Continuous Geometry

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Preface

Traditionally, projective geometry was based on fundamental concepts like points and lines, and it was thought that this was unavoidable. However, around 1935, it was shown by G. Birkhoff and K. Menger[1] that, from a lattice theoretic standpoint, projective geometries are irreducible finite dimensional complemented modular lattices. Here, just as points are 'included' in lines, 'order' becomes the fundamental concept and, due to the finite dimensionality restriction, objects like points and lines naturally arise. Thus if we eliminate the finite dimensionality condition, we would expect to get a new geometry that, in a lattice theoretic sense, has the same structure as a projective geometry but without points and lines. But constructing such a geometry is no easy matter. Still, J. von Neumann[2] accomplished this difficult task in 1936-1937. If we change the way we express dimension slightly, a linear subspace like a point or line in an $n-1$ dimensional projective geometry can take dimension values 0, $\frac{1}{n}$, $\frac{2}{n}$,..., $\frac{n-1}{n}$ and 1. Specifically, the dimension of the 'empty' subspace is 0, and the dimension of points and lines is $\frac{1}{n}$ and $\frac{2}{n}$ respectively. And the whole space has dimension 1. Von Neumann showed that sometimes elements of an irreducible continuous complemented modular lattice can take all real numbers from 0 to 1 as dimension values. In this case, there are elements having dimension arbitrarily close to 0 and so there is no such thing as a 'point'. In other words, this is a continuous geometry (in the strict sense). On the other hand, it is known that in a projective geometry of dimension at least 3, coordinates can be introduced by a certain skew field which makes the linear subspace lattice isomorphic to the lattice of right ideals of the matrix ring over this skew field. Von Neumann generalized this, proving that a complemented modular lattice of order at least 4 is isomorphic the the principal right ideals of a certain matrix ring. This is the essence of von Neumann's continuous geometry (in the broad sense). In other words, continuous geometry can be roughly divided into two parts, dimension theory and representation theory.

Von Neumann completely worked out the dimension theory in the irreducible case. However, in the reducible case, even though he developed some fundamental theorems, he was unable to attain the final goal of expressing dimension in some form. In 1943 Tsurane Iwamura[3] showed that dimension can be expressed by continuous functions on the Boolean space[†] representing the centre of a continuous complemented modular lattice. In doing so, he showed that reducible continuous complemented modular lattices are subdirect products of irreducible ones. Following this, Yukiyoshi Kawada, Kaneo Higuti and Yatarô Matusima[4] developed Iwa-

[1] [Bir35] and [Men36]. [ ] symbols indicate the references at the end of the book.

[2] [vN36a]—[vN37b] detail this work, which is also contained in his Princeton lecture notes [vN98].

[3] [Iwa43] and [Iwa44b].

[4] [KHM44].

[†] In modern terminology, these are called *Stone spaces* or *extremally disconnected* topological spaces.

mura's results even further. With this we are inclined to think that the dimension theory of continuous geometries has been more or less completed.

In the present book, we aim to describe this theory so that it can be easily understood with almost no background knowledge. In particular, we have described the general theory relating exclusively to lattices in detail in Chapter 1. Furthermore, to show the relationship of continuous geometry to projective geometry and quantum logic, we have added Chapter 3 and Chapter 12 respectively.

I express my heartfelt gratitude to both Tôzirô Ogasawara and Usa Sasaki for reading over the manuscript for this book and providing many valuable comments. I also express my deep gratitude to both Kenzirô Syôda and Syôkiti Iyanaga for their assistance in publishing this book.

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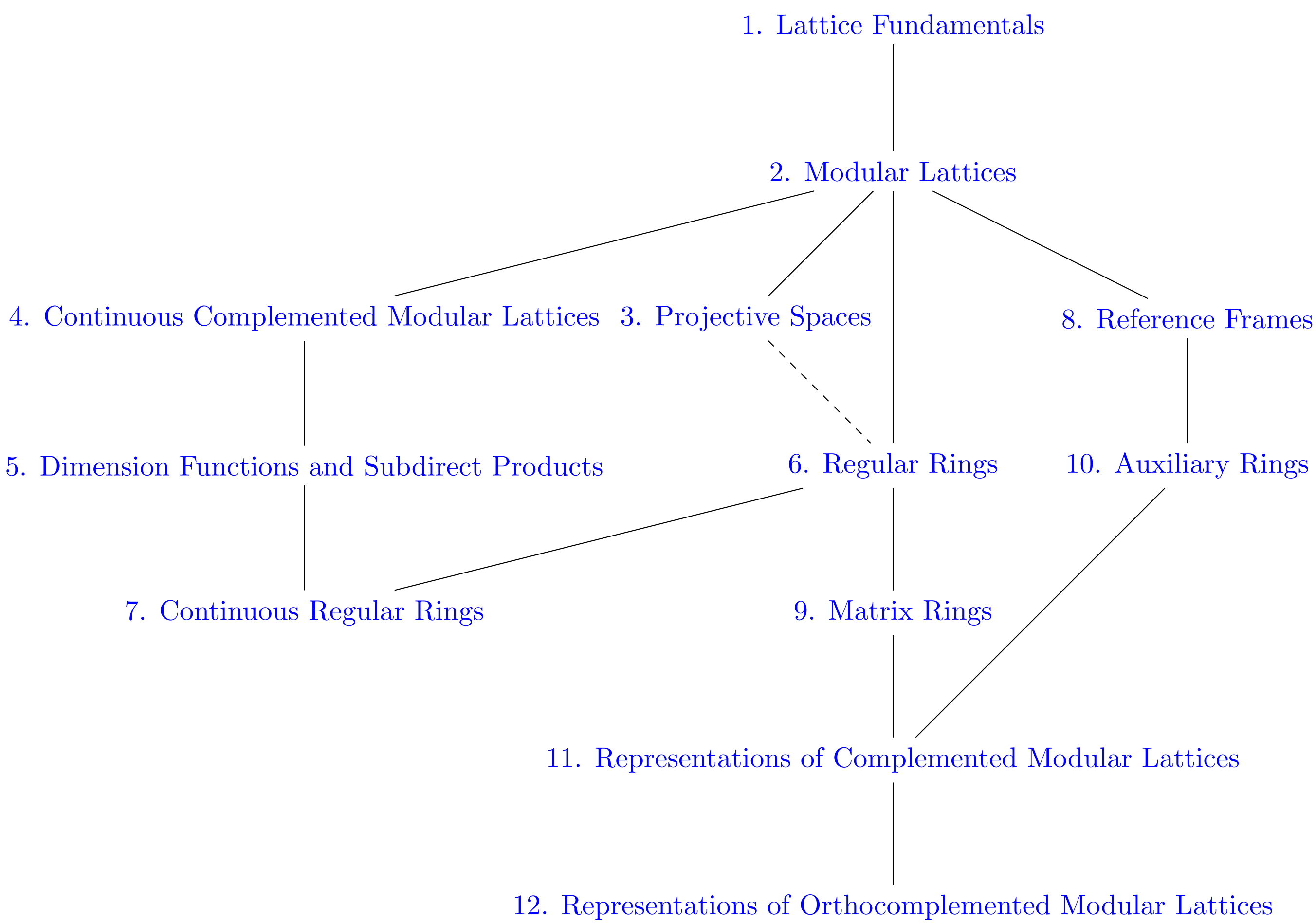
Figure 1: The relationship between each of the chapters is as follows.



Chapter 1

Lattice Fundamentals

1 Lattice Definitions

Definition 1.1. We denote the elements of a set L by $a, b, c, \ldots$, etc.. Given a binary relation $\leqq$ defined on L satisfying (1°), (2°), and (3°), we call L an *ordered set*.

(1°) for every $a \in L$, $a \leqq a$,

(2°) if $a \leqq b$ and $b \leqq a$ then $a = b$, and

(3°) if $a \leqq b$ and $b \leqq c$ then $a \leqq c$.

We also write $b \geqq a$ for $a \leqq b$ and say b *contains* a. When $a \leqq b$ and $a \neq b$, we write $a < b$.

When S is a subset of an ordered set L and there exists an element x of S such that $a \leqq x$, for all $a \in S$, we call x the *greatest element* of S. Likewise, when there exists an element y of S such that $a \geqq y$, for all $a \in S$, we call y the *least element* of S. When L has a greatest element we call this the *unit element* of L and denote it by 1. And when L has a least element we call this the *zero element* and denote it by 0.

In a lattice L having a zero element, if a is an element such that $a > x$ implies $x = 0$ then we call a an *atomic element* of L.

When S is a subset of L, we call an element a of S such that there is no $x \in S$ with $a < x$ a *maximal element* of S. And we call an element a of S such that there is no $x \in S$ with $a > x$ a *minimal element* of S.

