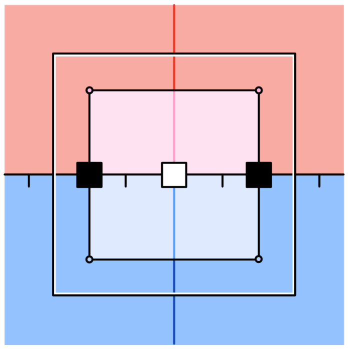



which coheres with the joins of $[\![\mathbb{X}]\!]$ and $[\![\mathbb{A}]\!]$.

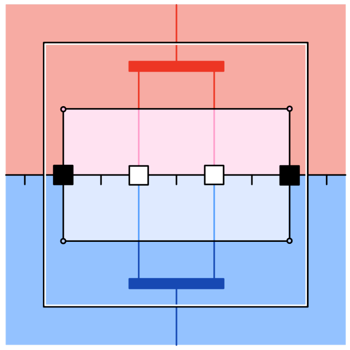



and the units of $[\![\mathbb{X}]\!]$ and $[\![\mathbb{A}]\!]$.

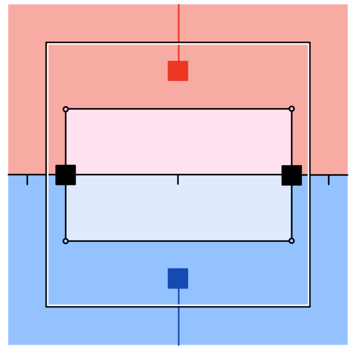




**vertical transformation**
$[\![f]\!] : f_0 \Rightarrow f_1$



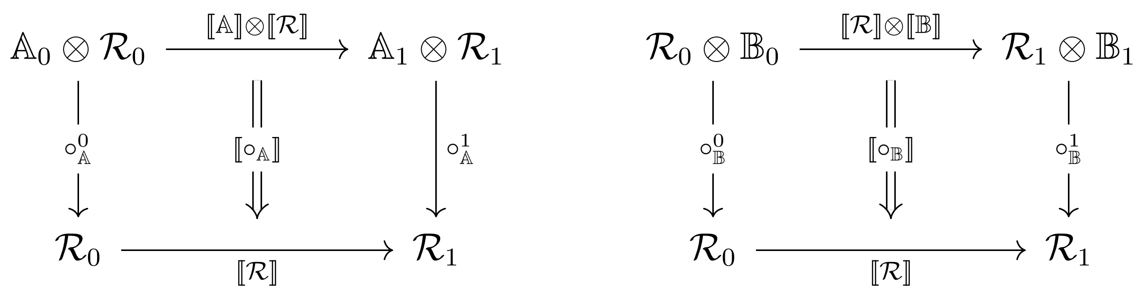
**join coherence**
$[\![\mathbb{X}]\!] \circ [\![\mathbb{X}]\!] \rightrightarrows [\![\mathbb{A} \circ \mathbb{A}]\!]$



**unit coherence**
$\mathrm{id}.[\![\mathbb{X}]\!] \rightrightarrows [\![\mathrm{id}.\mathbb{A}]\!]$

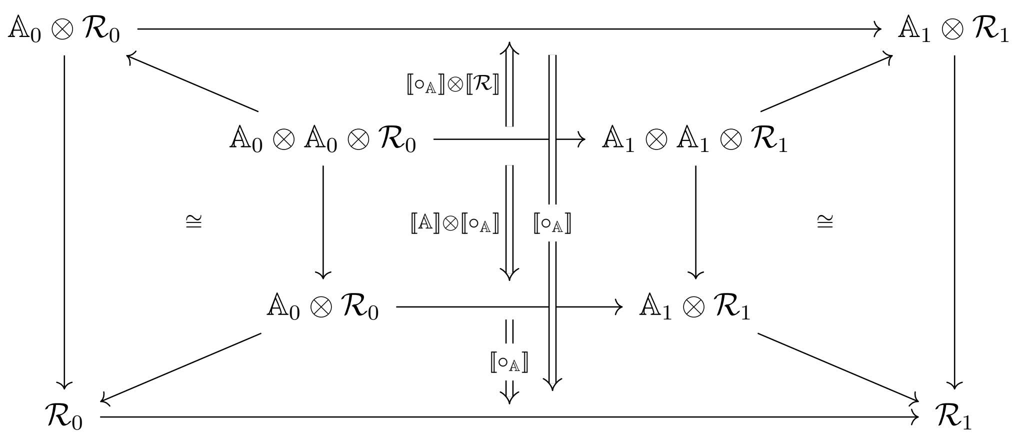
**Definition 64.** Let $\mathbb{A}_0, \mathbb{B}_0, \mathbb{A}_1, \mathbb{B}_1$ be bifibrant double categories, let $[\![\mathbb{A}]\!] : \mathbb{A}_0 \to \mathbb{A}_1$ and $[\![\mathbb{B}]\!] : \mathbb{B}_0 \to \mathbb{B}_1$ be double functors, and let $\mathcal{R}_0 : \mathbb{A}_0 \parallel \mathbb{B}_0$ and $\mathcal{R}_1 : \mathbb{A}_1 \parallel \mathbb{B}_1$ be h-profunctors.

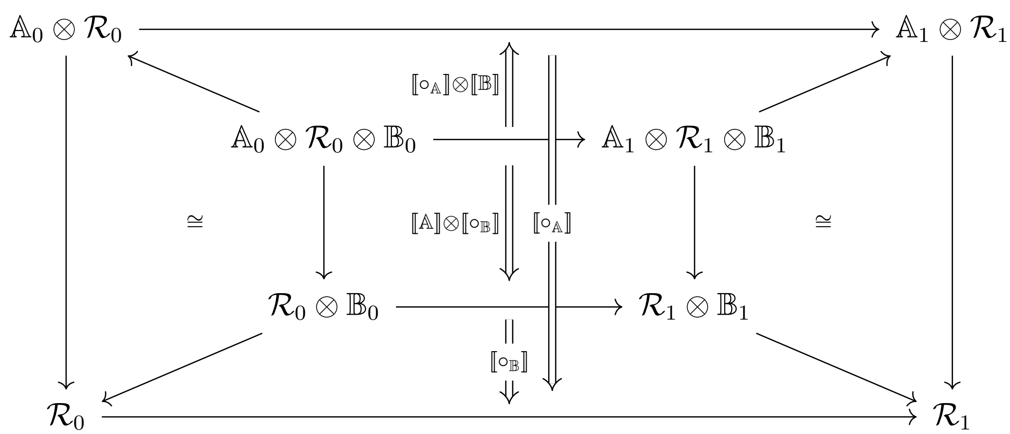
A **horizontal transformation**, i.e. **flow relation**, $[\![\mathcal{R}]\!]([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) : \mathcal{R}_0(\mathbb{A}_0, \mathbb{B}_0) \to \mathcal{R}_1(\mathbb{A}_1, \mathbb{B}_1)$ is a transformation of weak bimodules. Hence it is a matrix functor $[\![\mathcal{R}]\!] : \mathcal{R}_0 \to \mathcal{R}_1$ with invertible matrix transformations called **left and right join**

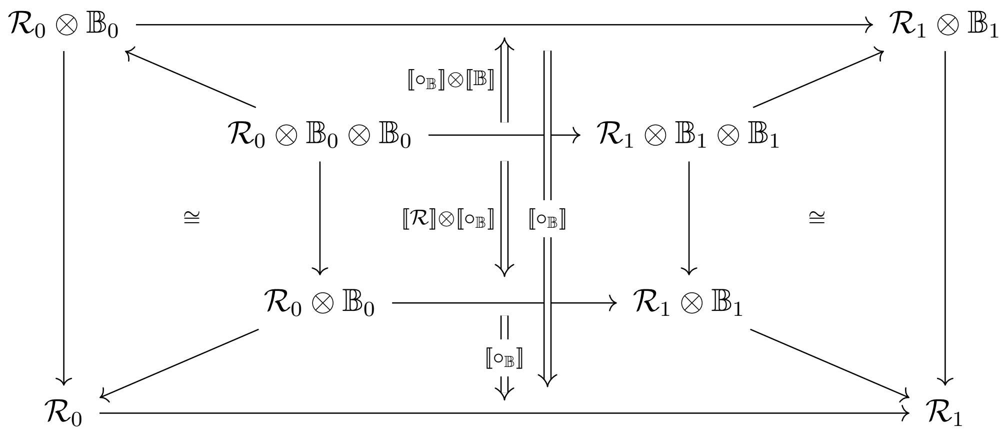



$[\![\circ_{\mathbb{A}}]\!] : [\![A_0]\!] \circ_1 [\![R_0]\!] \cong [\![A_0 \circ_0 R_0]\!]$ $\qquad$ $[\![\circ_{\mathbb{B}}]\!] : [\![R_0]\!] \circ_1 [\![B_0]\!] \cong [\![R_0 \circ_0 B_0]\!]$

