On doing category theory within set theoretic foundations

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Abstract

We give a quick survey of different approaches that have been proposed for modifying set theory so as to adequate for the purposes of category theory and its applications. We discuss modifications needed for categorical definitions that would allow one of them, ZFC with an elementary equivalent internal model, to be used in a way that allows all known applications.

1 Introduction

Ever since the introduction of category theory [2] it has been clear that the needs of category theory as applied in topology *etc.* cannot be accomodated within Zermelo-Frankel framework that is generally considered to be the foundational framework for all of mathematics. The responses to this have been various. One of the original proposals was to use the theory of sets and classes, as initiated by von Neuman [14] and subsequently expounded by Godel and Bernays. This is known as von Neuman-Godel-Bernays (vNGB) set theory. While this is probably adequate to prove the results needed to prove the theorems that can be stated without the language of categories, but proved using categorical machinary, it has two shortcoming. The less familiar one is stems from fact that quantification is allowed only over sets. In particular, mathematical induction cannot be applied to all statements. Allowing quantification over class variables gives us what is generally called Morse-Kelly (MK) set theory [9], [12].
The second shortcoming is shared by both vNGB and MK is that classes cannot be members of other classes. This means that large categories (categories that are proper classes) cannot be collected into categories, even if we are considering only finitely many of them. In algebraic geomtery in particular, this is felt to be a serious problem, though in topology this did not cause any misgivings, at least till recently. There have been a number of attempts to rectify this. We will discuss some of these in the next section. All of them suffer from one or more shortcomings. One of them, Zermelo-Frankel set theory with choice and the added assumption of a natural internal model V satisfying additonal properties is discussed in more detail in the third and fourth sections. This is closely related to the proposal of S. Feferman [4].

2 Various Approaches

One commonly used work-around was proposed by Grothendiek. This adds an additional axiom to ZFC, namely that every set is a member of an “universe” U which is closed under formation of elements, unions, power sets and replacement, and contains the smallest infinite ordinal. More precisely:

1. If $y \in x \in U$, then $y \in U$.
2. If $x \in U$, then $\{y \mid y \subset x\} \in U$.

3. If $x \in U$, then $\bigcup\{y \mid y \in x\} \in U$.
4. There is an $x \in U$ such that $\emptyset \in x$ and if $y \in x$, then $y \cup \{y\} \in x$.
5. If f is a function with range contained in U and domain an element of U, then the range of f is an element of U.

It is common to combine 3 and 5 into one: for f as in 5, require that $\bigcup\{y \mid \langle x, y\rangle \in f\} \in U$. We separate the two so that the role of the replacement axiom can be analyzed.
There are two objections to this approach. The first is that the addional axiom is much stronger than ZFC and its necessity is questionable. Second is that it does not really serve the purpose. It is customary to talk of the category of all topological spaces or all groups *etc.* In the above approach, we can only talk of category of spaces in a given universe. We then have to consider what happens if we take a different universe. Also, we cannot consider, for example, a homology theory as defined on all spaces as a functor, but must consider it as a collection of functors, one for each universe, which agree with each other. Apart from the complexity of the approach, it goes against how mathematicians actually view things.
This point is worth elaborating: Consider the Brown Representability Theorem. This is usually stated as follows: Every half-exact functor from the homotopy category of pointed connected topological spaces to the category of pointed sets that sends coproducts to products (upto isomorphism) is representable. The interpretation of this theorem depends on what we mean by a functor: We can stay within ZFC by taking a functor to mean a functional predicate from morphisms of the domain category to the codomain category, with categories themselves being defined by predicates. Alternatively, we can take a model of set theory and consider the categories of spaces and sets based on this model. These would be small categories in our meta theory and we can consider functors that exist in the meta theory. Then the truth of the statements depends on how big the model of set theory is in relation to the meta theory. Grothendieck universes are then really models (of MK system) which are assumed to be big enough for all functors that can be proved to exist in the meta theory. This seems to a stronger demand than necessary as functors that actually arise are built out of what is available in ZFC, without reference to the meta universe. Furthermore, mathemticians proceed as if the universe of mathematics is fixed for all time, and not subject to be change at whim as the use of multitudes of Grothendieck universes would suggest.
The latter objection can be met by using just one Grothendieck universe (see [11]). This fits in more with the next two approaches.
Systems in which classes can be members of other classes or sets have been proposed. One such is due to Ackermann [1]. This has the advantage of simplicity and a certain intuitive appeal. This turned out to be related to a system proposed by S. Feferman [4]. In the latter, we assume, in addition to the ZFC axioms, the existence of a set V that satisfies the reflection principle: For any formula ϕ that does not involve V and whose free variables are among $x_1, x_2, \ldots, x_n$,

$$\forall x_1 \in V\ \forall x_2 \in V\ \ldots \forall x_n \in V\ \phi(x_1, x_2, \ldots, x_n) \leftrightarrow \phi^V(x_1, x_2, \ldots, x_n)$$

where ϕ^V is the relativization of ϕ to V, obtained by replacing every occurance of $\exists x$ in ϕ by $\exists x \in V$, and every occurrance of $\forall x$ by $\forall x \in V$. A. Levy [10] showed that the reflection principle implies the replacement axiom schema in the presence of other ZF axioms, and that we get a model of Ackermann's theory by interpreting Ackermann's sets as elements of V and classes as general sets. W. Reinhardt [16] showed that with the added assumption of the axiom of regularity, Ackermann's sets satisfy the ZF axioms. So Ackermann's and Feferman's proposals are essentially equivalent. The main differences between them are what classes exist, and the philosophical stance we may adapt towards V. (See also [17], where a modification of Ackermann's system is proposed and is shown to imply ZF axioms for classes as well as sets, and also to be a conservative extention of ZF).