In addition to (1°), (2°) and (3°), when L satisfies

(4°) for any two elements a and b, at least one of $a \leqq b$ or $b \leqq a$ holds,

we call L a *totally ordered set*. In this case, for all a and b, precisely one of $a < b$, $a = b$ or $a > b$ always holds.

Definition 1.2. When S is a subset of an ordered set L and there exists an element x of L such that $a \leqq x$, for all $a \in S$, we call x an *upper bound* of S. If the set of all upper bounds has a least element, we call this the *join* (or *supremum*) of S and denote it by $\bigvee(a; a \in S)$. Likewise we define *lower bounds* of S, and when there is a greatest lower bound of S we call this the *meet* (or *infimum*) of S and denote it by $\bigwedge(a : a \in S)$.

When there is a join and meet of every non-empty finite subset S of L, we call L a *lattice*. When there is a join and meet of every non-empty countable subset S of L, we call L a *σ-complete lattice*. When there is a join and meet of every non-empty subset of L (irrespective of cardinality), we call L a *complete lattice*. When every non-empty subset with an upper bound has a join, and every non-empty subset with a lower bound has a meet, we call L a *conditionally complete lattice*. We define *conditionally σ-complete lattices* in the same way.

For L to be a lattice, it suffices that there exist joins and meets of any pair of elements. We denote the join of $S = \{a_1, \cdots, a_n\}$ by $a_1 \smile \cdots \smile a_n$ or $\bigvee_{i=1}^n a_i$. We denote the meet of $S = \{a_1, \cdots, a_n\}$ by $a_1 \frown \cdots \frown a_n$ or $\bigwedge_{i=1}^n a_i$. If $S = \{a_i; i = 1, 2, \cdots\}$, we denote the join and meet of S by $\bigvee_{1 \leqq i < \infty} a_i$ and $\bigwedge_{1 \leqq i < \infty} a_i$.

In a complete lattice, there is always a unit and zero element because $\bigvee(a; a \in L) = 1$ and $\bigwedge(a; a \in L) = 0$. When S is the empty set, every element of L is an upper bound so the join of the empty set is the zero element. Likewise, the meet of the empty set is the unit element.

Remark 1.1. By Definition 1.2, the following holds in any lattice L.

(1°) $a \smile a = a$ and $a \frown a = a$.

(2°) $a \smile b = b \smile a$ and $a \frown b = b \frown a$.

(3°) $a \smile (b \smile c) = (a \smile b) \smile c$ and $a \frown (b \frown c) = (a \frown b) \frown c$.

(4°) $a \smile (b \frown a) = a$ and $a \frown (b \smile a) = a$.

(5°) $a \leqq b$, $a \smile b = b$ and $a \frown b = a$ are equivalent.

Conversely, the following theorem holds.

Theorem 1.1. *Given operations $\smile$ and $\frown$ on a set L satisfying Remark 1.1 (2°), (3°) and (4°), L is a lattice in which $a \smile b$ and $a \frown b$ are the join and meet of a and b respectively.*

Proof.

(i) Replacing b with $a \smile b$ in the first equation in (4°), we see from the second equation in (4°) that $a \smile a = a$. Likewise, $a \frown a = a$. Thus (1°) holds.

(ii) If $a \smile b = b$ then, by the second equation in (4°), $a \frown b = a$. Conversely, if $a \frown b = a$ then, rewriting the first equation in (4°) as $b \smile (b \frown a) = b$, we see from (2°) that $a \smile b = a$. Thus $a \smile b = b$ and $a \frown b = a$ are equivalent, in which case we define $a \leqq b$.

(iii) As $a \smile a = a$, by (1°), $a \leqq a$. And if $a \leqq b$ and $b \leqq a$ then $a \smile b = b$ and $b \smile a = a$ so $a = b$. Next, if $a \leqq b$ and $b \leqq c$ then $a \smile b = b$ and $b \smile c = c$. Therefore $a \smile c = a \smile (b \smile c) = (a \smile b) \smile c = b \smile c = c$ so $a \leqq c$. Thus, by Definition 1.1, L is an ordered set.

(iv) By (4°) and (2°), $a \leqq a \smile b$ and $b \leqq a \smile b$. Furthermore, if $a \leqq c$ and $b \leqq c$ then $a \smile c = c$ and $b \smile c = c$. By (1°) and (3°), $a \smile b \smile c = c$ so $a \smile b \leqq c$. Therefore $a \smile b$ is the join of a and b. Likewise, $a \frown b$ is the meet of a and b. □

Remark 1.2. Generally, in any order theoretic statement, we can replace $\geqq$ with $\leqq$ to obtain another statement which we call its *dual*. In a lattice, we can replace $\geqq$, $\smile$, $\frown$, 0 and 1 with $\leqq$, $\frown$, $\smile$, 1, and 0 respectively to obtain the dual statement. As Remark 1.1 (2°), (3°) and (4°) are unchanged by these replacements, whenever a lattice theoretic statement holds, its dual also holds.

Theorem 1.2. *For an ordered set L to be a complete lattice, it suffices that either joins or meets exist for arbitrary subsets S.*

Proof. Now assuming that there are joins of arbitrary subsets[1], we prove that, given $S \leqq L^{\dagger}$, $\bigwedge(a; a \in S)$ exists. Let T be the set of all b such that $b \leqq a$, for all $a \in S$. By assumption, $\bigvee(b; b \in T)$ exists. However, for all $a \in S$, $\bigvee(b; b \in T) \leqq a$. Next, if $d \leqq a$, for all $a \in S$, then $d \in T$ so $\bigvee(b; b \in T) \geqq d$. Therefore, by Definition 1.2, $\bigwedge(a; a \in S)$ exists and coincides with $\bigvee(b; b \in T)$. Dually, we can say the same when meets exist. □

Definition 1.3. For a in a lattice L with 0 and 1, any a' satisfying

$$a \smile a' = 1, \quad \text{and} \quad a \frown a' = 0$$

is called a *complement* of a. When every element of L has a complement, we call L a *complemented lattice.*

Definition 1.4. In a lattice L, when

$$a \leqq c \quad \text{implies} \quad (a \smile b) \frown c = a \smile (b \frown c), \tag{1}$$

we call L a *modular lattice.* We call (1) the *modular law.*

When L is a complemented lattice and simultaneously a modular lattice, we call L a *complemented modular lattice.*

Definition 1.5. When the following holds, for all a, b and c in a lattice L,

$$(a \frown b) \smile c = (a \frown c) \smile (b \frown c) \quad \text{and} \tag{1}$$
$$(a \smile b) \frown c = (a \smile c) \frown (b \smile c), \tag{2}$$

we call L a *distributive lattice*[2]. We call (1) and (2) the *distributive law.* When L is a complemented lattice and simultaneously a distributive lattice, we call L a *Boolean lattice.*

Remark 1.3. All subsets of a fixed space, ordered by the inclusion relation on sets, form a complete Boolean lattice. Here joins and meets are unions and intersections respectively, the space is the unit element and the empty set is the zero element. Complements are complementary sets. Generally, lattices made up of sets, where joins and meets are unions and intersections respectively, are called *set lattices.* Set lattices are distributive lattices[3].

Remark 1.4. In Chapter 3 Remark 3.3[‡] we show that complemented modular lattices satisfying certain conditions are the same as projective geometries. Here the join of a point a and a line b not containing the point a is the plane defined by a and b. Meets are none other than intersections. If one applies the general properties of complemented modular lattices below to projective geometries, their geometric meaning becomes clear.

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[1] Thus, being the join of the empty subset, L has a zero element. So assuming that joins of arbitrary subsets of L exist is equivalent to assuming L has a zero element and joins of arbitrary non-empty subsets exist.

[2] See Remark 1.5.

[3] In this book, the union and intersection operations are expressed by $+$ or $\sum$ and $\cdot$ or $\prod$ respectively.

[†] i.e. $S \subseteq T$

[‡] Erroneous reference to Remark 2.3 in the original.

Definition 1.6. In a lattice L we denote the relation $(a \smile b) \frown c = (a \frown c) \smile (b \frown c)$ by $(a, b, c)D$. We denote the dual relation $(a \frown b) \smile c = (a \smile c) \frown (b \smile c)$ by $(a, b, c)D^*$.