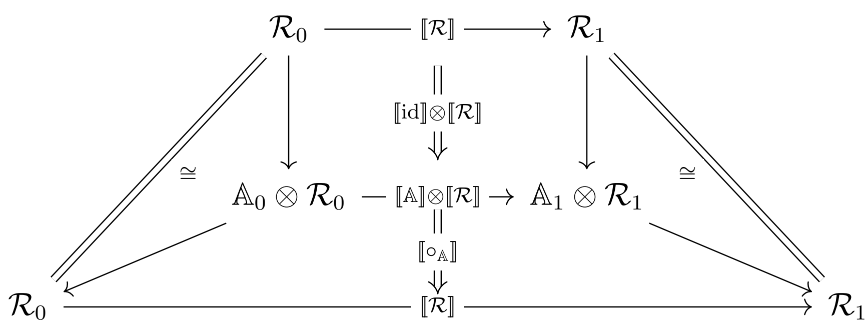
which coheres with the joins of $[\![\mathbb{A}]\!]$ and $[\![\mathbb{B}]\!]$, along the associators of $\mathcal{R}_0$ and $\mathcal{R}_1$

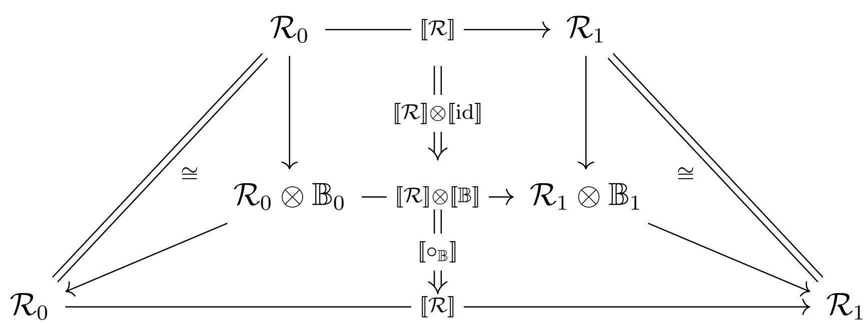


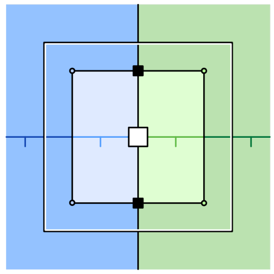





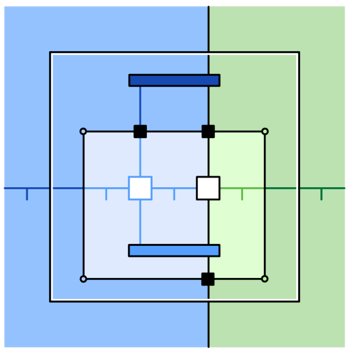

and the units of $[\![\mathbb{A}]\!]$ and $[\![\mathbb{B}]\!]$.



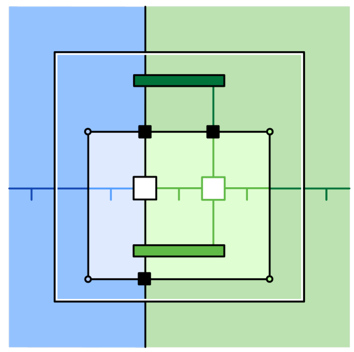





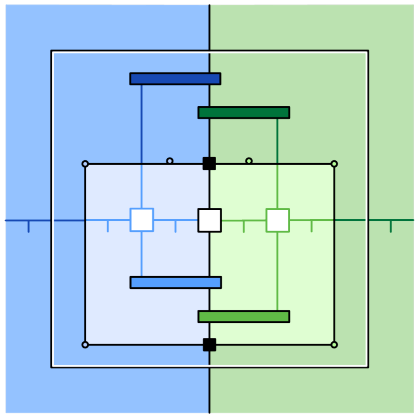
**horizontal transformation**
matrix functor
$[\![\mathcal{R}]\!]([\![\mathbb{A}]\!],[\![\mathbb{B}]\!]) : \mathcal{R}_0(\mathbb{A}_0,\mathbb{B}_0) \to \mathcal{R}_1(\mathbb{A}_1,\mathbb{B}_1)$



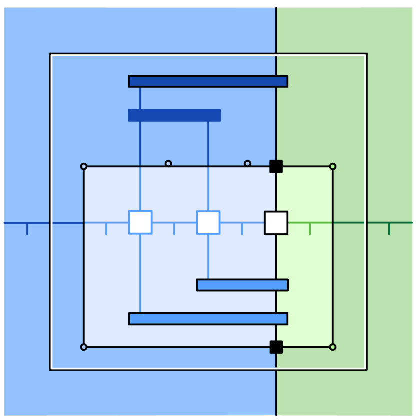
**left join**
matrix transformation
$[\![\circ_{\mathbb{A}}]\!] : [\![\mathbb{A}]\!] \circ [\![\mathcal{R}]\!] \cong [\![\mathbb{A} \circ \mathcal{R}]\!]$



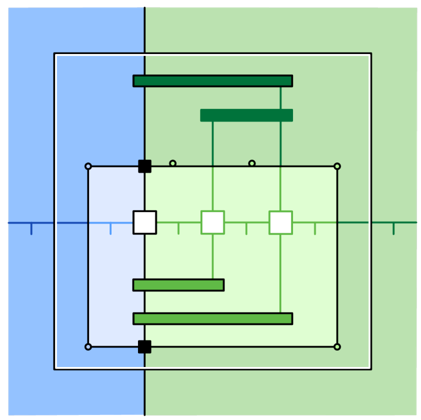
**right join**
matrix transformation
$[\![\circ_{\mathbb{B}}]\!] : [\![\mathcal{R}]\!] \circ [\![\mathbb{B}]\!] \cong [\![\mathcal{R} \circ \mathbb{B}]\!]$



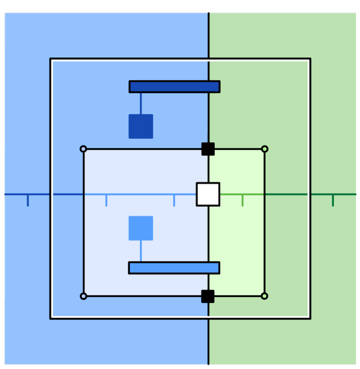
**center assoc. coherence**
equality
$[\![\mathbb{A}]\!] \circ ([\![\mathcal{R}]\!] \circ [\![\mathbb{B}]\!]) \rightrightarrows [\![(\mathbb{A} \circ \mathcal{R}) \circ \mathbb{B}]\!]$



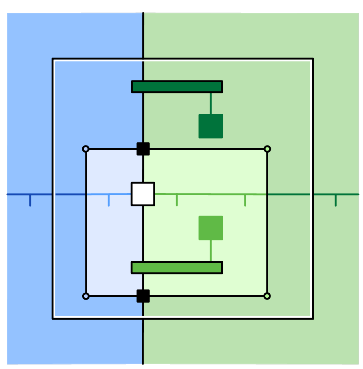
**left assoc. coherence**
equality
$([\![\mathbb{A}_1]\!] \circ [\![\mathbb{A}_2]\!]) \circ [\![\mathcal{R}]\!] \rightrightarrows [\![\mathbb{A}_1 \circ (\mathbb{A}_2 \circ \mathcal{R})]\!]$