Remark. The author does not understand the statement by Mac Lane [11, *p.* 195] that such a V may not be a model of ZFC. It may not the class of all sets of a model of MK axioms, but if we assume choice for all sets, theorems of ZFC will get reflected down to V.

This is also related to the "one Grothendieck universe" approach, with the differences being the properties of V we assume. In the next two sections, we will discuss what properties of V are really needed for various categorical constructions.

A very different approach, a modified form of Morse-Kelly theory of sets and classes, was proposed by Obershlep [15]. Obershlep showed that it is equiconsistent with Morse-Kelly set theory. In this theory, we can treat classes as if they are ur-elements, provided that we do not treat them as classes at the same time. To be more precise, any statement should be stratifiable in such a way that variables of stratum 0 are all set variables, those of stratum 1 appear only on the left side of $\in$ and those of stratum 2 appear only on the right. In categorical terms this would mean that categories of large categories must be treated only as 2-categories and we cannot refer to an object of an object or similar combinations. This is adequate for most purposes.

The final alternative, considered as the ideal by category theorists, is to have an alternative foundations, couched in terms of category theory rather than the language of sets (and classes). From the point of view of applications to topology, at least, this faces large obstacles: Constructions of localizations in homotopical algebra make use of transfinite iteration. It is not clear how to formulate these in categorical terms. Convenient categories of equivariant spectra make use special features of topological spaces that do not seem likely to be shared by categories of toposes or locales in a general topos satisfying the axiom of choice. Solving these problems looks remote at the moment.

3 Approaches based on an internal model

As we saw in the discussion of the approach based on Grothendieck universes, approaches based only on one kind of sets run counter to the way mathematicians view familiar categories. It would be better to have two kinds of sets one of which sufficient for the purposes of mathematics that can formulated without using categorical language, and the other is necessary only when we consider collections of all X, for various concepts X. Relationship between the two is easier if the relation $\in$ is same for both kinds.

We will assume only the Zermelo axioms, that is we will not assume the replacement axiom, to start with. This is so that we can compare various approaches. Next we assume that we have a fixed set V that is closed uder the formation of elements, pairs, power sets and unions, and contains the least infinite ordinal, that is V satisfies

1. If $y \in x \in V$, then $y \in V$.
2. If $x, y \in V$, then $\{x, y\} \in V$.
3. If $x \in V$, then $\{y \mid y \subset x\} \in V$.
4. If $x \in V$, then $\bigcup\{y \mid y \in x\} \in V$.
5. There is an $x \in V$ such that $\emptyset \in x$ and if $y \in x$, then $y \cup \{y\} \in x$.

Note that V need not satisfy 5 in the definition of a Grothendieck universe. We will call such a V a Zermelo universe, and elements of V **small sets**. V is to be thought of as the universe of objects of interest in mathematics. Sets not in V are to be thought of as objects of a meta theory, used when we wish to take a global view as in the category of all topological spaces *etc.*

A set map from a set A to a set B is a function whose domain is A and whose range is contained in B. A set map is small if, considered as a set, it is in V. Note that this does not require B to be small, though the domain of a small map must be small.

Categories and functors are defined as usual, but without any requirement of smallness of any component part. For the abstract theory of categories, it is convenient to identify objects with their identity morhpisms. This means that a category C will consist of a set which also we will denote by C, two maps src_C and tgt_C from C to itself, and a map from a subset of $C \times C$ to C giving the composition. These are to satisfy the usual identities (see, for example, [5, *p.* 3]).

This identification of objects with their identities is cumbersome to do while defining specific categories. In such cases we will simply describe the objects and maps. This is to be converted to the formal form as in [5, *pp.* 7–8]: The category itself will consist of elements of the form $\langle\langle x, y\rangle, f\rangle$ where x and y are objects and f is a map from x to y. The structure maps are defined in the evident fashion.

Let V be a Zermelo universe. Define $\mathrm{Sets}(V)$ to be the category of sets in V: More precisely, $\mathrm{Sets}(V)$ is constructed as above, using elements of V as objects and set maps as maps

In the same way, we can define other familiar concrete categories such as $\mathrm{Top}(V)$ the category of topological spaces in V and continuous maps, or $\mathrm{Cat}(V)$ the category of V-small categories and functors.

Proposition 3.1. *If V is a Zermelo universe, then* $\mathrm{Sets}(V)$ *is a topos with a natural number object.*

Proof. The usual construction of finite products, equalizers and hom-sets in the category of sets makes use of only finite iterations of subsets, power sets, pair sets and elements. Hence $\mathrm{Sets}(V)$ is closed under these constructions. These together with the power set suffice to give the topos structure. □

The fact that we do not assume any version of the replacement axiom means that some of the usual definitions are not strong enough, and need to be modified. The modified definitions can usually be found by adapting the definitions used for internal categories or fibered categories.