Theorem 1.3. *In a modular lattice, if $(a, b, c)D$ holds then relations like $(b, c, a)D$ and $(b, c, a)D^*$ all hold, irrespective of the position of a, b, and c.*

Proof. If $(a, b, c)D$ holds then, by the modular law,

$$(b \smile a) \frown (c \smile a) = \{(a \smile b) \frown c\} \smile a = \{(a \frown c) \smile (b \frown c)\} \smile a = (b \frown c) \smile a.$$

In other words,
$$(a, b, c)D \to (b, c, a)D^*. \tag{1}$$

Dually, by (1),

$$(a, b, c)D^* \to (b, c, a)D. \tag{2}$$

By (1) and (2)

$$(a, b, c)D \to (b, c, a)D^* \to (c, a, b)D \to (a, b, c)D^*. \tag{3}$$

By Definition 1.6, $(a, b, c)D = (b, a, c)D$. Thus, by (3),

$$(a, b, c)D \to (c, a, b)D \to (b, c, a)D = (c, b, a)D \to (a, c, b)D \to (b, a, c)D.$$

Also, by (3), $(a, b, c)D^*$ holds and, dually, every instance of D^* holds. □

Remark 1.5. In order for L to be a distributive lattice, it suffices for either (1) or (2) in Definition 1.5 to hold, for arbitrary a, b and c. This is because if (1), i.e. $(a, b, c)D$, holds then L is a modular lattice. Thus, by Theorem 1.3, $(a, b, c)D^*$, i.e. (2), holds.

Definition 1.7. When a subset L_0 of a lattice L containing a and b also contains $a \smile b$ and $a \frown b$, we call L a *sublattice*.

Definition 1.8. Assume $c \leqq d$ in a lattice L. We denote all x such that $c \leqq x \leqq d$, given the same ordering as in L, by $L(c, d)$.

$L(c, d)$ is a sublattice, with c and d being the zero element and unit element respectively.

Definition 1.9. When $c \leqq a \leqq d$ in a lattice L, we call b satisfying

$$a \smile b = d \quad \text{and} \quad a \frown b = c$$

a *relative complement* of a in d/c. When such relative complements exist for all a, c and d, we call L a *relatively complemented lattice.*

We call a relatively complemented distributive lattice with 0 a *generalized Boolean lattice.*

Remark 1.6. By Definition 1.8 a relative compelement of a in d/c is none other than a complement of a in $L(c, d)$.

Among relative complements, the $c = 0$ case is the most often considered. We call this the relative complement of a in d and write it as $a \oplus b = d$.

Generally, complements and relative complements are not unique.

Lemma 1.1. *For two elements a and b in a relatively complemented lattice with* 0,[†] *if* $b = (a \frown b) \oplus b_1$ *then* $a \smile b = a \oplus b_1$.

Proof. $a \smile b = a \smile (a \frown b) \smile b_1 = a \smile b_1$, and $b_1 \leqq b$ so $a \frown b_1 = a \frown b \frown b_1 = 0$. ☐

Lemma 1.2. *In a relatively complemented lattice with* 0 *and* 1,

(i) *if* $a \smile x = 1$ *then there is a complement* a' *of* a *such that* $a' \leqq x$.

(ii) *if* $a \frown x = 0$ *then there is a complement* a' *of* a *such that* $a' \geqq x$.

Proof.

(i) Taking a' such that $x = (a \frown x) \oplus a'$, by Lemma 1.1

$$1 = a \smile x = a \oplus a'.$$

(ii) Holds dually to (i). ☐

Lemma 1.3. *Complemented modular lattices are relatively complemented lattices.*

Proof. When $c \leqq a \leqq d$, take a' such that $a \oplus a' = 1$ and set $b = (a' \frown d) \smile c$. By the modular law,

$$a \smile b = a \smile (a' \frown d) \smile c = a \smile (a' \frown d) = (a \smile a') \frown d = d \quad \text{and}$$
$$a \frown b = a \frown \{(a' \frown d) \smile c\} = (a \frown a' \frown d) \smile c = c.$$

Thus b is a relative complment of a in d/c. ☐

Remark 1.7. By Lemma 1.3, if L is a complemented modular lattice then $L(c, d)$ is also a complemented modular lattice.

Theorem 1.4. *A necessary and sufficient condition for a lattice L to be a modular lattice is that whenever x and y are relative complements of a in d/c and $x \geqq y$, we must have $x = y$.*

Proof.

(i) Necessity. As $a \smile x = a \smile y = d$, $a \frown x = a \frown y = c$ and $x \geqq y$, by the modular law

$$x = (d \frown x) = (a \smile y) \frown x = (a \frown x) \smile y = c \smile y = y.$$

(ii) Sufficiency. When $a \leqq c$, set $x = (a \smile b) \frown c$ and $y = a \smile (b \frown c)$. $x \geqq y$ because $x \geqq a$ and $x \geqq b \frown c$. Also

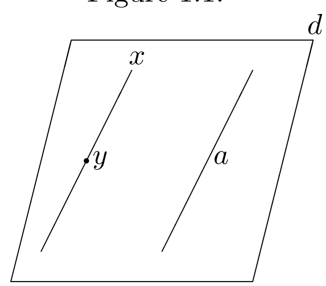
$$y \smile b = a \smile (b \frown c) \smile b = a \smile b = (a \smile b) \smile b \geqq x \smile b \geqq y \smile b$$

so $x \smile b = y \smile b$.

$$x \frown b = (a \smile b) \frown c \frown b = c \frown b = (b \frown c) \frown b \leqq y \frown b \leqq x \frown b$$

so $x \frown b = y \frown b$. Thus if we set $x \smile b = y \smile b = u$ and $x \frown b = y \frown b = v$ then x and y are relative complements of b in u/v with $x \geqq y$ so, by hypothesis, $x = y$, i.e. the modular law holds. ☐

Figure 1.1:



Remark 1.8. Consider a plane d in an affine geometry on which we take a straight line a and draw a striaght line x parallel to a through a point y not in a. Then x and y are relative complements of a in $d/0$ and $x > y$. Thus affine geometries are not modular lattices (see Figure 1.1).

Theorem 1.5. *A necessary and sufficient condition for a lattice L to be a distributive lattice is that if x and y are relative complements of a in d/c then we must have $x = y$.*

Proof.

(i) Necessity. As $a \smile x = a \smile y = d$ and $a \frown x = a \frown y = c$,

$$x = x \frown (a \smile x) = x \frown (a \smile y) = (x \frown a) \smile (x \frown y) = (a \frown y) \smile (x \frown y) = (a \smile x) \frown y = (a \smile y) \frown y = y.$$

(ii) Sufficiency[1]. By Theorem 1.4, L is a modular lattice. Next, for any $a, b, c \in L$, if we set

$$x = \{(a \smile b) \frown c\} \smile (a \frown b) \quad \text{and} \quad y = \{(a \smile c) \frown b\} \smile (a \frown c)$$

then, by the modular law,

$$\begin{aligned} x \smile a &= \{(a \smile b) \frown c\} \smile a = (a \smile b) \frown (a \smile c) \quad \text{and} \\ x \frown a &= \{((a \smile b) \frown c) \smile (a \frown b)\} \frown a = (a \frown c) \smile (a \frown b). \end{aligned}$$

The right hand side of the two equations above are symmetric in b and c so $x \smile a = y \smile a$ and $x \frown a = y \frown a$. Thus, by hypothesis, $x = y$. However

$$\begin{aligned} x \frown c &= [\{(a \smile b) \frown c\} \smile (a \frown b)] \frown c = \{(a \smile b) \frown c\} \smile (a \frown b \frown c) = (a \smile b) \frown c \quad \text{and} \\ y \frown c &= [\{(a \smile c) \frown b\} \smile (a \frown c)] \frown c = (b \frown c) \smile (a \frown c) \end{aligned}$$

so $(a, b, c)D$. Therefore, by Remark 1.5, L is a distributive lattice. □

Definition 1.10. In a lattice L, we call a pair of elements a and b related by $a \geqq b$ a *quotient* and denote it a/b. When $a \geqq a_1 \geqq b_1 \geqq b$, we call a_1/b_1 a *subquotient* of a/b. Two quotients that can be expressed in the form $a \smile b/a$ and $b/a \frown b$ are said to be *transformations* of each

[1] from [Oga48] p4.

†In fact this holds in any poset (necessarily with 0 to have $b = (a \frown b) \oplus b_1$) if $a \smile b$ exists.

other. Two quotients that are transformations of the same quotient are said to be mutually *perspective*. Two quotients that are connected by a finite sequence of successive quotient transformations are said to be *projective*. In particular, when L has a 0 and $a/0$ and $b/0$ are mutually perspective, we say the two elements a and b are *perspective* and denote this by $a \sim b$. Moreover, when $a/0$ and $b/0$ are mutually projective, we say the two elements a and b are *projective* and denote this by $a \approx b$.

Remark 1.9. In a lattice with 0, if $a \sim b$ then $a/0$ and $b/0$ are both tranformations of a certain quotient c/d. In other words, $a \oplus d = b \oplus d = c$ so a and b have a common complement d in c. Conversely, if two elements a and b have a common complement in some element c then $a \sim b$.