**right assoc coherence**
equality
$[\![\mathcal{R}]\!] \circ ([\![\mathbb{B}_1]\!] \circ [\![\mathbb{B}_2]\!]) \rightrightarrows [\![(\mathcal{R} \circ \mathbb{B}_1) \circ \mathbb{B}_2]\!]$



**left unit coherence**
equality
$\mathrm{id}.[\![\mathbb{A}]\!] \circ [\![\mathcal{R}]\!] \rightrightarrows [\![\mathrm{id}.\mathbb{A} \circ \mathcal{R}]\!]$



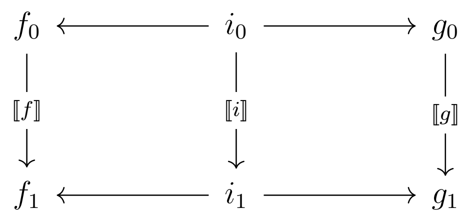
**right unit coherence**
equality
$[\![\mathcal{R}]\!] \circ \mathrm{id}.[\![\mathbb{B}]\!] \rightrightarrows [\![\mathcal{R} \circ \mathrm{id}.\mathbb{B}]\!]$

**Definition 65.** Let $i_0(f_0, g_0) : \mathcal{Q}_0(\mathbb{X}_0, \mathbb{Y}_0) \mid \mathcal{R}_0(\mathbb{A}_0, \mathbb{B}_0)$ and $i_1(f_1, g_1) : \mathcal{Q}_1(\mathbb{X}_1, \mathbb{Y}_1) \mid \mathcal{R}_1(\mathbb{A}_1, \mathbb{B}_1)$ be matrix profunctors. Let $[\![\mathbb{X}]\!] : \mathbb{X}_0 \to \mathbb{X}_1$ etc. be double functors, $[\![f]\!] : f_0 \Rightarrow f_1, [\![g]\!] : g_0 \Rightarrow g_1$ be vertical transformations, and $[\![\mathcal{Q}]\!]([\![\mathbb{X}]\!], [\![\mathbb{Y}]\!]) : \mathcal{Q}_0 \to \mathcal{Q}_1, [\![\mathcal{R}]\!]([\![\mathbb{A}]\!], [\![\mathbb{B}]\!]) : \mathcal{R}_0 \to \mathcal{R}_1$ be horizontal transformations.

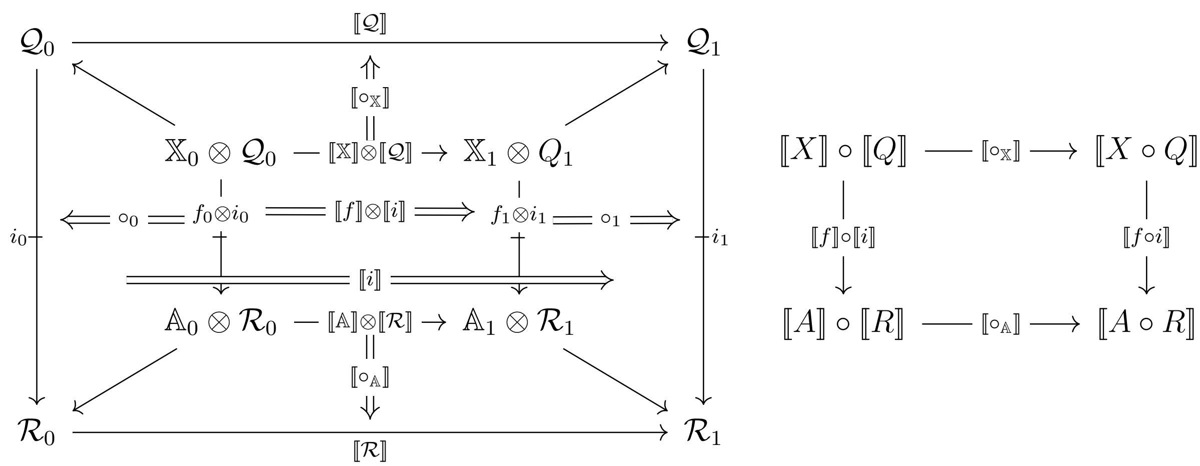
A **double transformation**, i.e. **flow inference** or simply **flow**

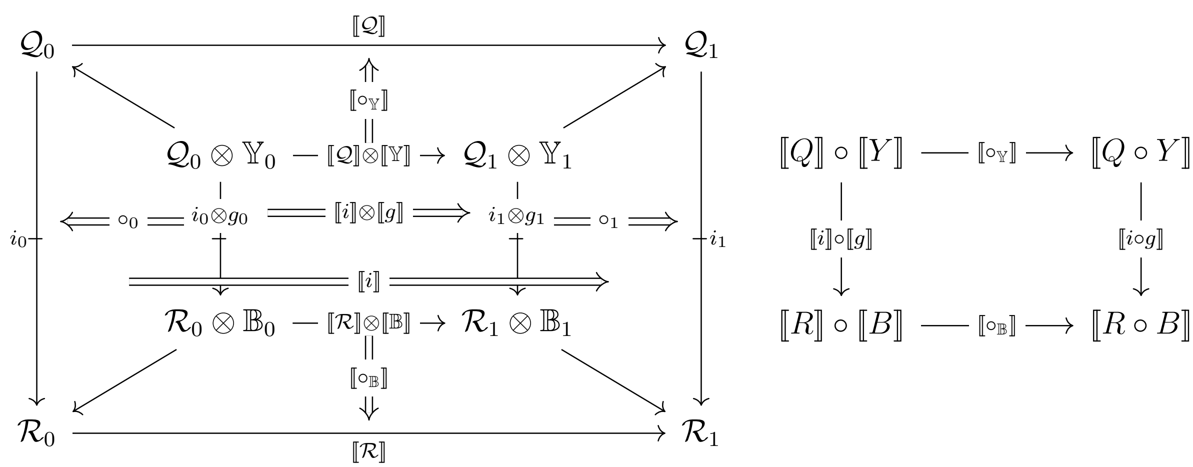
$$[\![i]\!]([\![f]\!], [\![g]\!]) : i_0(f_0, g_0) \Rightarrow i_1(f_1, g_1)$$

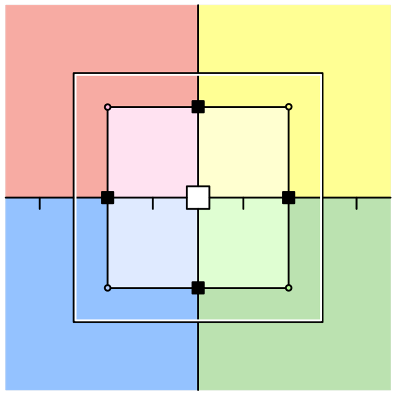
is a transformation of vertical bimodules. Hence it is a matrix transformation




which coheres with the left and right joins of the horizontal transformations.



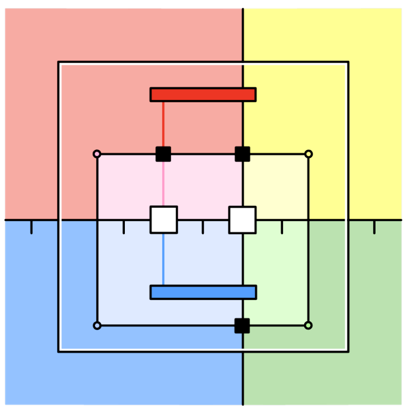





**double transformation**

matrix transformation

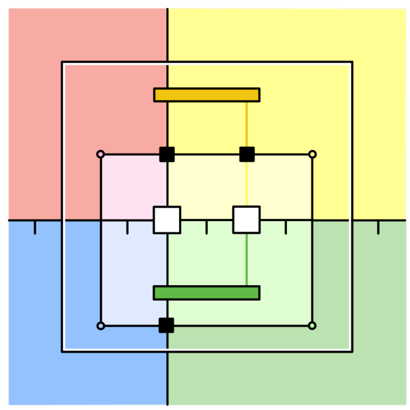
$[\![i]\!]([\![f]\!],[\![g]\!]) : i_0(f_0,g_0) \Rightarrow i_1(f_1,g_1)$