Let C be a category whose underlying set is contained in V. Define a category $\mathrm{Fam}_V(C)$ fibered over $\mathrm{Sets}(V)$ as follows: Objects of $\mathrm{Fam}_V(C)$ are *small set maps* $f : X \to \mathrm{Obj}(C)$. A morphism from $f : X \to \mathrm{Obj}(C)$ to $g : Y \to \mathrm{Obj}(C)$ is a pair $\langle h, q\rangle$ where h is a set map from X to Y and q is a small set map from X to C, such that for all $x \in X$, $q(x)$ is a morphism (of C) from $f(x)$ to $g(h(x))$. Define a functor Π_C from $\mathrm{Fam}_V(C)$ to $\mathrm{Sets}(V)$ sending $\langle h, q\rangle$ to h.

Remark. The definition above differs from the usual definition, in which only $X \in V$ and $f \subset V$ are required. The usual definition is inappropriate for two reasons: First, the family fibration of a topos is defined in terms of the morphisms of the topos rather than maps that may exist in a metatheory in which we choose to interpret the morphisms of the topos. This corresponds to $f \in V$. Secondly, any non-trivial use of the usual definition requires us to assume 5 of the definition of Grothendieck universes so that we can conclude that $f \in V$. However, that assumption is not used in any other material way. Hence there is no loss in limiting families to $f \in V$.

The standard proof, *mutatis mutandis*, applies to give the next propostion.

Proposition 3.2. $\mathrm{Fam}_V(C) \to \mathrm{Sets}(V)$ *is a split fibration.*

Definition 3.3. A category C is said to be locally small if the underlying set of C is contained in V and every small set of morphisms of C is contained in a small full subcategory of C.

Note. We do not require full subcategories to be replete.

Example. $\mathrm{Sets}(V)$ is locally small: Recall the morphisms of $\mathrm{Sets}(V)$ are of the form $\langle x, y\rangle, f\rangle$ where f is a set map from x to y. If F is a small set of morphisms of $\mathrm{Sets}(V)$, then it is easily verified that $\{\langle x, y\rangle, f\rangle \mid \exists z, g\langle x, z\rangle, g\rangle \in F \wedge \exists w, h\langle w, y\rangle, h\rangle \in F\}$ is a small full subcategory of $\mathrm{Sets}(V)$ that contains F.

We can verify that familiar concrete categories are locally small by using the method outlined below Definition 4.2.
The usual argument (see, for example, [8, p. 272]) proves the following proposition.

Proposition 3.4. *If C is locally small, then* $\mathrm{Fam}_V(C) \to \mathrm{Sets}(V)$ *is a locally small fibration.*

We say that a category has chosen equalizers if there is a function Eq that assigns to each parallel pair f, g of morphisms, an equalizer diagram $\bullet \xrightarrow{\mathrm{Eq}(f,g)} \bullet \rightrightarrows \bullet$ for f and g. We define the property of having chosen products *etc.* in a similar fashion.

Proposition 3.5. *Suppose that*

1. *Given any small set A of morphisms of C, the set of domains of A and the set of codomains of A are both small.*
2. *C has chosen equalizers of parallel pairs of maps such that if A is a small set of parallel pairs, then* $\{\mathrm{Eq}(f,g) \mid \langle f,g\rangle \in A\}$ *is small.*
3. *Suppose that C has chosen products of small sets of objects such that* $\{\mathrm{Prod}_{i\in I_j} x_i \mid j \in J\}$ *is small if* $\{\langle i, x_i\rangle \mid \exists j \in J : i \in I_j\}$ *is small.*

Then $\mathrm{Fam}_V(C) \to \mathrm{Sets}(V)$ *is a complete fibration.*

Proof. First we show that $\mathrm{Fam}_V(C)$ has equalizers and that they are preserved by the projection to $\mathrm{Sets}(V)$: Let $F : I \to \mathrm{Obj}(C)$ and $G : J \to \mathrm{Obj}(C)$ be two objects of $\mathrm{Fam}_V(C)$ and $\langle h,p\rangle$ and $\langle k,q\rangle$ two morphisms from F to G. Let $K = \{i \in I \mid h(i) = k(i)\}$. For any $i \in K$, $p(i)$ and $q(i)$ form a parallel pair, because they go from $F(i)$ to $G(h(i)) = G(k(i))$ Let $E(i)$ be the domain of their equalizer $\mathrm{Eq}(p(i), q(i))$. By (1) and (2), E is a small function and so it is an object of $\mathrm{Fam}_V(C)$. By (2), $r : K \to C$ given by $r(i) = \mathrm{Eq}(p(i), q(i))$ is small. It follows that $\langle K, r\rangle$ is in $\mathrm{Fam}_V(C)$. It is easy to verify that it is the equalizer of $\langle h,p\rangle$ and $\langle k,q\rangle$. Thus $\mathrm{Fam}_V(C)$ has equalziers and the projection Π_C to $\mathrm{Sets}(V)$ preserves them.
In a similar fashion, we can show that $\mathrm{Fam}_V(C)$ has finite products and that ΠC preserves them. To complete the proof, all we need to do is to show that for any $f : I \to J$ in $\mathrm{Sets}(V)$, there is a functor $\mathrm{Prod}_f : \Pi_C^{-1}(I) \to \Pi_C^{-1}(J)$ and that these satisfy the Beck-Chevelly conditions. The usual proof still works: Prod_f is defined on objects as follows: If $G : I \to \mathrm{Obj}(C)$ is a small map, $\mathrm{Prod}_f(G)(j)$ is the product of $\{G(i) \mid f(i) = j\}$. As $\{\langle i, G(i)\rangle \mid \exists j \in J : i \in f^{-1}(j)\} = G$ is small, (3) implies that $\mathrm{Prod}_f(G)$ is a small family. □