Lemma 1.4. *In a relatively complemented lattice*[†] *with* 0*, if there is an* x *such that*

$$a \smile x = b \smile x \quad \text{and} \quad a \frown x = b \frown x$$

then $a \sim b$*.*

Proof. If we take w such that $x = (a \frown x) \oplus w$ then, by Lemma 1.1, $a \smile x = a \oplus w$. But $x = (b \frown x) \oplus w$ so, likewise, $b \smile x = b \oplus w$. In other words, $a \oplus w = b \oplus w$ so $a \sim b$. □

Lemma 1.5. *In a relatively complemented*[‡] *lattice having a* 0*, a necessary and sufficient condition for* $a \approx b$ *is that there exist finitely many* $c_1, \cdots, c_n$ *such that*

$$a \sim c_1 \sim \cdots \sim c_n \sim b.$$

Proof. Sufficiency is clear. Next, to see necessity, consider two quotients $u/u \frown v$ and $u \smile v/v$ that are transformations of one another. In this situation, there exist r and s such that $u = (u \frown v) \oplus r$ and $u \smile v = v \oplus s$. Also $u/u \frown v$ and $r/0$, and $u \smile v/v$ and $s/0$ are, respectively, transformations of each other. However, by Lemma 1.1, $u \smile v = v \oplus r$. Therefore $r \sim s$. Next, if $a \approx b$ then $a/0$ and $b/0$ are connected by a finite sequence

$$a/0, \quad c_1/d_1, \quad \cdots, \quad c_n/d_n, \quad b/0$$

of successive quotient transformations. From above, there are transformations $r_1/0$ of c_1/d_1 and $s_1/0$ of c_2/d_2 with $r_1 \sim s_1$. Here $a/0$ and $r_1/0$ are transformations of c_1/d_1 so $a \sim r_1$, i.e. $a \sim r_1 \sim s_1$. If we continue in this way, we can connect a and b with a finite sequence of successively perspective elements. □

Lemma 1.6. *In a modular lattice with* 0*, if* $a \sim b$ *and* $a \leqq b$ *then* $a = b$*. Moreover, in a distributive lattice with* 0*, if* $a \sim b$ *then* $a = b$*.*

Proof. By Remark 1.9, if $a \sim b$ then there are c and d such that $a \oplus d = b \oplus d = c$. As a and b are complements of d in c, the lemma follows from Theorem 1.4 and Theorem 1.5. □

[†]Only the section $L(0, x)$ needs to be complemented.
[‡]Again, section complemented suffices.

Definition 1.11. When there is a map[†] $\phi : x \to \phi(x)$ from a lattice L_1 onto a lattice L_2 such that, for all x and y, we always have

$$\phi(x \smile y) = \phi(x) \smile \phi(y) \quad \text{and} \quad \phi(x \frown y) = \phi(x) \frown \phi(y),$$

we say L_2 is *homomorphic* to L_1. In particular, when the mapping is one-to-one, we say the two lattices are *isomorphic*.

When a mapping yields a one-to-one correspondence that does not change the order, the two lattices are isomorphic. This is because joins and meets are determined by the order.

If a map yields a one-to-one correspondence that reverses the order then joins and meets are interchanged by the mapping. In this case we say the two lattices are *dual isomorphic*. When the two lattices coincide, we call this correspondence a *self dual isomorphic correspondence*.

Remark 1.10. The relevant properties of modular, distributive, complemented and relatively complemented lattices are preserved by homomorphisms. For example, say $x \to x^* = \phi(x)$ is a homomorphism from a lattice L_1 onto a lattice L_2 and L_1 is a modular lattice. When $a^* \leqq c^*$ in L_2, take certain preimages a and c of a^* and c^* respectively. Now $\phi(a \smile c) = \phi(a) \smile \phi(c) = a^* \smile c^* = c^*$ so $a \smile c$ is a preimage of c^*. Thus if we replace c with $a \smile c$, we may assume that $a \leqq c$. If we take a preimage b of b^* then, as L_1 is a modular lattice, $(a \smile b) \frown c = a \smile (b \frown c)$. Thus $(a^* \smile b^*) \frown c^* = a^* \smile (b^* \frown c^*)$, i.e. L_2 is a modular lattice. The other properties are proved in the same way.

Theorem 1.6 (The Transformation Law[‡])**.** *In a modular lattice L, the maps S and T defined by*

$$\begin{array}{lll} x \in L(a, a \smile b) & \textit{implies} & Sx = x \frown b \ \textit{and} \\ y \in L(a \frown b, b) & \textit{implies} & Ty = y \smile a \end{array}$$

are order preserving maps between $L(a, a \smile b)$ and $L(a \frown b, b)$ such that S and T are inverse maps of each other. Thus $L(a, a \smile b)$ and $L(a \frown b, b)$ are isomorphic.

Proof. If $a \leqq x \leqq a \smile b$ then $a \frown b \leqq x \frown b \leqq (a \smile b) \frown b = b$. Thus $Sx \in L(a \frown b, b)$. Likewise $Ty \in L(a, a \smile b)$. Moreover, by the modular law, $TSx = (x \frown b) \smile a = x \frown (b \smile a) = x$. Likewise, $STy = y$. Thus S and T are inverse maps of each other. It is clear that they are order preserving. Therefore $L(a, a \smile b)$ and $L(a \frown b, b)$ are isomorphic. □

Definition 1.12. When $\leqq$ is a binary relation defined on a set D satisfying (1°), (2°) and (3°) below, we call D a *directed set*.

(1°) For all $\delta \in D$, $\delta \leqq \delta$.

(2°) If $\delta_1 \leqq \delta_2$ and $\delta_2 \leqq \delta_3$ then $\delta_1 \leqq \delta_3$.

(3°) For any two elements δ_1 and δ_2 of D, there exists an element δ_3 of D such that $\delta_1 \leq \delta_3$ and $\delta_2 \leqq \delta_3$.

[†] In the original, this map was required to be unique (which would imply L_1 and L_2 are singleton sets). This was either an oversight/misprint or, alternatively, the word for 'mapping' could be interpreted as 'image' so a more literal translation might be something like '...for each x there is a unique image $\phi(x)$...'.

[‡] More commonly known as 'The Diamond Isomorphism Theorem'

When a subset $\{a_\delta; \delta \in D\}$ of ordered set L is indexed by a directed set D and

$$\delta_1 < \delta_2 \quad \text{implies} \quad a_{\delta_1} \leqq a_{\delta_2},$$

we call $\{a_\delta; \delta \in D\}$ a *monotone increasing system* of L. Moreover, when

$$\delta_1 < \delta_2 \quad \text{implies} \quad a_{\delta_1} \geqq a_{\delta_2},$$

we call $\{a_\delta; \delta \in D\}$ a *monotone decreasing system* of L.

Definition 1.13.

(i) When there is a monotone increasing system $\{a_\delta; \delta \in D\}$ in a conditionally complete lattice L and $a = \bigvee(a_\delta; \delta \in D)$, we write $a_\delta \uparrow a$. Likewise, when there is a monotone decreasing $\{a_\delta; \delta \in D\}$ and $a = \bigwedge(a_\delta; \delta \in D)$, we write $a_\delta \downarrow a$.

(ii) More generally, when given a subset $\{a_\delta; \delta \in D\}$ of L indexed by a directed set D and there exist u_δ and v_δ such that $v_\delta \leqq a_\delta \leqq u_\delta$, $v_\delta \uparrow a$ and $u_\delta \downarrow a$, we say $a_\delta (\delta \in D)$ (0)-*converges* to a and write $\lim_\delta a_\delta = a$.

Definition 1.14. If, for all elements b in a complete lattice L,

$$a_\delta \uparrow a \quad \text{implies} \quad a_\delta \frown b \uparrow a \frown b$$

then we call L an *upper continuous lattice*. Dually, if

$$a_\delta \downarrow a \quad \text{implies} \quad a_\delta \smile b \downarrow a \smile b$$

then we call L a *lower continuous lattice*. When L is simultaneously an upper and lower continuous lattice we call L a *continuous lattice*[1†].

When L is a conditionally complete lattice satisfying the above conditions, we call L a *conditionally upper continuous lattice*, *conditionally lower continuous lattice* or *conditionally continuous lattice* respectively. We call a conditionally upper continuous relatively complemented modular lattice with 0 a *generalized upper continuous relatively complemented modular lattice*, and we call a conditionally continuous relatively complemented modular lattice with 0 a *generalized upper continuous relatively complemented modular lattice*.

If a generalized (upper) continuous complemented modular lattice has a unit element 1 then it is a (upper) continuous complemented modular lattice.

Remark 1.11. When (0)-$\lim_\delta a_\delta = a$, we certainly have $u_\delta \frown b \downarrow a \frown b$ and so being (conditionally) upper continuous means that (0)-$\lim_\delta a_\delta = a$ implies (0)-$\lim_\delta (a_\delta \frown b) = a \frown b$[‡]. Dually, being a (conditionally) lower continuous lattice means (0)-$\lim\limits_\delta a_\delta = a$ implies (0)-$\lim\limits_\delta (a_\delta \smile b) = a \smile b$. Thus, in a (conditionally) continuous lattice, the lattice operations $\smile$ and $\frown$ are continuous with respect to (0)-convergence.