**left join coherence**

equality

$[\![\mathbb{X}]\!] \circ [\![\mathcal{Q}]\!] \rightrightarrows [\![\mathbb{A} \circ \mathcal{R}]\!]$



**right join coherence**

equality

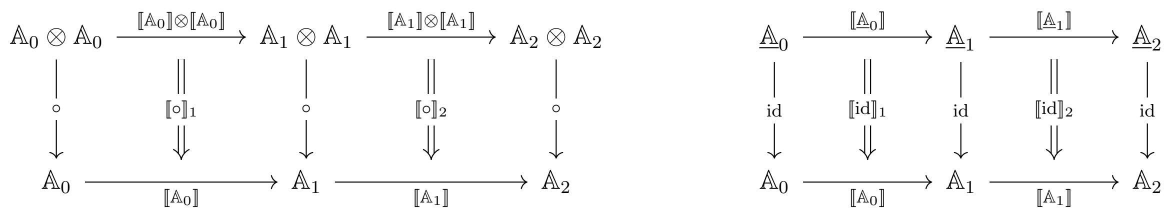
$[\![\mathcal{Q}]\!] \circ [\![\mathbb{Y}]\!] \rightrightarrows [\![\mathcal{R} \circ \mathbb{B}]\!]$

## 3.4 The metalogic of logics

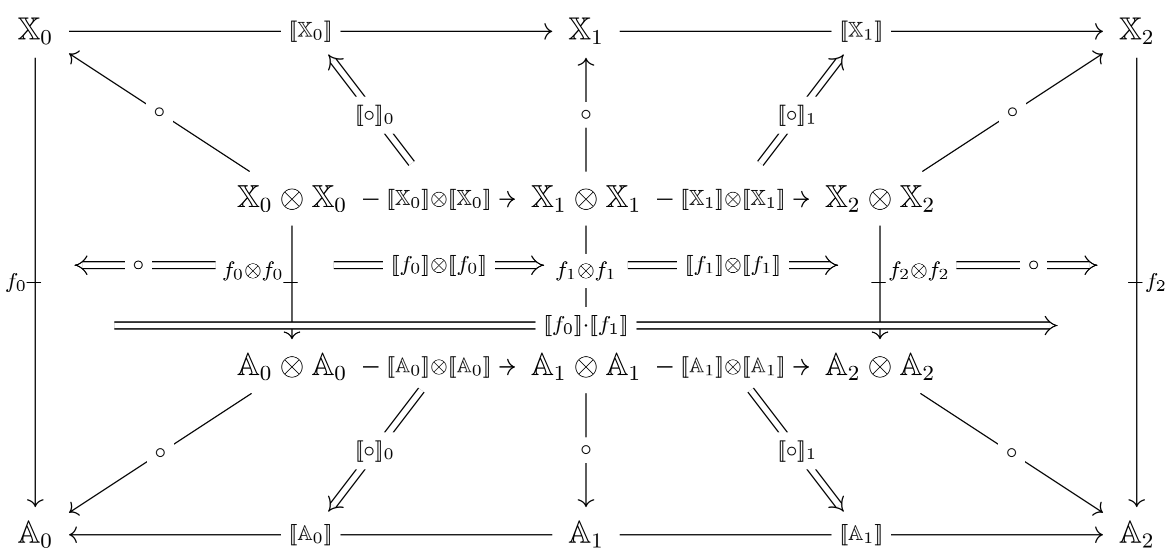
To show that logics form a metalogic, we simply compose structures and equations.

**Proposition 66.** Bifibrant double categories and double functors, vertical profunctors and vertical transformations form a double category, which we call $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$.

*Proof.* For double functors $[\![\mathbb{A}]\!]_1 : \mathbb{A}_0 \to \mathbb{A}_1$, $[\![\mathbb{A}]\!]_2 : \mathbb{A}_1 \to \mathbb{A}_2$, the composite $[\![[\![\mathbb{A}]\!]_1]\!]_2 : \mathbb{A}_0 \to \mathbb{A}_2$ is a double functor, with structure given by $[\![\circ]\!]_1 \circ [\![\circ]\!]_2$ and $[\![\mathrm{id}]\!]_1 \circ [\![\mathrm{id}]\!]_2$; these satisfy the coherence by composing equations. Composition is associative and unital.

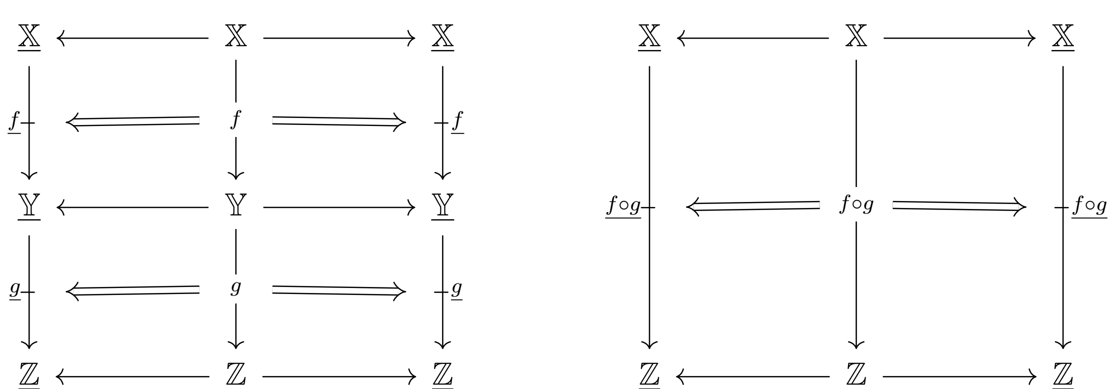


Composition of vertical transformations is given in the same way.

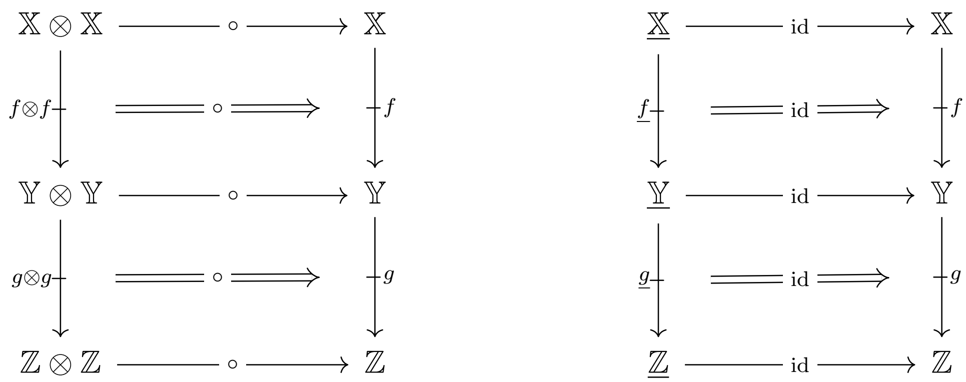


So it remains to define sequential composition of vertical profunctors, and verify that it is functorial, i.e. preserves composition of vertical transformations.

Consider the following sequential composite of vertical profunctors.

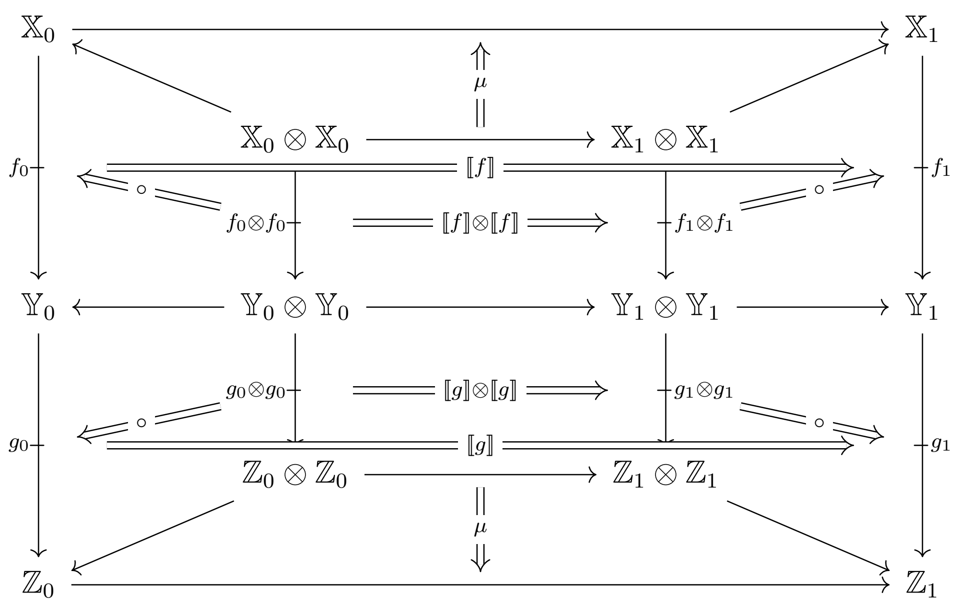


Just as for matrix profunctors, the equalities adjoined by the quotient are represented by squares in $\mathbb{Y}$. The sequential composite matrix profunctor $f \diamond g : \mathbb{X} | \mathbb{Z}$ is a vertical profunctor, with composition and unit given by sequentially composing that of $f$ and $g$.