The above propositions lend strong support to the thesis that the assumption of an internal Zermelo universe is sufficient for theorems of category theory that are formulated in the first order theory of elementary categories. To carry this out effectively, the following concepts will prove useful.

Definition 3.6. A category C is called slowly growing if the underlying set of C is contained in V and every small set of morphisms of C is contained in a small subcategory of C.

Definition 3.7. A set map F from a subset of V to a subset of V is called slowly growing if for any small subset A of the domain of F, $F(A)$ is small.

Thus we can talk of slowly growing functors or slowly growing natural transformations.

Definition 3.8. Let C be a slowly growing category. The category of small diagrams in C, $\mathrm{Dgrm}(C)$ is defined as follows: Objects are $\langle X, F\rangle$ where X is a small category and F is functor from X to C whose graph is small. A morphism from $\langle X, F\rangle$ to $\langle Y, G\rangle$ is a pair $\langle A, \alpha\rangle$ where A is a functor from Y to X and α is a natural transformation from $F \circ A$ to G such that $\{\alpha_x \mid x \in X\} \in V$.

Note that $\mathrm{Dgrm}(C)$ comes with a forgetful functor to $\mathrm{Cat}^{\mathrm{op}}(C)$.

Proposition 3.9. *If C is slowly growing, then* $\mathrm{Dgrm}(C)$ *is slowly growing.*

Proof. Given a small set P of morphisms of $\mathrm{Dgrm}(C)$, the set Q of all $\langle X, F\rangle$ which are either the domain or codomain of a morphism in P is also small. It follows that $\bigcup\{F(X) \mid \langle X, f\rangle \in P\}$ is small. This is contained in a small subcategory D of C, because C is slowly growing. Now consider the subcategory of $\mathrm{Dgrm}(C)$ whose objects are diagrams in Q, and whose morphisms are the $\langle A, \alpha\rangle$ with $\alpha(X) \subset D$. This is small and clearly contains P. □

Define the "big diagonal functor" Δ from $C \times \mathrm{Cat}^{\mathrm{op}}(V)$ to $\mathrm{Dgrm}(C)$ as follows: For $c \in C$ and a small category X, $\Delta(c, X)$ is the constant functor from X to C that maps all objects of X to c. Given $f : c \to d$ in C and $F : Y \to X$, $\Delta(f, F)$ corresponds to $< Y, \tilde{f} >$ where $\tilde{f}$ is the constant natural transformation with value f.

Proposition 3.10. *If C is a slowly growing category, then $\Delta : C \times \mathrm{Cat}^{op}(V) \to$ $\mathrm{Dgrm}(C)$ is a slowly growing functor.*

Proof. Given a small set of P morphisms of $C \times \mathrm{Cat}^{\mathrm{op}}(V)$, take a small subcategory D of C that contains all the first coordinates of elements of P. Let Q be the set of the second coordinates of the elements of P. Note that Q is small. The subcategory of $\mathrm{Dgrm}(C)$ constructed from D and Q as in the proof of Proposition 3.9 is small and contains $\Delta(P)$. □

Definition 3.11. We say that a slowly growing category C has slowly growing limits if Δ has an adjoint that is slowly growing and commutes with the projections to $\mathrm{Cat}^{\mathrm{op}}(C)$.

It is easy to verify that a category with slowly growing limits satisfies the hypotheses of Proposition 3.5. Conversely, a slowly growing category that satisfies (2) and (3) of Proposition 3.5 has slowly growing limits. This can be proved by the same technique as in the proof of Theorem 4.5 below. Then we can show that the $\mathrm{Sets}(V)$, the category of topological spaces in V, *etc.* have slowly growing limits.
Now it is an easy if tedious task to verify that the theorems of basic category theory remain valid for slowly growing categories if we replace "complete" by "has slowly grwoing limits", "cocomplete" by "has slowly growing colimits" (defined by dualizing the above) and define "locally small" as in Definition 3.3. In most cases, this can accomplished by simply noting that the result follows from the appropriate formulation for fibered categories.