Lemma 1.7. *If S is a subset with an upper bound in a conditionally upper continuous lattice L with 0 then the following statement (α) holds.*

[1] See Appendix §2.

[†] Not to be confused with the rather different notion later introduced by Dana Scott under the same name.
[‡] In essence, L is upper continuous precisely when $a \to a \frown b$ is continuous w.r.t the Scott topology on L.

(α) *If* $\bigvee(a; a \in \nu) \frown b = 0$, *for all finite subsets* ν *of* S, *then* $\bigvee(a; a \in S) \frown b = 0$.

Proof. The set D of all finite subsets of S, ordered by the inclusion relation, is a directed set. If we let $s_\nu = \bigvee(a; a \in \nu)$ and $s = \bigvee(a; a \in S)$ then $s_\nu \uparrow s$. However, by hypothesis, $s_\nu \frown b = 0$ and $s_\nu \frown b \uparrow s \frown b$ so $s \frown b = 0$. □

Remark 1.12. When L is a conditionally complete relatively complemented lattice with 0, if Lemma 1.7(α) holds then L is a conditionally upper continuous lattice. This is because, when $a_\delta \uparrow a$ and we assume

$$a \frown b = \bigvee(a_\delta \frown b; \delta \in D) \oplus c$$

then, for all $\delta \in D$,

$$a_\delta \frown b \frown c \leqq \bigvee(a_\delta \frown b; \delta \in D) \frown c = 0.$$

Now if we take any finite subset ν of D then there exists δ_0 such that $\delta \leqq \delta_0$, for all $\delta \in \nu$, so

$$\bigvee(a_\delta; \delta \in \nu) \frown b \frown c \leqq a_{\delta_0} \frown b \frown c = 0.$$

As Lemma 1.7(α) holds,

$$a \frown b \frown c = \bigvee(a_\delta; \delta \in D) \frown b \frown c = 0.$$

But $c \leqq a \frown b$ so $c = 0$. Therefore $a_\delta \frown b \uparrow a \frown b$.

Remark 1.13. Complete Boolean lattices are continuous lattices. To show this it suffices, by Remark 1.12, to note that Lemma 1.7(α) and its dual statement hold. For all finite subsets ν of S, $\bigvee(a; a \in \nu) \frown b = 0$ so, for all $a \in S$, $a \frown b = 0$. In other words, if we take a complement b' of b, $a \leqq b'$[1]. Therefore $\bigvee(a; a \in S) \leqq b'$, in other words $(a; a \in S) \frown b = 0$. Thus ($\alpha$) holds. Likewise, the dual statement to (α) also holds.

Therefore, for any subset S of a Boolean lattice,

$$\bigvee(a; a \in S) \frown b = \bigvee(a \frown b; a \in S)$$

and its dual statment hold. This is because if we take finite subsets ν of S and set $s_\nu = \bigvee(a; a \in \nu)$ then $s_\nu \uparrow \bigvee(a; a \in S)$ so

$$\bigvee(a \frown b; a \in \nu) = s_\nu \frown b \uparrow \bigvee(a; a \in S) \frown b.$$

Lemma 1.8. *If* $a_\delta \uparrow a$ *and* $b_\delta \uparrow b$ *in a conditionally upper continuous lattice,* $a_\delta \frown b_\delta \uparrow a \frown b$.

Proof. As $\bigvee(a_\delta \frown b_\delta; \delta \in D) \geqq \bigvee(a_\delta \frown b_\gamma; \gamma \in D) = a_\delta \frown b$,

$$\bigvee(a_\delta \frown b_\delta; \delta \in D) \geqq \bigvee(a_\delta \frown b; \delta \in D) = a \frown b.$$

On the other hand, $\bigvee(a_\delta \frown b_\delta; \delta \in D) \leqq a \frown b$ so $\bigvee(a_\delta \frown b_\delta; \delta \in D) = a \frown b$, i.e. $a_\delta \frown b_\delta \uparrow a \frown b$. □

Lemma 1.9. *Assume* $a_\delta \uparrow a$ *in a continuous lattice. If there are complements* a'_δ *of* a_δ *and* $a'_\delta \downarrow a'$ *then* a' *is a complement* a.

[1] In a Boolean lattice, $a \frown b = 0$ and $a \leqq b'$ are equivalent. See Lemma 3.5.

Proof. As
$$1 = a_\delta \smile a_\delta{}' \leqq a \smile a_\delta{}' \quad \text{and} \quad a \smile a_\delta{}' \downarrow a \smile a',$$
$a \smile a' = 1$. Also, as
$$0 = a_\delta \frown a_\delta{}' \geqq a_\delta \frown a' \quad \text{and} \quad a_\delta \frown a' \uparrow a \frown a',$$
$a \frown a' = 0$. Therefore a' is a complement of a. □

Definition 1.15. Say $S = \{a_1, \cdots, a_n\}$ is a set of finitely many elements of a lattice L with 0. If, for any two subsets S_1 and S_2 of S with no elements in common,
$$\bigvee(a; a \in S_1) \frown \bigvee(a; a \in S_2) = 0$$
then we call S an *independent system* and denote this by $(a, \cdots, a_n)\perp$, $(a_i; i = 1, \cdots, n)\perp$ or $(a; a \in S)\perp$.

In particular, when L is a conditionally complete lattice with 0, we can apply the above definition of an independent system to any subset S (of arbitrary cardinality) with an upper bound. (In this case the cardinality of S_1 and S_2 is also arbitrary)

By this definition, the order of the sequence $a_1, \cdots, a_n$ has no relation to $(a_1, \cdots, a_n)\perp$. Also, if $(a; a \in S)\perp$ and $S_1 \leqq S$ then $(a; a \in S_1)\perp$. And if $(a_\alpha; \alpha \in I)\perp$ and $b_\alpha \leqq a_\alpha (\alpha \in I)$ then $(b_\alpha; \alpha \in I)\perp$.

When $(a_1, \cdots, a_n)\perp$, we denote $a_1 \smile \cdots \smile a_n$ by $a_1 \oplus \cdots \oplus a_n$ or $\bigvee_{i=1}^n \oplus a_i$. Also, when $(a; a \in S)\perp$, we denote $\bigvee(a; a \in S)$ by $\bigvee(\oplus a; a \in S)$.

Theorem 1.7. *In a conditionally upper continuous lattice L with 0, a necessary and sufficient condition for a subset S of L with an upper bound to be an independent system is that all finite subsets of S are independent systems.*

Proof. Necessity is clear. Next assume all finite subsets ν of S are independent systems. Take any subsets S_1 and S_2 of S with no elements in common. When ν_1 and ν_2 are finite subsets of S_1 and S_2 respectively, by hypothesis $\nu_1 + \nu_2$ is an independent system so
$$\bigvee(a; a \in \nu_1) \frown \bigvee(a; a \in \nu_2) = 0.$$
This holds for all finite subsets ν_2 of S_2 so, by Lemma 1.7,
$$\bigvee(a; a \in \nu_1) \frown \bigvee(a; a \in S_2) = 0.$$
Again, this holds for all finite subsets ν_1 of S_1 so
$$\bigvee(a; a \in S_1) \frown \bigvee(a; a \in S_2) = 0.$$
Therefore, S is an independent system. □

Lemma 1.10. *In a conditionally upper continuous lattice with 0, if $(a_i; i = 1, 2, \cdots)\perp$ and we set $b_n = \bigvee_{n \leqq i < \infty} a_i$ then $b_n \downarrow 0$.*

Proof. If we set $b = \bigwedge_{1 \leqq n < \infty} b_n$ then
$$b \frown \bigvee_{1 \leqq i \leqq n} a_i \leqq b_{n+1} \frown \bigvee_{1 \leqq i \leqq n} a_i = 0.$$
But $\bigvee_{1 \leqq i \leqq n} a_i \uparrow b_1$ so, by upper continuity $b \frown b_1 = 0$. Thus, as $b \leqq b_1$, $b = 0$. □

Lemma 1.11. *In a modular lattice with* 0*, if* $(x_1 \smile \cdots \smile x_n) \frown a = 0$ *then*

$$(x_1 \frown \cdots \frown x_n) \smile a = (x_1 \smile a) \frown \cdots \frown (x_n \smile a).$$

Proof. If we set $x_1 \smile \cdots \smile x_n = b$ then, by Theorem 1.6, $Tx = x \smile a$ is an isomorphic map from $L(0, b)$ to $L(a, a \smile b)$ and $x_i \in L(0, b)(i = 1, \cdots, n)$ so

$$\begin{aligned}(x_1 \frown \cdots \frown x_n) \smile a &= T(x_1 \frown \cdots \frown x_n) = Tx_1 \frown \cdots \frown Tx_n \\ &= (x \smile a) \frown \cdots \frown (x_n \smile a)\end{aligned}$$

□

Theorem 1.8. *In a modular lattice with* 0*, a necessary and sufficient condition for* $(a_1, \cdots, a_n)\perp$ *is that*

$$(a_1 \smile \cdots \smile a_i) \frown a_{i+1} = 0 \quad (i = 1, \cdots, n-1)$$

holds.