Again, these satisfy the coherence simply by composing equations.

Sequential composition is functorial: let $[\![f]\!] : f_0 \Rightarrow f_1$ and $[\![g]\!] : g_0 \Rightarrow g_1$ be vertical transformations; then $([\![f]\!] \diamond [\![g]\!]) : (f_0 \diamond g_0) \Rightarrow (f_1 \diamond g_1)$ is defined by sequential composition.

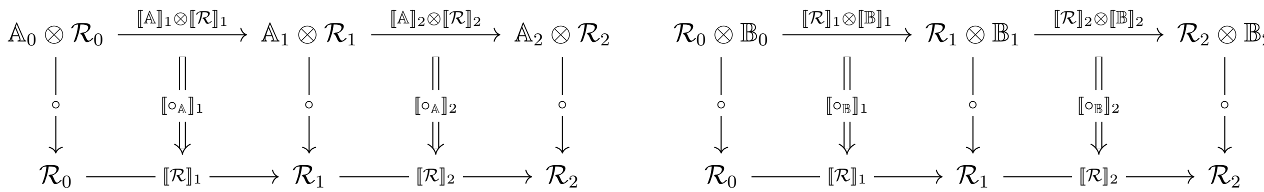



This preserves composition of transformations: picture two of the above such cubes, composed from left to right. So, sequential composition is functorial.

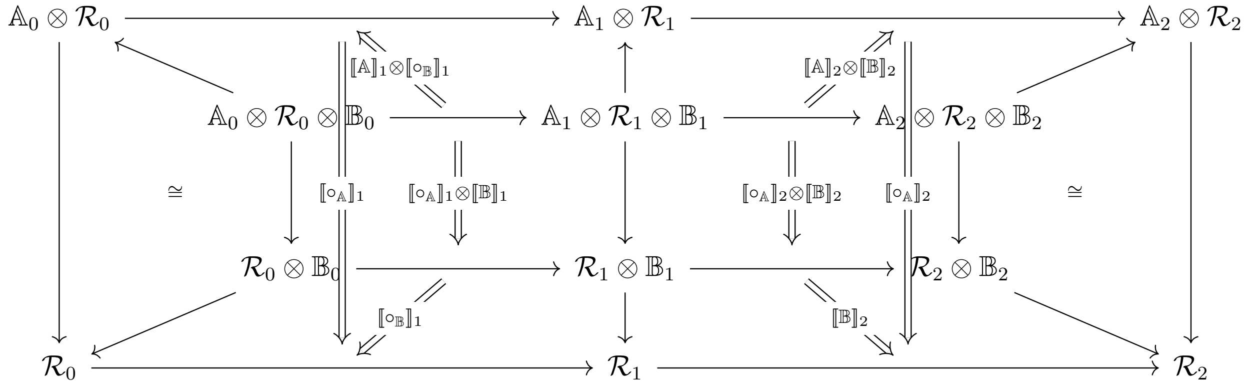
Hence bifibrant double categories and double functors, vertical profunctors and vertical transformations form a double category $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$. □

**Proposition 67.** Horizontal profunctors and horizontal transformations, double profunctors and double transformations form a double category, which we call $bf.\mathrm{DblProf}$.

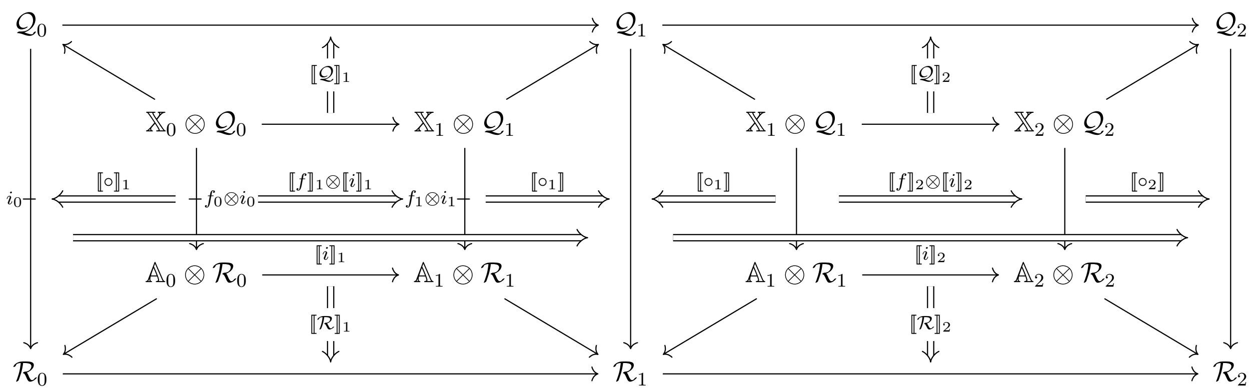
*Proof.* Composition of horizontal transformations $[\![\mathcal{R}]\!]_1 : \mathcal{R}_0 \to \mathcal{R}_1$ and $[\![\mathcal{R}]\!]_2 : \mathcal{R}_1 \to \mathcal{R}_2$ is defined by that of matrix functors, and that of the joins.



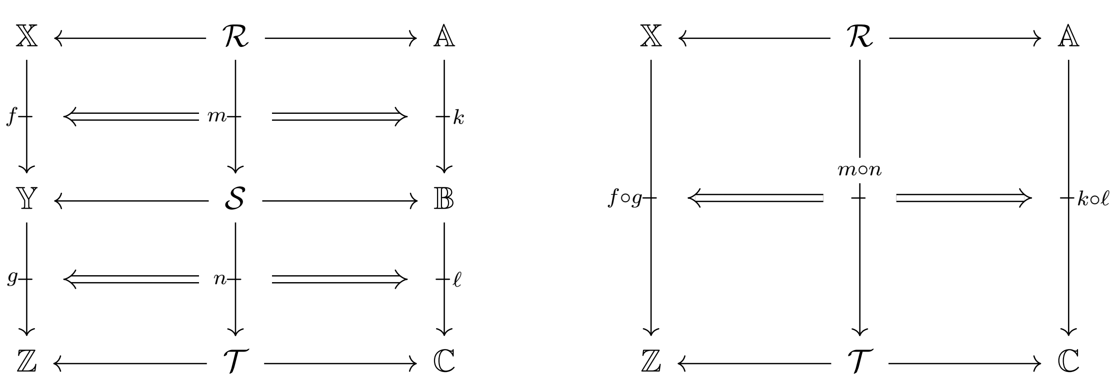
This coheres with the associators, again by composing equations.



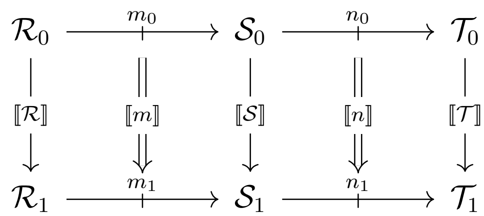
Composition of double transformations $[\![i]\!]_1 \cdot [\![i]\!]_2 : i_0 \Rightarrow i_2$ is defined by that of matrix transformations, and this coheres with the joins of $[\![[\![\mathcal{Q}]\!]_1]\!]_2$ and $[\![[\![\mathcal{R}]\!]_1]\!]_2$ by composing equations.