4 Additional Axioms

While the above is sufficient for many purposes, it is inadequate for applied category theory. Several constructions, such as constructing Postnikov stages by killing homotopy groups, construction of spectra representing (co)homology theories *etc.* proceed inductively and the final step is to take the (co)limit over natural numbers. That requires that the infinite diagram be small, and thus some version of the replacement axiom. Bousfield localization, in sufficient generality, requires such a construction for uncountable ordinals (this is easily seen in the proof presented in [3]). Thus we are led to consider the following forms of the replacement axiom as additional requirrements on V.
Weak Replacement: If $\phi(x, y)$ is a predicate that does not involve V and $\phi(x, y_1) \wedge \phi(x, y_2) \to y_1 = y_2$, then for any $z \in V$, $\{y \mid \exists x \in z \wedge \phi(x, y)\} \in V$.
Strong Replacement: Suppose that $F \subset V \times V$, and whenever $(x, y_1) \in F$ and $(x, y_2) \in F$, $y_1 = y_2$. Then for any $z \in V$, $\{y \mid \exists x \in z \wedge (x, y) \in F\} \in V$.

The requirement that V satisfy *Weak Replacement* amounts to requiring that V is a natural model of Zermelo-Fraenkel axioms. The proposal of S. Feferman [4] would give us just this form of replacement. On the other hand, *Strong Replacement* says that V is a Grothendieck universe. The desire for this seems to stem the fact that *Weak Replacement* seems to limit us to predicatively definable categories and functors, this is felt to be inadequate. We will argue that this is not so.
One source of impredicativity in applied category theory is how definitions are phrased. For example, the definition of Quillen model categories requires the existence of factorizations with certain properties. Constructions based on arbitarily choosing one such factoriation for each morphism would give us impredicative constructions. However, proofs of the existence of such factorizations in case of categories that actually arise in practice are often predicative. Hence it can be argued that all we need to do is to change the definitions so that existence over a large category is interpreted to mean having chosen representatives that are predicatively definable.
Unfortunately, in a few cases, proofs of existence make use of the axiom of choice. However, this is used in a limited way, by replacing a predicatively definable relation by a functional subset with the same domain, where the domain may be a large subset of V. We need a mechanism for dealing with these without invoking *Strong Replacement*.
For the rest of this section, we will assume that V satisfies *Weak Replacement*. Let $\mathrm{Ord}(V)$ denote the set of all ordinals of V.

Definition 4.1. A scale on a set X consists of a sequence X_α for $\alpha \in \mathrm{Ord}(V)$ such that each X_α is small, $X_\beta \subset X_\alpha$ if $\beta < \alpha$, $X_\alpha = \bigcup\{X_\beta \mid \beta < \alpha\}$ if α is a limit ordinal, $X = \bigcup X_\alpha$, and any small subset of X is contained in some X_α.

If C is a category, we will further require that each C_α to be a subcategory.
We recall the notion of rank of sets. Define V_α for ordinals α by transfinite induction as follows:

1. $V_0 = \emptyset$;
2. $V_{\alpha+1} = V_\alpha \cup \mathrm{PowerSet}(V_\alpha)$;
3. If α is a limit ordinal, then $V_\alpha = \bigcup\{V_\beta \mid \beta < \alpha\}$.

By transfinite induction we can show that $V_\alpha \subset V_\beta$ if $\alpha < \beta$. A set has rank α if it is in V_α, but not in V_β for $\beta < \alpha$.
It follows from the axiom of regularity that $V = \bigcup\{V_\beta \mid |\beta \in \mathrm{Ord}(V)\}$.

Remark. Readers unfamiliar with the notion are cautioned against trying to interpret rank in terms of cardinality. For example, $V_{\omega+\omega}$ has sets of cardinality $2^{\aleph_0}$, $2^{2^{\aleph_0}}$, …, but does not contain those cardinals themselves.

Remark. Zermelo universes are precisely sets of the form V_α, where α is a limit ordinal.

In case of functor categories, diagram categories, as well as familiar concrete categories have scales that make them uniformly slowly growing. Let C be a category with scale C_α. If A is a fixed small category, we will use $(C_\alpha)^A$ as the scale on C^A, the category of small functors from A to C. Give $\mathrm{Fam}_V(C)$ the scale $\{f : I \to \mathrm{Obj}(C_\alpha)) \mid \alpha > \text{rank of } I\}$
For the diagram category, let D_α consist of the full subcategory of $\mathrm{Dgrm}(C_\alpha)$ with objects all functors whose domain categories have rank less than α. Then D_α is a scale for $\mathrm{Dgrm}(C)$. If the C_α's are full subcategories, so are the D_α's.
A concrete category is usually defined as a category C with an *underlying set* functor U to “the” category of sets that is faithful, *i.e.*, 1-1 on morphisms with same domain and same codomain. This fails to capture the intuition that concrete categories can be defined in set-theoretic language. Making this precise does not

seem to be worth the trouble for our purposes. So we will impose a less explicit comdition which will need to be verified for each category individually.

Definition 4.2. A concrete category consists of a category C with a faithful functor U to $\mathrm{Sets}(V)$ such that for any set of morphisms A of C, $U(A)$ is small if and only if A is.