Proof. Necessity is clear from Definition 1.15. To prove sufficiency, we use mathematical induction. The $n = 2$ case is clear. Now assume $m < n$ and $(a_1, \cdots, a_m)\perp$. Take two subsets S_1 and S_2 of $\{a_1, \cdots, a_{m+1}\}$ that have no elements in common. When both S_1 and S_2 do not contain a_{m+1}, $(a_1, \cdots, a_m)\perp$ so $\bigvee(a; a \in S_2) \frown \bigvee(a; a \in S_1) = 0$. Next, when S_2 contains a_{m+1}, if we let S_2' be the set S_2 with a_{m+1} removed and let

$$b_1 = \bigvee(a; a \in S_1) \quad \text{and} \quad b_2 = \bigvee(a; a \in S_2')$$

then $b_1 \frown b_2 = 0$ and $(b_1 \smile b_2) \frown a_{m+1} = 0$. Then, by Lemma 1.11,

$$(b_1 \smile a_{m+1}) \frown (b_2 \smile a_{m+1}) = (b_1 \frown b_2) \smile a_{m+1} = a_{m+1}$$

so

$$b_1 \frown (b_2 \smile a_{m+1}) = b_1 \frown (b_1 \smile a_{m+1}) \frown (b_2 \smile a_{m+1}) = b_1 \frown a_{m+1} = 0,$$

i.e. $\bigvee(a; a \in S_1) \frown \bigvee(a; a \in S_2) = 0$ so $(a_1, \cdots, a_{m+1})\perp$. □

Lemma 1.12. *For two elements* a *and* b *in a relatively complemented modular lattice with* 0*, if we let* $a = (a \frown b) \oplus a_1$ *and* $b = (a \frown b) \oplus b_1$ *then*

$$a \smile b = (a \frown b) \oplus a_1 \oplus b_1.$$

Proof. As

$$(a \frown b) \frown a_1 = 0 \quad \text{and} \quad \{(a \frown b) \smile a_1\} \frown b_1 = a \frown b_1 = a \frown b \frown b_1 = 0,$$

$(a \frown b, a_1, b_1)\perp$, by Theorem 1.8. Also $a \smile b = (a \frown b) \smile a_1 \smile b_1$. □

Lemma 1.13. *For any two elements* a *and* b *and any complement* a' *of* a *in a complemented modular lattice, there is a complement* b' *of* b *satisfying the following condition.*

(α) $a' \frown b'$ *is a complement of* $a \smile b$ *and* $a' \smile b'$ *is a complement of* $a \frown b$[1].

[1] Due to Tôzirô Ogasawara.

Proof. As $(a \smile b) \smile a' = 1$, there exists c such that $(a \smile b) \oplus c = 1$ and $c \leqq a'$, by Lemma 1.2 (i). And if we apply Lemma 1.2 (i) to $L(0, a \smile b)$ then there exists b_1 such that $b \oplus b_1 = a \smile b$ and $b_1 \leqq a$. By Theorem 1.8, $b \oplus b_1 \oplus c = 1$. Therefore, if we let $b' = b_1 \oplus c$ then b' is a complement of b. As $a' \frown b_1 \leqq a' \frown a = 0$,

$$a' \frown b' = a' \frown (b_1 \smile c) = (a' \frown b_1) \smile c = c,$$

i.e. $a' \frown b'$ is a complement of $a \smile b$. Moreover, $a' \smile b' = a' \smile b_1 \smile c = a' \smile b_1$ so

$$(a' \smile b') \smile (a \frown b) = (a' \smile b_1) \smile (a \frown b) = a' \smile \{a \frown (b_1 \smile b)\} = a' \smile \{a \frown (a \smile b)\} = a' \smile a = 1 \quad \text{and}$$
$$(a' \smile b') \frown (a \frown b) = (a' \smile b_1) \frown (a \frown b) = \{(a' \frown a) \smile b_1\} \frown b = b_1 \frown b = 0.$$

Thus $a' \smile b'$ is a complement of $a \frown b$. Therefore b' is a complement of b satisfying (α). □

Definition 1.16. Say we are given a set S. If, to each subset X, there always corresponds a second subset $\bar{X}$ such that

(1°) $\bar{X} \geqq X$ and

(2°) $X \geqq Y$ implies $\bar{X} \geqq \bar{Y}$

holds, we call this correspondence $X \to \bar{X}$ a *closure operation*. In particular, we call a set such that $\bar{X} = X$ a *closed set* with respect to this closure operation. Apart from (1°) and (2°), when the closure operation $X \to \bar{X}$ also satisfies

(3°) $\bar{\bar{X}} = \bar{X}$,

we call this an *idempotent closure operation*.

Theorem 1.9. *All the closed sets with respect to a closure operation on a set S form a complete lattice when ordered by the set inclusion relation.*

Proof. If $(X_\alpha; \alpha \in I)$ is a non-empty family of closed sets and $P = \prod(X_\alpha; \alpha \in I)$ then $\bar{P} \leqq \bar{X}_\alpha = X_\alpha (\alpha \in I)$ so $\bar{P} \leqq P$. By Definition 1.16 (1°), P is a closed set so P is the meet of $\{X_\alpha; \alpha \in I\}$. As S itself is also a closed set, all the closed sets form a complete lattice, by Theorem 1.2. □

Remark 1.14. In Theorem 1.9, in particular case when the closure operation is idempotent, the join of $\{X_\alpha; \alpha \in I\}$, in the complete lattice formed by all closed sets with respect to the closure operation, is $\overline{Q}$, where $Q = \sum(X_\alpha; \alpha \in I)$. This is because $\overline{Q}$ is a closed set containing all the X_α and, whenever T is a closed set containing all the X_α, $T \geqq Q$ so $T = \overline{T} \geqq \overline{Q}$. Thus $\overline{Q}$ is the join of $\{X_\alpha; \alpha \in I\}$.

Lemma 1.14. *When there is a family of subsets Φ of a certain set S that satisfies the conditions*

(1°) *S belongs to Φ,*

(2°) *if we take any non-empty subset Φ_0 of Φ then the intersection $\prod(X; X \in \Phi_0)$ belongs to Φ, and*

(3°) *if we take any monotone increasing system $\{X_\delta; \delta \in D\}$ of Φ, the union $\sum(X_\delta; \delta \in D)$ belongs to Φ,*

Φ is an upper continuous lattice, when ordered by the set inclusion relation.

Proof. By (1°) and (2°), any subset Φ_0 of Φ has a meet so, by Theorem 1.2, Φ is a complete lattice. Here the meet is the intersection and, by (3°), the join of a monotonic increasing sequence $\{X_\delta; \delta \in D\}$ is the union $Q = \sum(X_\delta; \delta \in D)$, i.e. $X_\delta \uparrow Q$. Thus, for any $M \in \Phi$, the set operations

$$\sum(X_\delta \cdot M; \delta \in D) = \sum(X_\delta; \delta \in D) \cdot M$$

yield $\bigvee(X_\delta \frown M; \delta \in D) = Q \frown M$, i.e. $X_\delta \frown M \uparrow Q \frown M$. Therefore Φ is an upper continuous lattice. □

Remark 1.15. Given a family of subsets Φ of a set S satisfying Lemma 1.14 (1°) and (2°), if we set $\bar{X}$ to be the intersection of all sets in Φ containing X, for any subset X of S, then $X \to \bar{X}$ is an idempotent closure operation, and the collection of all closed subsets with respect to this closure operation coincides with Φ. This is because $X \to \bar{X}$ clearly satisfies Definition 1.16 (1°) and (2°) and, by Lemma 1.14 (2°), $\bar{X} \in \Phi$ so Definition 1.16 (3°) holds and $X = \bar{X}$ is equivalent to $X \in \Phi$.

Lemma 1.15. *Given a family of subsets Φ of a certain set S ordered by the set inclusion relation, if the condition*

(α) *whenever Ψ is a non-empty totally ordered subset of Φ, the union $\sum(X; X \in \Psi)$ belongs to Φ,*

holds then there is a maximal element in Φ.