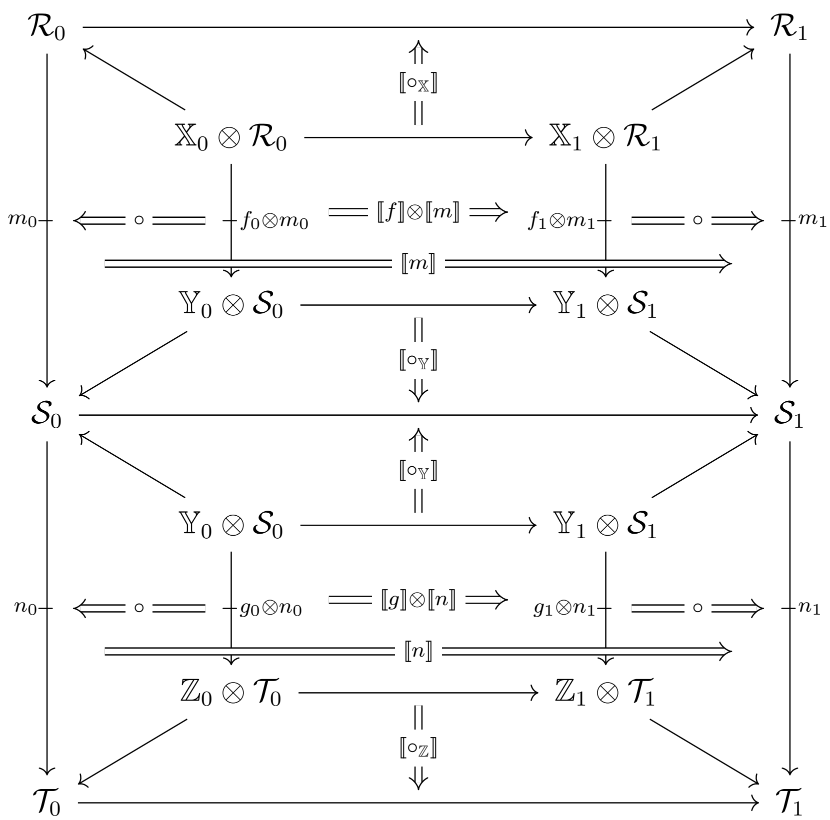
So it remains to define sequential composition of double profunctors, and verify that it is functorial. Because the reindexing quotient has already been imposed in $\mathrm{Mat}\mathbb{C}\mathrm{at}$ (44), composition of double profunctors is defined by that of matrix profunctors.



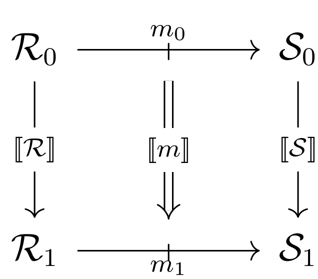
So the composite double transformation is given by the composite matrix transformation.




The coherence with the joins is given by composing equations.




Sequential composition of double transformations preserves transformation composition, because that of matrix transformations does. Thus, horizontal profunctors and transformations, double profunctors and transformations form a double category $bf.\mathrm{DblProf}$.




□

**Proposition 68.** $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at} \leftarrow bf.\mathrm{DblProf} \rightarrow bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$ is a fibered logic.

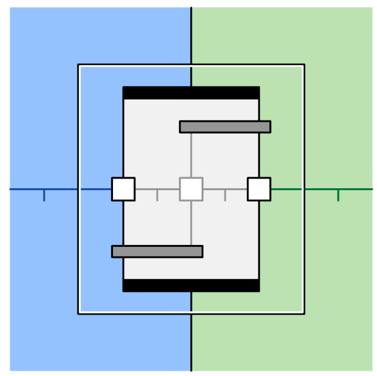
*Proof.* Substitution of double functors in a horizontal profunctor, and vertical transformations in a double profunctor, are defined by pullback. Sequential composition of vertical profunctors preserves this substitution, in the same way as for matrix profunctors. □

## Parallel composition

Composition of horizontal profunctors is defined in the same way as for matrix categories (2.5): by a codescent object, which adjoins a coherent associator for the middle action — in fact, all the proofs are essentially the same. The only difference is now $\mathbb{B}$ is a general bifibrant double category, rather than a weave double category $\langle\mathbb{B}\rangle$.

**Definition 69.** Let $\mathcal{R} : \mathbb{A} \| \mathbb{B}$ and $\mathcal{S} : \mathbb{B} \| \mathbb{C}$ be horizontal profunctors. We define the **parallel composite** $\mathcal{R} \otimes \mathcal{S} : \mathbb{A} \| \mathbb{C}$ as follows. To the composite matrix category $\mathcal{R} \otimes_{\mathbb{M}} \mathcal{S} : \mathbb{A} \| \mathbb{C}$ we adjoin for every $B : \mathbb{B}(\mathrm{B}_0, \mathrm{B}_1)$ an associator $\mathrm{B}_0.(R, B \circ S) \cong \mathrm{B}_1.(R \circ B, S)$.

$$\begin{array}{ccccc}
\mathcal{R} \otimes_{\mathbb{M}} (\mathbb{B} \otimes_{\mathbb{M}} \mathcal{S}) & \xrightarrow{\mathcal{R}\otimes_{\mathbb{M}}\circ_{\mathbb{B}}} & \mathcal{R} \otimes_{\mathbb{M}} \mathcal{S} & \overset{\iota}{\rightsquigarrow} & \\
\cong \downarrow & \Downarrow \alpha_{\mathcal{RS}} & & & (\mathcal{R} \otimes \mathcal{S})_\alpha \\
(\mathcal{R} \otimes_{\mathbb{M}} \mathbb{B}) \otimes_{\mathbb{M}} \mathcal{S} & \xrightarrow[\circ_{\mathbb{B}}\otimes_{\mathbb{M}}\mathcal{S}]{} & \mathcal{R} \otimes_{\mathbb{M}} \mathcal{S} & \overset{\iota}{\rightsquigarrow} &
\end{array}$$



This associator is natural by its universal construction, so for every square $b : \mathbb{B}(B_0, B_1)$ and $r : \mathcal{R}(R_0, R_1)$, $s : \mathcal{S}(S_0, S_1)$ the following commutes.

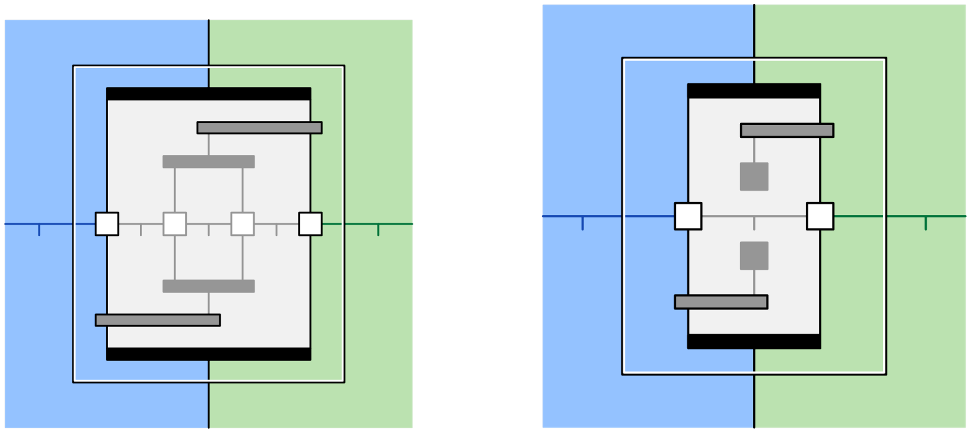
$$\begin{array}{ccc}
(R_0, B_0 \circ S_0) & \xrightarrow{\alpha_{\mathcal{RS}}} & (R_0 \circ B_0, S_0) \\
\downarrow (r, b \circ s) & & \downarrow (r \circ b, s) \\
(R_1, B_1 \circ S_1) & \xrightarrow{\alpha_{\mathcal{RS}}} & (R_1 \circ B_1, S_1)
\end{array}$$

Then we form the following coequifier, for reassociating a composite and a unit.