This applies to all familiar concrete categories: In all of them a morphism can be defined as consisting of set map with additional requirements on the domain, codomain and the function. This is sufficient to show that if A is a set, then so is $U(A)$. To prove the converse, we first observe that there is a predicate $obj(c, x)$ in $\in$-language that does not contain C such that $\{c \mid c \text{ is an object of } C \text{ and } Uc = x\} = \{c \mid obj(c, x)\}$. Now apply *Weak Replacement*.
To allow for such categories as modules over a fixed ring, or spaces with the action of a fixed group, we allow parameters in the predicate *obj* provided that C is not referred to and C corresponds to fixed values of the parameters. Note that such things as the category of all modules M over all rings R can be handled by taking the underlying set to be $M \coprod R = \{0\} \times M \cup \{1\} \times R$.
If C is a concrete category C with underlying set functor U, let $C_\alpha = \{f \mid Uf : a \to b, \operatorname{rank} a, \operatorname{rank} b < \alpha\}$ These are evidently full subcategories of C. This will be a scale because of the assumptions on U.

Definition 4.3. A collection F_i of functions from X_i to Y_i, with all X_i and Y_i having scales, is called uniformly slowly growing if there is an ordinal valued function $\phi(u_1, \ldots, u_n, \alpha)$ of ordinal α, definable in $\in$-language and depending on parameters u_i, and there exist small sets $a_1, \ldots a_n$ such that $F_i(X_{i,\alpha})$ is contained in $Y_{i,\beta}$ where $\beta = \phi(a_1, \ldots, a_n, \alpha)$.

Definable in $\in$-language means that the function can be defined using only $\in$ and $=$. For functions that make use of defined notions, this means that the definition of ϕ must not make use of V.
To simplify the notation, we will indicate only one parameter in such functions ϕ. This can always be arranged using n-tuples.
If the above condition holds, we say that ϕ dominates the growth rates of F_i.
Given a finite number of uniformly slowly growing collections of functions, we can find a single function $\phi(u, \alpha)$ that dominates the growth rates of the union of those collections: Rename the parameters if necessary to ensure that they are all distinct, form the tuple of all of them and then take the maximum of the dominating functions of the individual collections.
A uniformly slowly growing adjunction $F \vdash G$ consists of adjoint functors F and G such that the collection consisting of F, G, and the natural transformations $1 \to FG$ and $GF \to 1$ giving the adjunction is uniformly slowly growing.

Definition 4.4. Let C be a category with a scale and let $\Delta : C \times \mathrm{Cat}^{\mathrm{op}}(V) \to \mathrm{Dgrm}(C)$ be the big diagonal. We say that C has uniformly slowly growing limits if there is a uniformly slowly growing adjunction $\lim \vdash \Delta$.

We will denote the category $\bullet \rightrightarrows \bullet$ by $\mathbf{2}$.

Theorem 4.5. *Let C be a category with scale C_α. Suppose that there are uniformaly slowly growing adjunctions $\mathrm{Eq} \vdash \Delta_p$ and $\mathrm{Prod} \vdash \Delta_F$ to the constant digram functors $\Delta_p : C \to C^{\mathbf{2}}$ and $\Delta_F : C \to \mathrm{Fam}_V(C)$. Then C has uniformly slowly growing limits.*

Proof. The assumption implies that there is a predicatively defined function $\phi(u, \alpha)$ and a small set a such that

1. $\phi(u, \alpha) \geq \alpha$.

2. If f, $g \in C_\alpha$ is a parallel pair of morphims, then $\mathrm{Eq}(f,g)$ is an equalizer of f and g, and the map to the common domain of f and g is contained in $C_{\phi(a,\alpha)}$. Further, if $h \in C_\alpha$ and $fh = gh$, then h factors via $\mathrm{Eq}(f,g)$ in $C_{\phi(a,\alpha)}$. [The last point is crucial.]

3. If x is a function whose range is contained $\mathrm{Obj}(C_\alpha)$ and the domain I of x has rank at most α, then $\mathrm{Prod}(x)$ is the product of $x(i)$, and the projections are in $C_{\phi(a,\alpha)}$. Further, if c is an object of C_α and we are given morphisms $f_i : c \to x(i)$ in C_α, then there is a morphism $\langle f_i \rangle : c \to \mathrm{Prod}(x)$ in $C_{\phi(a,\alpha)}$ such that f_i equals $\langle f_i \rangle$ followed by the projection to $x(i)$.

Consider a small diagram F in D_α with domain category A. Thus A has rank less than α and $F(A)$ is contained in C_α. Let $\tilde{P}$ be F on objects considered as an object of $\mathrm{Fam}_V(C)$, and let $\tilde{Q}$ be the set map from the underlying set of A to $\mathrm{Obj}(C)$ that sends $h : a \to b$ to Fb. Our assumption on Prod and ϕ implies that $P = \mathrm{Prod}(\tilde{P})$ and $Q = \mathrm{Prod}(\tilde{Q})$, as well as the projections $p_a : P \to Fa$, $q_h : Q \to Fb$ are in $C_{\phi(a,\alpha)}$. It follows that there are morphisms s and t from P to Q in $C_{\phi(a,\phi(a,\alpha))}$ such that $h : a \to b$ is in A, then $q_h s = p_b$ and $q_h t = (Fh)p_a$. Then $\mathrm{Eq}(s,t)$ is in $C_{\phi(a,\phi(a,\phi(a,\alpha)))}$ and the usual construction of limits from products and equalizers shows that $\mathrm{Eq}(s,t)$ is the limit of F with structure maps $\mathrm{Eq}(s,t) \to P \xrightarrow{p_a} Fa$ which are also in $C_{\phi(a,\phi(a,\phi(a,\alpha)))}$. This gives us lim that is adjoint to the big diagonal and carries D_α into $C_{\phi(a,\phi(a,\phi(a,\alpha)))}$. □

The above proposition easily implies that familiar concrete categories that are complete in fact have uniformly slowly growing limits. Let us take the category of topological spaces as an example. Equalizers can be constructed as subspaces so do not even raise the rank. Products are constructed from cartesian products and hence raise the rank only by a finite ordinal.