Proof. By (α), the union of Ψ is its upper bound so, by Zorn's lemma[1], there is a maximal element in Φ. □

Remark 1.16. Conversely one can prove Zorn's lemma from Lemma 1.15. Specifically, the collection Φ of all totally ordered subsets of an ordered set L satisfies (α) so, by Lemma 1.15, there is a maximal element in Φ. As C is a totally ordered subset of L, by hypothesis it has an upper bound a in L. If a were not a maximal element of L then there would be an element b of L such that $a < b$. The set $C + \{b\}$[†] obtained by adding b to C also belongs to Φ and so contradicts the fact C is a maximal element of Φ. Thus a is a maximal element of L.

Thus Lemma 1.15 and Zorn's lemma are equivalent.

Definition 1.17. When J is a subset of a lattice L satisfying

(1°) $a, b \in J$ implies $a \smile b \in J$ and

(2°) $a \in J$ and $c \leqq a$ implies $c \in J$,

we call J an *ideal* of L. Dually when

(1′) $a, b \in J$ implies $a \frown b \in J$ and

(2′) $a \in J$ and $c \geqq a$ implies $c \in J$,

[1] See Appendix 1.

[†] $\{C, b\}$ in the original.

we call J a *dual ideal* of L.

$J(a) = \{x; x \leqq a\}$[1] is an ideal of L. We call this a *principal ideal* of L. Dually, we call $J(a) = \{x; x \geqq a\}$ a *dual principal ideal*.

When J is an ideal different from L and there is no ideal I such that $J < I < L$, we call J a *maximal ideal* of L. Dually we define *maximal dual ideals*.

Theorem 1.10. *The ideals J of a lattice L ordered by the set inclusion relation form an upper continuous lattice. L is isomorphic to the sublattice $\{J(a); a \in L\}$ of Φ via the map $a \to J(a)$. A necessary and sufficient condition for Φ to be a modular (or distributive) lattice is that L be a modular (or distributive) lattice*[2]*. Likewise, this also holds for dual ideals (although in this case L is dual isomorphic to the sublattice $\{J(a); a \in L\}$ of Φ via the map $a \to J(a)$).*

Proof.

(i) Considering ideals as subsets of L, it is clear that the collection Φ of all ideals of L satisfies Lemma 1.14 (1°) and (2°). Take a monotone increasing system of ideals $\{J_\delta; \delta \in D\}$ and set $I = \sum(J_\delta; \delta \in D)$. If $a, b \in I$ then there are δ_1 and δ_2 such that $a \in J_{\delta_1}$ and $b \in J_{\delta_2}$. If we take δ_3 such that $\delta_1 \leqq \delta_3$ and $\delta_2 \leqq \delta_3$ then $a, b \in J_{\delta_3}$. Thus $a \smile b \in J_{\delta_3}$ so $a \smile b \in I$. Moreover, if $a \in I$ and $c \leqq a$ then $a \in J_\delta$ for some δ. As $c \in J_\delta$, $c \in I$. Thus I an ideal of L, i.e. Lemma 1.14 (3°) holds so Φ is upper continuous.

The meet of two ideals I and J of L is their intersection. Next, if we set $K = \{c; c \leqq a \smile b, a \in I \text{ and } b \in J\}$[3] then K is clearly an ideal containing I and J. Conversely, any ideal containing I and J contains K so $I \smile J = K$.

(ii) For principal ideals $J(a) = \{x; x \leqq a\}$ of L, in Φ we have

$$J(a) \smile J(b) = J(a \smile b) \quad \text{and} \quad J(a) \frown J(b) = J(a \frown b). \tag{1}$$

This is because $J(a) \leqq J(a \smile b)$ and $J(b) \leqq J(a \smile b)$ so $J(a) \smile J(b) \leqq J(a \smile b)$. As $a \smile b \in J(a) \smile J(b)$, if $x \leqq a \smile b$ then $x \in J(a) \smile J(b)$, i.e. $J(a \smile b) \leqq J(a) \smile J(b)$ so the first equation in (1) holds. Next, $J(a) \geqq J(a \frown b)$ and $J(b) \geqq J(a \frown b)$ so $J(a) \frown J(b) \geqq J(a \frown b)$. If $x \in J(a) \frown J(b)$ then $x \in J(a)$ so $x \leqq a$. Likewise, $x \leqq b$ so $x \leqq a \frown b$. Thus $x \in J(a \frown b)$ so the second equation in (1) holds.

By (1), L is isomorphic to the sublattice $\{J(a); a \in L\}$ of Φ via the map $a \to J(a)$. Therefore if Φ is a modular lattice (or distributive lattice) then L is also a modular lattice (or distributive lattice).

(For dual ideals $J(a) = \{x; x \leqq a\}$, instead of (1),

$$J(a) \smile J(b) = J(a \frown b) \quad \text{and} \quad J(a) \frown J(b) = J(a \smile b)$$

holds.)

(iii) Assuming L is a modular lattice, if X, Y and Z are ideals such that $X \leqq Z$ then, for any $t \in (X \smile Y) \frown Z$, $t \in Z$ and there exist x and y such that $t \leqq x \smile y$, $x \in X$ and $y \in Y$. If we set $z_1 = x \smile t$ then $z_1 \in Z$ so

$$t \leqq (x \smile y) \frown z_1 = x \smile (y \frown z_1) \in X \smile (Y \frown Z).$$

[1] $\{x; x \leqq a\}$ denotes the set of all x such that $x \leqq a$.

[2] [Dil41] p329.

[3] K denotes the set of all c such that $c \leqq a \smile b$, for some $a \in I$ and $b \in J$.

Thus $(X \smile Y) \frown Z \leqq X \smile (Y \frown Z)$. Clearly $(X \smile Y) \frown Z \geqq X \smile (Y \frown Z)$ so Φ is a modular lattice.

(iv) Next, when L is a distributive lattice, if $t \in (X \smile Y) \frown Z$ then $t \in Z$ and there exist x and y such that $t \leqq x \smile y$, $x \in X$ and $y \in Y$. As

$$t = (x \smile y) \frown t = (x \frown t) \smile (y \frown t), \quad x \frown t \in X \frown Z \quad \text{and} \quad y \frown t \in Y \frown Z,$$

$t \in (X \frown Z) \smile (Y \frown Z)$. Thus $(X \smile Y) \frown Z \leqq (X \frown Z) \smile (Y \frown Z)$. Clearly $(X \smile Y) \frown Z \geqq (X \frown Z) \smile (Y \frown Z)$ so, by Remark 1.5, Φ is a distributive lattice. □

Lemma 1.16. *When S is a subset of a lattice L with 0 such that the meet of any finite ($\neq 0$) number of elements of S is not 0, there exists a maximal dual ideal of L containing S.*

Proof. If we let J be the collection of all a such that $a \geqq a_1 \frown \cdots \frown a_n$, for some arbitrary elements $a_1, \cdots, a_n$ of S, then J is clearly a dual ideal of L containing S but not 0. As there are dual ideals like this that contain S but not 0, we let Φ be the collection of all these dual ideals. Let Ψ be any non-empty totally ordered subset of Φ and set $I = \sum(J; J \in \Psi)$. If $a, b \in I$ then there exist J_1 and J_2 belonging to Ψ such that $a \in J_1$ and $b \in J_2$. As Ψ is totally ordered, $J_1 \leqq J_2$ or $J_2 \leqq J_1$. For example, in the former case, $a, b \in J_2$ so $a \frown b \in J_2$. Therefore $a \frown b \in I$. If $c \geqq a$ and $a \in I$ then there exists $J \in \Psi$ such that $a \in J$ which, as $c \in J$, means $c \in I$. Thus Lemma 1.15 (α) holds so there is a maximal element $\mathfrak{p}$ in Φ and $\mathfrak{p} < L$. If there were a dual ideal J such that $\mathfrak{p} < J < L$ then J would be a dual ideal containing S but not 0, contradicting the fact that $\mathfrak{p}$ is a maximal element of Φ. Thus $\mathfrak{p}$ is a maximal dual ideal[1]. □

2 Lattice Products and Direct Sums

3 The Centre of a Lattice

Lemma 3.5. *If we take a complement z' of a central element z in a lattice L with 0 and 1 then, when $a \in L$, $z \frown a = 0$ is equivalent to $a \leqq z'$.*

4 Lattice Congruences

5 Representing Lattices on Sets

6 Metric Lattices

[1] As in Theorem 1.10 and Lemma 1.16, there are many proofs in which we use Lemma 1.14 or Lemma 1.15. From now on we will not be making this explicit in every single such proof.

Chapter 2

General Properties of Modular Lattices

Chapter 3

Projective Spaces

1 Relatively Atomic Upper Continuous Lattices

2 Atomic Elements in Modular Lattices

Remark 2.3.

3 Projective Spaces

Remark 3.3.