$$\mathcal{R} \otimes_{\mathbb{M}} \mathbb{B} \otimes_{\mathbb{M}} \mathbb{B} \otimes_{\mathbb{M}} \mathcal{S} \underset{\mathrm{B}_2.((R\circ B_1)\circ B_2, S)}{\overset{\mathrm{B}_0.(R, B_1\circ(B_2\circ S))}{\rightrightarrows}} (\mathcal{R} \otimes \mathcal{S})_\alpha \overset{co.equif}{\rightsquigarrow} (\mathcal{R} \otimes \mathcal{S})_\beta$$

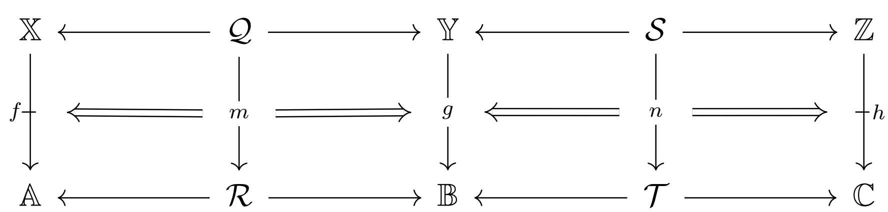
$$\mathcal{R} \otimes_{\mathbb{M}} \mathcal{S} \underset{\mathrm{B}.(R\circ U_{\mathrm{B}}, S)}{\overset{\mathrm{B}.(R, U_{\mathrm{B}}\circ S)}{\rightrightarrows}} (\mathcal{R} \otimes \mathcal{S})_\beta \overset{co.equif}{\rightsquigarrow} \mathcal{R} \otimes \mathcal{S}$$

This defines the parallel composite horizontal profunctor $\mathcal{R} \otimes \mathcal{S} : bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}(\mathbb{A}, \mathbb{C})$.

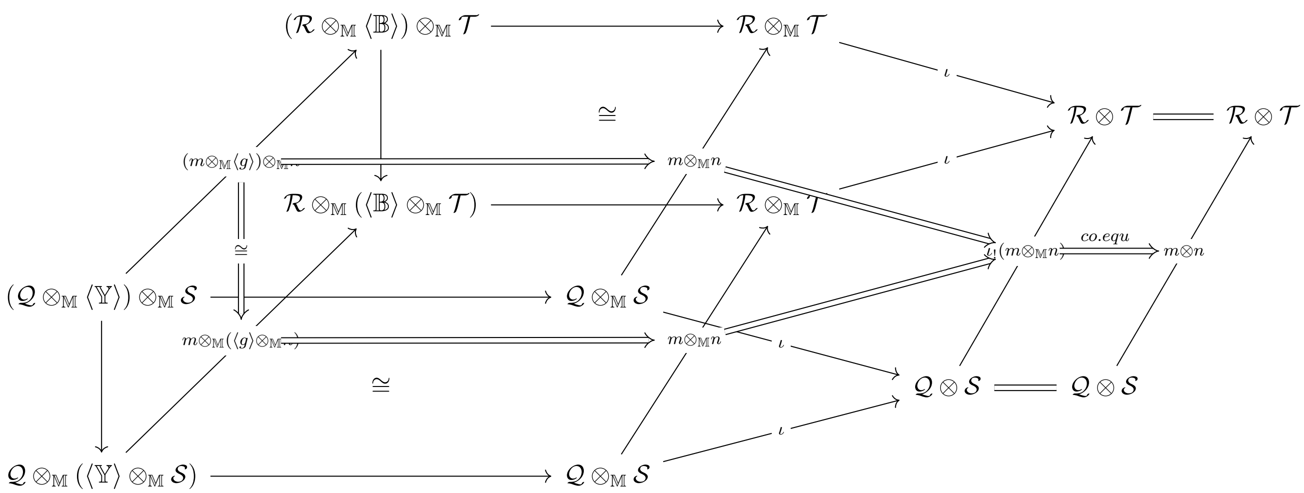


Next, we define parallel composition of double profunctors along vertical profunctors.

**Definition 70.** Let $m(f,g) : \mathcal{Q}(\mathbb{X}, \mathbb{Y}) \mid \mathcal{R}(\mathbb{A}, \mathbb{B})$ and $n(g,h) : \mathcal{S}(\mathbb{Y}, \mathbb{Z}) : \mathcal{T}(\mathbb{B}, \mathbb{C})$ be double profunctors, composable along the vertical profunctor $g : \mathbb{Y} \mid \mathbb{B}$.




The **parallel composite** $(m \otimes n)(f,h) : (\mathcal{Q} \otimes \mathcal{S})(\mathbb{X}, \mathbb{Z}) \mid (\mathcal{R} \otimes \mathcal{T})(\mathbb{A}, \mathbb{C})$ is defined as the following coequalizer.



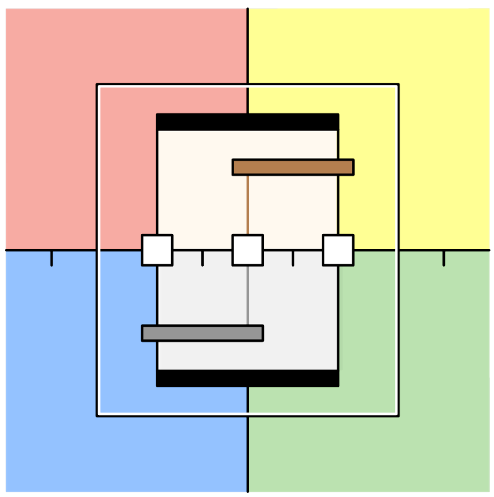

The profunctor $\iota_!(m \otimes_{\mathbb{M}} n)$ forms all composites of elements $\mathrm{g}.(m,n)$ and the morphisms of $\mathcal{Q} \otimes \mathcal{S}$ and $\mathcal{R} \otimes \mathcal{T}$. Then, the coequalizer imposes that the associators are natural with respect to the elements. So the elements of the composite $(m \otimes n)(f,h) : (\mathcal{Q} \otimes \mathcal{S})(\mathbb{X}, \mathbb{Z}) \mid (\mathcal{R} \otimes \mathcal{T})(\mathbb{A}, \mathbb{C})$ are composites of:

$$\begin{array}{rrl} \text{morphisms} & \mathrm{y}.(q,s): & (\mathcal{Q}\otimes\mathcal{S})(\mathrm{Y}_0.(Q_0,S_0),\mathrm{Y}_1.(Q_1,S_1)) \\ \text{associators} & \alpha_{\mathcal{QS}}: & (\mathcal{Q}\otimes\mathcal{S})(\mathrm{Y}_0.(Q,Y\circ S),\mathrm{Y}_1.(Q\circ Y,S)) \\ \text{elements} & \mathrm{g}.(m,n): & (m\circ_{\mathbb{M}} n)(\mathrm{Y}.(Q,S),\mathrm{B}.(R,T)) \\ \text{associators} & \alpha_{\mathcal{RT}}: & (\mathcal{R}\otimes\mathcal{T})(\mathrm{B}_0.(R,B\circ T),\mathrm{B}_1.(R\circ B,T)) \\ \text{morphisms} & \mathrm{b}.(r,t): & (\mathcal{R}\otimes\mathcal{T})(\mathrm{B}_0.(R_0,T_0),\mathrm{B}_1.(R_1,T_1)) \end{array}$$

such that for any $g:g(\mathrm{g}_0,\mathrm{g}_1)(Y,B)$ and $m:m(\mathrm{f},\mathrm{g}_0)$, $n:n(\mathrm{g}_1,\mathrm{h})$ the following commutes.

$$\begin{array}{ccc} \mathrm{Y}_0.(Q,Y\circ S) & \xrightarrow{\alpha_{\mathcal{QS}}} & \mathrm{Y}_1.(Q\circ Y,S) \\ \downarrow{\scriptstyle \mathrm{g}_0.(m,g\circ n)} & & \downarrow{\scriptstyle \mathrm{g}_1.(m\circ g,n)} \\ \mathrm{B}_0.(R,B\circ T) & \xrightarrow[\alpha_{\mathcal{RT}}]{} & \mathrm{B}_1.(R\circ B,T) \end{array}$$

The parallel composite matrix profunctor can be drawn as follows.