Theorem 4.6. *Let C be a category with a scale and uniformly slowly growing limits. Suppose that F is a uniformly slowly growing functor from C to itself and θ is a uniformly slowly growing natural transformation from F to 1_C. Then there are functors F^α for each small ordinal α and natural transformations θ^α_β from F^α to F^β for $\beta < \alpha$ such that*

1. $F^0 = 1_C$;
2. $F^{\alpha+1} = F \circ F^\alpha$;
3. *If α is a limit ordinal, then* $F^\alpha = \lim_{\beta<\alpha} F^\beta$;
4. $\theta^{\alpha+1}_\alpha = \theta \circ F^\alpha$;
5. *If $\gamma < \beta < \alpha$, then* $\theta^\alpha_\beta \theta^\beta_\gamma = \theta^\alpha_\gamma$.
6. *For any small ordinal α, $\{F^\beta \mid \beta < \alpha\}$ and $\{\theta^\beta_\gamma \mid \gamma < \beta < \alpha\}$ are uniformly slowly growing.*

Proof. Without loss of generality, we may assume that there is a predicatively definable function $\phi(u,\alpha)$ and a small set a such that $\alpha \le \phi(u,\alpha)$, $F(C_\alpha)$, $\{\theta(c) \mid c \in \mathrm{Obj}(C_\alpha)\}$ and limits of functors from a category of rank less than α with image contained in C_α are all contained in $C_{\phi(a,\alpha)}$. Define $\tilde{\phi}(u,\lambda,\alpha)$ by

1. $\tilde{\phi}(u,0,\lambda) = \lambda$;
2. $\tilde{\phi}(u,\alpha+1,\lambda) = \phi(u,\tilde{\phi}(u,\alpha,\lambda)$;
3. if α is a limit ordinal, then $\tilde{\phi}(u,\alpha,\lambda) = \tilde{\phi}(u,\max\{\tilde{\phi}(u,\beta,\lambda) \mid \beta < \alpha\})$.

Note that $\tilde{\phi}$ is predicatively definable.
Define F^α by transfinite induction using 1–3 of the proposition and θ^α_β by

1. $\theta^{\alpha+1}_\beta = (\theta \circ F^\alpha)\theta^\alpha_\beta$;

2. if α is a limit ordinal, then θ^α_β is $F^\alpha = \lim_{\beta<\alpha} F^\beta \to F^\beta$.

That the limit in 3 is small follows from the observation that $F^\alpha(C_\lambda)$ and $\theta^\alpha_\beta(\mathrm{Obj}(C_\lambda))$ are contained in $C_{\tilde{\phi}(a,\alpha,\lambda)}$: This is trivial if α is a successor ordinal. If it is a limit ordinal, the induction hypothesis implies that, when restricted to C_λ, limit diagram in 3 of the proposition is contained in C_γ where γ is $\max\{\tilde{\phi}(a,\beta,\lambda) \mid \beta < \alpha\}$. Hence that diagram is small and we can take the limit. Our assumption on ϕ then implies that the limit is in $C_{\tilde{\phi}(a,\alpha,\lambda)}$. □

Dualizing the above, we have categories with uniformly slowly growing colimits and transfinite composition of uniformly slowly growing functors F with uniformly slowly growing natural transformations $1 \to F$.
Finally, we show that predicatively definable relations contain uniformly slowly growing functions with the same domain.

Definition 4.7. A scale X_α on X is commensurate with rank if there are functions $\phi_1(u,\alpha)$ and $\phi_2(u,\alpha)$ definable in the $\in$-language and small a, b such that every element of X_α has rank less than $\phi_1(a,\alpha)$ and every element of X of rank less than α is in $X_{\phi_2(b,\alpha)}$.

Proposition 4.8. *Let X and Y be subsets of V, with scales that are commensurate with rank. Let $\theta(u_1,\ldots,u_n,w,x)$ and $\psi(u_1,\ldots,u_n,w,x,y)$ be predicates in ϵ-language with free variables u_1, $\ldots u_n$ and not involving X, Y or V such that*

$$\theta(u_1,\ldots,u_n,w,x) \leftrightarrow x \in X$$
$$(\theta(u_1,\ldots,u_n,w,x) \wedge \psi(u_1,\ldots,u_n,w,x,y)) \to y \in Y$$
$$\forall x \theta(u_1,\ldots,u_n,w,x) \to \exists y : \psi(u_1,\ldots,u_n,w,x,y)$$

are provable. Then for all small u_1, $\ldots u_n$ and small W, there exist set maps $F_w : X \to Y$ for $w \in W$ such that $\psi(u_1,\ldots,u_n,w,x,F_w(x))$ holds for all $x \in X$ and $w \in W$, and the collection $\{F_w \mid w \in W\}$ is uniformly slowly growing.