Chapter 4

Basic Properties of Continuous Complemented Modular Lattices

Chapter 5

Dimension Functions and Subdirect Product Decompositions in Continuous Complemented Modular Lattices

Chapter 6

Regular Rings

Chapter 7

Continuous Regular Rings

1 Rank in Continuous Regular Rings

2 Rank Functions on Irreducible Continuous Regular Rings

3 Subdirect Product Decompositions of Continuous Regular Rings

Chapter 8

Reference Frames in Complemented Modular Lattices

Chapter 9

Matrix Rings

1 Matrices over Regular Rings

2 Matrix Rings

3 Vector Spaces

Chapter 10

Auxiliary Rings of Complemented Modular Lattices

1 L-Number Multiplication

2 L-Number Addition

3 L-Number Distributivity

Chapter 11

Representations of Complemented Modular Lattices

1 The Isomorphism between $L(0, a_k)$ and $\overline{R}(\mathfrak{S}^L)$

2 $(\beta; \gamma^{(1)}, \ldots, \gamma^{(m-1)})$

3 The Isomorphism between L and $\overline{R}(\mathfrak{S}_n^L)$

Chapter 12

Representations of Orthocomplemented Modular Lattices

1 Orthocomplemented Modular Lattices

2 *-Regular Rings

3 Representations of Orthocomplemented Modular Lattices

Appendix

1 The Axiom of Choice, Well-Ordering Theorem and Zorn's Lemma

2 The Definition of a Continuous Lattice

In Chapter 1 Definition 1.14, a complete lattice L is defined to be upper continuous when the following holds.

(α) *Whenever $\{a_\delta; \delta \in D\}$ is a subset of L indexed by a directed set D,*

$$a_\delta \uparrow a \quad \textit{implies} \quad a_\delta \frown b \uparrow a \frown b.$$

However, J. von Neumann used (β) instead of (α).

(β) *Whenever Ω is a transfinite ordinal such that $\alpha < \beta < \Omega$ implies $a_\alpha \leqq a_\beta$,*

$$\bigvee(a_\alpha; \alpha < \Omega) \frown b = \bigvee(a_\alpha \frown b; \alpha < \Omega).$$

We can also consider the following.

(γ) *If $S \leqq L$ and, for all finite subsets ν of S,*

$$\bigvee(a; a \in \nu) \frown b = \bigvee(a \frown b; a \in \nu)$$

then $\bigvee(a; a \in S) \frown b = \bigvee(a \frown b; a \in S)$.

In a complete lattice L, (α), (β) and (γ) are equivalent[1]. In other words, we can use either (α), (β) or (γ) as the definition of an upper semicontinuous lattice.

(α)→(γ): If we let D be the set of all finite subsets ν of S, which we make a directed set by ordering by set inclusion, let $a' = \bigvee(a; a \in S)$ and let $s_\nu = \bigvee(a; a \in \nu)$ then $s_\nu \uparrow a'$.[†] Thus, by (α), $s_\nu \frown b \uparrow a \frown b$. But, by hypothesis, $s_\nu \frown b = \bigvee(a \frown b; a \in \nu)$ so (γ) holds.

(γ)→(β): If we let $\{a_{\alpha_1}, \cdots, a_{\alpha_n}\}$ be an aribtrary finite subset of $\{a_\alpha; \alpha < \Omega\}$ then it has a maximum, for example a_{α_n}. In this case, $\bigvee(a_{\alpha_i}; i = 1, \cdots, n) \frown b = a_{\alpha_n} \frown b = \bigvee(a_{\alpha_i} \frown b; i = 1, \cdots, n)$, so from ($\gamma$) we see that ($\beta$) holds.

[1] The equivalence of (β) and (γ) was shown in [Mae42]. And their equivalence to (α) was shown in [Sas48].

[†] a' was misprinted as a in the original.

In order to prove (β)→(α) we use the following lemma.

Lemma.[1] *If a directed set D is an infinite set then there is a transfinite sequence $\{D_\alpha; \alpha < \Omega\}$ of directed subsets of D having the following properties.*

(1°) $\overline{\overline{D}}_\alpha < \overline{\overline{D}}$, *($\overline{\overline{D}}_\alpha$ indicates the cardinality of D_α),*

(2°) $\alpha < \beta < \Omega$ *implies* $D_\alpha \leqq D_\beta$, *and*

(3°) $D = \sum_{\alpha<\Omega} D_\alpha$.

Proof.

(i) For any finite subset ν of D, there exists $\delta(\nu)$ such that $\delta \leqq \delta(\nu)$, for all $\delta \in \nu$. When the cardinality of D is countable, there exist ν_n $(n = 1, 2, \cdots)$ such that $\nu_1 < \cdots < \nu_n < \cdots$ and $D = \sum_{1 \leqq n < \infty} \nu_n$. If we add $\delta(\nu_1)$ to ν_1 and define this to be D_1, and add $\delta(\nu_{n+1} + D_n)$ to $\nu_{n+1} + D_n$ and define this to be D_{n+1} then each D_n is a directed subset of D of finite cardinality. In other words, (1°) holds. (2°) and (3°) are clear.

(ii) Next assume $\overline{\overline{D}} > \aleph_0$. For any $N \leqq D$,

$$F_1(N) = N + \{\delta(\nu); \nu \leqq N\}^{2\dagger}.$$

More generally, if we set $F_{n+1}(N) = F_1(F_n(N))$ then

$$F_1(N) \leqq F_2(N) \leqq \cdots .$$

Set $F_\omega(N) = \sum_{1 \leqq n < \infty} F_n(N)$. If a finite set ν satisfies $\nu \leqq F_\omega(N)$ then there exists n such that $\nu \leqq F_n(N)$ so $\delta(\nu) \in F_\omega(N)$. Thus $F_\omega(N)$ is a directed subset. Next, if N is a finite set then the cardinality of $\{\delta(\nu); \nu \leqq N\}$ is finite. If $\overline{\overline{N}} \geq \aleph_0$ then it is the same as $\overline{\overline{N}}$. Thus, in either case, $\overline{\overline{\{\delta(\nu); \nu \leqq N\}}} \leqq \overline{\overline{N}} \cdot \aleph_0{}^{\ddagger}$. Therefore $\overline{\overline{F_1(N)}} \leqq \overline{\overline{N}} \cdot \aleph_0$. In general $\overline{\overline{F_n(N)}} \leqq \overline{\overline{N}} \cdot \aleph_0$ so $\overline{\overline{F_\omega(N)}} \leqq \overline{\overline{N}} \cdot \aleph_0$. But $\overline{\overline{D}} > \aleph_0$ so

$$\overline{\overline{N}} < \overline{\overline{D}} \qquad \text{implies} \qquad \overline{\overline{F_\omega(N)}} < \overline{\overline{D}}. \tag{1}$$

By the well-ordering theorem, there is a transfinite sequence $\{N_\alpha; \alpha < \Omega\}$ of subsets of D such that

(1′) $\overline{\overline{N}}_\alpha < \overline{\overline{D}}$,

(2′) $\alpha < \beta < \Omega$ implies $N_\alpha < N_\beta$, and

(3′) $D = \sum_{\alpha<\Omega} N_\alpha$.

If we write $D_\alpha = F_\omega(N_\alpha)$ here then (1°) holds, by (1). And (2°) and (3°) are clear from (2′) and (3′). □

[1] [Iwa44a] [3]. See [Nak49] [2] p79.

[2] $\{\delta(\nu); \nu \leqq N\}$ denotes all $\delta(\nu)$ such that ν is a finite subset of N.

† Erroneous reference to footnote 1 in the original.

‡ Second ν was misprinted as δ in the original.

$(\beta)\to(\alpha)$: If we assume that (α) does not hold for some directed set then there exists one with the smallest cardinality $\aleph$ among such directed sets. Let this be D. As (α) holds for all finite directed sets, D is an infinite set. Thus, by the lemma, there is a transfinite sequence $\{D_\alpha; \alpha < \Omega\}$ such that $\overline{\overline{D}}_\alpha < \overline{\overline{D}}$, for each $\alpha < \Omega$. Therefore, by hypothesis, (α) holds for D_α so

$$\bigvee(a_\delta \frown b; \delta \in D_\alpha) = \bigvee(a_\delta; \delta \in D_\alpha) \frown b.$$

However, by (2°), if $\alpha < \beta < \Omega$ then $\bigvee(a_\delta; \delta \in D_\alpha) \leqq \bigvee(a_\delta; \delta \in D_\beta)$ so, by (β),

$$\bigvee_{\alpha<\Omega}\{\bigvee(a_\delta; \delta \in D_\alpha)\} \frown b = \bigvee_{\alpha<\Omega}\{\bigvee(a_\delta; \delta \in D_\alpha) \frown b\} = \bigvee_{\alpha<\Omega}\{\bigvee(a_\delta \frown b; \delta \in D_\alpha)\}.$$

Thus, by (3°), we see that (α) holds for D, a contradiction.

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