Parallel composition of horizontal profunctors and double profunctors is functorial in the same way as matrix categories and matrix profunctors, by functoriality of colimit.

Yet just as for matrix profunctors, parallel composition does not preserve sequential composition of horizontal profunctors. So following definition 55, bifibrant double categories form a metalogic.

**Theorem 71.** Bifibrant double categories form a metalogic.

Morphisms are double functors, vertical profunctors, and horizontal profunctors; squares are vertical transformations, horizontal transformations, and double profunctors; and cubes are double transformations.

$$bf.\mathrm{Dbl}\mathbb{C}\mathrm{at} \leftarrow bf.\mathrm{DblProf} \rightarrow bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$$

*Proof.* Let $\mathbb{DC}$ be the category of bifibrant double categories and double functors, and let $\mathbb{VP}$ be the category of vertical profunctors and vertical transformations; so $\mathbb{DC} \leftarrow \mathbb{VP} \rightarrow \mathbb{DC}$ is $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$.

Let $\mathbb{HP}$ be the category of horizontal profunctors and horizontal transformations, and let $\mathbb{DP}$ be the category of double profunctors and double transformations; so $\mathbb{HP} \leftarrow \mathbb{DP} \rightarrow \mathbb{HP}$ is $bf.\mathrm{DblProf}$.

As we showed, these form a fibered span of logics

$$\begin{array}{ccccc} \mathbb{DC} & \longleftarrow & \mathbb{HP} & \longrightarrow & \mathbb{DC} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{VP} & \longleftarrow & \mathbb{DP} & \longrightarrow & \mathbb{VP} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{DC} & \longleftarrow & \mathbb{HP} & \longrightarrow & \mathbb{DC} \end{array}$$

equipped with span functors for parallel composition and unit:

$$\begin{array}{ccc} \mathbb{HP} *_h \mathbb{HP} & \xrightarrow{\ \otimes\ } & \mathbb{HP} \\ \uparrow & & \uparrow \\ \mathbb{DP} *_h \mathbb{DP} & \xrightarrow{\ \otimes\ } & \mathbb{DP} \\ \downarrow & & \downarrow \\ \mathbb{HP} *_h \mathbb{HP} & \xrightarrow{\ \otimes\ } & \mathbb{HP} \end{array} \qquad \begin{array}{ccc} \mathbb{DC} & \xrightarrow{\mathbb{X}.\mathbb{X}(-,-)} & \mathbb{HP} \\ \uparrow & & \uparrow \\ \mathbb{VP} & \xrightarrow{f.f(-,-)} & \mathbb{DP} \\ \downarrow & & \downarrow \\ \mathbb{DC} & \xrightarrow[\mathbb{A}.\mathbb{A}(-,-)]{} & \mathbb{HP} \end{array}$$

with span transformations for left and right unitors, forming adjoint equivalences: for every horizontal profunctor $\mathcal{R} : \mathbb{A} \parallel \mathbb{B}$, its unitors and associators give the following horizontal transformations.

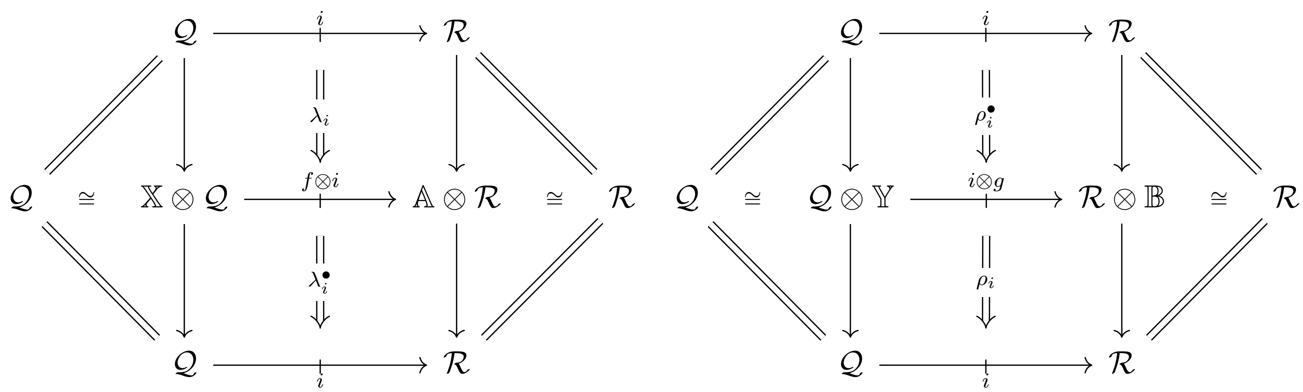
$$\begin{array}{ccccc} \mathcal{R} & & = & & \mathcal{R} \\ & \searrow^{\lambda^\circ} & \Downarrow \eta_\lambda & \nearrow^{\lambda^\bullet} & \\ & & \mathbb{A} \otimes \mathcal{R} & & \end{array} \qquad \begin{array}{ccccc} \mathcal{R} & & = & & \mathcal{R} \\ & \searrow^{\rho^\circ} & \Downarrow \eta_\rho & \nearrow^{\rho^\bullet} & \\ & & \mathcal{R} \otimes \mathbb{B} & & \end{array}$$

$$\eta_\lambda = \upsilon_{\mathbb{A}} : R \cong U_{\mathrm{A}} \circ R \qquad \eta_\rho = \upsilon_{\mathbb{B}} : R \cong R \circ U_{\mathrm{B}}$$

$$\begin{array}{ccccc} & & \mathcal{R} & & \\ & \nearrow^{\lambda^\bullet} & \Downarrow \varepsilon_\lambda & \searrow^{\lambda^\circ} & \\ \mathbb{A} \otimes \mathcal{R} & & = & & \mathbb{A} \otimes \mathcal{R} \end{array} \qquad \begin{array}{ccccc} & & \mathcal{R} & & \\ & \nearrow^{\rho^\bullet} & \Downarrow \varepsilon_\rho & \searrow^{\rho^\circ} & \\ \mathcal{R} \otimes \mathbb{B} & & = & & \mathcal{R} \otimes \mathbb{B} \end{array}$$

$$\varepsilon_\lambda = \alpha_{\mathbb{A}} : (U_{\mathrm{A}_0}, A \circ R) \cong (A, R) \qquad \varepsilon_\rho = \alpha_{\mathbb{B}} : (R \circ B, U_{\mathrm{B}_1}) \cong (R, B)$$

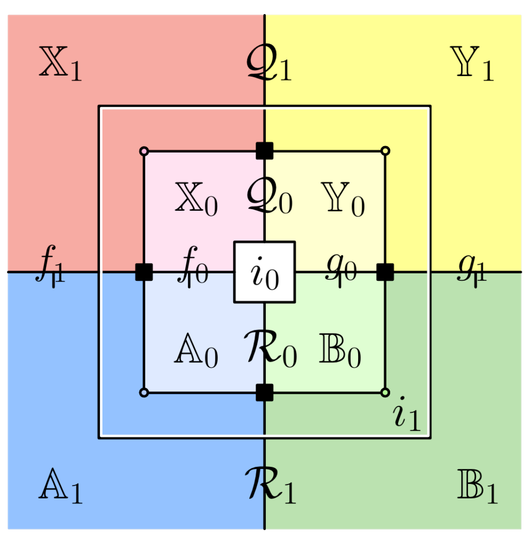
Just as in MatCat, the naturality of unitors with respect to elements of double profunctors gives that the above transformations cohere with the unitor transformations for double profunctors, as in a modification.



The associator is an isomorphism $\mathcal{R}\otimes(\mathcal{S}\otimes\mathcal{T}) \cong (\mathcal{R}\otimes\mathcal{S})\otimes\mathcal{T}$, with equality pentagonator.

The triangulator is given by the unitors, and its coherence follows from the naturality of the unitors with respect to the associator.

Hence $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$ is a metalogic, whose cubes are drawn as follows.



□

By the same reasoning as Theorem 57, $bf.\mathrm{Dbl}\mathbb{C}\mathrm{at}$ is closed, with the same extension and lift formulae, defining double co/limits with *fully general* weights. Details in next draft.

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