Proof. Fix small u_1, $\ldots$, u_n and W. For simplicity of notation, we will omit u_i's. By our assumptions, there are functions $\phi_1(u,\alpha)$ and $\phi_2(u,\alpha)$ definable in $\in$-language and small a, b such that every element of X_α has rank less than $\phi_1(a,\alpha)$ and every element of Y of rank less than α is in $Y_{\phi_2(b,\alpha)}$.
Let $G(z,x) = \{y \mid \psi(z,x,y) \wedge (\psi(z,x,v) \to (\mathrm{rank}(y) \le \mathrm{rank}(v)))\}$. For an ordinal α, define $\phi(Z,\alpha)$ to be the rank of $\bigcup\{G(z,x) \mid z \in Z \wedge (\mathrm{rank}(x) < \alpha) \wedge \theta(z,x)\}$. Fix a choice function for the non-empty subsets of V and let $F_w(x)$ be the chosen element of $G(w,x)$. If $x \in X_\alpha$, the rank of x is less than $\phi_1(a,\alpha)$, and so the rank of $F_w(x)$ is less than $\phi(W,\phi_1(a,\alpha))$. Our assumption then implies that $F_w(x) \in Y_\beta$ where $\beta = \phi_2(b,\phi(W,\phi_1(a,\alpha)))$. □

We will use Bousfield localization as an example of how these ideas can be used to avoid invoking *Strong Replacement*. First we need to modify the definition of Quillen model category to assume that we have chosen factorizations given by uniformly slowly growing functions. Other existence hypotheses in definitions and heorems need to be modified similarly. Then the standard construction of Bousfield localization L_f in combinatorial model categories is a (small) transfinite iterate of a uniformly slowly growing functor. So L_f is iteself uniformly slowly growing, and hence so is $\{L^n_f\}$. Thus the cosimplicial space $L^*_f X$ and its homotopy spectral sequence exist as small objects. Then we can check that this is adequate for known uses of Bousfield localizations to study homotopy theory of spaces.
Other constructions involving Quillen model categories (see [7] or [6]) or trianglulated categories ([13]) can be dealt in a similar way.

5 Concluding Remarks

The usual way in which category theory is used by working mathematicians leads to two desiderata for a foundation of mathematics. First, we should be able to talk about such categories as "the category of sets". This can be done using a "small" *vs*"large" distinction. Secondly, we should be able to use familiar constructions even on large categories.

If the only constructions we use on large categories are those that can be formulated categorically and proved in any (well-pointed or two-valued or satisfying the axiom of choice or ...) topos, then we just need to assume that the universe of large sets satifies Zermelo axioms. If we wish to make use of the full extent of impredicative functors, we need to assume that the universe of small sets is a Grothendieck universe, that is it satisfies the Zermelo axioms, as well as *Strong Replacement*. This is the idea behind the claim that "one universe is enough", as elaborated in [11]. On the other hand, there is good evidence that the functors actually encountered in practice can be accommadated using just *Weak Replacement*.

However, there is a potential problem with the one universe approach. If we prove a theorem about small categories, we would like to able to apply it to large categories without worrying about what axioms were used in the proof. For example, there have been proposals for constructing a "homotopy theory of homotopy theories", that is a model category structure on the category of all models of a suitable abstract homotopy theory. We would like to be able to apply theorems about model categories to this and derive consequences concerning concrete homotopy categories. This is not possible with the one universe approach.

This seems to be the main reason for Grothendieck's axiom, namely the assumption that the union of all Grothendieck universes contain all sets. A more natural approach would be the reflection principle. This allows us to transfer results about small categories to large categories more transparently. Note that Levy's results mentioned in the introduction tell us that the resulting system is not stronger than ZFC.

Another potential reason for assuming many universes is to be able to define basic notions as functors on large categories. We can assume an arbitarily long but finite chain of reflecting subuniveres and still have a conservative extention of ZFC ([10]). In practice even this may not be necessary: The large category used can be replaced by the corresponding category based on a Zermelo universe large enough to contain sufficiently many examples of the relevant types. For example, some structures are defined as functors on the category of *all* finite sets. We can instead define them as functors on Sets(V_ω), the category of *hereditarily finite* sets, without losing any of the technical advantages of using all finite sets. Categories of functors on all rings can be replaced by functors defined on the category of rings in V_α, where α is a limit ordinal greater than the rank of polynomial rings on an infinite number of variables over the algebraic closure of $\widehat{Q}_{(p)}((x_0, x_1, \dots))$.

It is worthwhile to recall that if ϕ is any sentence provable in ZFC, there exist a cofinal collection of Zermelo universes U contained in V such that the relativization of ϕ to U is true.

Thus, the results of this paper can be taken to be further evidence that ZFC axioms plus the existence of a set V satisfying the reflection principle is adequate for the current needs.

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